B. Sc. Mathematics (Hons.) Examination, 2023

(2nd Year, 1st Semester)

Theory of Real Function

Paper – Core 5

Time : 2 hours Full Marks : 40

Use separate Answer-script for each part.

(Symbols have usual meanings, if not mentioned otherwise)

Part – I (20 Marks)

Answer ***any four*** questions. 5×4=20

1. a) Let $I=(a,b)$ be a bounded open interval and $f:I\to\mathbb{R}$ be a monotone increasing function on *I*. If f is bounded above on *I*, then show that $\lim\limits_{x\to b-} f(x)=\sup\limits_{x\in(a,b)} f(x)$. 3

   b) Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ which is continuous exactly at two points. 2

2. a) Let *I* be an interval and $f:I\to\mathbb{R}$ be a non-constant continuous function on *I*. Prove that $f(I)$ is an interval. 3

   b) A real function f is continuous on [0, 2] and $f(0)=f(2)$. Prove that there exists at least a point *c* in [0, 1] such that $f(c)=f(c+1)$. 2

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3. a) Let $[a,b]$ be a closed and bounded interval and $f:[a,b]\to\mathbb{R}$ be continuous and injective on $[a,b]$. Prove that f is strictly monotone on $[a,b]$. 3

   b) Prove or disprove: If $f:\mathbb{R}\to\mathbb{R}$ is a continuous function such that $|f(x)-f(y)|\geq\frac{1}{2}|x-y|$ for all $x,y\in\mathbb{R}$, then f is both one-to-one and onto. 2

4. a) Let $D\subseteq\mathbb{R}$ and $f:D\to\mathbb{R}$ be uniformly continuous on D. If $\{x_n\}$ be a Cauchy sequence in D, then prove that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. 3

   b) Prove that the function $f(x)=\sin\frac{1}{x}$, $x\in(0,\,1)$ is not uniformly continuous on (0, 1). 2

5. Let $f:[0,\infty)\to\mathbb{R}$ be continuous on $[0,\infty)$ and $\lim\limits_{x\to\infty}f(x)=0$. Prove that f is uniformly continuous on $[0,\infty)$. 5

6. a) Let $D\subseteq\mathbb{R}$ be a compact set and a function $f:D\to\mathbb{R}$ be continuous on D. Prove that $f(D)$ is a compact set in $\mathbb{R}$. 3

   b) Find the points of discontinuity of the function $f(x)=[\sin x]$, $x\in\mathbb{R}$, where $[x]$ denotes the greatest integer not greater than x. 2

Part – II (20 Marks)

Answer ***any four*** questions. 4×5=20

1. A function f is defined in (–1, 1) by

$$f(x)=\begin{cases} x^p\sin\left(\dfrac{1}{x^q}\right), & \text{when } x\neq 0\\ 0 \quad , & \text{when } x=0\end{cases}$$

   Determine the conditions of p and q when f' is continuous and discontinuous at $x=0$.

2. If for a function $f:\mathbb{R}\to\mathbb{R}$, $f(x+y)=f(x)+f(y)$, $\forall x,y\in\mathbb{R}$ and f is derivable at some point $a\in\mathbb{R}$, then prove that f is derivable on $\mathbb{R}$.

3. State Rolle's theorem for polynomials and give also its geometrical interpretation. Use this theorem to show that the polynomial equation

   $c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0=0$ has at least one root between 0 and 1, if $c_0+\dfrac{c_1}{2}+\dfrac{c_2}{3}+\cdots+\dfrac{c_n}{n+1}=0$.

4. Using Mean value theorem to prove that if $\phi(x)=F\{f(x)\}$, $\phi'(x)=f'(x)\cdot F'\{f(x)\}$ assuming the derivatives to be continuous.

5. If $\phi(x)=f(x)+f(1-x)$, $x\in[0,1]$ and $f''(x)<0$ for all $x\in[0,1]$, show that ϕ is increasing on $[0,\frac{1}{2}]$ and decreasing on $[\frac{1}{2},\,1]$.

6. Expand $\log(1+x)$ in power of x, as an infinite series and mention the region for validity of expansion.