Ex/SC/MATH/UG/CORE/TH/03/2023

B. Sc. Mathematics (Hons.) Examination, 2023

(1st Year, 2nd Semester)

Real Analysis

Paper – Core-03

Time : Two hours Full Marks : 40

Use separate Answer script for each Part.

Symbols / Notations have their usual meanings.

Part – I (Marks: 24)

Answer ***any six*** questions.

1. Show that cardinality of $\mathbb{X}$ is strictly less than cardinality of $\mathcal{P}(\mathbb{X})$, where $\mathbb{X}$ is a set. 4
2. Show that $[a,b]$, $a<b$ and $[c,d]$, $c<d$ have the same cardinality. 4
3. Let F be an Archimedean ordered field. Show that if F satisfies least upper bound property then F satisfies Dedekind Completeness property. 4
4. State Archimedean property of real numbers. Using it prove the following: "Given any two real numbers a and b with $a<b$, there exists a rational number q such that $a<q<b$."
5. Show that
   i) the set of natural numbers $\mathbb{N}$ is closed in $\mathbb{R}$.
   ii) the set of rational numbers $\mathbb{Q}$ is neither closed nor open in $\mathbb{R}$. 2+2

6. Prove that in $\mathbb{R}$, finite union of closed sets is closed. Give an example to show that arbitrary union of closed sets may not be closed. 2+2

7. Let K_1 be a compact set in $\mathbb{R}$. If $K_n\,(n \geq 2)$ is a nested sequence of non-empty closed sets in $\mathbb{R}$ with

$$K_1 \supset K_2 \supset K_3 \supset \ldots$$

then show that

$$\bigcap_{n=1}^{\infty} K_n \neq \phi.$$ 4

8. Give an example of a closed and bounded set in $\mathbb{Q}$ which is not compact. Can you find such a set in $\mathbb{R}$? 3+1

9. If a set $\mathbb{K} \subset \mathbb{R}$ is compact in $\mathbb{R}$ then show that every sequence of elements of $\mathbb{K}$ has a subsequence converging to some element in $\mathbb{K}$. Hence show that $\mathbb{Q}$ is not compact in $\mathbb{R}$. 3+1

Part – II (Marks: 16)

Answer ***any four*** questions. 4×4=16

1. i) Prove that a bounded sequence $\{x_n\}$ is convergent if, and only if, $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$. 3

   ii) Prove/Disprove: If a sequence $\{x_n\}$ of real numbers is unbounded, then sequence $\{1/x_n\}$ is bounded. 1

2. i) If a sequence $\{x_n\}$ converges to l, then show that every subsequence of $\{x_n\}$ also converges to l. 2

   ii) Show by an example that if two subsequences of a sequence converge to the same limit l, then the sequence $\{x_n\}$ may not be convergent. 2

3. i) Prove that the sequence $\{x_n\}$, where $x_1 = 0$, $x_2 = 1$ and $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$ for all $n \geq 1$ is a Cauchy sequence. 3

   ii) Show that the sequence $\{(-1)^n\}$ is not a Cauchy sequence. 1

4. Discuss the convergence of the series

   i) $1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5} + \frac{1}{2}\cdot\frac{1}{3} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7} + \cdots,$.

   ii) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, $p > 0$ 2+2

5. i) If $\sum_{n=1}^{\infty} u_n$ be a convergent series of positive real numbers, then prove that the series $\sum_{n=1}^{\infty} u_n^2$ is convergent. 1

   ii) Test the convergence of the series

$$1 + \frac{x}{1!} + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \cdots,\ x > 0.$$ 3

6. Prove that the series

$$1 + \frac{\alpha^2}{1\cdot\beta} + \frac{\alpha^2(\alpha+1)^2}{1\cdot 2\cdot\beta(\beta+1)} + \frac{\alpha^2(\alpha+1)^2(\alpha+2)^2}{1\cdot 2\cdot 3\cdot\beta(\beta+1)(\beta+2)} + \cdots,\ \alpha,\beta > 0$$

is convergent if $\beta > 2\alpha$ and divergent if $\beta \leq 2\alpha$. 4