Study of Certain Ternary Algebraic Structures

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**STUDY OF CERTAIN TERNARY ALGEBRAIC STRUCTURES**" submitted by Smt. **Agni Roy** who got his name registered on **5th September, 2018 (Index No.: 152/18/MATHS/26)** for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon his own work under the supervision of **Prof. Sukhendu Kar** and that neither this thesis nor any part of it has been submitted for either any degree/ diploma or any other academic award anywhere before.

18/10/2022

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To my parents
Goutam Roy & Sibani Roy

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Abstract

Ternary algebraic structures is one of the fascinating concepts in modern Mathematics. This thesis deals with the study of ternary semigroups and ordered ternary semigroups. We discuss different kind of regularities, ideal theory in ternary semigroup, embedding of a ternary semigroup and some special type of ordered ternary semigroup. The thesis consists of 7 chapters. We shall give a brief structure of the thesis.

Chapter 1 discusses the background and motivation of the study. Also it gives all the required definition and results from ternary semigroups and ordered ternary semigroups that will be used throughout the thesis.

In **Chapter 2**, our focus is to characterize various kind of regularities in ordered ternary semigroup by different ideals. We show the way to get into some results of ordered ternary semigroup based on quasi-ideals, bi-ideals and semiprime ideals. We extend some results of ordered semigroup into ordered ternary semigroup under certain methodology. In particular, we characterize some properties of regular ordered ternary semigroup, left (resp. right) regular ordered ternary semigroup, completely regular ordered ternary semigroup and intra-regular ordered ternary semigroup by using quasi-ideal, bi-ideal and semiprime ideal of ordered ternary semigroup.

In **Chapter 3**, we study the notion of semigroup cover of ternary semigroup introduced by Santiago and Sri Bala [78] in 2010. We mainly study the connection between a ternary semigroup S and the semigroup cover $Q(S)$ of the ternary semigroup S by using various ideals. Moreover we characterize left and right regularity, complete regularity, intra-regularity in $Q(S)$ and investigate isomorphism problem

of ternary semigroup S and the corresponding semigroup cover $Q(S)$. We further introduce a partial order relation in $Q(S)$ and study various lattice structures.

In **Chapter 4**, we consider a power ternary semigroup $P(S)$ associated with a ternary semigroup S and study some properties of $P(S)$ by using the corresponding properties of S. After that we study the notion of ordered power ternary semigroup $\mathcal{P}(\mathcal{S})$ and our main aim is to establish some interconnection between the properties of a ternary semigroup S and the associated ordered ternary semigroup $\mathcal{P}(\mathcal{S})$. The purpose of this chapter is to give an overview of the results that are interesting from the algebraic point of view.

In **Chapter 5**, our focus is to characterize the structures of lattices in special class of regular ternary semigroup, called ternary semigroup of mappings and denoted by $T[X, Y]$. Then we discuss the isomorphism problem. We also derive simple conditions under which the converse is also true. Also we introduce a partial order relation in the ternary semigroup of mappings $T[X, Y]$. We also study the notion of ternary semigroup of isotone mappings. Further we present the characterization of regular, intra-regular and idempotent ordered ternary semigroup in ternary semigroup of isotone mappings.

In **Chapter 6**, we introduce the concept of right chain ordered ternary semigroup as a genralization of right chain ordered semigroup. Then we study the ideal theory of a right chain ordered ternary semigroup. Mainly we characterize them by using various ideals. A right chain ordered ternary semigroup is a ternary semigroup whose right ideals forms a chain. Our main aim to study right chain ordered ternary semigroup in terms of prime ideals, completely prime ideals and prime segment.

Chapter 7 is devoted to introduce the concept of (n, m, l)-ideal in ordered ternary semigroup. Also we characterize (n, m, l)-regular ordered ternary semigroups. We study the notion of quasi-prime, strongly quasi-prime, irreducible and strongly irreducible (n, m, l)-ideal in (n, m, l)-regular ordered ternary semigroup.

List of Symbols and Abbreviations

Symbols	Definitions
$\mathbb{N}$	Set of all Natural numbers
$\mathbb{Z}$	Set of all Integers
$\mathbb{Z}^+$	Set of all positive Integers
$\mathbb{Z}^-$	Set of all negative Integers
$\mathbb{N}_0$	Set of all non-negetive Integers
$\mathbb{Q}$	Set of all Rational numbers
$\mathbb{R}$	Set of all Real numbers
$\mathbb{C}$	Set of all Complex numbers
$card(A)$	Cardinality of a set 'A'
$\{\}$	The null set or empty set
$A \backslash B$	All elements which are in set A but not in set B.
$< a >$	Ideal of a ternary semigroup generated by 'a'
$I(a)$	Ideal of an ordered ternary semigroup generated by 'a'
$B(a)$	Bi-ideal of an ordered ternary semigroup generated by 'a'
$(H]$	Downward closure of H
$Q(S)$	The semigroup cover of a ternary semigroup S
$L(a, b)$	The left multiplication operators on S
$R(a, b)$	The right multiplication operators on S
$m(a, b)$	$\Big(L(a, b), R(b, a)\Big)$
$f : A \longrightarrow B$	A mapping f from A to B
$f(A)$	$\{f(x) : x \in A\}$
$S \cong T$	S is isomorphic to T

Symbols	**Definitions**
$P(S)$	The set of all non-empty subsets of a ternary semigroup
$\mathcal{P}(\mathcal{S})$	The ordered power ternary semigroup
$T[X,Y]$	The ternary semigroup of mappings
$O[X,Y]$	The ternary semigroup of isotone mappings
$a \wedge b$	The greatest lower bound of a and b
$a \vee b$	The least upper bound of a and b
$\inf A$	The greatest lower bound of a set A
$\sup A$	The least upper bound of a set A
$\bigcup\limits_{\alpha\in\Delta} A_\alpha$	$\{x : x \in A_\alpha$ for at least one $\alpha \in \Delta\}$
$\bigcap\limits_{\alpha\in\Delta} A_\alpha$	$\{x : x \in A_\alpha$ for all $\alpha \in \Delta\}$
'0'	The zero element in ternary semigroup S
$H(S_A)$	$\mathcal{H}$-ideal of right chain ordered ternary semigroup
$P(S_A)$	Associated prime right ideal of right chain ordered ternary semigroup
(R_1, R_2)	Prime right segment of right chain ordered ternary semigroup

Contents

Introduction, preliminaries and prerequisites

Chapter-1

Chapter 1

Introduction, preliminaries and prerequisites

1.1 Introduction

The modern presentation of abstract algebra begins with the simple abstract definition of algebraic structures. The results in binary algebraic structures may be extended to n-ary algebraic structures for arbitrary n but the transition from $n = 3$ to arbitrary n entails a great degree of complexity that makes it undesirable for exposition. For this reason, we shall confine ourselves in the proposed research work wholly to ternary algebraic structures. There are many topics in different areas of mathematics which remain to be disclosed in ternary algebraic structures. The main objective of this thesis is to extend different fundamental results of semigroups to ternary semigroups. Since there has been a remarkable growth of semigroup theory with manifold applications, it has been possible to study ternary semigroups to a good extent, the outcome of which is the present thesis.

First of all, we study the literature of ternary semigroup. The literature of ternary algebraic system dealing with ternary operation has a broad history. The introduction of mathematical literature of ternary algebraic system dated back to

1924. The notion of ternary algebraic system was first introduced by H. Prüfer [72] by the name 'Schar'. Later on W. Dörnte [27] further studied this type of algebraic system. The theory of ternary algebraic systems was introduced by Lehmer [64] in 1932. Also he explored the triplex systems. But earlier such structures were studied by Kasner who gave the concept of n-ary algebras. E. L. Post [71] later developed the theory of n-ary group to higher level of study. R. Kerner [54] contributed his ideas of ternary algebraic systems in mathematical physics. The notion of ternary semigroup was known to S. Banach. He showed, by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup. Later J. Los [66] proved that every ternary semigroup can be embedded in a semigroup. In 1953, the idea of semiheap was introduced and studied by V. V. Vagner [91]. M. L. Santiago further developed the theory of ternary semigroups and ternary semiheaps in his thesis. Study of this thesis develop ternary semigroup theory. In 1932, D. H. Lehmer [64] investigated certain ternary algebraic systems called triplexes which turn out to be commutative ternary groups. This ternary algebraic system has two types of associativity laws as follows :

(i) A nonempty set S together with a ternary operation denoted by juxtaposition, satisfying the associative law of 1st kind $(abc)de = a(bcd)e = ab(cde)$, for all $a, b, c, d, e \in S$ is said to be **Ternary Semigroup**.

(ii) A nonempty set S together with a ternary operation denoted by juxtaposition, satisfying the associative law of 2nd kind $(abc)de = a(dcb)e = ab(cde)$, for all $a, b, c, d, e \in S$ is said to be **Ternary Semiheap**.

We consider the set of integers $\mathbb{Z}$ which plays a major role in semigroup theory. If we consider the set $\mathbb{Z}^+$ the set of all positive integers subset of $\mathbb{Z}$, then we see that $\mathbb{Z}^+$ together with usual binary multiplication forms a semigroup. This is a natural example of binary semigroup. If we consider the set $\mathbb{Z}^-$ the set of all negetive integers subset of $\mathbb{Z}$, then we see that $\mathbb{Z}^-$ is not closed under the binary multiplication. But if we take the ternary multiplication '.' defined by $(a, b, c) \longrightarrow abc$ on $\mathbb{Z}^-$, then $\mathbb{Z}^-$ is closed under the ternary multiplication. Also $\mathbb{Z}^-$ satisfies the associative law

$(abc)de = a(bcd)e = ab(cde)$, for all $a, b, c, d, e \in \mathbb{Z}^-$. Thus we see that $\mathbb{Z}^-$ forms a ternary semigroup with respect to usual ternary multiplication of negetive integers. This is a natural examples of ternary semigroups that are not reducible to binary semigroups. Any semigroup can be made into a ternary semigroup in the natural way by defining the ternary multiplication $abc = (ab)c$.

In this thesis we study the notion of 'Ordered Ternary Semigroup' which disclosed a new field of vision in the research of Abstract Algebra. Ordered ternary semigroup bring the opportunity to study a partial order relation together with an associative ternary operation on the same set. The basic definition of ordered ternary semigroup has its origin in algebraic equations, computer science, economics and geometry because very similar techniques were found to be applicable in variety of situations. The formal definition of ordered ternary semigroup is as follows :

An ordered ternary semigroup $(S, ., \leq)$ is a partially ordered set $(S, \leq)$ with resepect to partial order '$\leq$' and at the same moment a ternary semigroup $(S, .)$ with resepect to ternary operation '.' such that for all $a, b, x, y \in S$ we have $a \leq b \Longrightarrow axy \leq bxy$, $xay \leq xby$, $xya \leq xyb$.

At the present time the theory of ordered ternary semigroups has an exceptional growth in reseach area. Many researchers have been taken interests to explore ordered ternary semigroup. Ordered ternary semigroups and ordered semigroups were studied by a number of authors in [16], [38], [65], [82]. N. Kehayapulu [46], [47] introduced and studied the notion of completely regular ordered semigroup. Also completely regular ordered semigroup was studied by D. M. Lee and S. K. Lee [63]. In 2012, Daddi and Pawar [25] studied the concept of ordered quasi-ideals and ordered bi-ideals in ordered ternary semigroup and also discussed about their properties. The result on the minimality and maximality theory of ordered quasi-ideal in ordered ternary semigroup was developed by Jailoka and Iampan [39].

In 1965, F.M. Sioson [87] developed the ideal theory and it is a key concept to study ternary semigroup and ordered ternary semigroup. Other than left and right ideal, Sioson also invented the idea of a new type of ideal which is known as the lateral ideal. Ideas of radicals, m-system, semiprimality, irreducbility, regularity in ternary semigroup is also developed by Sioson. Ideal theory in ternary semigroup was studied by Y. Sarala, A. Anjaneyulu and D. Madhusudhana Rao [79] and they described properties of prime ideals and primary ideals. A. Anjaneyulu [2], [3], [4] studied prime ideals and primary ideals and introduced the idea of primary decomposition in duo semigroup. Muhammad Shabir and Shahida Bashir [83] studied the notion of prime, semiprime and irreducible ideals in ternary semigroup. Recent study of semiprime ideal theory in commutative ternary semigroup was developed by G. Hanumanta Rao, A. Anjaneyulu and A. Gangadhar Rao in their paper [73]. Many authors like Bourne [7], H. Lal [62], V. L. Mannepalli, C. Nagore [69], M. Satyanarayana [80], [81] developed the ideal theory in commutative semigroups. H. J. Hoehnke [36] used the ideal theory in their work to develop the ideal theory of commutative semigroup. In 1993, R. D. Giri and A. K. Wazalwar [32] initiated the study of prime ideals and prime radicals in non-commutative semigroup.

In 1997, V. N. Dixit and S. Dewan [26] studied the notion of quasi and bi-ideal in ternary semigroup. Later on T. K. Dutta, S. Kar and B. K. Maity [29] illustrated the theory of ideal, quasi-ideal, bi-ideal in regular ternary semigroup and also developed some properties of intra-regular ternary semigroup. Further S. Kar and B. K. Maity [44] discussed over ideal theory of ternary semigroup. Congruence on ternary semigroup was studied by A. Chronowski [20], S. Kar and B. K. Maity [43]. Further bi-ideal was studied by R. A. Good and D. R. Hughes [33]. In ordered semigroup, the notion of bi-ideal and quasi-ideal was studied by N. Kehayopulu [48], [49], [50], [51]. The theory of minimal and maximal ideals in ordered semigroup was studied by Y. Cao and X. Xu [14] in 2000. In [15], they also characterized minimal and maximal left ideals in ordered semigroup. In 2002, the minimal and maximal ideal in ordered semigroup was developed by M. M. Arslanov and N. Kehayopulu [5]. In ordered ternary semigroup, ideal theory also plays an important role. V. Jyothi, Y. Sarala and D. Madhusudhana Rao [40] study the concept of semipseudo symmetric ideals in ordered ternary semigroups in 2014. Recenty in 2017, K. Hansda

[35] studied minimal bi-ideal in ordered semigroup.

In 1961, S. Lajos [58] introduced the concept of (m, n)-ideals in semigroup as a generelization of one-sided ideal. Further he studied (m, n)-ideal in [59], [60], [61]. T. Changphas [17] studied (m, n)-ideals of ordered semigroups. Later the theory of (m, n)-ideal in various algebric structures were studied by many authors like Muhammad Akram [1], Limpapat Bussaban [13], P. Luangchaisri [68], R. Mazurek [70], J. Sanborisoot [75] and so on.

A large number of authors make an attempt to study regularities in ternary semigroup. Perhaps the massive impact of regular semigroups have been convinced them to study ordered structure in regular ternary semigroup. M. L. Santiago [76], [77] investigated regular ternary semigroup, strongly regular ternary semigroup, completely regular ternary semigroup, clifford ternary semigroup, vagner ternary semigroup, inverse ternary semigroup. In the thesis of M. L. Santiago the notion of idempotent pair was used to show completely regular ternary semigroup as a disjoint union of ternary group and clifford ternary semigroup as a semilattice union of ternary groups. M. L. Santiago, S. Sri Bala [78] contributed their works to establish the theory of cover of a ternary semigroup and to develop its properties. Ternary semigroup and semiheaps also studied by W. A. Dudek [28] and A. Knoebel [55]. G. Sheeja [84], [85], [86] studied about ternary groups and developed the idea of simple ternary semigroup, 0-simple ternary semigroup, orthodox ternary semigroup etc. N. Kehayopulu [45], [46], [47], [52], [53] investigated regularities in ordered semigroup. D. N. Krgović [56] studied the notion of (m, n)-regular ordered semigroups. P. Luangchaisri and T. Changphas [67] also investigated (m, n)-regular and intra-regular ordered semigroups.

This thesis deals with the study of ternary semigroup and ordered ternary semigroup. So we need to know the basic definitions and results of ternary semigroups. Here in this chapter, I discuss briefly some important basic definitions and results that we need in the rest of my thesis.

1.2 Ternary Semigroup

In this section, we discuss some preliminary definitions and results of ternary semigroup which are relevant for this thesis. Most of the basic definitions and some results are taken from [29], [43], [44], [77]. Throughout this section S denotes a ternary semigroup.

Definition 1.2.1. *A nonempty set S together with a ternary operation $\cdot$ denoted by juxtaposition is said to be a ternary semigroup if it satisfies the ternary associative law*

$$ab(cde) = a(bcd)e = (abc)de \text{ for all } a, b, c, d, e \in S.$$

Example 1.2.2. *There are some examples of ternary semigroup.*

- *Set of all negative integers $\mathbb{Z}^-$ with usual ternary multiplication is a natural example of ternary semigroup.*
- *$\{i, -i\}$ forms a ternary semigroup with usual ternary multiplication where $i = \sqrt{-1}$.*
- *Consider the set $S_1 = \{r\sqrt{2} \ : \ r \in \mathbb{Q}\}$, where $\mathbb{Q}$ is the set of all rational numbers. Then S_1 forms a ternary semigroup with usual ternary multiplication.*

Definition 1.2.3. *A nonempty subset A of a ternary semigroup S is said to be ternary subsemigroup if A is itself a ternary semigroup w.r.t. ternary operation on A.*

A nonempty subset A of a ternary semigroup S is called a ternary subsemigroup of S if $A^3 = AAA \subseteq A$.

• For example, set of all negative integers w.r.t. usual ternary multiplication is a ternary subsemigroup of the set of all negative real numbers w.r.t. usual ternary multiplication.

Definition 1.2.4. *An element e of a ternary semigroup S is said to be an identity element of S if $eea = eae = aee$ for all $a \in S$.*

- -1 is an identity element of $\mathbb{Z}^-$.

Definition 1.2.5. *Let a be an element of a ternary semigroup S. An element b of S is said to be an inverse of a if $aba = a$ and $bab = b$.*

From the definition we can see that a is also an inverse of b.

Definition 1.2.6. *A ternary semigroup S is said to be inverse ternary semigroup if every element of S has unique inverse in S.*

- $\mathbb{Q}^-$, the set of all negative rational number is an example of inverse ternary semigroup w.r.t. usual ternary multiplication.

Definition 1.2.7. *A ternary semigroup S is said to be*

(i) *left cancellative if $abx = aby \Longrightarrow x = y$ for all $a, b, x, y \in S$.*

(ii) *right cancellative if $xab = yab \Longrightarrow x = y$ for all $a, b, x, y \in S$.*

(iii) *lateral cancellative if $axb = ayb \Longrightarrow x = y$ for all $a, b, x, y \in S$.*

(iv) *cancellative if S is left, right and lateral cancellative.*

Definition 1.2.8. *A ternary semigroup S is said to be commutative if $x_1x_2x_3 = x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$, where σ is a permutation of $\{1, 2, 3\}$.*

- Set of all negative integers with usual ternary multiplication is a commutative ternary semigroup.

Definition 1.2.9. *Let S be a ternary semigroup. An element $a \in S$ is said to be a regular element of S if there exists an element $x \in S$ such that $a = axa$.*

A ternary semigroup S is said to be regular ternary semigroup if every element of S is regular.

Definition 1.2.10. *An element a of a ternary semigroup S is said to be left (resp. right) regular element of S if there exists an element $x \in S$ such that $a = xaa$ (resp. $a = aax$).*

A ternary semigroup S is said to be left (resp. right) regular ternary semigroup if every element of S is left (resp. right) regular.

Definition 1.2.11. *An element a of a ternary semigroup S is said to be completely regular element of S is regular, left regular and right regular.*

A ternary semigroup S is said to be completely regular ternary semigroup if every element of S is completely regular.

Theorem 1.2.12. *Let S be a ternary semigroup S. The following conditions are equivalent :*

(i) S is completely regular,

(ii) $a \in a^2Sa^2$ for all $a \in S$.

Definition 1.2.13. *An element a of a ternary semigroup S is said to be intra-regular element of S is if there exist some elements $x, y \in S$ such that $a = xa^3y$.*

A ternary semigroup S is said to be intra-regular ternary semigroup if every element of S is intra-regular.

Definition 1.2.14. *An element a of a ternary semigroup S is said to be idempotent element of S if $a^3 = a$.*

A ternary semigroup S is said to be an idempotent ternary semigroup if every element of S is idempotent element.

An idempotent ternary semigroup is also known as ternary band.

Every idempotent element in a ternary semigroup is regular.

An idempotent ternary semigroup S is said to be strong idempotent ternary semigroup if $a^3 = a$ and $a^2b = ab^2$ for all $a, b \in S$.

- S_3 is the symmetric group of order 6. Then $T = \{(1\ \ 2), (1\ \ 3), (2\ \ 3)\}$ is an idempotent ternary semigroup with usual ternary composition.
- $\{0, -1\} \subset \mathbb{R}$ is an example of strong idempotent ternary semigroup w.r.t. usual ternary multiplication.

Definition 1.2.15. *An equivalence relation ρ on a ternary semigroup S is said to be ternary*

(*i*) *left congruence if $a\rho b \Longrightarrow (sta)\rho(stb)$ for all $a, b, s, t \in S$.*

(*ii*) *right congruence if $a\rho b \Longrightarrow (ast)\rho(bst)$ for all $a, b, s, t \in S$.*

(*iii*) *lateral congruence if $a\rho b \Longrightarrow (sat)\rho(sbt)$ for all $a, b, s, t \in S$.*

(*iv*) *congruence if $a\rho a'$, $b\rho b'$, $c\rho c' \Longrightarrow (abc)\rho(a'b'c')$ for all $a, a', b, b', c, c' \in S$.*

Proposition 1.2.16. *An equivalence relation ρ on a ternary semigroup S is a ternary congruence if and only if it is a ternary left, a ternary right, a ternary lateral congruence on S.*

Definition 1.2.17. *A pair (a, b) of elements in a ternary semigroup S is said to be an idempotent pair if $ab(abx) = abx$ and $(xab)ab = xab$ for all $x \in S$.*

Definition 1.2.18. *Two idempotent pairs (a, b) and (c, d) of a ternary semigroup S are said to be equivalent if $abx = cdx$ and $xab = xcd$ for all $x \in S$ and it is denoted by $(a, b) \sim (c, d)$.*

Definition 1.2.19. *A ternary semigroup S is said to be a ternary group if for $a, b, c \in S$, the equations $abx = c$, $axb = c$ and $xab = c$ have solutions in S.*

Remark 1.2.20. *In a ternary group S, the equations $abx = c$, $axb = c$ and $xab = c$ have unique solutions for all $a, b, c \in S$.*

Definition 1.2.21. *A ternary semigroup S is said to be*

(*i*) *left zero if $abc = a$ for all $a, b, c \in S$;*

(*ii*) *right zero if $abc = c$ for all $a, b, c \in S$;*

(*iii*) *lateral zero if $abc = b$ for all $a, b, c \in S$.*

Note 1.2.22. *Let S be a lateral zero ternary semigroup and $a, b, c \in S$. Thus $abc = b$. Therefore, $ac(abc) = acb \Longrightarrow (aca)bc = c \Longrightarrow cbc = c \Longrightarrow b = c$.*

Again $ac(abc) = acb \Longrightarrow a(cab)c = c \Longrightarrow aac = c \Longrightarrow a = c$. Thus $a = b = c$. So we conclude that lateral zero ternary semigroup is always singleton.

Definition 1.2.23. *A nonempty subset I of a ternary semigroup S is said to be*

(*i*) *a left ideal of S if $SSI \subseteq I$;*

(*ii*) *a right ideal of S if $ISS \subseteq I$;*

(*iii*) *a lateral ideal of S if $SIS \subseteq I$;*

(*iv*) *a two-sided ideal of S if I is both left and right ideal of S;*

(*v*) *an ideal of S if I is a left, right and lateral ideal of S.*

An ideal I of a ternary semigroup S is called a proper ideal if $I \neq S$.

Every ideal of S is a ternary subsemigroup of S. Thus ideals of a ternary semigroup S is also a ternary semigroup.

Proposition 1.2.24. *Let S be a ternary semigroup and $a \in S$. Then the principal*

(*i*) *left ideal generated by 'a' is given by $< a >_l = SSa \cup \{a\}$;*

(*ii*) *right ideal generated by 'a' is given by $< a >_r = aSS \cup \{a\}$;*

(*iii*) *lateral ideal generated by 'a' is given by $< a >_m = SaS \cup SSaSS \cup \{a\}$;*

(*iv*) *ideal generated by 'a' is given by $< a > = SSa \cup aSS \cup SaS \cup SSaSS \cup \{a\}$.*

Definition 1.2.25. *Let S be a ternary semigroup. Then S is called*

(*i*) *left simple if S has no non-trivial proper left ideal.*

(*ii*) *right simple if S has no non-trivial proper right ideal.*

(*iii*) *lateral simple if S has no non-trivial proper lateral ideal.*

(*iv*) *simple if S has no non-trivial proper ideal.*

A ternary semigroup S is simple if it is left simple, right simple and lateral simple.

Definition 1.2.26. *Let S be a ternary semigroup. An ideal I of S is said to be prime ideal if for any ideals A, B, C of S such that $ABC \subseteq I$ we have $A \subseteq I$ or $B \subseteq I$ or $C \subseteq I$.*

Definition 1.2.27. *Let S be a ternary semigroup. An ideal I of S is said to be semiprime ideal if for any ideal A of S such that $A^3 \subseteq I$ we have $A \subseteq I$.*

Definition 1.2.28. *Let S be a ternary semigroup. An ideal I of S is said to be completely prime ideal if for any elements a, b, c of S such that $abc \in I$ we have $a \in I$ or $b \in I$ or $c \in I$.*

Definition 1.2.29. *Let S be a ternary semigroup. An ideal I of S is said to be completely semiprime ideal if for an element a of S such that $a^3 \in I$ we have $a \in I$.*

In the following figure we made an conclusion pictorially for the four types of ideals which are defined above :

$$\begin{tikzcd}
& \boxed{\text{prime}} \arrow[dr] & \\
\boxed{\text{completely prime}} \arrow[ur] \arrow[dr] & & \boxed{\text{semiprime}} \\
& \boxed{\text{completely semiprime}} \arrow[ur] &
\end{tikzcd}$$

Definition 1.2.30. *Let S be a ternary semigroup. A nonempty subset Q of S is said to be quasi-ideal of S if $QSS \cap SQS \cap SSQ \subseteq Q$ and $QSS \cap SSQSS \cap SSQ \subseteq Q$.*

Definition 1.2.31. *A ternary subsemigroup B of a ternary semigroup S is said to be bi-ideal of S if $BSBSB \subseteq B$.*

Definition 1.2.32. *Let C be a non-empty subset of a ternary semigroup S. Then $C \cup CCC \cup CSCSC$ is the smallest bi-ideal of S containing C.*

Definition 1.2.33. *A ternary semigroup S is said to be a ternary semilattice if S is commutative, idempotent and satisfies the condition $x^2y = xy^2$ for all $x, y \in S$.*

Note 1.2.34. *A commutative strong idempotent ternary semigroup is a ternary semilattice.*

Example 1.2.35. *Some examples of ternary semilattice are as follows :*

- $S_1 = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \subseteq M_2(\mathbb{R})$, *w.r.t. ternary matrix multiplication.*

- $S_2 = \{-1, 0\} \subseteq \mathbb{R}$, *w.r.t. usual ternary multiplication.*

Definition 1.2.36. *An idempotent ternary semigroup S is said to be a rectangular ternary band if $aba = a$ for all $a, b \in S$.*

Although the definition of rectangular ternary band and rectangular band in binary are similar, but all the rectangular ternary bands are not rectangular bands in binary.

Example 1.2.37. *The following are examples of rectangular ternary bands which are not rectangular bands in binary.*

- $\left\{ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \right\} \subseteq M_2(\mathbb{R})$ *w.r.t. ternary matrix multiplication.*
- $\left\{ \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \right\} \subseteq M_2(\mathbb{R})$ *w.r.t. ternary matrix multiplication.*

Definition 1.2.38. *Let S_1 and S_2 be two ternary semigroups. A mapping $\psi : S_1 \longrightarrow S_2$ is said to be ternary homomorphism if $\psi(abc) = \psi(a)\psi(b)\psi(c)$ for all $a, b, c \in S_1$.*

If ψ is one-one then ψ is said to be a ternary monomorphism from S_1 to S_2.

If ψ is onto then ψ is said to be a ternary epimorphism from S_1 to S_2.

If ψ is both one-one and onto then ψ is said to be a ternary isomorphism from S_1 to S_2.

1.3 Ordered Ternary Semigroup

Now we briefly discuss the basic definitions and terminologies of ordered ternary semigroup. Most of the basic definitions and some results are taken from [16], [25], [38], [37], [39], [65]. Throughout this section S denotes an ordered ternary semigroup.

