Ex/SC/MATH/UG/CORE/TH/02/2023

B. Sc. Mathematics (Hons.) Examination, 2023

(1st Year, 1st Semester)

Group Theory - I

Paper – Core-02

Time : Two hours — Full Marks : 40

All questions carry equal marks.

Answer any **four** questions. — 4×10

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{R}$ be the set of all real numbers.

1. a) Define a group. Let G be a finite group and $a,b \in G$ be such that $b \neq e$, $a^3 = e$ and $aba^{-1} = b^2$. Find the order of b.

   b) Define a subgroup of a group. Let H be a subgroup of a group G. Show that for any $g \in G$, $K = gHg^{-1} = \{ghg^{-1} \mid h \in H\}$ is a subgroup of G and $|K| = |H|$.

2. a) In S_7, let $\alpha(2\ 3\ 7\ 4)(1\ 5)\alpha^{-1} = (4\ 6\ 3\ 2)(6\ 7)$. Then find α.

   b) Define the alternating group A_n. Let H be a normal subgroup of A_5 such that H contains a 3-cycle. Show that $H = A_5$.

3. a) Let G be a finite group of order $n > 1$ where n is not a prime. Show that G has a subgroup other than $\{e\}$ and G.

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   b) Let G be a group which has finite number of subgroups. Show that G is finite.

4. a) Define a cyclic group. Prove that every subgroup of a cyclic group is cyclic.

   b) Let G be cyclic group with more than one elements. Prove that $G \times G$ is not cyclic.

5. a) Let G be a group and H be a subgroup of G. Define the left coset aH of H in G for any $a \in G$. Prove that $aH = H \Leftrightarrow a \in H$ and $aH = bH \Leftrightarrow a^{-1}b \in H$ for any $a, b \in G$.

   b) Define a normal subgroup of a group. Let G be a group and H be a subgroup of G such that $aba^{-1}b^{-1} \in H$ for all $a, b \in G$. Prove that H is normal in G.

6. a) Define homomorphisms of groups. Let G be a group and H be a normal subgroup of G. Prove that there exists an onto homomorphism $\phi : G \to G/H$ such that $H = \ker\phi$.

   b) Show that $GL(n,\mathbb{R})/SL(n,\mathbb{R}) \cong \mathbb{R}^*$ for all $n \in \mathbb{N}$, $n > 1$.