Definition 1.3.1. *A ternary semigroup $(S, .)$ is called an ordered ternary semigroup or a partially ordered ternary semigroup (in short po-ternary semigroup) if there is a partial order " $\leq$ " on S such that*

$$x \leq y \Longrightarrow xx_1x_2 \leq yx_1x_2,\ x_1xx_2 \leq x_1yx_2,\ x_1x_2x \leq x_1x_2y \text{ for all } x, y, x_1, x_2 \in S.$$

Example 1.3.2. *There are some examples of ordered ternary semigroups :*

- *Let $\mathbb{Z}^-$ be the set of all negetive integers. Then $(\mathbb{Z}^-, ., \leq)$ is an ordered ternary semigroup, where '.' is the usual ternary multiplication and '$\leq$' is the usual partial order on $\mathbb{Z}^-$.*
- *Let $\mathbb{N}$ be the set of all natural number. Then $(\mathbb{N}, ., \leq)$ is an ordered ternary semigroup, where '$\leq$' is the usual less than or equal to partial order on $\mathbb{N}$ and the ternary multiplication '.' is defined by $abc = a + b + c$.*
- *Let S be a commutative ternary semigroup. Let $I(S)$ be the set of all ideals of S. Then $(I(S), ., \leq)$ is an ordered ternary semigroup where, $\leq$ is the set inclusion and the multiplication '.' is defined by $IJK = \{ijk \mid i \in I, j \in J, k \in K\}$.*

Definition 1.3.3. *Let $(S, ., \leq)$ be an ordered ternary semigroup. Every ternary subsemigroup with the parital order '$\leq$' defined on S is an ordered ternary semigroup. If '$\leq_A$' is partial order relation on A, then $\leq_A = \leq \cap (A \times A)$. So $(A, ., \leq_A)$ is called ordered ternary subsemigroup of $(S, ., \leq)$.*

- For an ordered ternary semigroup $(S, ., \leq)$ and a subset H of S, we denote by $(H]$ the subset of S defined by :

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}$$

The set $(H]$ is called the downward closure of H and H is called downward closed if $(H] = H$ i.e. $a \in H,\ b \in S$ such that $b \leq a \Longrightarrow b \in H$. Clearly $(H] \subseteq S$.

• Let A be an ordered ternary subsemigroup of the ordered ternary semigroup S then the set $(H]_A \subseteq A$ defined by

$$(H]_A := \{t \in A \mid t \leq h \text{ for some } h \in H\}$$

The set $(H]_A \subseteq A$ is called downward closure of H in A. Now $(H]_A = H$ implies that $a \in H, b \in A$ such that $b \leq a \Longrightarrow b \in H$

Definition 1.3.4. *A nonempty subset A of an ordered ternary semigroup S is called*

(i) a left ideal of S, if (1) $SSA \subseteq A$ and (2) $(A] = A$

(ii) a right ideal of S, if (1) $ASS \subseteq A$ and (2) $(A] = A$,

(iii) a lateral ideal of S, if (1) $SAS \subseteq A$ and (2) $(A] = A$,

(iv) an ideal of S if it is a left ideal, right ideal and lateral ideal of S.

Definition 1.3.5. *For an ordered ternary semigroup S and $a \in S$, we denote by $R(a)$ (resp. $L(a)$, $M(a)$) the right (resp. left, lateral) ideal of S generated by the element a and $I(a)$ denotes the ideal generated by the element a. Thus*

(i) left ideal generated by 'a' is given by $L(a) = (a \cup SSa]$;

(ii) right ideal generated by 'a' is given by $R(a) = (a \cup aSS]$;

(iii) lateral ideal generated by 'a' is given by $M(a) = (a \cup SaS \cup SSaSS]$;

(iv) ideal generated by 'a' is given by $I(a) = (a \cup SSa \cup aSS \cup SaS \cup SSaSS]$.

Definition 1.3.6. *Let $(S, ., \leq)$ be an ordered ternary semigroup. A nonempty subset Q of S is called a quasi-ideal of S, if*

(i) $(SSQ] \cap (SQS] \cap (QSS] \subseteq Q$,

(ii) $(SSQ] \cap (SSQSS] \cap (QSS] \subseteq Q$ and

(iii) $(Q] = Q$.

• Every left, right and lateral ideal of an ordered ternary semigroup S is a quasi-ideal of S. But the converse does not hold, in general.

Definition 1.3.7. *Let $(S, ., \leq)$ be an ordered ternary semigroup. A subsemigroup B of S is called a bi-ideal of S, if (i) $BSBSB \subseteq B$ and (ii) $(B] = B$.*

Definition 1.3.8. *For an ordered ternary semigroup S and $a \in S$, the bi-ideal generated by the element a is given by $B(a) = (a \cup a^3 \cup aSaSa]$.*

- Every quasi-ideal of an ordered ternary semigroup S is a bi-ideal of S. Since every left, right and lateral ideal of an ordered ternary semigroup S is a quasi-ideal of S, it follows that every left, right and lateral ideal of an ordered ternary semigroup S is a bi-ideal of S.

Note 1.3.9. *In ordered ternary semigroup, the definition of prime ideal, semiprime ideal, completely prime ideal, completely semiprime ideal, left simple ideal, right simple ideal, simple ideal are same as ternary semigroups.*

Definition 1.3.10. *Let $(S_1, . \leq_1)$ and $(S_2, ., \leq_2)$ be two ordered ternary semigroups. A mapping $f : S_1 \longrightarrow S_2$ is said to be an isotone mapping if $a \leq_1 b$ implies that $f(a) \leq_2 f(b)$ for all $a, b \in S$.*

Definition 1.3.11. *Let $(S_1, . \leq_1)$ and $(S_2, ., \leq_2)$ be two ordered ternary semigroups. A mapping $g : S_1 \longrightarrow S_2$ is said to be an ordered ternary homomorphism if g is an isotone mapping and*

$$g(abc) = g(a)g(b)g(c) \text{ for all } a, b, c \in S_1.$$

If g is one-one then g is said to be an ordered ternary monomorphism from S_1 to S_2.

If g is onto then g is said to be an ordered ternary epimorphism from S_1 to S_2.

If g is both one-one and onto then g is said to be an ordered ternary isomorphism from S_1 to S_2. Two ordered ternary semigroups are called isomorphic if there is an ordered ternary isomorphism between them.

1.4 Some results on ordered ternary semigroups

In this section S denotes an ordered ternary semigroup. We give all the required results on ordered ternary semigroup that will be used throughout the rest of the thesis.

Proposition 1.4.1. *[16] Let $(S, ., \leq)$ be an ordered ternary semigroup and A be an ordered ternary subsemigroup of S. Then for any ideal I of S, $A \cap I$ is an ideal of S.*

Proposition 1.4.2. *[16] Let $(S, ., \leq)$ be an ordered ternary semigroup and $\{I_\lambda\}_{\lambda\in\Delta}$ is family of non-trivial ideals of S. Then $\bigcap_{\lambda\in\Delta} I_\lambda$ and $\bigcup_{\lambda\in\Delta} I_\lambda$ are ideals of S.*

Next we have the following result which we will often use in this thesis.

Proposition 1.4.3. *[25] Let $(S, ., \leq)$ be an ordered ternary semigroup. Then the followings hold :*
(i) $A \subseteq (A]$ for any non-empty subset A of S,
(ii) If $A, B \subseteq S$ such that $A \subseteq B$ then $(A] \subseteq (B]$,
(iii) $((A]] = (A]$ for any $A \subseteq S$,
(iv) $(A](B](C] \subseteq (ABC]$ for all $A, B, C \subseteq S$,
(v) $((A](B](C]] = ((A](B]C] = (AB(C]] = (ABC]$ for all $A, B, C \subseteq S$,
(vi) $(A \cup B] = (A] \cup (B]$ for all $A, B \subseteq S$,
(vii) $(A \cap B] \subseteq (A] \cap (B]$ for all $A, B \subseteq S$,
In particular, if A and B are ideals in S, then $(A \cap B] = (A] \cap (B]$,
(viii) If $\{A_\lambda\}_{\lambda\in\Delta}$ is family of non empty subsets of S, then $(\bigcup_{\lambda\in\Delta} A_\lambda] = \bigcup_{\lambda\in\Delta}(A_\lambda]$ and $(\bigcap_{\lambda\in\Delta} A_\lambda] \subseteq \bigcap_{\lambda\in\Delta}(A_\lambda]$,
(ix) $(SSA]$, $(ASS]$, $(SAS \cup SSASS]$ are left, right and lateral ideal in S respectively.
(x) $((A^{(2n-1)}]^{(2m-1)}] = (A^{(2n-1)(2m-1)}]$,
(xi) If $x, y \in S$ and $x \leq y$, then $(xAA] \subseteq (yAA]$ and $(AAx] \subseteq (AAy]$

Lemma 1.4.4. *Let $(S, ., \leq)$ be a ordered ternary semigroup. The following are equivalent:*

(i) $(A^3] = A$ *for every ideal* A *of* S.

(ii) $A \cap B \cap C = (ABC]$ *for all ideals* A, B, C *of* S.

(iii) $I(a) \cap I(b) \cap I(c) = (I(a)I(b)I(c)]$ *for all* $a, b, c \in S$.

(iv) $I(a) = (I(a)^3]$ *for all* $a \in S$.

Proof. $(i) \implies (ii)$ Let us assume that (i) holds. Suppose that A, B, C are ideals of S. Then $(ABC] \subseteq (SSC] \subseteq (C] = C$. Similarly, $(ABC] \subseteq A$ and $(ABC] \subseteq B$. Thus $(ABC] \subseteq A \cap B \cap C$. Since $(A^3] = A$ for every ideal A of S, $A \cap B \cap C = ((A \cap B \cap C)^3] = ((A \cap B \cap C)(A \cap B \cap C)(A \cap B \cap C)] \subseteq (ABC]$. Thus, $A \cap B \cap C = (ABC]$ for all ideals A, B, C of S.

$(ii) \implies (iii)$ Let $a, b, c \in S$. Since $I(a)$, $I(b)$, $I(c)$ are ideals in S then by (ii), we have $I(a) \cap I(b) \cap I(c) = (I(a)I(b)I(c)]$.

$(iii) \implies (iv)$ It is obvious. We take $I(a) = I(b) = I(c)$.

$(iv) \implies (i)$ Let a be an element of S. Then $I(a) = (I(a)^3] = (I(a)I(a)I(a)] = ((I(a)^3](I(a)^3](I(a)^3]] = (I(a)^3 I(a)^3 I(a)^3]$. Now $I(a)^3 \subseteq SI(a)S = S(a \cup SSa \cup aSS \cup SaS \cup SSaSS]S \subseteq (SaS \cup SSSaS \cup SaSSS \cup SSaSS \cup SSSaSSS] \subseteq (SaS \cup SSaSS]$. Thus $a \in I(a) \subseteq ((SaS \cup SSaSS](SaS \cup SSaSS](SaS \cup SSaSS]] \subseteq (SaSSaSSaS \cup SaSSaSaSS \cup SaSaSaS \cup SaSaSSaSS \cup SSaSaSSaS \cup SSaSaSaSS \cup SSaSSaSaS \cup SSaSSaSSaSS]$.

Let A be an ideal in S. Let $x \in (A^3]$. Then $x \leq a_1a_2a_3$ for some $a_1, a_2, a_3 \in A$. Now $a_1a_2a_3 \in AAA \subseteq ASS \subseteq A$. Hence $x \in (A] = A$. Thus $(A^3] \subseteq A$. Again let $y \in A \subseteq S$. Then $y \in (SySSySSyS \cup SySSySySS \cup SySySyS \cup SySySSySS \cup SSySySSyS \cup SSySySySS \cup SSySSySyS \cup SSySSySSySS] \subseteq (SASSASSAS \cup SASSASASS \cup SASASAS \cup SASASSASS \cup SSASASSAS \cup SSASASASS \cup SSASSASAS \cup SSASSASSASS] \subseteq (A^3]$. Thus $A \subseteq (A^3]$. Therefore, $(A^3] = A$. $\square$

Theorem 1.4.5. *[74] An ordered ternary semigroup* S *is left (resp. right, lateral) simple if and only if* $(aSS] = S$ $\Big($*resp.* $(SSa] = S, (aSa] = S\Big)$ *for all* $a \in S$. *Again* S *is simple if and only if* $(aSS] = S$, $(SSa] = S$ *and* $(aSa] = S$ *for all* $a \in S$.

Corollary 1.4.6. *An ordered ternary semigroup S is left (resp. right, lateral) simple if and only if for every $a, b, c \in S$ there exists $x \in S$ such that $b \leq xac$ (resp. $b \leq acx$, $b \leq axc$).*

Theorem 1.4.7. *[74] An ordered ternary semigroup S is left (resp. right, lateral) simple if and only if $(abS] = S$ $\Big($resp. $(Sab] = S, (aSb] = S\Big)$ for all $a, b \in S$. Again S is simple if and only if $(abS] = S$, $(Sab] = S$ and $(aSb] = S$ for all $a, b \in S$.*

Corollary 1.4.8. *An ordered ternary semigroup S is left (resp. right, lateral) simple if and only if for every $a, b \in S$ there exist $x, y \in S$ such that $b \leq xya$ (resp. $b \leq axy$, $b \leq xay$).*

Theorem 1.4.9. *Let S be an ordered ternary semigroup. Then S is left and right simple if and only if S does not contain proper bi-ideals.*

Proof. Let S be left and right simple ordered ternary semigroup and B be a bi-ideal in S. Let $b \in B$. Then by Theorem 1.4.5, we have $S = (bSS] = (bS(SSb]] = (bSSSb] = (bS(bSS]Sb] = (bSbSSSb] \subseteq (bSbSb] \in (BSBSB] \subseteq (B] = B$ Thus S does not contain proper bi-ideal.

Conversely, suppose that S does not contain proper bi-ideals. Let L be a left ideal and R be a right ideal in S. Since every left ideal and right ideal are bi-ideal of S, then we have $L = S$ and $R = S$. Thus S is left and right simple. □

Corollary 1.4.10. *Let S be an ordered ternary semigroup. Then S is lateral simple if and only if S does not contain proper bi-ideals.*

Theorem 1.4.11. *[16] For every left ideal L, lateral ideal M and right ideal R of an ordered ternary semigoup S, $R \cap M \cap L$ is a quasi-ideal of S.*

From the above theorem we have the following corollary :

Corollary 1.4.12. *Let S be an ordered ternary semigoup and Q be a quasi-ideal of S. Then $Q = R(Q) \cap M(Q) \cap L(Q)$.*

On Regularities In Ordered Ternary Semigroups

Chapter-2

Chapter 2

On regularities in ordered ternary semigroups

2.1 Introduction

In this chapter, we study the notion of certain special classes regular ordered ternary semigroups. We develop some results of ordered semigroup into ordered ternary semigroup under certain methodology. In particular, we characterize some properties of regular ordered ternary semigroup, left (resp. right) regular ordered ternary semigroup, completely regular ordered ternary semigroup and intra-regular ordered ternary semigroup by using ideal, quasi-ideal, bi-ideal, completely prime ideal and semiprime ideal of ordered ternary semigroup.

Throughout this chapter, S denotes an ordered ternary semigroup.

2.2 Left regular and right regular ordered ternary semigroup

In this section, we characterize left regular ordered ternary semigroup by using properties of various ideals.

Definition 2.2.1. *An ordered ternary semigroup S is said to be left regular if $A \subseteq (SA^2]$ for every $A \subseteq S$.*

An element a of an ordered ternary semigroup S is said to be left regular if there exists an element $x \in S$ such that $a \leq xaa$. If all elements of S are left regular then S is said to be a left regular ordered ternary semigroup.

Definition 2.2.2. *An ordered ternary semigroup S is said to be right regular if $A \subseteq (A^2S]$ for every $A \subseteq S$.*

An element a of an ordered ternary semigroup S is said to be right regular if there exists an element $x \in S$ such that $a \leq aax$. If all elements of S are right regular then S is said to be a right regular ordered ternary semigroup.

Lemma 2.2.3. *Let S be a left (resp. right) regular ordered ternary semigroup and L be a lateral ideal of S, then L is left (resp. right) regular.*

Proof. Let L be a lateral ideal of an ordered ternary semigroup S. Let $A \subseteq L$. Since S is left regular, $A \subseteq (SAA] \subseteq (SA(SAA]] \subseteq (SASAA] \subseteq (SLSAA] \subseteq (LAA]$. Thus L is left regular. $\square$

Theorem 2.2.4. *Let S be an ordered ternary semigroup. Then the followings are equivalent :*
(i) S is left regular,
(ii) $L(a) \subseteq L(a^3)$ for every $a \in S$,
(iii) $L(a) = L(a^3)$ for every $a \in S$.

Proof. (i)$\Longrightarrow$(ii) Let $a \in S$. Since S is left regular $a \in (Sa^2]$. Let $x \in L(a) = (a \cup SSa] = (a] \cup (SSa]$. If $x \in (a]$ then $x \leq a \leq xa^2 \leq x(xa^2)a = xxa^3 \in SSa^3$. Thus $x \in (SSa^3] \subseteq (SSa^3 \cup a^3]$. Again if $x \in (SSa]$ then $x \in (SS(Sa^2]] = (SSSa^2] \subseteq (Sa^2] \subseteq (S(Sa^2]a] = (SSa^3] \subseteq (SSa^3 \cup a^3]$. Thus in both cases $x \in L(a^3)$. Therefore, $L(a) \subseteq L(a^3)$.
(ii) $\Longrightarrow$ (iii) Let $a \in S$. Now $a^3 \in SSa \subseteq (SSa] \subseteq L(a)$. Thus $L(a^3) \subseteq L(a)$. Hence $L(a) = L(a^3)$.

(iii) $\Longrightarrow$ (i) Let a be an element of S such that $L(a) = L(a^3)$. Thus $a \in L(a^3) = (a^3 \cup SSa^3]$. Therefore, either $a \leq a^3 \in Sa^2$ or there exixts $x, y \in S$ such that $a \leq xya^3 \in SSSa^2 \subseteq Sa^2$. Hence S is left regular. $\square$

Theorem 2.2.5. *Let S be an ordered ternary semigroup. Then the followings are equivalent :*

(i) S is right regular,

(ii) $R(a) \subseteq R(a^3)$ for every $a \in S$,

(iii) $R(a) = R(a^3)$ for every $a \in S$.

Proof. Proof is similar to the above Theorem 2.2.4. $\square$

Theorem 2.2.6. *Let S be an ordered ternary semigroup such that S is a union of left regular ternary subsemigroups of S, then S is left regular.*

Proof. Suppose that S is a union of left regular ternary subsemigroups of S. Then $S = \bigcup_{i \in \Delta} S_i$ where $\{S_i | i \in \Delta\}$ is a family of left regular ternary subsemigroups of S. Let $a \in S$. Then $a \in S_j$ for some $j \in \Delta$. Since S_j is a left regular ternary subsemigroup of S, we have $a \in (S_j a^2]$. Since $S_j \subseteq S$, then $S_j a^2 \subseteq Sa^2 \Longrightarrow (S_j a^2] \subseteq (Sa^2]$. Thus $a \in (Sa^2]$ and hence S is left regular ordered ternary semigroup. $\square$

Similar results hold if we replace 'left' by 'right'.

Theorem 2.2.7. *Let S be an ordered ternary semigroup such that S is a union of left simple ternary subsemigroups of S, then S is left regular.*

Proof. Let $S = \bigcup_{j \in \Delta} S_j$, where S_j is a left simple subsemigroup of S for every $j \in \Delta$. Let T be a left ideal of S such that $a^3 \in T$ for some $a \in S$. Since $a \in S$, $a \in S_j$ for some $j \in \Delta$. Now $a^3 \in T$ and $a^3 \in S_j S_j S_j \subseteq S_j$. Thus $a^3 \in T \cap S_j$. So $T \cap S_j \neq \{\}$. Now $S_j S_j (T \cap S_j) = S_j S_j T \cap {S_j}^3 \subseteq SST \cap S_j \subseteq T \cap S_j$. Again let $x \in S_j$ such that $x \in (T \cap S_j]$. Then $x \leq y$ for some $y \in T \cap S_j \subseteq T$. Thus $x \in (T] = T$. Hence $x \in T \cap S_j$ and so $(T \cap S_j] = T \cap S_j$. Thus $T \cap S_j$ is a left ideal in S_j. Since S_j is left simple we have $T \cap S_j = S_j$. Thus $S_j \subseteq T$ and hence $a \in T$. Thus T is left simple. Since T is arbitrary left ideal of S, every left ideal of S is left simple. Now S is also a left ideal. Thus $S = (aaS]$ for all $a \in S$ by Theorem 1.4.7. Hence S is left regular. $\square$

Similarly we have the following corollary.

Corollary 2.2.8. *Let S be an ordered ternary semigroup such that S is a union of right simple ternary subsemigroups of S, then S is right regular.*

Theorem 2.2.9. *Let S be an ordered ternary semigroup. Then S is left (resp. right) regular if and only if every left (resp. right) ideal of S is semiprime.*

Proof. Let S be a left regular ordered ternary semigroup and L be a left ideal of S. Let $A^3 \subseteq L$ for some left ideal A of S. Since S is left regular, we have $A \subseteq (SA^2] \subseteq (S(SA^2]A] = (S(SA^2)A] = (SSA^3] \subseteq (SSL] \subseteq (L] = L$. Thus L is semiprime.

Conversely, suppose that every left ideal of S is semiprime. Let $A \subseteq S$. Then $SS(SAA] \subseteq (S](S](SAA] \subseteq (SSSAA] \subseteq (SAA]$ and $((SAA]] = (SAA]$. Therefore, $(SAA]$ is a left ideal of S. Now $A^3 = AAA \subseteq SAA \subseteq (SAA] = (SA^2]$. Since every left ideal of S is semiprime, we have $A \subseteq (SA^2]$. Thus S is a left regular ordered ternary semigroup.

Similarly, we can also prove the same for right ideal of S. $\square$

Theorem 2.2.10. *Let S be an ordered ternary semigroup. Then the followings are equivalent:*

(i) If L_1, L_2 and L_3 are left ideals of S, then $(L_1L_2L_3] = (L_{\sigma(1)}L_{\sigma(2)}L_{\sigma(3)}]$ for every permutation σ of $\{1,2,3\}$ and $({L_1}^3] = (L_1]$.

(ii) If L_1, L_2 and L_3 are left ideals of S, then $L_1 \cap L_2 \cap L_3 = (L_1L_2L_3]$.

(iii) S is left regular and left simple.

Proof. $(i) \Longrightarrow (ii)$ Let L_1, L_2 and L_3 are left ideals of S. Thus, $(L_1L_2L_3] \subseteq (SSL_3] \subseteq (L_3] = L_3$, $(L_2L_3L_1] \subseteq (SSL_1] \subseteq (L_1] = L_1$ and $(L_3L_1L_2] \subseteq (SSL_2] \subseteq (L_2] = L_2$. By (i) $(L_1L_2L_3] = (L_2L_3L_1] = (L_3L_1L_2]$. Thus $(L_1L_2L_3] \subseteq L_1 \cap L_2 \cap L_3$. Now

$(L_1L_2L_3] \neq \{\}$. Therefore, $L_1 \cap L_2 \cap L_3$ is a left ideal of S and by hypothesis $L_1 \cap L_2 \cap L_3 = ((L_1 \cap L_2 \cap L_3)^3] = ((L_1 \cap L_2 \cap L_3)(L_1 \cap L_2 \cap L_3)(L_1 \cap L_2 \cap L_3)] \subseteq (L_1L_2L_3]$ and hence $(L_1L_2L_3] = L_1 \cap L_2 \cap L_3$.

$(ii) \implies (iii)$ Let $a \in S$. Then by (ii) $L(a) = L(a) \cap L(a) \cap L(a) = (L(a)L(a)L(a)] = (L(a)^3]$. Also $L(a) = S \cap S \cap L(a) = (SSL(a)] = (SSa]$. Again $(aSS] \subseteq (L(a)SS] = L(a) \cap S \cap S = L(a) \subseteq (SSa]$. Thus $a \in L(a) = (L(a)^3] \subseteq ((SSa]^3] = ((SSa)^3] = (SSaSSaSSa] = (SS(aSS](aSS]a] \subseteq (SS(SSa](SSa]a] \subseteq (SSSSSSSa^2] \subseteq (Sa^2]$. Thus S is left regular.

Let $a \in S$. Then $a \in L(a) \subseteq (SSa]$. Thus $S \subseteq (SSa]$. Also $(SSa] \subseteq (SSS] \subseteq S$. Thus $S = (SSa]$. Hence S is left simple.

$(iii) \implies (i)$ Let S be a left regular and left simple ordered ternary semigroup. Now $SS(L_1L_2L_3] \subseteq (S](S](L_1L_2L_3] \subseteq (SSL_1L_2L_3] \subseteq (L_1L_2L_3]$ and $((L_1L_2L_3]] = (L_1L_2L_3]$. Thus $(L_1L_2L_3]$ is a left ideal of S. Similarly $(L_2L_3L_1]$ and $(L_3L_1L_2]$ are left ideals of S. Let $x \in (L_1L_2L_3]$. So, $x \leq abc$ where $a \in L_1$, $b \in L_2$, $c \in L_3$. Now $abc \in S$, $bca \in S$. Since S is simple by Corollary 1.4.8 there exists $y, z \in S$ such that $abc \leq yzbca \in SSL_2L_3L_1 \subseteq L_2L_3L_1$. Thus $x \in (L_2L_3L_1]$. Hence $(L_1L_2L_3] \subseteq (L_2L_3L_1]$. Similarly, we can prove that $(L_2L_3L_1] \subseteq (L_1L_2L_3]$. Thus $(L_1L_2L_3] = (L_2L_3L_1]$. Proceeding in the same manner we can show that, $(L_1L_2L_3] = (L_{\sigma(1)}L_{\sigma(2)}L_{\sigma(3)}]$ for every permutation σ of $\{1, 2, 3\}$. For the second part, let $a \in L_1 \subseteq S$. Since S is left regular, $a \in (Sa^2] \subseteq (S(Sa^2]a] = (SSa^3] \subseteq (SSL_1L_1L_1] \subseteq (L_1L_1L_1] \subseteq ({L_1}^3]$. Also $({L_1}^3] \subseteq (SSL_1] \subseteq (L_1] = L_1$. This completes the proof. $\square$

Theorem 2.2.11. *Let S be an ordered ternary semigroup. Then the followings are equivalent:*

(i) If R_1, R_2 and R_3 are right ideals of S, then $(R_1R_2R_3] = (R_{\sigma(1)}R_{\sigma(2)}R_{\sigma(3)}]$ for every permutation σ of $\{1, 2, 3\}$ and $({R_1}^3] = (R_1]$.

(ii) If R_1, R_2 and R_3 are left ideals of S, then $R_1 \cap R_2 \cap R_3 = (R_1R_2R_3]$.

(iii) S is right regular and right simple.

Proof. Proof is similar to Theorem 2.2.10 □

2.3 Regular ordered ternary semigroups

In this section, we discuss the behaviour and properties of ideals, quasi-ideals, bi-ideals, semiprime ideals on regular ordered ternary semigroup.

Definition 2.3.1. *An ordered ternary semigroup S is said to be regular if $A \subseteq (ASA]$ for every $A \subseteq S$.*

An element a of an ordered ternary semigroup S is said to be regular if there exists an element $x \in S$ such that $a \leq axa$. If all elements of S are regular then S is said to be a regular ordered ternary semigroup.

Lemma 2.3.2. *Let S be a regular ordered ternary semigroup and I be a lateral ideal of S, then I is regular.*

Proof. Let I be a lateral ideal of a regular ordered ternary semigroup S. Let $A \subseteq I$. Since S is regular, $A \subseteq (ASA]$. Now $A \subseteq (ASA] \subseteq (AS(ASA]] = (ASASA] = (A(SAS)A] \subseteq (A(SIS)A] \subseteq (AIA]$. Consequently, I is a regular ordered ternary semigroup. □

Corollary 2.3.3. *In a regular ordered ternary semigroup S every ideal of S is regular.*

Theorem 2.3.4. *[65, N. Lekkoksung] In a regular ordered ternary semigroup S, the following are equivalent :*

(i) S is regular;

(ii) $(RML] = R \cap M \cap L$ where R, M, L are right ideal, lateral ideal and left ideal of S respectively.

Theorem 2.3.5. *[65, N. Lekkoksung] In a regular ordered ternary semigroup S, the following are equivalent :*

(i) S is regular;

(ii) for every bi-ideal B of S, $(BSBSB] = B$;

(iii) for every quasi-ideal Q of S, $(QSQSQ] = Q$.

Theorem 2.3.6. *An ordered ternary subsemigroup B of a regular ordered ternary semigroup S is a bi-ideal of S if and only if $B = (BSB]$.*

Proof. Let S be a regular ordered ternary semigroup and $B \subseteq S$. Let $B = (BSB]$. Then $B = (BSB] = (BS(BSB]] = (BS(BSB)] = (BSBSB]$. Thus $BSBSB \subseteq (BSBSB] = B$. It remains to show that $(B] = B$. Let $x \in (B]$. Then $x \in ((BSB]] = (BSB] = B$. Thus $(B] \subseteq B$. Hence B is a bi-ideal of S.

Conversely, let B be any bi-ideal of a regular ordered ternary semigroup S. Since S is regular and $B \subseteq S$ we have $B \subseteq (BSB]$. Again $(BSB] \subseteq (BS(BSB]] = (BS(BSB)] = (BSBSB] \subseteq (B] = B$. Thus $B = (BSB]$. $\square$

Theorem 2.3.7. *In a regular ordered ternary semigroup S, every bi-ideal of S is a quasi-ideal of S.*

Proof. Let B be a bi-ideal of a regular ordered ternary semigroup S. Then $BSBSB \subseteq B$ and $(B] = B$. Now $SS(SSB] \subseteq (S](S](SSB] \subseteq (SSSSB] \subseteq (SSB]$ and $((SSB]] = (SSB]$. Hence $(SSB]$ is a left ideal of S. Also $(BSS]SS \subseteq (BSS](S](S] \subseteq (BSSSS] \subseteq (BSS]$ and $((BSS]] = (BSS]$. Thus $(BSS]$ is a right ideal of S. Again $S(SBS \cup SSBSS]S \subseteq (S](SBS \cup SSBSS](S] \subseteq (SSBSS \cup SSSBSSS] \subseteq (SSBSS \cup SBS]$ and $((SBS \cup SSBSS]] = (SBS \cup SSBSS]$. So $(SBS \cup SSBSS]$ is a lateral ideal of S. From Theorem 2.3.5, we have $(BSS] \cap (SBS \cup SSBSS] \cap (SSB] = ((BSS](SBS \cup SSBSS](SSB]] = ((BSS)(SBS \cup SSBSS)(SSB)] = (BSSSBSSSB \cup BSSSSBSSSSB] \subseteq (BSBSB \cup BSSBSSB] \subseteq (BSBSB \cup BSB] = (BSBSB] \cup (BSB] = B \cup B = B$, by using Theorem 2.3.5 and Theorem 2.3.6. Consequently, B is a quasi-ideal of S. $\square$

Definition 2.3.8. *An ordered ternary semigroup S is called commutative if $x_1x_2x_3 = x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$ for every permutation σ of $\{1, 2, 3\}$ and $x_1, x_2, x_3 \in S$.*

Theorem 2.3.9. *Let S be a commutative ordered ternary semigroup. Then S is regular if and only if every ideal of S is semiprime.*

Proof. Let S be a commutative regular ordered ternary semigroup and I be any ideal of S. Let $A^3 \subseteq I$ for $A \subseteq S$. Since S is regular and $A \subseteq S$ we have $A \subseteq (ASA] = (AAS] \subseteq (A(ASA]S] = (A(ASA)S] = (A(AAS)S] = ((AAA)SS] = (A^3SS] \subseteq (ISS] \subseteq (I] = I$. Thus I is a semiprime ideal of S.

Conversely, we assume that every ideal of commutative ordered ternary semigroup S is semiprime. Let $A \subseteq S$. Then $(ASA]$ is an ideal of S.

Case 1 : If $(ASA] = (S] = S$, we get our conclusion.

Case 2 : If $(ASA] \neq S$. Then by hypothesis, $(ASA]$ is a semiprime ideal of S. Now $A^3 = AAA \subseteq ASA \subseteq (ASA]$ implies that $A \subseteq (ASA]$. Consequently, S is regular. □

Definition 2.3.10. *Let S be an ordered ternary semigroup. A nonempty subset B_w of S is called a weak bi-ideal of S, if*
(i) $bSbSb \subseteq B_w$ for all $b \in B_w$ and
(ii) $(B_w] = B_w$.

Clearly, we have the following results :

Lemma 2.3.11. *Every bi-ideal of an ordered ternary semigroup S is a weak bi-ideal of S.*

Lemma 2.3.12. *The intersection of arbitrary set of weak bi-ideals of a ordered ternary semigroup S is either empty or a weak bi-ideal of S.*

Theorem 2.3.13. *Let S be an ordered ternary semigroup. Then S is regular if and only if $B_w = (\bigcup_{b \in B_w} bSbSb]$ for any weak bi-ideal B_w of S.*

Proof. Let S be a regular ordered ternary semigroup and B_w be any weak bi-ideal of S. Then $bSbSb \subseteq B_w$ for all $b \in B_w$. So $\bigcup_{b \in B_w} bSbSb \subseteq B_w$. This implies that $(\bigcup_{b\in B_w} bSbSb] \subseteq (B_w] = B_w$. Let $b \in B_w$. Since S is regular, there exists $x \in S$ such that $b \leq bxb$. So $b \leq bxb \leq bxbxb \in bSbSb \subseteq \bigcup_{b\in B_w} bSbSb$. Therefore, $b \in (\bigcup_{b\in B_w} bSbSb]$. Thus $B_w \subseteq (\bigcup_{b\in B_w} bSbSb]$. Hence $B_w = (\bigcup_{b\in B_w} bSbSb]$.

Conversely, let $B_w = (\bigcup_{b\in B_w} bSbSb]$, where B_w is a weak bi-ideal of S. Let R be a right ideal, M be a lateral ideal and L be a left ideal of S. Since every left, right and lateral ideal of an ordered ternary semigroup S is a bi-ideal of S, it follows that every left, right and lateral ideal of an ordered ternary semigroup S is a weak bi-ideal of S. So R, M, L are weak bi-ideals of S. Thus by Lemma 2.3.12, $R\cap M\cap L$ is a weak bi-ideal of S. Clearly, $(RML] \subseteq R\cap M\cap L$. Now let $a \in R\cap M\cap L$. Since $R\cap M\cap L$ is weak bi-ideal of S, by hypothesis we have $R \cap M \cap L = (\bigcup_{x\in R\cap M\cap L} xSxSx]$. Then $a \leq xs_1xs_2x$ for some $x \in R \cap M \cap L$ and $s_1, s_2 \in S$. So $a \leq xs_1xs_2ys_3ys_4y$ for some $x, y \in R \cap M \cap L$ and $s_1, s_2, s_3, s_4 \in S$. This implies that $a \in (RML]$. Thus $R\cap M \cap L \subseteq (RML]$ and hence $(RML] = R\cap M\cap L$. Consequently, S is a regular ordered ternary semigroup by Theorem 2.3.4. □

2.4 Completely regular ordered ternary semigroups

In this section, we introduce and study completely regular ordered ternary semigroup. We also characterize completely regular ordered ternary semigroup by using quasi-ideals, bi-ideals and semiprime ideals.

Definition 2.4.1. *An ordered ternary semigroup S is said to be completely regular if it is regular, left regular and right regular i.e. $A \subseteq (ASA]$, $A \subseteq (SA^2]$ and $A \subseteq (A^2S]$ for every $A \subseteq S$.*

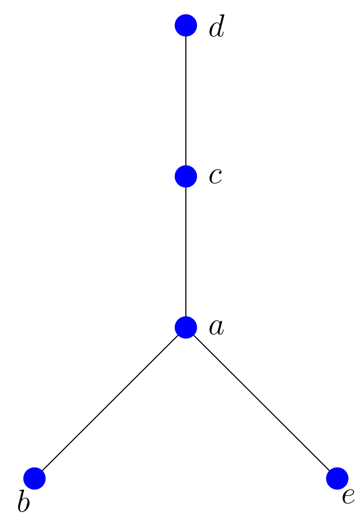
Example 2.4.2. *Let $S = \{a, b, c, d, e\}$ be an ordered ternary semigroup with the ternary operation defined on S as $abc = a * (b * c)$, where the binary operation * is* *defined by*

$*$	a	b	c	d	e
a	a	a	c	d	a
b	a	b	c	d	a
c	a	a	c	d	a
d	a	a	c	d	a
e	a	a	c	d	e

and the order defined as $\leq \; := \{(a,a),(a,c),(a,d),(b,b),(b,d),(b,a),(b,c),$
$(c,c),(c,d),(d,d),(e,a),(e,c),(e,d),(e,e)\}.$

Now we have the covering relation " $\prec$ " *and the figure of* S *as follows :*

$\prec \; = \{(a,c),(b,a),(c,d),(e,a)\}$



Then S *is a completely regular ordered ternary semigroup.*

Theorem 2.4.3. *In an ordered ternary semigroup* S, *the following conditions are equivalent :*

(i) S *is completely regular;*

(ii) $A \subseteq (A^2SA^2]$ *for every* $A \subseteq S$.

Proof. $(i) \Longrightarrow (ii)$.

Let S be a completely regular ordered ternary semigroup. Then for any $A \subseteq S$, we have $A \subseteq (ASA] = ((A^2S]S(SA^2]] = ((A^2S)S(SA^2)] = (A^2SSSA^2] \subseteq (A^2SA^2]$.

$(ii) \Longrightarrow (i)$.

Let $A \subseteq S$. Then $A \subseteq (A^2SA^2] = (A(ASA)A] \subseteq (ASA]$, $A \subseteq (A^2SA^2] = ((A^2S)A^2] \subseteq (SA^2]$ and $A \subseteq (A^2SA^2] = (A^2(SA^2)] \subseteq (A^2S]$. This implies that S is regular, left regular and right regular. Consequently, S is completely regular ordered ternary semigroup. $\square$

In the following result we provide another characterization of completely regular ordered ternary semigroup in terms of quasi-ideal.

Theorem 2.4.4. *Let S be an ordered ternary semigroup. Then S is completely regular if and only if every quasi-ideal of S is a completely regular subsemigroup of S.*

Proof. Let S be a completely regular ordered ternary semigroup and Q be a quasi-ideal in S. Since $\{\} \neq Q \subseteq S$ and $Q^3 \subseteq QSS \cap SQS \cap SSQ \subseteq (QSS] \cap (SQS] \cap (SSQ] \subseteq Q$, Q is a subsemigroup of S. Let $A \subseteq Q \subseteq S$. We have to show that Q is completely regular. Since S is completely regular and $A \subseteq S$, we have $A \subseteq (ASA] = ((A^2S]S(SA^2]] = ((A^2S)S(SA^2)] = (A^2SSSA^2] \subseteq (A^2SA^2] = (A(ASA)A] \subseteq (A(ASA]SAA] = (A(ASA)SAA] = (A(ASASA)A]$. Now $ASASA \subseteq SSASS \subseteq SSQSS$, $ASASA \subseteq SSA \subseteq SSQ$ and $ASASA \subseteq ASS \subseteq QSS$. Therefore, $ASASA \subseteq SSQ \cap SSQSS \cap QSS \subseteq (SSQ] \cap (SSQSS] \cap (QSS] \subseteq Q$. Hence $A \subseteq (AQA]$. Again $A \subseteq (ASA] \subseteq (AS(SA^2]] = (AS(SA^2)] \subseteq (ASS(SA^2]A] = (ASS(SA^2)A] = ((ASSSA)A^2] \subseteq ((ASA)A^2] \subseteq (AS(ASA]A^2] = (AS(ASA)A^2] = ((ASASA)A^2] \subseteq (QA^2]$ and $A \subseteq (ASA] \subseteq ((A^2S]SA] = ((A^2S)SA] \subseteq (A(A^2S]SSA] = (A(A^2S)SSA] = (A^2(ASSSA)] \subseteq (A^2(ASA)] \subseteq (A^2(ASA]SA] = (A^2(ASA)SA] = (A^2(ASASA)] \subseteq (A^2Q]$. Thus Q is regular, left regular and right regular. Consequently, Q is completely regular subsemigroup of S.

Conversely, suppose that every quasi-ideal of S is a completely regular subsemigroup of S. Since S itself a quasi-ideal in S, S is completely regular. $\square$

Theorem 2.4.5. *Let S be an ordered ternary semigroup. Then S is left regular and right regular if and only if every quasi-ideal of S is semiprime.*

Proof. Let S be a left regular and right regular ordered ternary semigroup and Q be a quasi-ideal of S. Let $A \subseteq S$ and $A^3 \subseteq Q$. Since S is left regular and right regular, $A \subseteq (SA^2]$ and $A \subseteq (A^2S]$. Now $A \subseteq (SA^2] \subseteq (S(SA^2]A] = (S(SA^2)A] = (SSA^3] \subseteq (SSQ]$, $A \subseteq (A^2S] \subseteq (A(A^2S]S] = (A(A^2S)S] = (A^3SS] \subseteq (QSS]$ and $A \subseteq (SA^2] \subseteq (SA(A^2S]] = (SA^3S] \subseteq (SQS]$. Therefore, $A \subseteq (SSQ] \cap (SQS] \cap (QSS] \subseteq Q$. Hence Q is semiprime.

Conversely, suppose that every quasi-ideal of S is semiprime. Since every right ideal and left ideal of S is a quasi-ideal of S, every right ideal and left ideal are semiprime. Now by using Theorem 2.2.9, we find that S is left regular and right regular. □

From Theorem 2.4.5, we have the following result :

Corollary 2.4.6. *If S is a completely regular ordered ternary semigroup then quasi-ideals of S are semiprime.*

The converse of the above result does not hold. This follows from the following example :

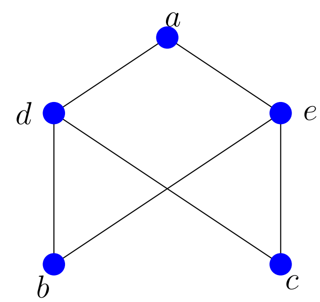
Example 2.4.7. *Let $S = \{a, b, c, d, e\}$ be an ordered ternary semigroup with ternary operation product defined on S by $abc = a*(b*c)$, where binary operation * is defined as*

*	a	b	c	d	e
a	a	e	e	a	e
b	d	b	b	d	b
c	d	b	b	d	b
d	d	b	b	d	b
e	a	e	e	a	e

and the order defined by $\leq\ := \{(a,a),(b,a),(b,b),(b,d),(b,e),(c,a),$
$(c,c),(c,d),(c,e),(d,d),(d,a),(e,a),(e,e)\}$.

We give the covering relation " $\prec$ *" and the figure of* S *as follows :*

$$\prec\ = \{(b,d),(b,e),(c,d),(c,e),(d,a),(e,a)\}$$



Then S *is a left regular and right regular ordered ternary semigroup. So every quasi-ideal of* S *is semiprime by Theorem 2.4.5 but* S *is not completely regular. In fact it is not regular since* $c \in S$ *is not regular.*

In the following result we represent a completely regular ordered ternary semigroup in terms of bi-ideal.

Theorem 2.4.8. *An ordered ternary semigroup* S *is completely regular if and only if every bi-ideal of* S *is semiprime.*

Proof. Let S be a completely regular ordered ternary semigroup and B be any bi-ideal of S. Let $A \subseteq S$ and $A^3 \subseteq B$. Since S is completely regular ordered ternary semigroup and $A \subseteq S$ we have,
$A \subseteq (A^2SA^2] \subseteq (A(A^2SA^2]S(A^2SA^2]A] = (A(A^2SA^2)S(A^2SA^2)A] = ((A^3SA^2S)(A^2S)A^3] \subseteq ((A^3SA^2S)(A^2SA^2](A^2SA^2]SA^3] = ((A^3SA^2S)(A^2SA^2)(A^2SA^2)SA^3] = (A^3(SA^2SA^2S)A^3(ASA^2S)A^3] \subseteq (BSBSB] \subseteq (B] = B$. Therefore B is semiprime.

Conversely, suppose that every bi-ideal of S is semiprime. Let $\{\} \neq A \subseteq S$. Then we have $A^2SA^2 \subseteq S$ i.e. $(A^2SA^2] \subseteq (S] = S$. Now $(A^2SA^2]S(A^2SA^2]S(A^2SA^2] \subseteq (A^2SA^2](S](A^2SA^2](S](A^2SA^2] \subseteq (A^2SA^2SA^2SA^2SA^2SA^2] \subseteq (A^2SA^2]$. Again we have $((A^2SA^2]] = (A^2SA^2]$. Thus $(A^2SA^2]$ is a bi-ideal in S. Now $A^9 = AAAAAAAAA = AA(AAAAA)AA \subseteq AASAA = A^2SA^2 \subseteq (A^2SA^2]$. By hypothesis, since every bi-ideal is semiprime, $A^9 = (A^3)^3 \subseteq (A^2SA^2] \Longrightarrow A^3 \subseteq (A^2SA^2] \Longrightarrow A \subseteq (A^2SA^2]$. Since A is arbitrary, $A \subseteq (A^2SA^2]$ for every $A \subseteq S$. Hence S is completely regular ordered ternary semigroup. □

Theorem 2.4.9. *Let S be a commutative ordered ternary semigroup. Then the bi-ideals of S are completely prime if and only if the bi-ideals of S form a chain and S is completely regular.*

Proof. Let the bi-ideals of S are completely prime. Let A and B are bi-ideals of S. Then $(BAB] \subseteq S$. Now $(BAB]S(BAB]S(BAB] \subseteq (BAB](S](BAB](S](BAB] \subseteq (BABSBABSBAB] \subseteq (BASSSASSSAB] \subseteq (BASASAB] \subseteq (BAB]$ and $((BAB]] = (BAB]$. Thus $(BAB]$ is a bi-ideal of S. Now $BAB \subseteq (BAB]$. Since $(BAB]$ is a completely prime bi-ideal of S, then $(BAB]$ is prime bi-ideal of S. Therefore, we have either $A \subseteq (BAB]$ or $B \subseteq (BAB]$. If $A \subseteq (BAB]$, then $A \subseteq (B(BAB]B] = (B(ABB]B] \subseteq (BSBSB] \subseteq (B] = B$. If $B \subseteq (BAB]$, then $B \subseteq (BA(BAB]] = (BA(BBA]] = (BABBA] \subseteq (BAB(BAB]A] = (ABBBABA] = (ABABA] \subseteq (A] = A$. Thus for any two bi-ideals A and B we have either $A \subseteq B$ or $B \subseteq A$. Thus the bi-ideals of S form a chain.
Again let $a \in S$. Then $(a^2Sa^2]S(a^2Sa^2]S(a^2Sa^2] \subseteq (a^2Sa^2Sa^2Sa^2Sa^2Sa^2] \subseteq (a^2Sa^2]$. So, $(a^2Sa^2]$ is a bi-ideal in S. Now $a^3aa = a^5 \in (a^2Sa^2]$ which implies that $a^3 \in (a^2Sa^2]$ or $a \in (a^2Sa^2]$. Again $a^3 \in (a^2Sa^2] \Longrightarrow a \in (a^2Sa^2]$. Thus S is completely regular.

For the converse part, let the bi-ideals of S form a chain and S is completelely regular. Now let $a \in S$ such that $a^3 \in B$ where B is a bi-ideal of S. Now $a \in (a^2Sa^2] \subseteq (a(a^2Sa^2]Sa^2] = (a^3Sa^2Sa^2] = (a^3Sa(a^2Sa^2]S(a^2Sa^2]a] = (a^3Sa^3Sa^2Sa^2Sa^3] \subseteq (a^3Sa^3Sa^3] \subseteq (BSBSB] \subseteq (B] = B$. Then B is completely semiprime. Now we have to prove that $(B(x)B(y)B(z)] = B(x) \cap B(y) \cap B(z)$. Let $p \in B(x) \cap B(y) \cap B(z) \subseteq B(x)$. Similarly $p \in B(y)$ and $p \in B(z)$. Thus $p^3 \in B(x)B(y)B(z) \subseteq$

$(B(x)B(y)B(z)]$. Now $(B(x)B(y)B(z)]$ is also a bi-ideal in S which is completely semiprime. Hence $p \in (B(x)B(y)B(z)]$. Thus $B(x) \cap B(y) \cap B(z) \subseteq (B(x)B(y)B(z)]$. Again let $q \in (B(x)B(y)B(z)]$. Then $q \leq x_1y_1z_1$ for some $x_1 \in B(x)$, $y_1 \in B(y)$ and $z_1 \in B(z)$. Then $q^3 \leq (x_1y_1z_1)(x_1y_1z_1)(x_1y_1z_1) = (x_1y_1z_1)(y_1x_1z_1)(y_1z_1x_1) \in B(x)B(y)B(z)B(y)B(x)B(z)B(y)B(z)B(x) \subseteq B(x)SB(x)SB(x) \subseteq B(x)$. Thus $q^3 \in (B(x)] = B(x)$ and so $q \in B(x)$. Thus $(B(x)B(y)B(z)] \subseteq B(x)$. Similarly, $(B(x)B(y)B(z)] \subseteq B(y)$ and $(B(x)B(y)B(z)] \subseteq B(z)$. Hence $(B(x)B(y)B(z)] \subseteq B(x) \cap B(y) \cap B(z)$. So, $(B(x)B(y)B(z)] = B(x) \cap B(y) \cap B(z)$.

Now let $a, b, c \in S$ such that $abc \in B$. For $a, b \in S$ either $B(a) \subseteq B(b)$ or $B(b) \subseteq B(a)$. For $b, c \in S$ either $B(b) \subseteq B(c)$ or $B(c) \subseteq B(b)$. For $c, a \in S$ either $B(c) \subseteq B(a)$ or $B(a) \subseteq B(c)$.

Case 1: If $B(a) \subseteq B(b)$, $B(a) \subseteq B(c)$ and $B(b) \subseteq B(c)$, then $a \in B(a) = B(a) \cap B(a) = B(a) \cap (B(b) \cap B(c)) = B(a) \cap B(b) \cap B(c) = (B(a)B(b)B(c)] \subseteq (B] = B$.

Case 2: If $B(a) \subseteq B(b)$, $B(a) \subseteq B(c)$ and $B(c) \subseteq B(b)$. This proof is similar to Case 1.

Case 3: If $B(b) \subseteq B(c)$, $B(b) \subseteq B(a)$ and $B(c) \subseteq B(a)$, then $b \in B(b) = B(b) \cap B(b) = B(b) \cap (B(c) \cap B(a)) = B(a) \cap B(b) \cap B(c) = (B(a)B(b)B(c)] \subseteq (B] = B$.

Case 4: If $B(b) \subseteq B(c)$, $B(b) \subseteq B(a)$ and $B(a) \subseteq B(c)$. This proof is similar to Case 3.

Case 5: If $B(c) \subseteq B(a)$, $B(c) \subseteq B(b)$ and $B(a) \subseteq B(b)$, then $c \in B(c) = B(c) \cap B(c) = B(c) \cap (B(a) \cap B(b)) = B(a) \cap B(b) \cap B(c) = (B(a)B(b)B(c)] \subseteq (B] = B$.

Case 6: If $B(c) \subseteq B(a)$, $B(c) \subseteq B(b)$ and $B(b) \subseteq B(a)$. This proof is similar to Case 5.

Thus in all cases we have, either $a \in B$ or $b \in B$ or $c \in B$. Hence B is completely prime bi-ideal in S. □

2.5 Intra-regular ordered ternary semigroups

In this section, we mainly characterize intra-regular ordered ternary semigroup by using properties of ideals.

Definition 2.5.1. *An ordered ternary semigroup S is called intra-regular if for every $a \in S$, there exist $x, y \in S$ such that $a \leq xa^3y$ or equivalently, $a \in (Sa^3S]$ for all $a \in S$.*

In otherwords, an ordered ternary semigroup S is intra-regular if $A \subseteq (SA^3S]$ for every $A \subseteq S$.

Now we have the following result :

Lemma 2.5.2. *If S is a left (resp. right) regular ordered ternary semigroup, then S is intra-regular.*

Proof. Let S be left regular ordered ternary semigroup and $A \subseteq S$. Then $A \subseteq (SA^2] \subseteq (S(SA^2]A] = (S(SA^2)A] \subseteq (SS(SA^2]AA] = (SS(SA^2)AA] = (SSSA^3A] \subseteq (SSSA^3S] \subseteq (SA^3S]$. Thus S is intra-regular.

Similarly, we can prove the result for right regular ordered ternary semigroup. □

But the converse of the above result is not true.

In the following, we give an example of an intra-regular ordered ternary semigroup which is not left regular ordered ternary semigroup.

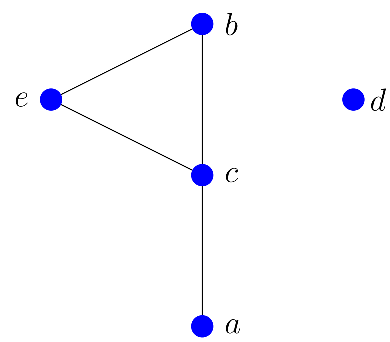
Example 2.5.3. *Let $S = \{a, b, c, d, e\}$ be an ordered ternary semigroup with ternary operation defined on S by $abc = a * (b * c)$, where the binary operation * is defined as*

$*$	a	b	c	d	e
a	a	b	a	d	a
b	a	b	a	d	a
c	a	b	a	d	a
d	a	b	a	d	a
e	a	b	a	d	a

and the order defined by $\leq\ := \{(a,a),(a,b),(a,c),(a,e),(b,b),(c,c),$
$(c,b),(c,e),(d,d),(e,b),(e,e)\}.$

We give the covering relation " $\prec$ *" and the figure of* S *as follows :*

$$\prec\ = \{(a,c),(c,b),(c,e),(e,b)\}$$



Then $(S,.,\leq)$ *is an intra-regular ordered ternary semigroup but not left regular, since* c *and* e *are not left regular elements of* S.

Now we can easily prove the following result :

Theorem 2.5.4. *In an intra-regular ordered ternary semigroup* S*,* $L\cap M\cap R\subseteq (LMR]$*, where* L*,* M*,* R *are left ideal, lateral ideal and right ideal of* S *respectively.*

Clearly, every ideal of an ordered ternary semigroup S is also a lateral ideal of S. Certainly a lateral ideal of S is not necessarily an ideal of S. But in particular, for intra-regular ordered ternary semigroup S we have the following result :

Theorem 2.5.5. *Let* S *be an intra-regular ordered ternary semigroup. Then a non-empty subset* I *of* S *is an ideal of* S *if and only if* I *is a lateral ideal of* S.

Proof. Clearly, if I is an ideal of S, then I is a lateral ideal of S.

Conversely, assume that I is a lateral ideal of an intra-regular ordered ternary semigroup S. Then $SIS\subseteq I$ and $(I]=I$. Since S is intra-regular and $I\subseteq S$ we have $I\subseteq (SI^3S]$.

Now $SSI \subseteq (SSI] \subseteq (SS(SI^3S]] = (SS(SI^3S)] = (SSSI^3S] \subseteq (SSS(SI^3S]I^2S] = (SSS(SI^3S)I^2S] = ((SSSSI)I(ISIIS)] \subseteq (SIS] \subseteq (I] = I$ and $ISS \subseteq (ISS] \subseteq ((SI^3S]SS] = ((SI^3S)SS] = (SI^3SSS] \subseteq (SI^2(SI^3S]SSS] = (SI^2(SI^3S)SSS] = ((SIISI)I(ISSSS) \subseteq (SIS] \subseteq (I] = I$. Thus I is a left ideal as well as a right ideal of S. Consequently, I is an ideal of S. □

Lemma 2.5.6. *Let S be an intra-regular ordered ternary semigroup and I be a lateral ideal of S then I is intra-regular.*

Proof. Let S be an intra-regular ordered ternary semigroup and I be a lateral ideal of S. Let $A \subseteq I \subseteq S$. Since S is intra-regular, it follows that $A \subseteq (SA^3S]$. Now we have $A \subseteq (SA^3S] \subseteq (S(SA^3S](SA^3S](SA^3S]S] = (S(SA^3S)(SA^3S)(SA^3S)S] = ((SSA^3SS)A^3(SSA^3SS)] \subseteq ((SSSASSS)A^3(SSSASSS)] \subseteq ((SAS)A^3(SAS)] \subseteq ((SIS)A^3(SIS)] \subseteq (IA^3I]$.

Consequently, I is an intra regular ordered ternary semigroup. □

Similarly, by Lemma 2.5.6 we can prove the following result :

Corollary 2.5.7. *Let S be an intra-regular ordered ternary semigroup and I be an ideal of S then I is intra-regular.*

Theorem 2.5.8. *Let S be an intra-regular ordered ternary semigroup. Let I be an ideal of S and J be an ideal of I. Then J is an ideal of the entire ordered ternary semigroup S.*

Proof. It is sufficient to show that J is a lateral ideal of S. Now $J \subseteq I \subseteq S$ and $SJS \subseteq SIS \subseteq I$. We have to show that $SJS \subseteq J$. From Corollary 2.5.7, it follows that I is an intra-regular ordered ternary semigroup. Also $SJS \subseteq I$. So we have $(SJS) \subseteq (I(SJS)^3I] = (I(SJS)(SJS)(SJS)I] = ((ISJSS)J(SSJSI)] \subseteq ((ISISS)J(SSISI)] \subseteq ((IIS)J(SII)] \subseteq ((ISS)J(SSI)] \subseteq (IJI] \subseteq (J] = J$. Consequently, J is a lateral ideal of S. □

Theorem 2.5.9. *Let S be an ordered ternary semigroup. Then S is intra-regular if and only if every ideal of S is semiprime.*

Proof. Let S be an intra-regular ordered ternary semigroup and I be an ideal of S. Let $A^3 \subseteq I$ for $A \subseteq S$. Since S is intra-regular ordered-ternary semigroup, we have $A \subseteq (SA^3S] \subseteq (SIS] \subseteq (I] = I$. Hence I is a semiprime ideal of S.

Conversely, suppose that every ideal of S is semiprime. Let $A \subseteq S$. Since $A^3 \subseteq I(A^3)$ and by hypothesis $I(A^3)$ is a semiprime ideal of S, so $A \subseteq I(A^3)$.
Now $I(A^3) = (A^3 \cup SSA^3 \cup SA^3S \cup SSA^3SS \cup A^3SS] = (A^3] \cup (SSA^3] \cup (SA^3S] \cup (SSA^3SS] \cup (A^3SS]$.

Case 1: If $A \subseteq (A^3]$. Then $A \subseteq (A(A^3]A] = (A(A^3)A] \subseteq (SA^3S]$.

Case 2: If $A \subseteq (SSA^3]$ then $A^3 \subseteq (SSA^3]A^2$. Hence $A \subseteq (SS(SSA^3]A^2] = (SS(SSA^3)A^2] = (SSSSAAAAA] \subseteq (SSSSSAAAS] = (SSSSSA^3S] \subseteq (SA^3S]$.

Case 3: If $A \subseteq (SA^3S]$ we get our conclusion.

Case 4: If $A \subseteq (SSA^3SS]$ then $A^3 \subseteq A(SSA^3SS]A$. Hence $A \subseteq (SSA(SSA^3SS]ASS] = (SSA(SSA^3SS)ASS] = (SSASSA^3SSASS] \subseteq (SSSSSA^3SSSSS] \subseteq (SA^3S]$.

Case 5: If $A \subseteq (A^3SS]$ then $A^3 \subseteq A^2(A^3SS]$. Hence $A \subseteq (A^2(A^3SS]SS] = (A^2(A^3SS)SS] = (AAAAASSSS] \subseteq (SAAASSSSS] = (SA^3SSSSS] \subseteq (SA^3S]$.

In each cases above we have seen that S is intra-regular. Consequently, S is an intra-regular ordered ternary semigrouop. □

Semigroup Cover Of Ternary Semigroup

Chapter-3

Chapter 3

Semigroup cover of ternary semigroup

3.1 Introduction

In this chapter, our main aim is to study the relation between a ternary semigroup S and the semigroup cover $Q(S)$ of the ternary semigroup S by their ideals, bi-ideals, quasi-ideals, prime ideals, completely prime ideals, semiprime ideals, completely semiprime ideals. Then we discuss about left regularity, right regularity, complete regularity and intra-regularity of S and $Q(S)$ by using these ideals. Then we investigate the isomorphism problem between two ternary semigroups and their corresponding semigroup covers. In the last section, we introduce a partial order in $Q(S)$ by help of the partial ordered defined in S and discuss lattice structure in between S and $Q(S)$. M. L Santiago and S. Sri Bala [78] introduced the notion of semigroup cover of ternary semigroup in the following way.

For a ternary semigroup S, a semigroup $Q(S)$ was constructed in such a way that S is embedded in $Q(S)$ as a ternary subsemigroup. The construction of the semigroup cover of a ternary semigroup is as follows : For $a, b \in S$, suppose that $L(a, b)$ and $R(a, b)$ are left and right multiplication operators on S given by $L(a, b)c = abc$

and $R(a,b)c = cba$ for all $c \in S$. The condition of associativity is equivalent to either of the following :

$$L(a,b)L(c,d) = L(a,bcd) = L(abc,d)$$
$$R(a,b)R(c,d) = R(a,dcb) = R(cba,d).$$

Put $m(a,b) = \Big(L(a,b), R(b,a)\Big)$. Let $M = \{m(a,b) : a,b \in S\}$ and define a product on M by the following way:

$$m(a,b)m(c,d) = \Big(L(a,b)L(c,d), R(d,c)R(b,a)\Big)$$

Thus $m(a,b)m(c,d) = m(abc,d) = m(a,bcd)$. Moreover M satisfies the associative law for binary multiplication. Thus M can be made into a semigroup. Consider the set $Q(S) = S \cup M$, where S and M are two disjoint sets. Define multiplication on $Q(S)$ as follows :

$$ab = \begin{cases} m(a,b) & \text{if } a,b \in S \\ m(a_1,a_2)m(b_1,b_2) & \text{if } a = m(a_1,a_2),\ b = m(b_1,b_2) \in M \\ L(a_1,a_2)b & \text{if } a = m(a_1,a_2) \in M,\ b \in S \\ R(b_2,b_1)a & \text{if } a \in S,\ b = m(b_1,b_2) \in M \end{cases} \tag{3.1}$$

This product is associative in $Q(S)$. Thus $Q(S)$ is a semigroup and $M = S^2$ in $Q(S)$. For all $a,b,c \in S$, we have $a(bc) = am(b,c) = R(c,b)a = abc$. The mapping $f : S \longrightarrow Q(S)$ defined by $f(a) = a$ is a monomorphism. Thus the ternary semigroup S is embedded in the semigroup $Q(S)$ as a ternary subsemigroup of $Q(S)$. The semigroup $Q(S)$ is called the “Semigroup Cover” of the ternary semigroup S.

The semigroup cover $Q(S)$ is commutative if and only if the ternary semigroup S is commutative [77].

Throughout this section S denotes a ternary semigroup and $Q(S)$ denotes the semigroup cover of corresponding ternary semigroup S.

3.2 Ideals of ternary semigroup S and semigroup cover $Q(S)$

In this section, we characterize a ternary semigroup S and the semigroup cover $Q(S)$ of the ternary semigroup S by using their different ideals.

Proposition 3.2.1. *[77, Santiago] If I is a left (resp. right) ideal in a ternary semigroup S, then $I \cup SI$ (resp. $I \cup IS$) is a left (resp. right) ideal in the semigroup cover $Q(S)$. If I is an ideal in S, then $I \cup SI \cup IS$ is an ideal in $Q(S)$.*

Proposition 3.2.2. *[77, Santiago] If J is a left (resp. right) ideal in the semigroup cover $Q(S)$, then $J \cap S$ is a left (resp. right) ideal in the ternary semigroup S. Moreover if J is an ideal in $Q(S)$, then $J \cap S$ is an ideal in S.*

Theorem 3.2.3. *Let S be a ternary semigroup. Then every ideal in S is a prime ideal of S if every ideal in $Q(S)$ is a prime ideal of $Q(S)$. If S is a left zero ternary semigroup, then every ideal in $Q(S)$ is a prime ideal of $Q(S)$ if every ideal in S is a prime ideal of S.*

Proof. First suppose that every ideal of $Q(S)$ is a prime ideal of $Q(S)$. Then for every ideal I of $Q(S)$, $JK \subseteq I$ implies that $J \subseteq I$ or $K \subseteq I$ for any ideal J, K of $Q(S)$. Let P be an ideal of S such that $ABC \subseteq P$ for some ideal A, B, C of S. We have to show that $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$. Since A, B, C are ideals in S, by Proposition 3.2.1, we have $A \cup SA \cup AS$, $B \cup SB \cup BS$, $C \cup SC \cup CS$ are ideals in $Q(S)$. Take $\alpha(A) = A \cup SA \cup AS$, $\alpha(B) = B \cup SB \cup BS$, $\alpha(C) = C \cup SC \cup CS$.

Then, $\alpha(A)\alpha(B)$

$= (A \cup SA \cup AS)(B \cup SB \cup BS)$

$= AB \cup ASB \cup ABS \cup SAB \cup SASB \cup SABS \cup ASB \cup ASSB \cup ASBS$

$\subseteq AB \cup ASB \cup ABS \cup SAB \cup AB \cup SABS \cup ASB \cup AB \cup AB$

$= AB \cup ASB \cup ABS \cup SAB \cup SABS$.....................(1)

Similarly, $\alpha(C)\alpha(A) \subseteq CA \cup CSA \cup CAS \cup SCA \cup SCAS$....................(2)

Now from (1) and (2) we have we have,

$\alpha(A)\alpha(B)\alpha(C)\alpha(A)$
$\subseteq (AB \cup ASB \cup ABS \cup SAB \cup SABS)(CA \cup CSA \cup CAS \cup SCA \cup SCAS)$
$= ABCA \cup ABCSA \cup ABCAS \cup ABSCA \cup ABSCAS \cup ASBCA \cup ASBCSA \cup ASBCAS \cup ASBSCA \cup ASBSCAS \cup ABSCA \cup ABSCSA \cup ABSCAS \cup ABSSCA \cup ABSSCAS \cup SABCA \cup SABCSA \cup SABCAS \cup SABSCA \cup SABSCAS \cup SABSCA \cup SABSCSA \cup SABSCAS \cup SABSSCA \cup SABSSCAS$
$\subseteq ABCS \cup ABCSS \cup ABCSS \cup ABSCS \cup ABSCSS \cup ASBCS \cup ASBCSS \cup ASBCSS \cup ASBSCS \cup ASBSCSS \cup ABSCS \cup ABSCSS \cup ABSCSS \cup ABSSCS \cup ABSSCSS \cup SABCS \cup SABCSS \cup SABCSS \cup SABSCS \cup SABSCSS \cup SABSCS \cup SABSCSS \cup SABSCSS \cup SABSSCS \cup SABSSCSS$
$\subseteq ABCS \cup ABC \cup ABC \cup ABC \cup ABCS \cup ASBCS \cup ASBC \cup ASBC \cup ASBC \cup ASBCS \cup ABC \cup ABCS \cup ABCS \cup ABCS \cup ABC \cup SABCS \cup SABC \cup SABC \cup SABC \cup SABCS \cup SABC \cup SABCS \cup SABCS \cup SABCS \cup SABC$
$\subseteq PS \cup P \cup P \cup P \cup PS \cup ASBCS \cup ASBC \cup P \cup PS \cup PS \cup PS \cup P \cup SPS \cup SP \cup SP \cup SP \cup SPS \cup SP \cup SPS \cup SPS \cup SPS \cup SP$
$\subseteq P \cup PS \cup SP \cup ASBCS \cup ASBC.$

Hence, $\Big(\alpha(A)\alpha(B)\alpha(C)\alpha(A)\Big)^2 = \Big(\alpha(A)\alpha(B)\alpha(C)\alpha(A)\Big)\Big(\alpha(A)\alpha(B)\alpha(C)\alpha(A)\Big)$
$\subseteq (P \cup PS \cup SP \cup ASBCS \cup ASBC)(P \cup PS \cup SP \cup ASBCS \cup ASBC)$
$\subseteq PP \cup PSP \cup PPS \cup PASBCS \cup PASBC \cup SPP \cup SPSP \cup SPPS \cup SPASBCS \cup SPASBC \cup PSP \cup PSSP \cup PSPS \cup PSASBCS \cup PSASBC \cup ASBCSP \cup ASBCSSP \cup ASBCSPS \cup ASBCSASBCS \cup ASBCSASBC \cup ASBCP \cup ASBCSP \cup ASBCPS \cup ASBCASBCS \cup ASBCASBC$
$\subseteq PS \cup PSS \cup PSS \cup PSSSSS \cup PSSSS \cup SSP \cup SSSP \cup SSPS \cup SPSSSSS \cup SPSSSS \cup SSP \cup SSSP \cup SSPS \cup PSSSSSS \cup PSSSSS \cup SSSSSP \cup SSSSSSP \cup SSSSSPS \cup SSSSSASBCS \cup SSSSSASBC \cup SSSSP \cup SSSSSP \cup SSSSPS \cup SSSSASBCS \cup SSSSASBC$

$\subseteq PS \cup SP \cup P \cup SASBCS \cup SASBC \cup SABCS \cup SABC$
$\subseteq PS \cup SP \cup P \cup PS \cup P \cup SPS \cup SP$
$\subseteq P \cup SP \cup PS.$

Since P is an ideal in S, we have $P \cup SP \cup PS$ is an ideal in $Q(S)$ and since every ideal of $Q(S)$ is prime, $P \cup SP \cup PS$ is a prime ideal in $Q(S)$. Also $\alpha(A)\alpha(B)$, $\alpha(C)\alpha(A)$, $\alpha(A)\alpha(B)\alpha(C)\alpha(A)$ are ideals in Q(S) since $\alpha(A)$, $\alpha(B)$ and $\alpha(C)$ are ideals in $Q(S)$. So, $\Big(\alpha(A)\alpha(B)\alpha(C)\alpha(A)\Big)^2 \subseteq P \cup SP \cup PS$ implies that $\alpha(A)\alpha(B)\alpha(C)\alpha(A) \subseteq P \cup SP \cup PS$. Similarly, $\alpha(A)\alpha(B)\alpha(C)\alpha(A) \subseteq P \cup SP \cup PS$ implies that $\alpha(A)\alpha(B) \subseteq P \cup SP \cup PS$ or $\alpha(C)\alpha(A) \subseteq P \cup SP \cup PS$. So, $\alpha(A)\alpha(B) \subseteq P \cup SP \cup PS$ implies that $\alpha(A) \subseteq P \cup SP \cup PS$ or $\alpha(B) \subseteq P \cup SP \cup PS$ and $\alpha(C)\alpha(A) \subseteq P \cup SP \cup PS$ implies that $\alpha(C) \subseteq P \cup SP \cup PS$ or $\alpha(A) \subseteq P \cup SP \cup PS$. If $\alpha(A) = A \cup SA \cup AS \subseteq P \cup SP \cup PS$ then $A \subseteq P \cup SP \cup PS$ which implies that $A \subseteq P$ or $A \subseteq SP$ or $A \subseteq PS$. If $A \subseteq SP$ then $A \subseteq SS = M$, which is a contradiction. Since A is an ideal of S then $A \subseteq S$, so A cannot be a subset of M. Thus $A \nsubseteq SP$. Similarly, $A \nsubseteq PS$. Hence we get $A \subseteq P$. Again we can show that $B \subseteq P$ or $C \subseteq P$. Therefore, P is a prime ideal of S and hence every ideal of S is prime ideal.

Conversely, let every ideal of S is prime ideal and S is a left zero ternary semigroup. Let R be an ideal of $Q(S)$ such that $AB \subseteq R$ for ideals A, B of $Q(S)$. Since A and B are ideals in $Q(S)$, by Proposition 3.2.2 we have $A \cap S$ and $B \cap S$ are ideals in S. Now, $(A \cap S)(B \cap S)(B \cap S) = ABB \cap ABS \cap ASB \cap ASS \cap SBB \cap SBS \cap SSB \cap SSS \subseteq ABQ(S) \cap ABQ(S) \cap AQ(S)B \cap A \cap Q(S)BB \cap Q(S)BQ(S) \cap B \cap S \subseteq AB \cap AB \cap AB \cap A \cap B \cap B \cap B \cap S \subseteq R \cap (A \cap B) \cap S \subseteq R \cap Q(S) \cap S = R \cap S$. Since R is an ideal in $Q(S)$, by Proposition 3.2.2 we have $R \cap S$ is an ideal in S. Thus, $(A \cap S)(B \cap S)(B \cap S) \subseteq R \cap S$ implies that $A \cap S \subseteq R \cap S$ or $B \cap S \subseteq R \cap S$ i.e. $A \cap S \subseteq R$ or $B \cap S \subseteq R$. Again since A and B are ideals of $Q(S)$ either $A, B \subseteq S$ or $A, B \subseteq M = S^2$. Then we have the following four cases:

Case 1 : Let $A, B \subseteq S$. Then $A \cap S = A$ and $B \cap S = B$. Thus, $A \cap S \subseteq R$ or $B \cap S \subseteq R$ implies that $A \subseteq R$ or $B \subseteq R$.

Case 2 : Let $A, B \subseteq M = S^2$. Let $a = m(a_1, a_2) \in A$ and $b = m(b_1, b_2) \in B$ where $a_1, a_2, b_1, b_2 \in S$. Since S is a left zero ternary semigroup $a_2 = a_2xy$ for all $x, y \in S$. Take $a_2 = a_2b_1b_2$. Thus $a = m(a_1, a_2) = m(a_1, a_2b_1b_2) = m(a_1, a_2)m(b_1, b_2) \in AB \subseteq R \Longrightarrow A \subseteq R$. Again Take $b_2 = b_2a_1a_2$. Then $b = m(b_1, b_2a_1a_2) = m(b_1, b_2)m(a_1, a_2) = m(b_1, b_2)m(a_1, a_2b_1b_2) = m(b_1, b_2)m(a_1, a_2)m(b_1, b_2) \in BAB \subseteq BR \subseteq Q(S)R \subseteq R \Longrightarrow B \subseteq R$.

Case 3 : Let $A \subseteq S, B \subseteq M = S^2$, By case 1 we can say that if $A \subseteq S$, then $A \subseteq R$. Let $a \in A, b = m(b_1, b_2) \in B$ where $b_1, b_2 \in S$. Take $b_1 = b_1b_2a$ and $a = aab_1$. Then $b = m(b_1, b_2) = m(b_1b_2a, b_2) = m(b_1, b_2)m(a, b_2) = m(b_1, b_2)m(aab_1, b_2) = m(b_1, b_2)m(a, a)m(b_1, b_2) = m(b_1, b_2)aam(b_1, b_2) \in BAAB \subseteq Q(S)Q(S)R \subseteq R \Longrightarrow B \subseteq R$.

Case 4 : Let $A \subseteq M = S^2, B \subseteq S$. Let $a = m(a_1, a_2) \in A$ and $b \in B$ where $a_1, a_2 \in S$. Take $a_2 = a_2bb$. Thus $a = m(a_1, a_2) = m(a_1, a_2bb) = m(a_1, a_2)m(b, b) = m(a_1, a_2)bb \in ABB \subseteq RQ(S) \subseteq R \Longrightarrow A \subseteq R$ and by case 1 we can say that if $B \subseteq S$, then $B \subseteq R$.

In the above four cases we have either $A \subseteq R$ or $B \subseteq R$. Hence R is a prime ideal in $Q(S)$ and so every ideal of $Q(S)$ is prime ideal. □

Theorem 3.2.4. *Let S be a ternary semigroup. Then every ideal of S is semiprime if every ideal of $Q(S)$ is semiprime. Moreover if S is a left zero ternary semigroup, then every ideal of $Q(S)$ is semiprime if every ideal of S is semiprime.*

Proof. Suppose that every ideal of $Q(S)$ is semiprime. Then for every ideal I of $Q(S)$, $J^2 \subseteq I$ implies that $J \subseteq I$ for any ideal J of $Q(S)$. Let P be an ideal of S such that $A^3 \subseteq P$ for an ideal A of S. We have to show that $A \subseteq P$. Since A is an ideal in S, by Proposition 3.2.1, we have $A \cup SA \cup AS$ is an ideal in $Q(S)$. Take $\alpha(A) = A \cup SA \cup AS$.

Then $\alpha(A)^2 = (A \cup SA \cup AS)(A \cup SA \cup AS)$

$= AA \cup ASA \cup AAS \cup SAA \cup SASA \cup SAAS \cup ASA \cup ASSA \cup ASAS$

$\subseteq AA \cup ASS \cup ASS \cup SSA \cup AA \cup SAAS \cup SSA \cup AA \cup AA$

$\subseteq AA \cup A \cup A \cup A \cup AA \cup SAAS \cup A \cup AA \cup AA$

$= A \cup AA \cup SAAS.$

Again $\alpha(A)^4 = \alpha(A)^2\alpha(A)^2 \subseteq (A \cup AA \cup SAAS)(A \cup AA \cup SAAS)$

$= AA \cup AAA \cup ASAAS \cup AAA \cup AAAA \cup AASAAS \cup SAASA \cup SAASAA \cup SAASSAAS$

$\subseteq AA \cup P \cup SSAAS \cup P \cup SSAA \cup SASSAS \cup SAASS \cup SASSAS \cup SAAAAS$ $[Since, A^3 \subseteq P]$

$\subseteq AA \cup P \cup AAS \cup P \cup AA \cup AA \cup SAA \cup AA \cup SSAASS$

$\subseteq AA \cup P \cup AAS \cup P \cup AA \cup AA \cup SAA \cup AA \cup AA$

$= P \cup A \cup AAS \cup SAA.$

Now $\alpha(A)^8 = \alpha(A)^4\alpha(A)^4 \subseteq (P \cup A \cup AAS \cup SAA)(P \cup A \cup AAS \cup SAA)$

$= PP \cup PAA \cup PAAS \cup PSAA \cup AAP \cup AAAA \cup AAAAS \cup AASAA \cup AASP \cup AASAA \cup AASAAS \cup AASSAA \cup SAAP \cup SAAAA \cup SAAAAS \cup SAASAA$

$\subseteq SP \cup PSS \cup PSSS \cup PSSS \cup SSP \cup AAAS \cup AAASS \cup AASSA \cup SSSP \cup AASSA \cup ASSAAS \cup AAAS \cup SSSP \cup SSAAA \cup SAAASS \cup SASSAA$

$\subseteq SP \cup P \cup PS \cup PS \cup P \cup PS \cup PSS \cup AAA \cup SP \cup AAA \cup AAAS \cup AAAS \cup SP \cup AAA \cup SPSS \cup SAAA$

$\subseteq SP \cup P \cup PS \cup PS \cup P \cup PS \cup PSS \cup P \cup SP \cup P \cup PS \cup PS \cup SP \cup P \cup SP \cup SP$

$= P \cup SP \cup PS.$

Since P is an ideal in S, we have $P \cup SP \cup PS$ is an ideal in $Q(S)$ and since every ideal of $Q(S)$ is semiprime, $P \cup SP \cup PS$ is a semiprime ideal in $Q(S)$. Also $\alpha(A)^2$, $\alpha(A)^4$ are ideals in $Q(S)$, since $\alpha(A)$ is an ideal in $Q(S)$. So $\alpha(A)^8 = \alpha(A)^4\alpha(A)^4 \subseteq P \cup SP \cup PS$ implies that $\alpha(A)^4 \subseteq P \cup SP \cup PS$. Similary, $\alpha(A)^4 = \alpha(A)^2\alpha(A)^2 \subseteq P \cup SP \cup PS$ implies that $\alpha(A)^2 \subseteq P \cup SP \cup PS$ and $\alpha(A)^2 = \alpha(A)\alpha(A) \subseteq P \cup SP \cup PS$ implies that $\alpha(A) \subseteq P \cup SP \cup PS$. Thus $A \subseteq \alpha(A) \subseteq P \cup SP \cup PS$. Now $A \subseteq P \cup SP \cup PS$ implies that $A \subseteq P$ or $A \subseteq SP$ or $A \subseteq PS$. But $A \subseteq SP$ implies that $A \subseteq SS = M$, which is a contradiction. Since A is an ideal of S, so $A \subseteq S$. Thus A cannot be a subset of $S^2 = M$. Thus $A \nsubseteq SP$. Similary, $A \nsubseteq PS$. Hence we get that $A \subseteq P$ and so P is a semiprime ideal of S. Therefore, every ideal of S

is semiprime.

Conversely, suppose that S is a left zero ternary semigroup and every ideal of S is a semiprime ideal of S. Let R be an ideal of $Q(S)$ such that $A^2 \subseteq R$ for an ideal A of $Q(S)$. Since A is an ideal in $Q(S)$, by Proposition 3.2.2, we have $A \cap S$ is an ideal in S. Now $(A \cap S)^3 = (A \cap S)(A \cap S)(A \cap S) = AAA \cap AAS \cap SAA \cap SAS \cap ASA \cap ASS \cap SSA \cap SSS \subseteq A^3 \cap AAQ(S) \cap Q(S)AA \cap Q(S)AQ(S) \cap AQ(S)A \cap AQ(S)Q(S) \cap Q(S)Q(S)A \cap S \subseteq A^3 \cap A \cap S \subseteq A^3 \cap S$. So $A^2 \subseteq R \Longrightarrow A^3 = A^2A \subseteq RQ(S) \subseteq R$. Thus $(A \cap S)^3 \subseteq R \cap S$. Since R is an ideal in $Q(S)$, by Proposition 3.2.2, we have $R \cap S$ is an ideal in S which is semiprime. Thus $A \cap S \subseteq R \cap S \subseteq R$. Again since A is an ideal of $Q(S)$, either $A \subseteq S$ or $A \subseteq S^2$. Then we have the following two cases :

Case 1 : If $A \subseteq S$, then $A \cap S = A$. Thus $A \cap S \subseteq R$ implies that $A \subseteq R$.

Case 2 : Let $A \subseteq M = S^2$. Let $a = m(a_1, a_2) \in A$, where $a_1, a_2 \in S$. Since S is a left zero ternary semigroup, $a_2 = a_2xy$ for all $x, y \in S$. Thus $a = m(a_1, a_2) = m(a_1, a_2a_1a_2) = m(a_1, a_2)m(a_1, a_2) \in AA = A^2$. So $A \subseteq A^2$. Hence $A^2 \subseteq R$ implies that $A \subseteq R$.

Thus in the above two cases, we have $A \subseteq R$. Hence R is a semiprime ideal in $Q(S)$ and so every ideal of $Q(S)$ is semiprime. □

Theorem 3.2.5. *Let S be a ternary semigroup. Then every ideal of S is completely prime if every ideal of $Q(S)$ is completely prime. Moreover if S is a left zero ternary semigroup, then every ideal of $Q(S)$ is completely prime if every ideal of S is completely prime.*

Proof. First, suppose every ideal of $Q(S)$ is completely prime. Then for every ideal I of $Q(S)$, $xy \in I$ implies that $x \in I$ or $y \in I$ for any elements x, y of $Q(S)$. Let P be an ideal of S such that $abc \in P$ for $a, b, c \in S$. We have to show that $a \in P$ or $b \in P$ or $c \in P$. Since P is an ideal in S, by Proposition 3.2.1 we have $P \cup SP \cup PS$ is an ideal in $Q(S)$. Now, $abc \in P \Longrightarrow abca \in PS \subseteq P \cup SP \cup PS$. By assumption, $P \cup SP \cup PS$ is a completely prime ideal in $Q(S)$. Then, $abca \in P \cup SP \cup PS$ for $ab, ca \in SS = S^2 \subseteq Q(S)$ implies that $ab \in P \cup SP \cup PS$ or $ca \in P \cup SP \cup PS$.

Again $ab \in P \cup SP \cup PS$ for $a, b \in S \subseteq Q(S)$ implies that $a \in P \cup SP \cup PS$ or $b \in P \cup SP \cup PS$ and $ca \in P \cup SP \cup PS$ for $c, a \in S \subseteq Q(S)$ implies that $c \in P \cup SP \cup PS$ or $a \in P \cup SP \cup PS$. Now, $a \in P \cup SP \cup PS$ implies that $a \in P$ or $a \in SP$ or $a \in PS$. But $a \in SP \Longrightarrow a \in SS = M$, which is a contradiction. Because we take a is an element of S. Thus $a \notin SP$. Similarly, $a \notin PS$. Hence $a \in P$. In the similar manner we can show that $b \in P \cup SP \cup PS$ implies that $b \in P$ and $c \in P \cup SP \cup PS$ implies that $c \in P$. Therefore, P is a completely prime ideal of S and hence every ideal of S is completely prime.

Conversely, let every ideal of S is completely prime ideal and S is a left zero ternary semigroup. Let T be an ideal of $Q(S)$ such that $ab \in T$ for $a, b \in Q(S)$. Since T is an ideal in $Q(S)$, by Proposition 3.2.2 we have $T \cap S$ is an ideal in S. We have $ab \in T$. Then $abb \in TQ(S) \subseteq T$. Now, a and b are elements of $Q(S)$. Hence either $a, b \in S$ or $a, b \in M = S^2$. Thus we have the following four cases:

<u>Case 1</u> : If $a, b \in S$. Then $abb \in SSS \subseteq S$. Hence $abb \in T \cap S$. Since $T \cap S$ is a completely prime ideal in S, then $abb \in T \cap S \Longrightarrow a \in T \cap S$ or $b \in T \cap S$ i.e. $a \in T$ or $b \in T$.

<u>Case 2</u> : If $a, b \in S^2 = M$. Let $a = m(a_1, a_2), b = m(b_1, b_2)$ where $a_1, a_2, b_1, b_2 \in S$. Then $ab = m(a_1, a_2)m(b_1, b_2) = m(a_1, a_2b_1b_2) = m(a_1, a_2) = a$. Hence $ab \in T \Longrightarrow a \in T$. Also, $bab = m(b_1, b_2)m(a_1, a_2)m(b_1, b_2) = m(b_1, b_2a_1a_2)m(b_1, b_2) = m(b_1, b_2)m(b_1, b_2) = m(b_1, b_2b_1b_2) = m(b_1, b_2) = b$. Since $ab \in T$, then $bab \in Q(S)T \subseteq T$. Hence $b \in T$.

<u>Case 3</u> : If $a \in S, b \in S^2 = M$. Let $b = m(b_1, b_2)$ where $b_1, b_2 \in S$. Then $ab = am(b_1, b_2) = (R(b_2, b_1)a) = ab_1b_2 = a$. Thus $ab \in T$ implies that $a \in T$. Also $baab = m(b_1, b_2)aam(b_1, b_2) = m(b_1, b_2)m(a, a)m(b_1, b_2) = m(b_1, b_2aa)m(b_1, b_2) = m(b_1, b_2)m(b_1, b_2) = m(b_1, b_2b_1b_2) = m(b_1, b_2) = b$. Since $ab \in T$, then $baab \in Q(S)Q(S)T \subseteq Q(S)T \subseteq T$. Thus $b \in T$.

<u>Case 4</u> : If $a \in S^2 = M, b \in S$. Let $a = m(a_1, a_2)$ where $a_1, a_2 \in S$. Then $abb = m(a_1, a_2)bb = m(a_1, a_2)m(b, b) = m(a_1, a_2bb) = m(a_1, a_2) = a$. Thus, $abb \in T \Longrightarrow a \in T$. Also $ab = m(a_1, a_2)b = (L(a_1, a_2)b) = a_1a_2b \in SSS \subseteq S$

and so $ab \in T \cap S \Longrightarrow a_1a_2b \in T \cap S$. Thus $a_1 \in T \cap S$ or $a_2 \in T \cap S$ or $b \in T \cap S$. i.e. $a_1 \in T$ or $a_2 \in T$ or $b \in T$.

So, in all four cases we have either $a \in T$ or $b \in T$. Hence, T is a completely prime ideal in $Q(S)$ and so every ideal of $Q(S)$ is completely prime. □

Corollary 3.2.6. *Let S be a ternary semigroup. Then every left (resp. right) ideal of S is completely prime if every left (resp. right) ideal of $Q(S)$ is completely prime. Moreover if S is a left zero ternary semigroup, then every left (resp. right) ideal of $Q(S)$ is completely prime if every left (resp. right) ideal of S is completely prime.*

Theorem 3.2.7. *Let S be a ternary semigroup. Then every ideal of S is completely semiprime if every ideal of $Q(S)$ is completely semiprime. Moreover if S is a left zero ternary semigroup, then every ideal of $Q(S)$ is completely semiprime if every ideal of S is completely semiprime.*

Proof. First, suppose every ideal of $Q(S)$ is completely semiprime. Then for every ideal I of $Q(S)$, $x^2 \in I$ implies $x \in I$ for any elements x of $Q(S)$. Let P be an ideal of S such that $a^3 \in P$ for some $a \in S$. We have to show that $a \in P$. Since P is an ideal in S, by Proposition 3.2.1 we have $P \cup SP \cup PS$ is an ideal in $Q(S)$. Now, $a^3 \in P \Longrightarrow a^4 \in PS \subseteq P \cup SP \cup PS$. By assumption $P \cup SP \cup PS$ is a completely semiprime ideal in $Q(S)$. So, $a^4 = (a^2)^2 \in P \cup SP \cup PS$ for $a^2 \in SS = S^2 \subseteq Q(S)$ implies that $a^2 \in P \cup SP \cup PS$. Again $a^2 \in P \cup SP \cup PS$ for $a \in S \subseteq Q(S)$ implies that $a \in P \cup SP \cup PS$. Now, $a \in P \cup SP \cup PS$ implies that $a \in P$ or $a \in SP$ or $a \in PS$. But $a \in SP \Longrightarrow a \in SS = M$, which is a contradiction. Because we take a is an element of S. Thus $a \notin SP$. Similarly $a \notin PS$. Hence $a \in P$. Therefore, P is a completely semiprime ideal of S and hence every ideal of S is completely semiprime.

Conversely, let every ideal of S is completely semiprime ideal and S is a left zero ternary semigroup. Let R be an ideal of $Q(S)$ such that $a^2 \in R$ for $a \in Q(S)$. Since R is an ideal in $Q(S)$, by Proposition 3.2.2 we have $R \cap S$ is an ideal in S. We have $a^2 \in R$. Then $a^3 \in RQ(S) \subseteq R$. Now, a is an element of $Q(S)$ either $a \in S$ or

$a \in S^2 = M$. Thus we have the following two cases:

Case 1 : If $a \in S$, then $a^3 \in SSS \subseteq S$. Hence $a^3 \in R \cap S$. Since $R \cap S$ is a completely semiprime ideal in S, then $a^3 \in R \cap S \Longrightarrow a \in R \cap S$ i.e. $a \in R$.

Case 2 : If $a \in S^2 = M$, then $a = m(a_1, a_2)$ where $a_1, a_2 \in S$. Then $a^2 = m(a_1, a_2)m(a_1, a_2) = m(a_1, a_2a_1a_2) = m(a_1, a_2) = a$. Thus, $a^2 \in R \Longrightarrow a \in R$. So, in both cases we have $a \in R$. Hence, R is a completely semiprime ideal in $Q(S)$ and so every ideal of $Q(S)$ is completely semiprime. □

Corollary 3.2.8. *Let S be a ternary semigroup. Then every left (resp. right) ideal of S is completely semiprime if every left (resp. right) ideal of $Q(S)$ is completely semiprime. Moreover if S is a left zero ternary semigroup, then every left (resp. right) ideal of $Q(S)$ is completely semiprime if every left (resp. right) ideal of S is completely semiprime.*

Theorem 3.2.9. *[77, Santiago] A ternary semigroup S is left (resp. right) regular if and only if every left (resp. right) ideal of S is completely semiprime.*

Corollary 3.2.10. *A semigroup S is left (resp. right) regular if and only if every left (resp. right) ideal of S is completely semiprime.*

Theorem 3.2.11. *A ternary semigroup S is left (resp. right) regular if the semigroup $Q(S)$ is left (resp. right) regular. Moreover if S is a left zero ternary semigroup, then $Q(S)$ is left (resp. right) regular if the semigroup S is left (resp. right) regular.*

Proof. Let $Q(S)$ be a left regular semigroup. Then by Corollary 3.2.10, every left ideal of $Q(S)$ is completely semiprime. Therefore, every left ideal of the ternary semigroup S is completely semiprime, by Corollary 3.2.8. Thus by Theorem 3.2.9, S is a left regular ternary semigroup.

Conversely, suppose that S is a left zero ternary semigroup and also a left regular ternary semigroup. Then by Therorem 3.2.9, every left ideal of S is completely semiprime. Since S is left zero ternary semigroup and every left ideal is completely semiprime, so every left ideal of $Q(S)$ is completely semiprime, by Corollary 3.2.10 and hence $Q(S)$ is left regular, by Corollary 3.2.10.

Similarly, we can prove the result for right ideal. $\square$

Theorem 3.2.12. *[29, Dutta] A ternary semigroup S intra-regular if and only if every ideal of S is completely semiprime ideal.*

Corollary 3.2.13. *A semigroup S is intra-regular if and only if every ideal of S is completely semiprime.*

Finally, we prove when $Q(S)$ will be intra-regular ternary semigroup.

Theorem 3.2.14. *A ternary semigroup S is intra-regular if the semigroup $Q(S)$ is intra-regular. Moreover if S is a left zero ternary semigroup, then $Q(S)$ is intra-regular if S is intra-regular.*

Proof. Let $Q(S)$ be an intra-regular semigroup and I be an ideal in $Q(S)$. Let $a^2 \in I$ for an element a of $Q(S)$. Since $Q(S)$ is intra-regular and $a \in Q(S)$, there exist $x, y \in Q(S)$ such that $a = xa^2y \in Q(S)IQ(S) \subseteq I$. Thus I is a completely semiprime ideal in $Q(S)$. Since I is an arbitrary ideal, every ideal of $Q(S)$ is completely semiprime. By Theorem 3.2.7, every ideal of S completely semiprime. Hence S is a intra-regular ternary semigroup, by Theorem 3.2.12.

Conversely, suppose that S is a left zero ternary semigroup and also an intra-regular ternary semigroup. Since S is an intra-regular ternary semigroup by Theorem 3.2.12, every ideal of S is completely semiprime ideal. Therefore, every ideal of $Q(S)$ is completely semiprime, by Theorem 3.2.7. Since $Q(S)$ is a semigroup and every ideal of $Q(S)$ is completely semiprime, by Corollary 3.2.13, it follows that $Q(S)$ is an intra-regular semigroup. $\square$

3.3 Bi-ideals and quasi-ideals of S and $Q(S)$

The aim of this section is to characterize a ternary semigroup S and the corresponding semigroup cover $Q(S)$ of the ternary semigroup S by their bi-ideals and quasi-ideals.

Proposition 3.3.1. *Let S be a ternary semigroup. If B is a bi-ideal in S, then $B \cup BSB$ is a bi-ideal in S and $BSSB$ is a bi-ideal in M. Moreover $B \cup BSB \cup BSSB$ is a bi-ideal in $Q(S)$.*

Proof. Let B be a bi-ideal in S, then $BSBSB \subseteq B$. Now $B \cup BSB \subseteq S$. So $(B \cup BSB)S(B \cup BSB)S(B \cup BSB) = (BSB \cup BSBSB \cup BSBSB \cup BSBSBSB)S(B \cup BSB) \subseteq (BSB \cup B \cup B \cup BSB)S(B \cup BSB) = (B \cup BSB)S(B \cup BSB) \subseteq (B \cup BSB)$ and $(BSSB)M(BSSB) = (BSSB)SS(BSSB) = BSSBSSBSSB \subseteq BSSB$. Thus $B \cup BSB$ is a bi-ideal in S and $BSSB$ is a bi-ideal in M.
Now $B \cup BSB \cup BSSB \subseteq S \cup SSS \cup SSSS \subseteq S \cup S^2 = S \cup M = Q(S)$.
Then $(B \cup BSB \cup BSSB)Q(S)(B \cup BSB \cup BSSB)$
$= (B \cup BSB \cup BSSB)(S \cup S^2)(B \cup BSB \cup BSSB)$
$= BSB \cup BSBSB \cup BSBSSB \cup BS^2B \cup BS^2BSB \cup BS^2BSSB \cup BSBSB \cup BSBSBSB \cup BSBSBSSB \cup BSBS^2B \cup BSBS^2BSB \cup BSBS^2BSSB \cup BSSBSB \cup BSSBSBSB \cup BSSBSBSSB \cup BSSBS^2B \cup BSSBS^2BSB \cup BSSBS^2BSSB$
$\subseteq BSB \cup B \cup BSSB \cup BSSB \cup BSSB \cup BSB \cup B \cup BSB \cup BSSB \cup BSSB \cup BSSB \cup BSB \cup BSSB \cup BSSB \cup BSB \cup BSB \cup BSB \cup BSSB$
$\subseteq B \cup BSB \cup BSSB$.
Hence $B \cup BSB \cup BSSB$ is a bi-ideal in $Q(S)$. □

Proposition 3.3.2. *Let A be a non-empty subset of a ternary semigroup S. Then $ASA \cup ASSA$ is a bi-ideal in $Q(S)$.*

Proof. Let A be a non-empty subset of a ternary semigroup S. Then $ASA \cup ASSA \subseteq Q(S)$. Now we have $(ASA \cup ASSA)Q(S)(ASA \cup ASSA) = (ASA \cup ASSA)(S \cup S^2)(ASA \cup ASSA) = ASASASA \cup ASASASSA \cup ASAS^2ASA \cup ASAS^2ASSA \cup ASSASASA \cup ASSASASSA \cup ASSAS^2ASA \cup ASSAS^2ASSA \subseteq ASA \cup ASSA \cup ASSA \cup ASA \cup ASSA \cup ASA \cup ASA \cup ASSA = ASA \cup ASSA$.
Therefore, $ASA \cup ASSA$ is a bi-ideal in $Q(S)$. □

Proposition 3.3.3. *If B is a bi-ideal of $Q(S)$, then $B \cap S$ is a bi-ideal of S.*

Proof. Let B be a bi-ideal of $Q(S)$. Then B is a subsemigroup $Q(S)$ and $BQ(S)B \subseteq B \Longrightarrow B(S \cup M)B \subseteq B \Longrightarrow BSB \cup BMB \subseteq B \Longrightarrow BSB \cup BSSB \subseteq B$.
Now, $(B \cap S)S(B \cap S)S(B \cap S)$
$= (BSB \cap BSS \cap SSB \cap SSS)S(B \cap S)$
$= BSBSB \cap BSBSS \cap BSSSB \cap BSSSS \cap SSBSB \cap SSBSS \cap SSSSB \cap SSSSS$
$\subseteq BSB \cap BSS \cap BSB \cap BSS \cap SSB \cap SSBSS \cap SSB \cap S$
$\subseteq B \cap S$.
Hence $B \cap S$ is a bi-ideal of S. □

Theorem 3.3.4. *Let S be a ternary semigroup. Then every bi-ideal of S is semiprime if every bi-ideal of $Q(S)$ is semiprime. Moreover if S is a left zero ternary semigroup, then every bi-ideal of $Q(S)$ is semiprime if every bi-ideal of S is semiprime.*

Proof. Suppose that every bi-ideal of $Q(S)$ is semiprime. Let B be a bi-ideal of S such that $X^3 \subseteq B$ for $X \subseteq S$. Since B is a bi-ideal of S, B^3 is also a bi-ideal of S. Thus $B^3 \cup B^3SB^3 \cup B^3SSB^3$ is a bi-ideal in $Q(S)$, by Proposiotion 3.3.1. We have to show that $X \subseteq B$. Now $X^3 \subseteq B \Longrightarrow X^9 \subseteq B^3$. Then $X^{16} = X^7X^9 \subseteq S^7B^3 \subseteq SB^3$. Again $X^{16} = X^9X^7 \subseteq B^3S^7 \subseteq B^3S$ and hence $X^{32} = X^{16}X^{16} \subseteq B^3SSB^3 \subseteq B^3 \cup B^3SB^3 \cup B^3SSB^3$. Since every bi-ideal of $Q(S)$ is semiprime, so $B^3 \cup B^3SB^3 \cup B^3SSB^3$ is a semiprime bi-ideal of $Q(S)$. Thus $X^{32} \subseteq B^3 \cup B^3SB^3 \cup B^3SSB^3$ implies that $X^{16} \subseteq B^3 \cup B^3SB^3 \cup B^3SSB^3$. Proceeding in this manner, we get $X \subseteq B^3 \cup B^3SB^3 \cup B^3SSB^3$. But $X \subseteq B^3SSB^3$ implies that $X \subseteq S^2 = M$, which is a contradiction. Thus $X \subseteq B^3 \cup B^3SB^3$ which implies that $X \subseteq B^3$ or $X \subseteq B^3SB^3$. Since B is a bi-ideal of S, B is a subsemigroup of S. Thus $X \subseteq B^3 \subseteq B$. If $X \subseteq B^3SB^3$, then $X \subseteq B^3SB^3 \subseteq BSBSBB^2 \subseteq B^3 \subseteq B$. Hence we get $X \subseteq B$ and so B is a semiprime bi-ideal of S. Therefore, every bi-ideal of S is semiprime.

Conversely, suppose that S is a left zero ternary semigroup and every bi-ideal of S is semiprime. Let P be a bi-ideal of $Q(S)$ such that $Y^2 \subseteq P$ for $Y \subseteq Q(S)$. Since

P is a bi-ideal in $Q(S)$, by Proposition 3.3.3, we have $P \cap S$ is bi-ideal in S. Now $Y^2 \subseteq P \Longrightarrow Y^9 = Y^2YY^2Y^2Y^2 \subseteq PQ(S)PPP \subseteq PQ(S)PQ(S)P \subseteq PQ(S)P \subseteq P$. Thus $Y^9 \subseteq P$. Again since $Y \subseteq Q(S)$ either $Y \subseteq S$ or $Y \subseteq M = S^2$. Then we have the following two cases :

<u>Case 1</u> : If $Y \subseteq S$, then $Y^9 \subseteq S^9 \subseteq S$. Thus, $Y^9 \subseteq P \cap S$. Since P is bi-ideal in $Q(S)$, by Proposition 3.3.3 we have $P \cap S$ is bi-ideal in S which is semiprime. Then $Y^9 \subseteq P \cap S$ implies that $Y^3 \subseteq P \cap S$ and $Y^3 \subseteq P \cap S$ implies that $Y \subseteq P \cap S$. Hence $Y \subseteq P$.

<u>Case 2</u> : If $Y \subseteq M = S^2$. Let $y = m(y_1, y_2) \in Y$ where $y_1, y_2 \in S$. Since S is a left zero ternary semigroup $y_2 = y_2xz$ for all $x, z \in S$. Thus $y = m(y_1, y_2) = m(y_1, y_2y_1y_2) = m(y_1, y_2)m(y_1, y_2) \in YY = Y^2$. Thus $Y \subseteq Y^2$. Hence $Y^2 \subseteq P$ implies $Y \subseteq P$.

Thus in the above two cases, we have $Y \subseteq P$. Hence P is a semiprime bi-ideal of $Q(S)$ and so every bi-ideal of $Q(S)$ is semiprime. □

Similarly, we have the following result :

Theorem 3.3.5. *Let S be a ternary semigroup. Then every bi-ideal of S is completely semiprime if every bi-ideal of $Q(S)$ is completely semiprime. Moreover if S is a left zero ternary semigroup, every bi-ideal of $Q(S)$ is completely semiprime if every bi-ideal of S is completely semiprime.*

Proof. First, suppose that every bi-ideal of $Q(S)$ is completely semiprime. Let B be a bi-ideal of S such that $a^3 \in B$ for $a \in S$. We have to show that $a \in B$. Now, $a^3 \in B \Longrightarrow a^9 \in B^3$ and B^3 is also bi-ideal in S. Since B is a bi-ideal in S, by Proposition 3.3.1 we have $B^3 \cup B^3SB^3 \cup B^3SSB^3$ is a bi-ideal in $Q(S)$, which is a completely semiprime bi-ideal in $Q(S)$. Now, $a^{16} = a^9a^7 \in B^3S^7$ and $a^{16} = a^7a^9 \in S^7B^3$. Thus $a^{32} = a^{16}a^{16} \in B^3SSB^3 \subseteq B^3 \cup B^3SB^3 \cup B^3SSB^3$ which implies that $a^{16} \in B^3 \cup B^3SB^3 \cup B^3SSB^3$. In this similar manner we get $a \in B^3 \cup B^3SB^3 \cup B^3SSB^3$ which implies that $a \in B^3$ or $a \in B^3SB^3$ or $a \in B^3SSB^3$. But $a \in B^3SSB^3 \Longrightarrow a \in SS = M$, which is a contradiction. Because we take a is an element of S. Thus $a \notin B^3SSB^3$. Hence $a \in B^3 \cup B^3SB^3 \subseteq B \cup BSBSBB^2 \subseteq B \cup B^3 \subseteq B$. Therefore, B is a completely semiprime bi-ideal of S and hence every bi-ideal of S is completely semiprime.

Conversely, suppose that every bi-ideal of S is completely semiprime and S is left zero ternary semigroup. Let R be a bi-ideal of $Q(S)$ such that $b^2 \in R$ for $b \in Q(S)$. Since R is a bi-ideal in $Q(S)$, by Proposition 3.3.3 $R \cap S$ is a bi-ideal in S. Now, $b^9 \subseteq RRQ(S)RR \subseteq R$. Since $b \in Q(S)$, either $b \in S$ or $b \in M = S^2$. Then we have two cases:

Case 1 : If $b \in S$, then $b^9 \in S$ and so $b^9 \in R \cap S$. Since $R \cap S$ is a completely semiprime quasi-ideal in $Q(S)$ we have $b^3 \in R \cap S$. Again $b^3 \in R \cap S \Longrightarrow b \in R \cap S$. Hence $b \in R$.

Case 2 : If $b \in M = S^2$, then $b = m(b_1, b_2)$ where $b_1, b_2 \in S$. Now $b^2 = m(b_1, b_2)m(b_1, b_2) = m(b_1, b_2b_1b_2)$. Since S is a left zero ternary semigroup, we have $b_2b_1b_2 = b_2$. So $b^2 = m(b_1, b_2) = b$. Thus $b^2 \in R \Longrightarrow b \in R$. Thus in both cases we have $b \in R$. Therefore, R is a completely semiprime bi-ideal of $Q(S)$ and hence every bi-ideal of $Q(S)$ is completely semiprime. □

Theorem 3.3.6. *[29, Dutta] A ternary semigroup S is completely regular if and only if every bi-ideal of S is a completely semiprime bi-ideal of S.*

Corollary 3.3.7. *A semigroup S is completely regular if and only if every bi-ideal of S is a completely semiprime bi-ideal of S.*

Theorem 3.3.8. *A ternary semigroup S is completely regular if $Q(S)$ is completely regular. Moreover if S is a left zero ternary semigroup, then $Q(S)$ is completely regular if S is completely regular.*

Proof. Suppose that $Q(S)$ is a completely regular semigroup. Then by Corollary 3.3.7, every bi-ideal of $Q(S)$ is completely semiprime. Therefore, every bi-ideal of the ternary semigroup S is completely semiprime, by Theorem 3.3.5. Thus S is completely regular ternary semigroup, by Theorem 3.3.6.

Conversely, suppose that S is a left zero ternary semigroup and also a completely regular ternary semigroup. Then by Therorem 3.3.6, every bi-ideal of S is completely semiprime. Since S is left zero ternary semigroup and every bi-ideal is completely semiprime, so every bi-ideal of $Q(S)$ is completely semiprime, by Theorem 3.3.5 and hence $Q(S)$ is completely regular, by Corollary 3.3.7. $\square$

Theorem 3.3.9. *[44, Kar] Let S be a ternary semigroup. Then S has no proper bi-ideal if and only if S is a ternary group.*

Corollary 3.3.10. *Let S be a semigroup. Then S has no proper bi-ideal if and only if S is a group.*

Theorem 3.3.11. *Let S be a ternary semigroup. Then S has no proper bi-ideal if and only if $Q(S)$ has no proper bi-ideal.*

Proof. Suppose that S has no proper bi-ideal. For any bi-ideal B in $Q(S)$, $B \cap S$ is a bi-ideal in S by Proposition 3.3.3. Then $B \cap S = S$ which implies that $S \subseteq B$. Again since B is a bi-ideal in $Q(S)$, so BSB is also a bi-ideal in $Q(S)$ and hence $BSB \cap S$ is a bi-ideal in S. Thus $BSB \cap S = S$ which implies that $S \subseteq BSB$. So $S^2 \subseteq BSBB \subseteq BSQ(S)B = BS(S \cup M)B = BS(S \cup S^2)B = BSSB \cup BSSSB \subseteq BSSB \cup BSB \subseteq B(S^2 \cup S)B = B(M \cup S)B = BQ(S)B \subseteq B$. Hence $Q(S) = S \cup M = S \cup S^2 \subseteq B$. Therefore, $Q(S)$ has no proper bi-ideal.

For the converse part, let P be a bi-ideal in S. Then by Proposition 3.3.1 $P \cup PSP \cup PSSP$ is a bi-ideal in $Q(S)$. Let $P_1 = P \cup PSP \cup PSSP$. Since $Q(S)$ has no proper bi-ideal, we have $P_1 = Q(S)$. Since P is a bi-ideal in S, PSP is also a bi-ideal in S. Then $PSP \cup (PSP)S(PSP) \cup (PSP)SS(PSP)$ is a bi-ideal in $Q(S)$. Let $P_2 = PSP \cup PSPSPSP \cup PSPSSPSP$. Since $Q(S)$ has no proper bi-ideal, we have $P_2 = Q(S)$. Now $Q(S) \cap S = (S \cup M) \cap S = S$. But $Q(S) \cap S = P_1 \cap S = (P \cup PSP \cup PSSP) \cap S = P \cup PSP$ and $Q(S) \cap S = P_2 \cap S = (PSP \cup PSPSPSP \cup PSPSSPSP) \cap S = PSP \cup PSPSPSP$. Thus $P \cup PSP = PSP \cup PSPSPSP \subseteq PSP \cup PSP = PSP$. Hence $P \subseteq PSP$ and so

$PSP \subseteq PS(PSP) = PSPSP \subseteq P$. Thus $S = P \cup PSP \subseteq P \cup P = P$. Therefore, S does not have any proper bi-ideal. $\square$

Theorem 3.3.12. *A ternary semigroup S is a ternary group if and only if $Q(S)$ is a group.*

Proof. Let S be a ternary group. Then S has no proper bi-ideal. By Theorem 3.3.11, $Q(S)$ has no proper bi-ideal. Since $Q(S)$ is a semigroup, by Corollary 3.3.10, we have $Q(S)$ is a group.

For the converse part, let $Q(S)$ be a group. Then $Q(S)$ has no proper bi-ideal which implies that S has no proper bi-ideal. By Theorem 3.3.9, it follows that S is a ternary group. $\square$

Proposition 3.3.13. *Let S be a ternary semigroup and Q be a quasi-ideal of S. Then $SSQ \cup QSS \cup SQS \cup SSQSS$ is also a quasi-ideal of S and $(QS \cap SQ) \cup SQSS \cup SSQS$ is a quasi-ideal of M and $Q \cup (QS \cap SQ) \cup SSQ \cup QSS \cup SQSS \cup SSQS \cup SQS \cup SSQSS$ is a quasi-ideal of $Q(S)$.*

Proof. Let $Q \subseteq S$ be a quasi-ideal in a ternary semigroup S. Then $Q^{'} = Q \cup SSQ \cup QSS \cup SQS \cup SSQSS \subseteq S$.

Now, $SSQ^{'} \cap SQ^{'}S \cap Q^{'}SS$

$= SS(Q \cup SSQ \cup QSS \cup SQS \cup SSQSS) \cap S(Q \cup SSQ \cup QSS \cup SQS \cup SSQSS)S \cap (Q \cup SSQ \cup QSS \cup SQS \cup SSQSS)SS$

$\subseteq SS(Q \cup SSQ \cup QSS \cup SQS \cup SSQSS)$

$\subseteq SSQ \cup SSSSQ \cup SSQSS \cup SSSQS \cup SSSSQSS$

$\subseteq SSQ \cup SSQSS \cup SQS \cup SSQSS$

$\subseteq Q \cup SSQ \cup QSS \cup SQS \cup SSQSS = Q^{'}$.

Thus $Q^{'} = Q \cup SSQ \cup QSS \cup SQS \cup SSQSS$ is a quasi-ideal in S.

Let $K = (QS \cap SQ) \cup SQSS \cup SSQS \subseteq M$. Thus we have,

$KM = ((QS \cap SQ) \cup SQSS \cup SSQS)M = ((QSM \cap SQM) \cup SQSSM \cup SSQSM) = ((QSSS \cap SQSS) \cup SQSSSS \cup SSQSSS) \subseteq ((QS \cap SQSS) \cup SQSS \cup SSQS) \subseteq$

$QS \cup SQSS \cup SSQS$.

Also $MK = M((QS \cap SQ) \cup SQSS \cup SSQS) = ((MQS \cap MSQ) \cup MSQSS \cup MSSQS) = ((SSQS \cap SSSQ) \cup SSSQSS \cup SSSSQS) \subseteq ((SSQS \cap SQ) \cup SQSS \cup SSQS) \subseteq SQ \cup SQSS \cup SSQS$.

Thus we have, $KM \cap MK$

$$\begin{aligned} &\subseteq (QS \cup SQSS \cup SSQS) \cap (SQ \cup SQSS \cup SSQS) \\ &= (SQSS \cup SSQS) \cup (QS \cap SQ) \\ &= (QS \cap SQ) \cup SQSS \cup SSQS \\ &= K \end{aligned}$$

Therefore, $K = (QS \cap SQ) \cup SQSS \cup SSQS$ is a quasi-ideal in M.

Now $(Q' \cup K) \subseteq S \cup M = Q(S)$. Thus we have

$$\begin{aligned} &(Q' \cup K)Q(S) \\ &= (Q \cup (QS \cap SQ) \cup SSQ \cup QSS \cup SQSS \cup SSQS \cup SQS \cup SSQSS)(M \cup S) \\ &= (QM \cup QS \cup (QSM \cap SQM) \cup (QSS \cap SQS) \cup SSQM \cup SSQS \cup QSSM \cup \\ &\quad QSSS \cup SQSSM \cup SQSSS \cup SSQSM \cup SSQSS \cup SQSM \cup SQSS \cup \\ &\quad SSQSSM \cup SSQSSS) \\ &= (QSS \cup QS \cup (QSSS \cap SQSS) \cup (QSS \cap SQS) \cup SSQSS \cup SSQS \cup QSSSS \\ &\quad \cup QSSS \cup SQSSSS \cup SQSSS \cup SSQSSS \cup SSQSS \cup SQSSS \cup SQSS \\ &\quad \cup SSQSSSS \cup SSQSSS) \\ &\subseteq (QSS \cup QS \cup (QS \cap SQSS) \cup (QSS \cap SQS) \cup SSQSS \cup SSQS \cup SQSS \cup SQS) \\ &= (SSQSS \cup SQSS \cup SSQS \cup SQS) \cup ((QSS \cup QS \cup (QS \cap SQSS) \cup (QSS \cap SQS)) \\ &\subseteq (SSQSS \cup SQSS \cup SSQS \cup SQS) \cup ((QSS \cup QS \cup QS \cup (QSS \cap SQS)) \\ &= (SSQSS \cup SQSS \cup SSQS \cup SQS) \cup ((QSS \cup QS \cup (QSS \cap SQS)) \end{aligned}$$

Also we have,

$Q(S)(Q^{'} \cup K)$

$= (M \cup S)(Q \cup (QS \cap SQ) \cup SSQ \cup QSS \cup SQSS \cup SSQS \cup SQS \cup SSQSS)$

$= (MQ \cup SQ \cup (MQS \cap MSQ) \cup (SQS \cap SSQ) \cup MSSQ \cup SSSQ \cup MQSS \cup$
$SQSS \cup MSQSS \cup SSQSS \cup MSSQS \cup SSSQS \cup MSQS \cup SSQS \cup$
$MSSQSS \cup SSSQSS)$

$= (SSQ \cup SQ \cup (SSQS \cap SSSQ) \cup (SQS \cap SSQ) \cup SSSSQ \cup SSSQ \cup SSQSS$
$\cup SQSS \cup SSSQSS \cup SSQSS \cup SSSSQS \cup SSSQS \cup SSSQS \cup SSQS$
$\cup SSSSQSS \cup SSSQSS)$

$\subseteq (SSQ \cup SQ \cup (SSQS \cap SQ) \cup (SQS \cap SSQ) \cup SSQSS \cup SQSS \cup SSQS \cup SQS)$

$= (SSQSS \cup SQSS \cup SSQS \cup SQS) \cup (SSQ \cup SQ \cup (SSQS \cap SQ) \cup (SQS \cap SSQ))$

$\subseteq (SSQSS \cup SQSS \cup SSQS \cup SQS) \cup (SSQ \cup SQ)$

Now

$(Q^{'} \cup K)Q(S) \cap Q(S)(Q^{'} \cup K)$

$= \Big((SSQSS \cup SQSS \cup SSQS \cup SQS) \cup ((QSS \cup QS \cup (QSS \cap SQS))\Big) \cap$
$\Big((SSQSS \cup SQSS \cup SSQS \cup SQS) \cup (SSQ \cup SQ)\Big)$

$= (SSQSS \cup SQSS \cup SSQS \cup SQS) \cup (QSS \cap SSQ) \cup (QSS \cap SQ) \cup (QS \cap SSQ)$
$\cup (QS \cap SQ) \cup (QSS \cap SQS \cap SSQ) \cup (QSS \cap SQS \cap SQ)$

$\subseteq (SSQSS \cup SQSS \cup SSQS \cup SQS) \cup QSS \cup QSS \cup SSQ \cup (QS \cap SQ) \cup Q \cup QSS$

$= (SSQSS \cup SQSS \cup SSQS \cup SQS) \cup QSS \cup Q \cup SSQ \cup (QS \cap SQ)$

$= Q \cup (QS \cap SQ) \cup SSQ \cup QSS \cup SQSS \cup SSQS \cup SQS \cup SSQSS$

$= Q^{'} \cup K.$

Therefore, $Q \cup (QS \cap SQ) \cup SSQ \cup QSS \cup SQSS \cup SSQS \cup SQS \cup SSQSS$ is a quasi-ideal in $Q(S)$. $\square$

Proposition 3.3.14. *Let A be a non-empty subset of a ternary semigroup S. Then*

(i) $SA \cup SSA$ is a quasi-ideal of $Q(S)$,

(ii) $AS \cup ASS$ is a quasi-ideal of $Q(S)$,

(iii) $SAS \cup SSASS \cup SASS \cup SSAS$ is a quasi-ideal of $Q(S)$.

Proof. (i) Let A be a non-empty subset of a ternary semigroup S. Then $SA \cup SSA \subseteq Q(S)$. Now $SS(SA \cup SSA) \cap S(SA \cup SSA)S \cap (SA \cup SSA)SS \subseteq SS(SA \cup SSA) = SSSA \cup SSSSA \subseteq SA \cup SSA$. Therefore, $SA \cup SSA$ is a quasi-ideal in $Q(S)$.

Similarly, we can prove (ii) and (iii). $\square$

Proposition 3.3.15. *Let K be a quasi-ideal of $Q(S)$. Then $K \cap S$ is a quasi-ideal of S.*

Proof. Let K be a quasi-ideal in $Q(S)$, then $K \cap S$ is a non-empty subset in S and $KQ(S) \cap Q(S)K \subseteq K \implies K(M \cup S) \cap (M \cup S)K \subseteq K \implies (KM \cup KS) \cap (MK \cup SK) \subseteq K \implies (KSS \cup KS) \cap (SSK \cup SK) \subseteq K \implies (KSS \cap SSK) \cup (KSS \cap SK) \cup (KS \cap SSK) \cup (KS \cap SK) \subseteq K$. Thus $(KSS \cap SSK) \subseteq K$. Now in S,

$$\begin{aligned}
&SS(K \cap S) \cap S(K \cap S)S \cap (K \cap S)SS \\
&= SSK \cap SSS \cap SKS \cap SSS \cap KSS \cap SSS \\
&\subseteq SSK \cap S \cap SKS \cap S \cap KSS \cap S \\
&= SSK \cap SKS \cap KSS \cap S \\
&\subseteq (SSK \cap KSS) \cap S \\
&\subseteq K \cap S
\end{aligned}$$

Hence $K \cap S$ is a quasi-ideal in S. $\square$

Theorem 3.3.16. *Let S be a ternary semigroup. Then every quasi-ideal of S is semiprime if every quasi-ideal of $Q(S)$ is semiprime. Moreover if S is a left zero* *ternary semigroup, then every quasi-ideal of $Q(S)$ is semiprime if every quasi-ideal of S is semiprime.*

Proof. First suppose that every quasi-ideal of $Q(S)$ is semiprime. Let K be a quasi-ideal of S such that $A^3 \subseteq K$ for any non-empty subset A of S. We have to show that $A \subseteq K$. Now $A^3 \subseteq K \Longrightarrow A^4 \subseteq KS \subseteq KS \cup KSS$. Since $K \subseteq S$, by Proposition 3.3.14, we have $KS \cup KSS$, $SK \cup SSK$ and $SKS \cup SSKSS \cup SKSS \cup SSKS$ are quasi-ideals in $Q(S)$. By assumption, $KS \cup KSS$ is a semiprime quasi-ideal of $Q(S)$. So $A^4 = (A^2)^2 \subseteq KS \cup KSS$ for $A^2 \subseteq SS = S^2 \subseteq Q(S)$ implies that $A^2 \subseteq KS \cup KSS$. Again $A^2 \subseteq KS \cup KSS$ for $A \subseteq S \subseteq Q(S)$ implies that $A \subseteq KS \cup KSS$. Now $A \subseteq KS \cup KSS$ implies that $A \subseteq KS$ or $A \subseteq KSS$. But $A \subseteq KS \Longrightarrow A \subseteq S.S = M$, which contradicts the fact that A is a subset of S. So $A \nsubseteq KS$. Thus $A \subseteq KSS$. Similarly, we can show that $A \subseteq SSK$ and $A \subseteq SKS \cup SSKSS$. Hence $A \subseteq KSS \cap SSK \cap (SKS \cup SSKSS) \subseteq K$. Therefore, K is semiprime and hence every quasi-ideal of S is semiprime.

Conversely, suppose that S is a left zero ternary semigroup and every quasi-ideal of S is semiprime. Let R be quasi-ideal of $Q(S)$ such that $B^2 \subseteq R$ for $B \subseteq Q(S)$. Since R is a quasi-ideal in $Q(S)$, $R \cap S$ is a quasi-ideal in S by Proposition 3.3.15. Now $B^3 \subseteq RQ(S) \cap Q(S)R \subseteq R$. We take B is a non-empty subset of $Q(S) = S \cup M$. Then either $B \subseteq S$ or $B \subseteq M = S^2$. Therefore, we have the following two cases :

Case 1 : If $B \subseteq S$, then $B^3 \subseteq S$ and so $B^3 \subseteq R \cap S$. Since R is a semiprime quasi-ideal in $Q(S)$, we have $B \subseteq R \cap S$. Hence $B \subseteq R$.

Case 2: Let $B \subseteq M = S^2$. Let $b = m(b_1, b_2) \in B$, where $b_1, b_2 \in S$. Now $b^2 = m(b_1, b_2)m(b_1, b_2) = m(b_1, b_2 b_1 b_2) = m(b_1, b_2) = b \in B$. Then $B^2 \subseteq R \Longrightarrow B \subseteq R$.

Thus in both cases we have $B \subseteq R$. Therefore, R is a semiprime quasi-ideal of $Q(S)$ and hence every quasi-ideal of $Q(S)$ is semiprime. □

Similarly, we have the following result :

Theorem 3.3.17. *Let S be a ternary semigroup. Then every quasi-ideal of S is completely semiprime if every quasi-ideal of $Q(S)$ is completely semiprime. Moreover* *if S is a left zero ternary semigroup, then every quasi-ideal of $Q(S)$ is completely semiprime if every quasi-ideal of S is completely semiprime.*

Proof. First, suppose every quasi-ideal of $Q(S)$ is completely semiprime. Let K be a quasi-ideal of S such that $a^3 \in K$ for $a \in S$. We have to show that $a \in K$. Now, $a^3 \in K \Longrightarrow a^4 \in KS \subseteq KS \cup KSS$. Since K is a quasi-ideal in S, by Proposition 3.3.14 we have $KS \cup KSS$, $SK \cup SSK$ and $SKS \cup SSKSS \cup SSKS \cup SKSS$ is a quasi-ideal in $Q(S)$. Then $KS \cup KSS$ is a completely semiprime quasi-ideal in $Q(S)$. So, $a^4 = (a^2)^2 \in KS \cup KSS$ for $a^2 \in S.S = S^2 \subseteq Q(S)$ implies that $a^2 \in KS \cup KSS$. Again $a^2 \in KS \cup KSS$ for $a \in S \subseteq Q(S)$ implies $a \in KS \cup KSS$. Now, $a \in KS \cup KSS$ implies that $a \in KS$ or $a \in KSS$. But $a \in KS \Longrightarrow a \in S.S = M$, which is a contradiction. Because we take a is an element of S. So $a \notin KS$. Thus $a \in KSS$. Similarly, $a \in SSK$ and $a \in SKS \cup SSKSS$. Hence $a \in K$. Therefore, K is a completely semiprime quasi-ideal of S and hence every quasi-ideal of S is completely semiprime.

Conversely, let every quasi-ideal of S is completely semiprime and S is a left zero ternary semigroup. Let R be quasi-ideal of $Q(S)$ such that $b^2 \in R$ for any element b of $Q(S)$. Since R is a quasi-ideal in $Q(S)$, by Proposition 3.3.15 $R \cap S$ is a quasi-ideal in S. Now, $b^3 \in RQ(S) \cap Q(S)R \subseteq R$. Since $b \in Q(S)$, either $b \in S$ or $b \in M = S^2$. Then we have two cases:

<u>Case 1</u> : If $b \in S$, Then $b^3 \in S$ and so $b^3 \in R \cap S$. Since $R \cap S$ is a completely semiprime quasi-ideal in S we have $b \in R \cap S$. Hence $b \in R$.

<u>Case 2</u> : If $b \in M = S^2$. Let $b = m(b_1, b_2)$ where $b_1, b_2 \in S$. Now we have, $b^3 = m(b_1, b_2)m(b_1, b_2)m(b_1, b_2) = m(b_1, b_2b_1b_2)m(b_1, b_2) = m(b_1, b_2)m(b_1, b_2) = m(b_1, b_2b_1b_2) = m(b_1, b_2) = b$. Hence we get, $b = b^3 \in R$. Thus in both cases we have $b \in R$. Therefore R is a completely semiprime quasi-ideal of $Q(S)$ and hence every quasi-ideal of $Q(S)$ is completely semiprime. □

Theorem 3.3.18. *Let S be a ternary semigroup. Then S has no proper quasi-ideal if and only if $Q(S)$ has no proper quasi-ideal.*

Proof. Suppose S has no proper quasi-ideal. Let K be a quasi-ideal in $Q(S)$. Then by Proposition 3.3.15 we have $K \cap S$ is a quasi-ideal in S. Thus $K \cap S = S$. So $S \subseteq K$ and $S^2 \subseteq KK \subseteq KQ(S)$. Also $S^2 \subseteq KK \subseteq Q(S)K$. Thus $S^2 \subseteq KQ(S) \cap Q(S)K \subseteq K$. Hence $S \cup S^2 = S \cup M \subseteq K$ i.e. $Q(S) \subseteq K$. Therefore, $Q(S)$ has no proper quasi-ideal.

Conversely, let $Q(S)$ has no proper quasi-ideal. Let Q be a quasi-ideal in S. Then $Q \subseteq S$. By Lemma 3.3.14 we have $SQ \cup SSQ$, $QS \cup QSS$, $SQS \cup SSQSS \cup SQSS \cup SSQS$ are all quasi-ideals in $Q(S)$. Let $Q_1 = SQ \cup SSQ$, $Q_2 = QS \cup QSS$, $Q_3 = SQS \cup SSQSS \cup SQSS \cup SSQS$. Since $Q(S)$ has no proper quasi-ideal, $Q_1 = Q(S)$, $Q_2 = Q(S)$, $Q_3 = Q(S)$. Now $Q(S) \cap S = (S \cup M) \cap S = S$. But $Q(S) \cap S = Q_1 \cap S = (SQ \cup SSQ) \cap S = SSQ$. Again $Q(S) \cap S = Q_2 \cap S = (QS \cup QSS) \cap S = QSS$ and $Q(S) \cap S = Q_3 \cap S = (SQS \cup SSQSS \cup SQSS \cup SSQS) \cap S = SQS \cup SSQSS$. Then $S = SSQ = QSS = (SQS \cup SSQSS)$. Hence $S = SSQ \cap QSS \cap (SQS \cup SSQSS) \subseteq Q$. Thus S has no proper quasi-ideal. □

3.4 Isomorphism problem of S and $Q(S)$

Theorem 3.4.1. *Let S_1 and S_2 be two ternary semigroups. If $S_1 \cong S_2$, then $Q(S_1) \cong Q(S_2)$.*

Proof. Let S_1 and S_2 be two ternary semigroups such that $S_1 \cong S_2$. Then there exists an ternary isomorphism $f : S_1 \longrightarrow S_2$. Let us define a mapping $\phi : Q(S_1) \longrightarrow Q(S_2)$ by

$$\phi(a) = \begin{cases} f(a) & \text{if } a \in S_1 \\ f(a_1)f(a_2) & \text{if } a = m_1(a_1, a_2) \in M_1 \text{ or } a = a_1a_2 \in {S_1}^2. \end{cases}$$

First we have to show that the mapping is well defined. For this, let $a = b$ for $a, b \in Q(S_1)$. If $a, b \in S_1$, then $a = b \implies f(a) = f(b) \implies \phi(a) = \phi(b)$. If $a = m_1(a_1, a_2), b = m_1(b_1, b_2) \in {S_1}^2$, where $a_1, a_2, b_1, b_2 \in S_1$. Then $a = b \implies m_1(a_1, a_2) = m_1(b_1, b_2)$. This implies that $m_1(a_1, a_2)c = m_1(b_1, b_2)c$ for all $c \in S_1$

i.e. $a_1a_2c = b_1b_2c$ for all $c \in S_1$. Let $c_1 \in S_1$. Then we have,

$$\begin{aligned}
& a_1a_2c_1 = b_1b_2c_1 \\
& \Longrightarrow f(a_1a_2c_1) = f(b_1b_2c_1) \\
& \Longrightarrow f(a_1)f(a_2)f(c_1) = f(b_1)f(b_2)f(c_1) \\
& \Longrightarrow m_2(f(a_1), f(a_2))f(c_1) = m_2(f(b_1), f(b_2))f(c_1)
\end{aligned}$$

So, $m_2(f(a_1), f(a_2))f(c) = m_2(f(b_1), f(b_2))f(c)$ for all $c \in S_1$. Since f is onto $f(S_1) = S_2$. Thus, $m_2(f(a_1), f(a_2))d = m_2(f(b_1), f(b_2))d$ for all $d \in S_2$. Similarly, $dm_2(f(a_1), f(a_2)) = dm_2(f(b_1), f(b_2))$ for all $d \in S_2$. Hence $m_2(f(a_1), f(a_2)) = m_2(f(b_1), f(b_2)) \Longrightarrow f(a_1)f(a_2) = f(b_1)f(b_2) \Longrightarrow \phi(a) = \phi(b)$. Thus ϕ is a well defined mapping.

Now for any $a, b \in Q(S_1)$, we have to show that $\phi(ab) = \phi(a)\phi(b)$.

Case 1 : Let $a, b \in S_1$. Then $ab \in S_1S_1 \subseteq {S_1}^2$. So $\phi(ab) = f(a)f(b) = \phi(a)\phi(b)$.

Case 2 : Let $a, b \in {S_1}^2$. Let $a = m_1(a_1, a_2)$, $b = m_1(b_1, b_2)$, where $a_1, a_2, b_1, b_2 \in S_1$. Then $ab \in {S_1}^2{S_1}^2 \subseteq {S_1}^2$. So $\phi(ab) = \phi(m_1(a_1, a_2)m_1(b_1, b_2)) = \phi(m_1(a_1, a_2b_1b_2)) = \phi(a_1a_2b_1b_2) = f(a_1)f(a_2b_1b_2) = f(a_1)f(a_2)f(b_1)f(b_2) = \phi(a)\phi(b)$.

Case 3 : Let $a \in S_1$ and $b \in {S_1}^2$. Let $b = m_1(b_1, b_2)$, where $b_1, b_2 \in S_1$. Then $ab \in S_1{S_1}^2 \subseteq S_1$. So $\phi(ab) = f(ab) = f(am_1(b_1, b_2)) = f(R(b_2, b_1)a) = f(ab_1b_2) = f(a)f(b_1)f(b_2) = \phi(a)\phi(b)$.

Case 4 : Let $a \in {S_1}^2$ and $b \in S_1$. Let $a = m_1(a_1, a_2)$, where $a_1, a_2 \in S_1$. Then $ab \in {S_1}^2S_1 \subseteq S_1$. So $\phi(ab) = f(ab) = f(m_1(a_1, a_2)b) = f(L(a_1, a_2)b) = f(a_1a_2b) = f(a_1)f(a_2)f(b) = \phi(a)\phi(b)$.

It remains to show that ϕ is a bijective mapping. Let $\phi(a) = \phi(b)$ for $a, b \in Q(S_1)$. If $a, b \in S_1$, then $\phi(a) = f(a)$ and $\phi(b) = f(b)$. Thus $\phi(a) = \phi(b) \Longrightarrow f(a) = f(b) \Longrightarrow a = b$ (since f is one-one). Let $a, b \in {S_1}^2$. Let $a = m_1(a_1, a_2)$, $b = m_1(b_1, b_2)$, where $a_1, a_2, b_1, b_2 \in S_1$. Now we have $\phi(a) = \phi(b) \Longrightarrow \phi(m_1(a_1, a_2)) = \phi(m_1(b_1, b_2)) \Longrightarrow \phi(a_1a_2) = \phi(b_1b_2) \Longrightarrow f(a_1)f(a_2) = f(b_1)f(b_2) \Longrightarrow m_2(f(a_1), f(a_2)) = m_2(f(b_1), f(b_2)) \Longrightarrow m_2(f(a_1), f(a_2))d = m_2(f(b_1), f(b_2))d$ for all $d \in S_2$. Since

f is onto, for all $d \in S_2$, there exists $c \in S_1$ such that $f(c) = d$. Thus we have $m_2(f(a_1), f(a_2))f(c) = m_2(f(b_1), f(b_2))f(c)$ for all $c \in S_1$ which implies that $f(a_1)f(a_2)f(c) = f(b_1)f(b_2)f(c)$ for all $c \in S_1$. Since f is a ternary isomorphism, $f(a_1)f(a_2)f(c) = f(b_1)f(b_2)f(c) \Longrightarrow f(a_1a_2c) = f(b_1b_2c) \Longrightarrow a_1a_2c = b_1b_2c \Longrightarrow m_1(a_1, a_2)c = m_1(b_1, b_2)c$ for all $c \in S_1$. Similarly, $cm_1(a_1, a_2) = cm_1(b_1, b_2)$ for all $c \in S_1$. Hence $m_1(a_1, a_2) = m_1(b_1, b_2)$ which implies that $a = b$. Let $a \in S_1$, $b \in {S_1}^2$. Let $b = m_1(b_1, b_2)$, where $b_1, b_2 \in S_1$. Then $a \neq b$. Now $\phi(a) = f(a) \in S_2$ and $\phi(b) = f(m(b_1, b_2)) = f(b_1)f(b_2) \in {S_2}^2 = M_2$. Thus $\phi(a) \neq \phi(b)$. Therefore, ϕ is one-one.

Now let $b \in Q(S_2) = S_2 \cup M_2$. For any $y \in S_2$, there exists $x \in S_1$ such that $f(x) = y$. If $b \in S_2$, then there exists $a \in S_1$ such that $f(a) = b$ i.e. $\phi(a) = b$ for some $a \in S_1 \subseteq Q(S_1)$. Let $b \in M_2 = {S_2}^2$. Then $b = m_2(b_1, b_2)$, where $b_1, b_2 \in S_2$. Now $m_2(b_1, b_2) = b_1b_2 = f(a_1)f(a_2) = \phi(a)$ for some $a = a_1a_2 \in {S_1}^2 = M_1 \subseteq Q(S_1)$. Hence ϕ is onto. Therefore, $\phi : Q(S_1) \longrightarrow Q(S_2)$ is an isomorphism. $\square$

Remark 3.4.2. *Let S_1 and S_2 be two ternary semigroups such that $Q(S_1) \cong Q(S_2)$. Then $S_1 \cong S_2$ is not necessarily true, in general.*

We give the following example.

Example 3.4.3. *Let $S_1 = \{1, -1, i, -i\}$ be a semigroup. A semigroup is always a ternary semigroup. Thus S_1 is also a ternary semigroup. In that case, ${S_1}^2 = S_1$. Then $Q(S_1) = S_1 \cup {S_1}^2 = \{1, -1, i, -i\}$. Let us take another ternary semigroup $S_2 = \{i, -i\}$. Thus ${S_2}^2 = \{m(i, i), (-i, -i), (i, -i), m(-i, i)\}$. Since $m(i, i) = m(-i, -i)$ and $m(i, -i) = m(-i, i)$ then $Q(S_2) = \{i, -i, m(i, i), m(i, -i)\}$. Let us define a mapping ψ from $Q(S_1)$ to $Q(S_2)$ by $\psi(1) = m(i, -i)$, $\psi(-1) = m(i, i)$, $\psi(i) = i$, $\psi(-i) = -i$. Hence $\psi : Q(S_1) \longrightarrow Q(S_2)$ is an isomorphism. But there is no bijection from S_1 to S_2 and so S_1 is not isomorphic to S_2.*

However we have the following results, in particular :

We find some class of ternary semigroups in which the above result holds. For this, we first need the following lemma :

Lemma 3.4.4. *If f is a homomorphism from $Q(S_1)$ to $Q(S_2)$ then the restriction of f on S_1 is also a ternary homomorphism from S_1 to $Q(S_2)$, considering $Q(S_2)$ as ternary semigroup.*

Proof. Let f be a homomorphism from $Q(S_1)$ to $Q(S_2)$ and f^* be the restriction of f on S_1. Then for all $x, y \in Q(S_1)$ we have $f(xy) = f(x)f(y)$. Let $a, b, c \in S_1$. Since $S_1 \subseteq Q(S_1)$ we have $a, b, c \in Q(S_1)$. Now in $Q(S_1)$, $abc = a(bc) = am(b, c)$. Thus $f(abc) = f(am(b, c)) = f(a)f(m(b, c)) = f(a)f(bc) = f(a)f(b)f(c)$. Hence $f(abc) = f(a)f(b)f(c)$ for all $a, b, c \in S_1$. Thus $f^* : S_1 \longrightarrow Q(S_2)$ is a ternary homomorphism. □

Theorem 3.4.5. *Let S_1 and S_2 be two ternary semigroups such that S_1 is a left zero ternary semigroup. If $Q(S_1) \cong Q(S_2)$ then $S_1 \cong S_2$.*

Proof. Let f be an isomorphism from $Q(S_1)$ to $Q(S_2)$. Then by Lemma 3.4.4 f is a homomorphism from S_1 to $Q(S_2)$ and f is a bijection from $Q(S_1)$ to $Q(S_2)$. We have to show that $f(S_1) = S_2$. Now $S_1 \subseteq Q(S_1)$. So $f(S_1) \subseteq Q(S_2) = S_2 \cup M_2$. Then $f(S_1) \subseteq S_2 \cup {S_2}^2$. Let $a, b \in S_1$ such that $f(a) \in S_2$ and $f(b) \in {S_2}^2$. Since S_1 is left zero ternary semigroup, $aab = a$. Then $f(aab) = f(a) \in S_2$. Also by the above Lemma 3.4.4 we have, $f(aab) = f(a)f(a)f(b) \in S_2 S_2 {S_2}^2 \subseteq {S_2}^2$. Thus for the element $aab \in S_1$ we get, $f(aab) \in S_2 \cap {S_2}^2$, which is a contradiction. Therefore, either $f(S_1) \subseteq S_2$ or $f(S_1) \subseteq {S_2}^2$.

Let $f(S_1) \subseteq {S_2}^2$. Let $a, b \in S_1$. Then $f(a), f(b) \in {S_2}^2$. Thus $f(ab) = f(a)f(b) \in {S_2}^2 {S_2}^2 \subseteq {S_2}^2$. Hence $f(m(a, b)) \in {S_2}^2$ for all $a, b \in S_1$. Thus $f(M_1) = f({S_1}^2) \subseteq {S_2}^2$ and so $f(Q(S_1)) \subseteq {S_2}^2$. Thus $f(Q(S_1)) \subseteq {S_2}^2$ implies that for any $d \in S_2$ there is no element $c \in Q(S_1)$ such that $f(c) = d$, which contradicts the fact that f is a bijection from $Q(S_1)$ to $Q(S_2)$. So our assumption is is not true. Hence $f(S_1) \nsubseteq {S_2}^2$. Therefore, we have $f(S_1) \subseteq S_2$.

Again since $f : Q(S_1) \longrightarrow Q(S_2)$ is an isomorphism. Then $f^{-1} : Q(S_2) \longrightarrow Q(S_1)$ is also an isomorphism. Thus $f^{-1}(S_2) \subseteq S_1$. Hence $f(f^{-1}(S_2)) \subseteq f(S_1) \subseteq$

S_2. Therefore, $f(S_1) = S_2$. Hence the restriction of f on S_1 i.e. f^* is a ternary isomorphism from S_1 to S_2. $\square$

Theorem 3.4.6. *Let S_1 be a ternary semilattice and S_2 be a ternary semigroup such that $Q(S_1) \cong Q(S_2)$. Then $S_1 \cong S_2$.*

Proof. Let f be an isomorphism from $Q(S_1)$ to $Q(S_2)$. Then f^* the restriction of f on S_1 is a ternary homomorphism from S_1 to $Q(S_2)$ by Lemma 3.4.4. We have to show that $f(S_1) = S_2$. Let $a, b \in S_1$ such that $f(a) \in {S_2}^2$ and $f(b) \in S_2$. Since S_1 is a ternary semilattice we have $a^2b = ab^2$. Thus $f(a^2b) = f(ab^2)$. Now $f(a^2b) = f(a)f(a)f(b) \in {S_2}^2{S_2}^2S_2 \subseteq S_2$ and $f(ab^2) = f(a)f(b)f(b) \in {S_2}^2S_2S_2 \subseteq {S_2}^2$. Thus $f(a^2b) \neq f(ab^2)$ which implies that $a^2b \neq ab^2$. This contradicts our assumption that S_1 is a ternary semilattice. Therefore, either $f(S_1) \subseteq S_2$ or $f(S_1) \subseteq {S_2}^2$. By previous Theorem 3.4.5 we can say that $f(S_1) \nsubseteq {S_2}^2$. Then $f(S_1) \subseteq S_2$. Now proceeding in the similar way we can say that $f(S_1) = S_2$ and hence f^* is a ternary isomorphism from S_1 to S_2. $\square$

Theorem 3.4.7. *Let S_1 be a ternary rectangular band and S_2 be a ternary semigroup. Then $Q(S_1) \cong Q(S_2)$ implies that $S_1 \cong S_2$.*

Proof. Suppose that $f : Q(S_1) \longrightarrow Q(S_2)$ be an isomorphism. Then by Lemma 3.4.4 we have $f^* = f|_{S_1}$ is a ternary homomorphism from S_1 to $Q(S_2)$. Since S_1 is a ternary rectangular band then $a = aba$ for all $a, b \in S_1$. Let $a, b \in S_1$ such that $f(a) \in S_2$ and $f(b) \in {S_2}^2$. Then $f(aba) = f(a) \in S_2$. Also by the Lemma 3.4.4 $f(aba) = f(a)f(b)f(a) \in S_2{S_2}^2S_2 \subseteq {S_2}^2$. Hence $f(aba) = f(a) \in S_2 \cap {S_2}^2$, which is a contradiction. Therefore, either $f(S_1) \subseteq S_2$ or $f(S_1) \subseteq {S_2}^2$. Proceeding in the same manner as Theorem 3.4.5 we can say that $f(S_1) = S_2$ and so f^* the restriction of f on S_1 is a ternary isomorphism from S_1 to S_2. $\square$

3.5 Lattice structures in ordered ternary semigroup S and its cover $Q(S)$

Let S be a ternary semigroup and '$\leq$' is a partial order on S. Then we can define a partial order '$\leq_Q$' on $Q(S) = S \cup M$ as follows:

$$\begin{cases} \text{For all } a, b \in S,\ a \leq_Q b \text{ if and only if } a \leq b\text{'}, \\ \text{For all } a, b \in M,\ a \leq_Q b \text{ if and only if } a_1a_2x \leq b_1b_2x \text{ and } xa_1a_2 \leq xb_1b_2\ \forall\ x \in S. \end{cases}$$

Let $m(a_1, a_2) \in M = S^2$. Then we have $a_1a_2x, xa_1a_2 \in S$ for all $x \in S$. Now $a_1a_2x \leq a_1a_2x$ and $xa_1a_2 \leq xa_1a_2$ for all $x \in S$. Hence $m(a_1, a_2) \leq_Q m(a_1, a_2)$. Thus $\leq_Q$ is reflexive.

Let $m(a_1, a_2), m(b_1, b_2) \in M = S^2$ such that $m(a_1, a_2) \leq_Q m(b_1, b_2)$ and $m(b_1, b_2)$ $\leq_Q m(a_1, a_2)$. Now $m(a_1, a_2) \leq_Q m(b_1, b_2) \Longrightarrow a_1a_2x \leq b_1b_2x$ and $xa_1a_2 \leq xb_1b_2$ for all $x \in S$ and $m(b_1, b_2) \leq_Q m(a_1, a_2) \Longrightarrow b_1b_2x \leq a_1a_2x$ and $xb_1b_2 \leq xa_1a_2$ for all $x \in S$. Since $\leq$ is anti-symmetric we have $a_1a_2x = b_1b_2x$ and $xa_1a_2 = xb_1b_2$ for all $x \in S$. Thus we have $m(a_1, a_2)x = a_1a_2x = b_1b_2x = m(b_1, b_2)x$ and $xm(a_1, a_2) = xa_1a_2 = xb_1b_2 = xm(b_1, b_2)$ which implies that $m(a_1, a_2) = m(b_1, b_2)$. Therefore $\leq_Q$ is anti-symmetric.

Let $m(a_1, a_2), m(b_1, b_2), m(c_1, c_2) \in M = S^2$ such that $m(a_1, a_2) \leq_Q m(b_1, b_2)$ and $m(b_1, b_2) \leq_Q m(c_1, c_2)$. Now $m(a_1, a_2) \leq_Q m(b_1, b_2) \Longrightarrow a_1a_2x \leq b_1b_2x$ and $xa_1a_2 \leq xb_1b_2$ for all $x \in S$ and $m(b_1, b_2) \leq_Q m(c_1, c_2) \Longrightarrow b_1b_2x \leq c_1c_2x$ and $xb_1b_2 \leq xc_1c_2$ for all $x \in S$. Thus we have $a_1a_2x \leq c_1c_2x$ and $xa_1a_2 \leq xc_1c_2$ for all $x \in S$(Since $\leq$ is transitive). Hence $m(a_1, a_2) \leq_Q m(c_1, c_2)$ and so $\leq_Q$ is transitive. Therefore $\leq_Q$ is a partial order relation in $M = S^2$. By definition $\leq_Q$ is also a partial order relation in S. Hence $\leq_Q$ is a partial order relation in $S \cup M = Q(S)$.

Theorem 3.5.1. *A ternary semigroup S is an ordered ternary semigroup with respect to $\leq$ if and only if $Q(S)$ is an ordered semigroup with respect to $\leq_Q$.*

Proof. First suppose that S is an ordered ternary semigroup with respect to $\leq$.

Then for $a, b \in S$, $a \leq b \Longrightarrow xya \leq xyb$, $xay \leq xby$, $axy \leq bxy$ for all $x, y \in S$. Let $a, b \in Q(S)$ such that $a \leq_Q b$. Then we have to show that for any two elements $a, b \in Q(S)$, $a \leq_Q b$ imples that $ax \leq_Q bx$ and $xa \leq_Q xb$ for all $x \in Q(S)$. Then we have the following two cases:

Case 1: Let $a, b \in S$ such that $a \leq_Q b$. Then $a \leq_Q b \Leftrightarrow a \leq b$. Then for all $x, y \in S$ we have $xya \leq xyb$, $xay \leq xby$, $axy \leq bxy$. Also $a \leq b \Longrightarrow yxa \leq yxb$, $yax \leq ybx$, $ayx \leq byx$. Thus $xay \leq xby$ and $yxa \leq yxb$ for all $x, y \in S \Longrightarrow m(x, a) \leq_Q m(x, b) \Longrightarrow xa \leq_Q xb$. Similarly $axy \leq bxy$ and $yax \leq ybx$ for all $x, y \in S \Longrightarrow m(a, x) \leq_Q m(b, x) \Longrightarrow ax \leq_Q bx$. Hence $a \leq_Q b \Longrightarrow ax \leq_Q bx$ and $xa \leq_Q xb$ for all $x \in S$.

Again $a \leq b \Longrightarrow axy \leq bxy \Longrightarrow axy \leq_Q bxy \Longrightarrow am(x, y) \leq_Q bm(x, y)$ and $a \leq b \Longrightarrow xya \leq xya \Longrightarrow xya \leq_Q xya \Longrightarrow m(x, y)a \leq_Q m(x, y)b$. Hence $a \leq_Q b \Longrightarrow am(x, y) \leq_Q bm(x, y)$ and $m(x, y)a \leq_Q m(x, y)b$ for all $m(x, y) \in S^2 = M$.

Case 2: Let $a, b \in M = S^2$ such that $a \leq_Q b$. Let $a = m(a_1, a_2)$, $b = m(b_1, b_2)$ where $a_1, a_2, b_1, b_2 \in S$. Now $m(a_1, a_2) \leq_Q m(b_1, b_2) \Longrightarrow a_1a_2z \leq b_1b_2z$ for all $z \in S \Longrightarrow a_1a_2z \leq_Q b_1b_2z$ for all $z \in S$(Since $a_1a_2z, b_1b_2z \in S$) $\Longrightarrow m(a_1, a_2)z \leq_Q m(b_1, b_2)z$ for all $z \in S$. Similarly, $m(a_1, a_2) \leq_Q m(b_1, b_2) \Longrightarrow za_1a_2 \leq zb_1b_2$ for all $z \in S \Longrightarrow za_1a_2 \leq_Q zb_1b_2$ for all $z \in S$(Since $za_1a_2, zb_1b_2 \in S$) $\Longrightarrow zm(a_1, a_2) \leq_Q zm(b_1, b_2)$ for all $z \in S$. Hence for all $a, b \in M$, $a \leq_Q b \Longrightarrow za \leq_Q zb$ and $az \leq_Q bz$ for all $z \in S$.

Again for all $x, y, z \in S$, $a_1a_2z \leq b_1b_2z \Longrightarrow xya_1a_2z \leq xyb_1b_2z$(Since $a_1a_2z, b_1b_2z \in S$ and S is an ordered ternary semigroup). Then $m(x, ya_1a_2)z \leq m(x, yb_1b_2)z$ for all $z \in S$ and $ya_1a_2 \leq yb_1b_2 \Longrightarrow zxya_1a_2 \leq zxyb_1b_2 \Longrightarrow zm(x, ya_1a_2) \leq zm(x, yb_1b_2)$. Thus we have $m(x, ya_1a_2) \leq_Q m(x, yb_1b_2) \Longrightarrow m(x, y)m(a_1, a_2) \leq_Q m(x, y)m(b_1, b_2)$ for all $m(x, y) \in M = S^2$. Similarly, for all $x, y, z \in S$, $za_1a_2 \leq zb_1b_2 \Longrightarrow za_1a_2xy \leq zb_1b_2xy \Longrightarrow zm(a_1, a_2xy) \leq zm(a_1, a_2xy)$ for all $z \in S$ and $a_1a_2x \leq b_1b_2x \Longrightarrow a_1a_2xyz \leq b_1b_2xyz \Longrightarrow m(a_1, a_2xy)z \leq m(b_1, b_2xy)z$. Then we have $m(a_1, a_2xy) \leq_Q m(b_1, b_2xy) \Longrightarrow m(a_1, a_2)m(x, y) \leq_Q m(b_1, b_2)m(x, y)$ for all $m(x, y) \in M = S^2$. Hence for all $a, b \in M$, $a \leq_Q b$ implies that $za \leq_Q zb$ and $az \leq_Q bz$ for all $z \in S$.

Thus in both cases we have $a \leq_Q b$ imples that $ax \leq_Q bx$ and $xa \leq_Q xb$ for all $x \in Q(S)$. Hence $Q(S)$ is an ordered ternary semigroup with respect to the partial order '$\leq_Q$'.

For the converse part, let $Q(S)$ is an ordered ternary semigroup with respect to the partial ordered '$\leq_Q$'. Then $a \leq_Q b$ in $Q(S) \Longrightarrow xa \leq_Q xb$ and $ax \leq_Q bx$ for all $x \in Q(S)$. Let $a, b \in S$ such that $a \leq_Q b$. Then $ax \leq_Q bx$ for all x in S. Since $ax, bx \in Q(S)$, $ax \leq_Q bx \Longrightarrow axy \leq_Q bxy \Longrightarrow axy \leq bxy$ for all $y \in S$. Again $xa \leq_Q xb \Longrightarrow xay \leq_Q xby \Longrightarrow xay \leq xby$ for all $x, y \in S$. Also $a \leq_Q b$ implies that $m(x,y)a \leq_Q m(x,y)b \Longrightarrow xya \leq_Q xyb \Longrightarrow xya \leq xyb$ and $am(x,y) \leq_Q bm(x,y) \Longrightarrow axy \leq_Q bxy \Longrightarrow axy \leq bxy$ for all $m(x,y) \in S^2 = M$. Therefore S is an ordered ternary semigroup with respect to '$\leq$'. $\square$

Note 3.5.2. *We cannot define a partial order between the element of S and $M = S^2$. Otherwise if we take $a \in S$ and $b \in M = S^2$ such that $a \leq_Q b$ and $b \leq_Q a$ then by anti-symmetric properties of '$\leq_Q$' we have $a = b$, which contradicts the fact that S and M are disjoint sets.*

Theorem 3.5.3. *Let S be an ordered ternary semigroup. Then S is a lattice w.r.t. '$\leq$' if and only if $Q(S)$ is a lattice w.r.t. '$\leq_Q$'.*

Proof. Suppose $Q(S)$ be a lattice with respect to $\leq_Q$. Let $a, b \in S$. Thus $\inf\{a, b\} = a \wedge b$ and $\sup\{a, b\} = a \vee b$ exists in $Q(S)$. Let $c, d \in Q(S)$ such that $c = a \wedge b$ and $d = a \vee b$. If $c \in M = S^2$ then $c = m(e, f)$ for some $e, f \in S$. Thus $a \wedge b = m(e, f)$ implies that $m(e, f) \leq_Q a$ and $m(e, f) \leq_Q b$, which is a contradiction (since '$\leq_Q$' is a partial order relation, $m(e, f) \nleq_Q a, b$). Thus $c \notin M = S^2$. Hence $c \in S$. Thus $c = \inf\{a, b\}$ exisxts in S. Similarly, we can show that $d = \sup\{a, b\}$ exists in S. Therefore, S is also a lattice.

Conversely, let S be a lattice with respect to '$\leq$'. Then for all $a, b \in S$ we have $\inf\{a, b\} = a \wedge b$ and $\sup\{a, b\} = a \vee b$ exists in S. We have to show that for any two elements $x, y \in Q(S)$, $x \wedge y$ and $x \vee y$ exists in $Q(S)$. Since S is a lattice, it is sufficient to prove that for all $x, y \in S^2$, $x \wedge y$ and $x \vee y$ exists in S^2.

Let $x = m(a_1, b_1), y = m(a_2, b_2) \in S^2 = M \subseteq Q(S)$ where $a_1, b_1, a_2, b_2 \in S$. Let $a = a_1 \wedge a_2$ and $b = b_1 \wedge b_2$. Then $a \leq a_1 \Longrightarrow abx \leq a_1bx$ and $b \leq b_1 \Longrightarrow a_1bx \leq a_1b_1x$ (since S is a partially ordered ternary semigroup). Thus $abx \leq a_1b_1x$ for all $x \in S$. Similarly, $a \leq a_1 \Longrightarrow xab \leq xa_1b$ and $b \leq b_1 \Longrightarrow xa_1b \leq xa_1b_1$. Thus $xab \leq xa_1b_1$. Hence $m(a,b) \leq_Q m(a_1, b_1)$. In the similar way we can show that, $m(a,b) \leq_Q m(a_2, b_2)$. Thus $m(a,b)$ is a lower bound of $m(a_1, b_1)$ and $m(a_2, b_2)$. Thus $\inf\{m(a_1, b_1), m(a_2, b_2)\}$ also exists in $Q(S)$. Therefore $Q(S)$ is a lattice. □

Note 3.5.4. *However, $m(a,b) = m(a_1 \wedge a_2, b_1 \wedge b_2)$ may not be the greatest lower bound of $m(a_1, b_1)$ and $m(a_2, b_2)$.*

In the followings example we can show that $m(a_1 \wedge a_2, b_1 \wedge b_2) \neq m(a_1, b_1) \wedge m(a_2, b_2)$.

Example 3.5.5. *Let $S = \mathbb{Z}$ = the set of all integers. Since all binary semigroups are ternary semigroups as well, we consider $\mathbb{Z}$ is a ternary semigroup here. Let us define a partial order relation on $\mathbb{Z}$ defined by $a \leq b$ if and only if a divides b. Thus $(\mathbb{Z}, ., \leq)$ is a lattice. Let $m(4,2), m(3,4) \in \mathbb{Z}^2$ where $2, 3, 4 \in \mathbb{Z}$. Thus $4 \wedge 3 = 1$ and $2 \wedge 4 = 2$. Thus $m(4 \wedge 3, 2 \wedge 4) = m(1,2)$. Again $m(2,2) \leq_Q m(4,2)$ since $2.2.x = 4x$ divides $4.2.x = 8x$ for all $x \in \mathbb{Z}$. Similarly, $m(2,2) \leq_Q m(3,4)$. Thus $m(2,2)$ is also a lower bound of $\{m(4,2), m(3,4)\}$. But $m(1,2) \leq_Q m(2,2)$. So, $m(1,2)$ is not the $\inf\{m(4,2), m(3,4)\} = m(4,2) \wedge m(3,4)$. Therefore, $m(4 \wedge 3, 2 \wedge 4) \neq m(4,2) \wedge m(3,4)$.*

Next we have the following theorem:

Theorem 3.5.6. *The ordered ternary semigroup S is a complete lattice if and only if $Q(S)$ is a complete lattice.*

Proof. *Let $Q(S)$ be a complete lattice with respect to $\leq_Q$. Let $A = \{a_\alpha : \alpha \in I\}$ be any non empty subset of S, I being an index set. Since $Q(S)$ is a complete lattice and $A \subseteq S \subseteq Q(S)$, $\inf A$ and $\sup A$ exists in $Q(S)$. Thus $\bigwedge\limits_{\alpha \in I} a_\alpha$ and $\bigvee\limits_{\alpha \in I} a_\alpha$*

exist in $Q(S)$. Let $c, d \in Q(S) = S \cup M$ such that $c = \underset{\alpha\in I}{\wedge} a_\alpha$ and $d = \underset{\alpha\in I}{\vee} a_\alpha$. If $c \in M = S^2$ then $c = m(a,b)$ for some $a, b \in S$. Thus $\underset{\alpha\in I}{\wedge} a_\alpha = m(a,b)$ implies that $m(a,b) \leq_Q a_\alpha$ for each $\alpha \in I$ where I is an index set, which is a contradiction [since $a_\alpha \in S$, and $m(a,b) \in M, a \nleq m(a,b)$]. Thus $c \notin M = S^2$. Hence $c \in S$. Therefore, $c = \underset{\alpha\in I}{\wedge} a_\alpha$ exisxts in S. Similarly, we can show that $d = \underset{\alpha\in I}{\vee} a_\alpha$ exists in S. Therefore, S is also a complete lattice.

Conversely, let S be a complete lattice with respect to '$\leq$'. Since S is a complete lattice, then for all non-empty subsets of S both infimum and supremum exist in S. We have to show that for any non-empty subset of $Q(S)$ both infimum and supremum exist in $Q(S)$. Now it is sufficient to prove that for all non-empty subsets of S^2 both infimum and supremum exists in S^2. Let $X = \{m(a_\beta, b_\beta) : \beta \in I\}$ where $a_\beta, b_\beta \in S$, I being an index set. Since S is a lattice by Theorem 3.5.3, $Q(S)$ is a lattice. Let $a = \underset{\beta\in I}{\wedge} a_\beta$ and $b = \underset{\beta\in I}{\wedge} b_\beta$. Then $a \leq a_\beta$ and $b \leq b_\beta$ for all $\beta \in I$. Hence $abx \leq a_\beta bx$ and $\Longrightarrow a_\beta bx \leq a_\beta b_\beta x$ for all $\beta \in I$ [since S is an ordered ternary semigroup]. Thus $abx \leq a_\beta b_\beta x$ for all $x \in S$ and for all $\beta \in I$. Similarly, $a \leq a_\beta \Longrightarrow xab \leq xa_\beta b$ and $b \leq b_\beta \Longrightarrow xa_\beta b \leq xa_\beta b_\beta$. So, $xab \leq xa_\beta b_\beta$. Thus $m(a,b) \leq_Q m(a_\beta, b_\beta)$ for all $\beta \in I$. Hence $m(a,b)$ is a lower bound of $m(a_\beta, b_\beta)$ for all $\beta \in I$. Thus $inf\{m(a_\beta, b_\beta) : \beta \in I\} = infX$ also exists in $Q(S)$. Therefore, $Q(S)$ is a complete lattice.

Note 3.5.7. *However from the previous example 3.5.5 we arrived at the conclusion that $m(\underset{\beta\in I}{\wedge} a_\beta, \underset{\beta\in I}{\wedge} b_\beta) \neq \underset{\beta\in I}{\wedge} m(a_\beta, b_\beta)$. Hence $m(a,b) = m(\underset{\beta\in I}{\wedge} a_\beta, \underset{\beta\in I}{\wedge} b_\beta)$ may not be the infimum of $inf\{m(a_\beta, b_\beta) : \beta \in I\}$.*

Theorem 3.5.8. *The ordered ternary semigroup S is a modular lattice if and only if $Q(S)$ is a modular lattice.*

Proof. Let S be an ordered ternary semigroup such that S a modular lattice. Then $a \leq b$ implies that $a \vee (x \wedge b) = (a \vee x) \wedge b$ where a, b, x are arbitrary elements of S. Let $m(a_1, b_1), m(a_2, b_2) \in M = S^2$ such that $m(a_1, b_1) \leq_Q m(a_2, b_2)$.

Then $a_1b_1x \leq a_2b_2x$ and $xa_1b_1 \leq xa_2b_2$ for all $x \in S$. Let $m(a_3, b_3)$ be an arbitrary element in M. Since $a_1b_1x, a_2b_2x, a_3b_3x \in S$, then $a_1b_1x \leq a_2b_2x$ implies that $a_1b_1x \vee (a_3b_3x \wedge a_2b_2x) = (a_1b_1x \vee a_3b_3x) \wedge a_2b_2x$ for all $x \in S$.
$\Longrightarrow m(a_1, b_1)x \vee (m(a_3, b_3)x \wedge m(a_2, b_2)x) = (m(a_1, b_1)x \vee m(a_3, b_3)x) \wedge m(a_2, b_2)x$ for all $x \in S$.
$\Longrightarrow \big(m(a_1, b_1) \vee (m(a_3, b_3) \wedge m(a_2, b_2))\big)x = \big((m(a_1, b_1) \vee m(a_3, b_3)) \wedge m(a_2, b_2)\big)x$ for all $x \in S$.
Similarly, we have $xa_1b_1, xa_2b_2, xa_3b_3$ are all elements of S. Then $xa_1b_1 \leq xa_2b_2$ implies that $xa_1b_1 \vee (xa_3b_3 \wedge xa_2b_2) = (xa_1b_1 \vee xa_3b_3) \wedge xa_2b_2$ for all $x \in S$.
$\Longrightarrow xm(a_1, b_1) \vee (xm(a_3, b_3) \wedge xm(a_2, b_2)) = (xm(a_1, b_1) \vee xm(a_3, b_3)) \wedge xm(a_2, b_2)$ for all $x \in S$.
$\Longrightarrow x\big(m(a_1, b_1) \vee (m(a_3, b_3) \wedge m(a_2, b_2))\big) = x\big((m(a_1, b_1) \vee m(a_3, b_3)) \wedge m(a_2, b_2)\big)$ for all $x \in S$. Therefore, $m(a_1, b_1) \vee (m(a_3, b_3) \wedge m(a_2, b_2)) = (m(a_1, b_1) \vee m(a_3, b_3)) \wedge m(a_2, b_2)$. Hence $Q(S) = S \cup M$ is a modular lattice.

Conversely, let $Q(S)$ be a modular lattice. Let $a, b \in S$ such that $a \leq b$. Since $a, b \in Q(S)$, then $a \leq b \Longrightarrow a \leq_Q b$. So $a \vee (x \wedge b) = (a \vee x) \wedge b$ for all $x \in Q(S)$. Thus for any $c \in S$ we have $a \vee (c \wedge b) = (a \vee c) \wedge b$. Hence S is a modular lattice. $\square$

In the similar way we can prove the following theorem:

Theorem 3.5.9. *The ordered ternary semigroup S is a distributive lattice if and only if $Q(S)$ is a distributive lattice.*

Proof. Let S be an ordered ternary semigroup such that S a diatributive lattice. Then for all $a, b, c \in S$ we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Let $m_1, m_2, m_3 \in Q(S)$. If $m_1, m_2, m_3 \in S$ then our proof is done. If $m_1 = m(a_1, b_1), m_2 = m(a_2, b_2), m_3 = m(a_3, b_3) \in M = S^2 \subseteq Q(S)$ then we have $\Big(m(a_1, b_1) \wedge (m(a_2, b_2) \vee m(a_3, b_3))\Big)x = m(a_1, b_1)x \wedge (m(a_2, b_2)x \vee m(a_3, b_3)x) = a_1b_1x \vee (a_2b_2x \wedge a_3b_3x)$ for all $x \in S$. Since $a_1b_1x, a_2b_2x, a_3b_3x \in S$, $a_1b_1x \vee (a_2b_2x \wedge a_3b_3x) = (a_1b_1x \vee a_2b_2x) \wedge (a_1b_1x \vee a_3b_3x)$ for all $x \in S$. Hence

$$\Big(m(a_1,b_1)\wedge(m(a_2,b_2)\vee m(a_3,b_3))\Big)x$$
$$=(a_1b_1x\vee a_2b_2x)\wedge(a_1b_1x\vee a_3b_3x)\ \text{ for all }\ x\in S$$
$$=(m(a_1,b_1)x\wedge m(a_2,b_2)x)\vee(m(a_1,b_1)x\wedge m(a_3,b_3)x)\ \text{ for all }\ x\in S$$
$$=\Big((m(a_1,b_1)\wedge m(a_2,b_2))\vee(m(a_1,b_1)\wedge m(a_3,b_3))\Big)x$$

Similarly, we have $xa_1b_1, xa_2b_2, xa_3b_3 \in S$. So, $xa_1b_1 \vee (xa_2b_2 \wedge xa_3b_3) = (xa_1b_1 \vee xa_2b_2) \wedge (xa_1b_1 \vee xa_3b_3)$ for all $x \in S$ implies that $x\Big(m(a_1,b_1) \wedge (m(a_2,b_2) \vee m(a_3,b_3))\Big) = x\Big((m(a_1,b_1) \wedge m(a_2,b_2)) \vee (m(a_1,b_1) \wedge m(a_3,b_3))\Big)$. Hence $Q(S) = S \cup M$ is a distributive lattice.

For the converse part, let a, b, c are any three elements of S. Since $Q(S)$ is a distributive lattice and $a, b, c \in Q(S)$ we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Hence S is a distributive lattice. $\square$

Ordered Power Ternary Semigroups

Chapter-4

Chapter 4

Ordered power ternary semigroups

4.1 Introduction

The motivation for constructing power ternary semigroup of a ternary semigroup came from the concept of power semigroup of a semigroup. For a ternary semigroup S, one may define a ternary semigroup on the power set $P(S)$. If S is a ternary semigroup, then the ternary product of non-empty subsets of S can be defined in a natural way to produce a ternary semigroup, which is called the power semigroup of S, we simply denote it by $P(S)$.

There are various ways to lift a relation from a base set to its power set. Power ternary semigroup $P(S)$ of ternary semigroup S is an appropriate ternary semigroup defined on the power set $P(S)$ as a generalization of power semigroup of a semigroup. The main object of this paper has two aspects. We characterize power ternary semigroup with the help of corresponding ternary semigroup and discuss the connection between them. The rest of the paper deals with ordered power ternary semigroup. We define a partial order in power ternary semigroup in a natural way. Ordered power ternary semigroups are closely related to power ternary semigroups. In this chapter, we study some properties of ordered power ternary semigroups and discuss the connection between ternary semigroup and ordered power ternary semigroup.

The study of power semigroup of a semigroup was initiated by Tamura and Shafer [89] in 1967. Many authors [90], [88], [34], [57] studied power semigroups and its properties. T. Dutta, S. Kar and K. Das [30] studied the notion of power ternary semiring of a ternary semiring. S. Kar and I. Dutta [41] extend the notion of power semigroups to power ternary semigroups.

In this chapter, S denotes a ternary semigroup.

4.2 Power ternary semigroup $P(S)$ of a ternary semigroup S

In this section, we consider the correlation between ternary semigroup S and the corresponding power ternary semigroup $P(S)$.

If S is a ternary semigroup and $P(S)$ be the set of all non-empty subsets of S, then $P(S)$ forms a ternary semigroup with respect to the ternary multiplication defined as follows :

$$ABC = \{abc : a \in A,\ b \in B,\ c \in C\} \text{ for all } A,\ B,\ C \in P(S).$$

We call $P(S)$, the power ternary semigroup of all non-empty subsets of a ternary semigroup S.

Now we have the following results regarding some properties of a ternary semigroup S and the corresponding properties of the power ternary semigroup $P(S)$.

Theorem 4.2.1. *A ternary semigroup S is commutative if and only if the power ternary semigroup $P(S)$ is commutative.*

Proof. First suppose that S be a commutative ternary semigroup. Then for all $a, b, c \in S$, we have $abc = acb = bac = bca = cab = cba$. Let $A, B, C \in P(S)$ and $x \in ABC$. Then $x = a_1b_1c_1$ for some $a_1 \in A$, $b_1 \in B$, $c_1 \in C$. Since S is commutative, it follows that $x = a_1b_1c_1 = a_1c_1b_1 \in ACB$. Hence $ABC \subseteq ACB$. Similarly, $ACB \subseteq ABC$. Thus $ABC = ACB$. Continuing in this way we can show that $ABC = ACB = BAC = BCA = CAB = CBA$ for all $A, B, C \in P(S)$. Hence $P(S)$ is a commutative ternary semigroup.

Conversely, let $P(S)$ be a commutative ternary semigroup. Let $a, b, c \in S$. Then $abc \in \{abc\} = \{a\}\{b\}\{c\}$. Since $P(S)$ is commutative, $\{a\}\{b\}\{c\} = \{a\}\{c\}\{b\} = \{acb\}$. Hence $abc \in \{acb\}$ and so $abc = acb$. Continuing in this way we get $abc = acb = bac = bca = cab = cba$ for all $a, b, c \in S$. Thus S is a commutative ternary semigroup. $\square$

Theorem 4.2.2. *Let S be a ternary semigroup and A be a non-empty subset of S. Then A is a ternary subsemigroup of S if and only if $P(A)$ is a ternary subsemigroup of $P(S)$.*

Proof. First let us consider A be a subsemigroup of S. Then $A^3 \subseteq A$. Since $A \subseteq S$, it follows that $P(A) \subseteq P(S)$. Let $X \in P(A)^3$. Thus $X \subseteq A^3 \subseteq A$ and so $X \in P(A)$. Hence $P(A)^3 \subseteq P(A)$ and $P(A)$ is a ternary subsemigroup of $P(S)$. For the converse part, let $P(A)$ be a ternary subsemigroup of $P(S)$ for some non empty subset A of S. Let $x \in A^3$. Then $x = a_1a_2a_3$ for some $a_1, a_2, a_3 \in S$. Now $\{x\} = \{a_1a_2a_3\} = \{a_1\}\{a_2\}\{a_3\} \in P(A)^3 \subseteq P(A)$. Thus $x \in A$ and hence A is a ternary subsemigroup of S. $\square$

Theorem 4.2.3. *Let S be a ternary semigroup and I be a non-empty subset of S. Then I is an ideal of S if and only if $P(I)$ is an ideal of the power ternary semigroup $P(S)$.*

Proof. Let us consider I be an ideal of S. Then $P(I) \subseteq P(S)$ and $P(I)P(J)P(K) = \{ABC : A \subseteq I, B \subseteq J, C \subseteq K\}$ for all $I, J, K \subseteq S$. Let $ABC \in P(S)P(S)P(I)$. Thus $ABC \subseteq SSI \subseteq I$ and so $ABC \in P(I)$. Hence $P(S)P(S)P(I) \subseteq P(I)$. Similarly, we can show that $P(S)P(I)P(S) \subseteq P(I)$ and $P(I)P(S)P(S) \subseteq P(I)$. Hence $P(I)$ is an ideal of the power ternary semigroup $P(S)$.

For the converse part, let $P(I)$ be an ideal of $P(S)$ for some non-empty subset I of S. Let $x \in SSI$. Then $x = s_1s_2i$ for some $s_1, s_2 \in S$ and $i \in I$. Now

$\{x\} = \{s_1 s_2 i\} = \{s_1\}\{s_2\}\{i\} \in P(S)P(S)P(I) \subseteq P(I)$. Thus $\{x\} \in P(I)$ and so $\{x\} \subseteq I$. Hence $x \in I$. Thus $SSI \subseteq I$. Similarly, we can prove that $SIS \subseteq I$ and $ISS \subseteq I$. Hence I is an ideal of S. □

Note 4.2.4. *However, we notice that not all ideal of $P(S)$ is in the form $P(I)$ for some ideal I of S.*

We give the following example:

Example 4.2.5. *Let $S = \mathbb{Z}_0^-$. Then $P(-3\mathbb{Z}_0^-\mathbb{Z}_0^-)\backslash P(\{2k+1\}_{k=-1,-2,-3,....})$ is an ideal of $P(\mathbb{Z}_0^-)$. This ideal cannot be written in the form $P(A)$ for any non empty subest A of S, so we conclude that the proposed ideal can not be written in the form $P(I)$ for some ideal I of $\mathbb{Z}_0^-$.*

Theorem 4.2.6. *Let S be a ternary semigroup and B be a non-empty subset of S. Then B is a bi-ideal of S if and only if $P(B)$ is a bi-ideal of the power ternary semigroup $P(S)$.*

Proof. Assume that B is a bi-ideal of S. Then $B^3 \subseteq B$ and $BSBSB \subseteq B$. Since B is a ternary subsemigroup of S, $P(B)$ is a ternary subsemigroup of $P(S)$, by Theorem 4.2.2. Let $A \in P(B)P(S)P(B)P(S)P(B)$. Then $A = B_1 S_1 B_2 S_2 B_3$ for some $B_1, B_2, B_3 \in P(B)$ and $S_1, S_2 \in P(S)$. Thus $A \subseteq BSBSB \subseteq B$. This implies that $A \in P(B)$. So $P(B)P(S)P(B)P(S)P(B) \subseteq P(B)$ and hence $P(B)$ is a bi-ideal of the power ternary semigroup $P(S)$. For the converse part, let $P(B)$ be a bi-ideal of $P(S)$. Then $P(B)$ is a ternary subsemigroup of $P(S)$ and hence B is a ternary subsemigroup of S, by Theorem 4.2.2. Let $y \in BSBSB$. Then $y = b_1 s_1 b_2 s_2 b_3$ for some $b_1, b_2, b_3 \in B$ and $s_1, s_2 \in S$. Now $\{y\} = \{b_1 s_1 b_2 s_2 b_3\} = \{b_1\}\{s_1\}\{b_2\}\{s_2\}\{b_3\} \in P(B)P(S)P(B)P(S)P(B) \subseteq P(B)$. Thus $\{y\} \in P(B)$ and so $\{y\} \subseteq B$. This shows that $y \in B$ and hence $BSBSB \subseteq B$. Consequently, B is a bi-ideal of S. □

Note 4.2.7. *However not all bi-ideal of $P(S)$ is in the form $P(B)$ for some bi-ideal B of S. In the previous example 4.2.5 we have already seen that each ideal of $P(S)$*

not in the form $P(I)$ for some ideal I of S. Since every ideal is a bi-ideal, this is also an example of our conclusion.

Theorem 4.2.8. *Let S be a ternary semigroup and Q be a non-empty subset of S. Then Q is a quasi-ideal of S if and only if $P(Q)$ is a quasi-ideal of the power ternary semigroup $P(S)$.*

Proof. First let us consider Q be a quasi-ideal of S. Then $Q^3 \subseteq Q$ and $SSQ \cap (SQS \cup SSQSS) \cap QSS \subseteq Q$. Let $A \in P(S)P(S)P(Q) \cap (P(S)P(Q)P(S) \cup P(S)P(S)P(Q)P(S)P(S)) \cap P(Q)P(S)P(S)$. Thus $A \in P(S)P(S)P(Q)$. Then $A = S_1S_2Q_1$ for some $Q_1 \in P(Q)$ and $S_1, S_2 \in P(S)$. Thus $A \subseteq SSQ$. Similarly, we can show that $A \subseteq (SQS \cup SSQSS)$ and $A \subseteq QSS$. This imlies that $A \subseteq SSQ \cap (SQS \cup SSQSS) \cap QSS \subseteq Q$. So $A \in P(Q)$ and hence $P(Q)$ is a quasi-ideal of the power ternary semigroup $P(S)$.

For the converse part, let $P(Q)$ be a quasi-ideal of $P(S)$. Let $y \in SSQ \cap (SQS \cup SSQSS) \cap QSS$. Then $y \in SSQ$ and $y = s_1s_2q_1$ for some $s_1, s_2 \in S$ and $q_1 \in Q$. Now $\{y\} = \{s_1s_2q_1\} = \{s_1\}\{s_2\}\{q_1\} \in P(S)P(S)P(Q)$. Hence $\{y\} \in P(S)P(S)P(Q)$. Similarly, we can prove that $\{y\} \in P(S)P(Q)P(S) \cup P(S)P(S)P(Q)P(S)P(S)$ and $\{y\} \in P(Q)P(S)P(S)$. So, $\{y\} \in P(S)P(S)P(Q) \cap \Big(P(S)P(Q)P(S) \cup P(S)P(S)P(Q)P(S)P(S)\Big) \cap P(Q)P(S)P(S)$
$\subseteq P(Q)$ and so $y \in Q$. Thus $SSQ \cap (SQS \cup SSQSS) \cap QSS \subseteq Q$. Consequently, it follows that Q is a quasi-ideal of S. □

Theorem 4.2.9. *Let S be a ternary semigroup and J be a non-empty subset of S. Then J is a completely prime ideal of S if and only if $P(J)$ is a completely prime ideal of $P(S)$.*

Proof. First suppose that J be a completely prime ideal of S. Then J is an ideal of S and hence $P(J)$ be an ideal of $P(S)$, by Theorem 4.2.3. It remains to show that $P(J)$ is completely prime. Let $X, Y, Z \in P(S)$ such that $XYZ \in P(J)$. Suppose that $Y, Z \notin P(J)$. Then there exist some $y \in Y$ and $z \in Z$ such that $y \notin J$ and

$z \notin J$. Let x be an arbitrary element of X. Now $\{xyz\} = \{x\}\{y\}\{z\} \subseteq XYZ \subseteq J$. Thus $xyz \in J$. Since J is completely prime ideal of S and $y, z \notin J$, we must have $x \in J$. Thus $X \in P(J)$ and so $P(J)$ is a completely prime ideal of $P(S)$.

Conversely, suppose that $P(J)$ is a completely prime ideal of $P(S)$. Let $abc \in J$ for some $a, b, c \in S$. Now $\{abc\} \in P(J)$ and so $\{a\}\{b\}\{c\} \in P(J)$. Since $P(J)$ is a completely prime ideal of $P(S)$, so $\{a\} \in P(J)$ or $\{b\} \in P(J)$ or $\{c\} \in P(J)$ i.e. $\{a\} \subseteq J$ or $\{b\} \subseteq J$ or $\{c\} \subseteq J$. Thus $a \in J$ or $b \in J$ or $c \in J$. Hence J is completely prime ideal of S. □

Theorem 4.2.10. *Let S be a ternary semigroup and K be a non-empty subset of S. Then K is a completely semiprime ideal of S if and only if $P(K)$ is a completely semiprime ideal of $P(S)$.*

Proof. Let K be a completely semiprime ideal of S. Then K is an ideal of S and hence $P(K)$ is an ideal of $P(S)$, by Theorem 4.2.3. We have to show that $P(K)$ is completely semiprime. Let $X \in P(S)$ such that $X^3 \in P(K)$. Suppose $x \in X$ such that $x^3 \in X^3 \subseteq K$. Since K is completely semiprime, we have $x \in K$ i.e. $X \subseteq K$. Thus $X \in P(K)$.

For converse part, let $P(K)$ be a completely semiprime ideal of $P(S)$. Let $a^3 \in K$ for some $a \in S$. Now $\{a^3\} \in P(K)$ and so $\{a^3\} = \{aaa\} = \{a\}\{a\}\{a\} = \{a\}^3 \in P(K)$. Since $P(K)$ is a completely semiprime ideal of $P(S)$, so $\{a\} \in P(K)$ i.e. $\{a\} \subseteq K$. Thus $a \in K$. Hence K is completely semiprime ideal of S. □

Now it can be easily proved the following result :

Theorem 4.2.11. *If the power ternary semigroup $P(S)$ is idempotent then the ternary semigroup S is idempotent.*

But the converse of the above Theorem 4.2.11 is not true. From following example, we see that $P(S)$ is not be an idempotent ternary semigroup though the ternary semigroup S is idempotent.

Example 4.2.12. *Let* $S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

Then S forms a ternary semigroup together with the ternary matrix multiplication operation. Clearly, every element of S is an idempotent element, thus S is an idempotent ternary semigroup. But $P(S)$ is not an idempotent power ternary semigroup.

For let $A = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \in P(S)$.

Then $A^3 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. *But* $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \notin A$.

Hence A is not an idempotent element and so $P(S)$ is not an idempotent ternary semigroup.

But in particular, we have the following result :

Theorem 4.2.13. *Let S be a ternary semigroup. Then $P(S)$ is an idempotent ternary semigroup if and only if S is an idempotent ternary semigroup in which every non-empty subset of S is a ternary subsemigroup of S.*

Proof. Let S be an idempotent ternary semigroup in which every non-empty subset of S is a ternary subsemigroup of S and $A \in P(S)$. Since $A \subseteq S$, by our hypothesis, A is a ternary subsemigroup of S i.e. $A^3 \subseteq A$. Let $x \in A$. Then $x = a_1$ fome some $a_1 \in A \subseteq S$. Since S is an idempotent ternary semigroup $x = a_1 = a_1^3 \in A^3$. Thus $A \subseteq A^3$. Hence $A^3 = A$ and so $P(S)$ is an idempotent ternary semigroup.

Conversely, suppose that $P(S)$ is an idempotent ternary semigroup and $a \in S$. Then $\{a\} \in P(S)$. Since $P(S)$ is an idempotent ternary semigroup, $\{a\}^3 = \{a\}$ i.e. $a^3 = a$. Hence S is an idempotent ternary semigroup. Let $S_1 \subseteq S$. Then ${S_1}^3 = S_1 \subseteq S_1$, since $P(S)$ is idempotent ternary semigroup. Thus S_1 is a ternary subsemigroup of S. $\square$

Note that if S is a regular ternary semigroup then the power ternary semigroup $P(S)$ may not be a regular ternary semigroup.

We explain it by the following examples :

Example 4.2.14. *Let $S = \mathbb{Z}^- \times \mathbb{Z}^- = \{(a,b) : a,b \in \mathbb{Z}^-\}$, where $\mathbb{Z}^-$ be the set of all negative integers. Then S forms a ternary semigroup together with the multiplication defined as $(a,b)(c,d)(e,f) = (a,f)$ for all $(a,b),(c,d),(e,f) \in \mathbb{Z}^- \times \mathbb{Z}^-$. Clearly, S is a regular ternary semigroup but $P(S)$ is not a regular ternary semigroup. Let $X = \{(a,b),(c,d)\} \in P(S)$ for some $(a,b),(c,d) \in S$. Then $XYX = \{(a,b),(c,d)\}\{(x,y)\}\{(a,b),(c,d)\} = \{(a,b),(a,d),(c,b),(c,d)\} \neq X$ for any $\{(x,y)\} \in P(S)$. Hence X is not regular element and so $P(S)$ is not a regular ternary semigroup.*

Example 4.2.15. *Let $S = \{q\sqrt{2} : q \in \mathbb{Q}_0{}^-\}$. Then with usual ternary multiplication, S forms a regular ternary semigroup. But $P(S)$ is not regular ternary semigroup.*

Moreover, we have the following result :

Theorem 4.2.16. *Let S be a ternary semigroup in which every non-empty subset of S is a left ideal of S. Then S is a regular ternary semigroup if and only if $P(S)$ is a regular ternary semigroup.*

Proof. Let S be a regular ternary semigroup and $A \in P(S)$. Since $A \subseteq S$, A is a left ideal of S i.e. $SSA \subseteq A$. Let $x \in A$. Then $x = a_1$ fome some $a_1 \in A \subseteq S$. Since S is regular ternary semigroup $a_1 = a_1 s a_1$ for some $s \in S$. So we find that $x = a_1 s a_1 \in ASA$. Thus $A \subseteq ASA$. Now $ASA \subseteq SSA \subseteq A$. Hence $ASA = A$ and so $P(S)$ is regular ternary semigroup.

For the converse part, suppose that $P(S)$ is a regular ternary semigroup and $a \in S$. Then $\{a\} \in P(S)$. Since $P(S)$ is regular ternary semigroup, $\{a\} = \{a\}X\{a\}$ for some $X \in P(S)$ i.e. $a = axa$ for some $x \in X \subseteq S$. Hence S is a regular ternary semigroup. $\square$

However, if $P(S)$ is regular, each non-empty subset of S may not be a left ideal of S. Let us give an example :

Example 4.2.17. *Let $S = \{i, -i\}$ be a ternary semigroup. Then $P(S)$ is a regular ternary semigroup. But $A = \{i\} \subseteq S$ is not a left ideal of S.*

Similarly, we can prove the following results :

Corollary 4.2.18. *Let S be a ternary semigroup in which every non-empty subset of S is a right ideal of S. Then S is a regular ternary semigroup if and only if $P(S)$ is a regular ternary semigroup.*

Corollary 4.2.19. *Let S be a ternary semigroup in which every non-empty subset of S is a lateral ideal of S. Then S is a regular ternary semigroup if and only if $P(S)$ is a regular ternary semigroup.*

Let S be a ternary semigroup. If S is a completely regular ternary semigroup then the power ternary semigroup $P(S)$ is not necessarily a completely regular ternary semigroup.

We explain it by the following example :

Example 4.2.20. *Let $S = \{\pm 1, \pm i\}$. Then S forms a ternary semigroup w.r.t. the multiplication. Futhermore, S is a completely regular ternary semigroup. But $P(S)$ is not a completely regular ternary semigroup. Let $A = \{1, -i\} \in P(S)$. Then there exists no $X \in P(S)$ such that $A = A^2XA^2$.*

Theorem 4.2.21. *Let S be a ternary semigroup in which every non-empty subset of S is a left ideal of S. Then S is a completely regular ternary semigroup if and only if $P(S)$ is a completely regular ternary semigroup.*

Proof. Suppose that S be a completely regular ternary semigroup and $A \in P(S)$. Let $x \in A$. Then $x = a_1$ fome some $a_1 \in A \subseteq S$. Since S is completely regular ternary semigroup, so $a_1 = a_1^2 s a_1^2$ for some $s \in S$. Thus $x \in A^2SA^2$. This shows that $A \subseteq A^2SA^2$. Since $A \subseteq S$, A is a left ideal of S i.e. $SSA \subseteq A$. Now

$A^2SA^2 \subseteq SSSAA \subseteq SAA \subseteq SSA \subseteq A$. Hence $A^2SA^2 = A$ and so $P(S)$ is a completely regular ternary semigroup.

For the converse part, suppose that $P(S)$ is a completely regular ternary semigroup and $a \in S$. Then $\{a\} \in P(S)$. Since $P(S)$ is a completely regular ternary semigroup, $\{a\}^2X\{a\}^2 = \{a\}$ for some $X \in P(S)$ i.e. $a^2xa^2 = a$ for some $x \in X \subseteq S$. Hence S is a completely regular ternary semigroup. $\square$

Similarly, we can prove the following results :

Corollary 4.2.22. *Let S be a ternary semigroup in which every non-empty subset of S is a right ideal of S. Then S is a completely regular ternary semigroup if and only if $P(S)$ is a completely regular ternary semigroup.*

Corollary 4.2.23. *Let S be a ternary semigroup in which every non-empty subset of S is a lateral ideal of S. Then S is a completely regular ternary semigroup if and only if $P(S)$ is a completely regular ternary semigroup.*

Notice that if S is an intra-regular ternary semigroup then $P(S)$ is not an intra-regular ternary semigroup.

Let us give an example :

Example 4.2.24. *Let $S = \{\pm 1, \pm i\}$. Then S forms a ternary semigroup with respect to the multiplication. Also S is an intra-regular ternary semigroup but $P(S)$ is not an intra-regular ternary semigroup.*

Theorem 4.2.25. *Let S be a ternary semigroup in which every non-empty subset of S is an ideal of S. Then S is an intra-regular ternary semigroup if and only if $P(S)$ is an intra-regular ternary semigroup.*

Proof. Let S be an intra-regular ternary semigroup and $A \in P(S)$. Since $A \subseteq S$, A is an ideal of S. Let $a \in A \subseteq S$. Since S is intra-regular ternary semigroup, $a = xa^3y$ for some $x, y \in S$. So we find that $a \in SA^3S$. Thus $A \subseteq SA^3S$. Now $SA^3S \subseteq SSSAS \subseteq SAS \subseteq A$. Hence $SA^3S = A$ and so $P(S)$ is an intra-regular ternary semigroup.

For the converse part, let $P(S)$ be an intra-regular ternary semigroup $a \in S$. Then $\{a\} \in P(S)$. Since $P(S)$ is an intra-regular ternary semigroup, $X\{a\}^3Y = \{a\}$ for some $X, Y \in P(S)$ i.e. $xa^3y = a$ for some $x \in X \subseteq S$ and $y \in Y \subseteq S$. Hence S is an intra-regular ternary semigroup. □

4.3 Ordered power ternary semigroup $\mathcal{P}(\mathcal{S})$ of a ternary semigroup S

In this section, our main focus is to describe and characterize ordered power ternary semigroup $\mathcal{P}(\mathcal{S})$ of a ternary semigroup S. Furthermore, we investigate the connection between a ternary semigroup S and its corresponding ordered power ternary semigroup $\mathcal{P}(\mathcal{S})$.

Let us define a partially order relation on the ternary semigroup $P(S)$ in the natural way as follows:

$$\text{“}A \leq B \; if \; and \; only \; if \; A \subseteq B\text{” for all } A,\; B \in P(S).$$

Then $(P(S), ., \leq)$ becomes a partially ordered ternary semigroup or ordered ternary semigroup. The partially ordered ternary semigroup $(P(S), ., \leq)$ is called the ordered power ternary semigroup of S and we simply denote it by $\mathcal{P}(\mathcal{S})$.

Definition 4.3.1. Let $(S, ., \leq)$ be a partially ordered ternary semigroup. An element a of S is said to be ordered idempotent if $a \leq a^3$ and S is said to be an ordered idempotent ternary semigroup if every element of S is ordered idempotent.

Theorem 4.3.2. *A ternary semigroup S is idempotent if and only if the ordered power ternary semigroup $\mathcal{P}(\mathcal{S})$ is ordered idempotent ternary semigroup.*

Proof. First, let us consider S be an idempotent ternary semigroup and let $A \in \mathcal{P}(\mathcal{S})$. Suppose $a \in A \subseteq S$. Since S is idempotent semigroup $a = a^3 \in A^3$. Thus $A \subseteq A^3$. Thus $A \leq A^3$ and $\mathcal{P}(\mathcal{S})$ is ordered idempotent ternary semigroup.

Conversely, let $\mathcal{P}(\mathcal{S})$ be ordered idempotent ternary semigroup. Let $a \in S$. Thus $\{a\} \in \mathcal{P}(\mathcal{S})$. Let us denote $\{a\}$ by A i.e. $A = \{a\}$. Since $\mathcal{P}(\mathcal{S})$ is idempotent then $A \leq A^3$ i.e. $A \subseteq A^3$. Hence $a \in A^3 = \{a\}^3 = \{a^3\}$. Thus $a = a^3$ and so S is idempotent ternary semigroup. □

Definition 4.3.3. An ordered idempotent ternary semigroup S is said to be left zero if for every $a, b, c \in S$, there exists $x, y \in S$ such that $a \leq axybc$.

An ordered idempotent ternary semigroup S is said to be right zero if for every $a, b, c \in S$, there exists $x, y \in S$ such that $a \leq cbxya$.

Theorem 4.3.4. *An idempotent ternary semigroup S is left simple if and only if the ordered power ternary semigroup $\mathcal{P}(\mathcal{S})$ is a left zero ordered idempotent ternary semigroup.*

Proof. Let S be a left simple idempotent ternary semigroup. Let $A, B, C \in \mathcal{P}(\mathcal{S})$. Thus for $a \in A, b \in B, c \in C$ there is $x \in S$ such that $a = xbc$. Let us denote them by x_{abc}. Take $X = \{x_{abc} : a \in A, b \in B, c \in C\}$. Hence $A \leq XBC$. Since S is idempotent ternary semigroup, by theorem 4.3.2 $\mathcal{P}(\mathcal{S})$ is an ordered idempotent ternary semigroup. Thus $A \leq A^3$ gives $A \leq AXBCXBC$. Let $Y = BCX \subseteq S$. Hence $A \leq AXYBC$. Thus for $A, B, C \in \mathcal{P}(\mathcal{S})$ there exists $X, Y \in \mathcal{P}(\mathcal{S})$ such that $A \leq AXYBC$. Hence $\mathcal{P}(\mathcal{S})$ is a left zero ordered idempotent ternary semigroup.

Conversely let $\mathcal{P}(\mathcal{S})$ is left zero ordered idempotent ternary semigroup. Let $A = \{a\}, B = \{b\} \in \mathcal{P}(\mathcal{S})$. Then for any $C = \{c\} \in \mathcal{P}(\mathcal{S})$ there is $X, Y \in \mathcal{P}(\mathcal{S})$ such that $C \leq CXYBA$. Thus $C \subseteq CXYBA$. Hence $c \in CXYBA = CXYB\{a\} \subseteq SSa$. Thus $S = SSa$ and so S is a left simple idempotent ternary semigroup. □

Corollary 4.3.5. *An idempotent ternary semigroup S is right simple if and only if the ordered power ternary semigroup $\mathcal{P}(\mathcal{S})$ is a right zero ordered idempotent ternary semigroup.*

Definition 4.3.6. A partially ordered ternary semigroup S is called an ordered ternary band if S is an ordered idempotent ternary semigroup.

Corollary 4.3.7. *A ternary semigroup S is a ternary band if and only if the ordered power ternary semigroup $\mathcal{P(S)}$ is an ordered ternary band.*

Definition 4.3.8. An ordered ternary semigroup S is called an ordered ternary rectangular band if S is an ordered idempotent ternary semigroup and $a \leq aba$ for all $a, b \in S$.

Example 4.3.9. *Let $S = \{a, b, c\}$ be an ordered ternary semigroup with the ternary operation . on S as $abc = a * (b * c)$ where the binary operation * is defined as*

$*$	a	b	c
a	a	a	a
b	a	b	b
c	a	c	c

and the order defined as $\leq \,:= \{(a, a), (b, a), (b, b), (c, a), (c, c)\}$
This is an ordered ternary rectangular band.

Theorem 4.3.10. *A ternary semigroup S is a ternary rectangular band if and only if the ordered power ternary semigroup $\mathcal{P(S)}$ is an ordered ternary rectangular band.*

Proof. First let us assume that, S is a ternary rectangular band. Since S is idempotent ternary semigroup by Theorem 4.3.2 $\mathcal{P(S)}$ is ordered idempotent ternary semigroup. It remains to show that $A \leq ABA$ for all $A, B \in \mathcal{P(S)}$. Let $a \in A$ and $b \in B$ for any $A, B \in \mathcal{P(S)}$. Since S is ternary rectangular band, $a = aba \in ABA$. Thus $A \subseteq ABA$. Hence $A \leq ABA$ for all $A, B \in \mathcal{P(S)}$. Hence $\mathcal{P(S)}$ is ordered ternary rectangular band.

Conversely, Let $a, b \in S$. Thus $\{a\}, \{b\} \in \mathcal{P(S)}$. Let $A = \{a\}$, $B = \{b\}$. Since $\mathcal{P(S)}$ is a ordered ternary rectangular band $A \leq ABA \Longrightarrow \{a\} \subseteq \{a\}\{b\}\{a\} \Longrightarrow a = aba$. Since $\mathcal{P(S)}$ is ordered idempotent ternary semigroup by Theorem 4.3.2 S is also idempotent ternary semigroup. Hence S is a ternary rectangular band. $\square$

Theorem 4.3.11. *A ternary semigroup S is regular if and only if the ordered power ternary semigroup $\mathcal{P(S)}$ is regular.*

Proof. Let us consider S be a regular ternary semigroup. Let $A \in \mathcal{P}(\mathcal{S})$. Then for every $a \in A$ there exists $x_a \in S$ such that $a = ax_aa$. Let $X = \{x_a : a \in A\}$. Then $X \in \mathcal{P}(\mathcal{S})$ such that $A \subseteq AXA$ i.e. $A \leq AXA$. Thus $A \in (A\mathcal{P}(\mathcal{S})A]$ for all $A \in \mathcal{P}(\mathcal{S})$. Hence $\mathcal{P}(\mathcal{S})$ is regular.

Conversely, suppose that $\mathcal{P}(\mathcal{S})$ be a regular ordered power ternary semigroup. Let $a \in S$. Then for $A = \{a\} \in \mathcal{P}(\mathcal{S})$ there exists $X \in \mathcal{P}(\mathcal{S})$ such that $A \leq AXA$ and so $A \subseteq AXA$. Thus for $a \in S$ there exists $x \in X \subseteq S$ such that $a = axa$. Hence S is regular ternary semigroup. □

Corollary 4.3.12. *A ternary semigroup S is left (resp. right) regular if and only if the ordered power ternary semigroup $\mathcal{P}(\mathcal{S})$ is left (resp. right) regular.*

Theorem 4.3.13. *A ternary semigroup S is completely regular if and only if the ordered power ternary semigroup $\mathcal{P}(\mathcal{S})$ is completely regular.*

Proof. Let S be a completely regular ternary semigroup. Let $A \in \mathcal{P}(\mathcal{S})$. Then for every $a \in A$ there exists $x_a \in S$ such that $a = a^2x_aa^2$. Let $X = \{x_a : a \in A\}$. Then $X \in \mathcal{P}(\mathcal{S})$ such that $A \subseteq A^2XA^2$ i.e. $A \leq A^2XA^2$. Thus $A \in (A^2\mathcal{P}(\mathcal{S})A^2]$ for all $A \in \mathcal{P}(\mathcal{S})$. Hence $\mathcal{P}(\mathcal{S})$ is completely regular.

Conversely, suppose that $\mathcal{P}(\mathcal{S})$ be a completely regular ordered power ternary semigroup. Let $a \in S$. Then for $A = \{a\} \in \mathcal{P}(\mathcal{S})$ there exists $X \in \mathcal{P}(\mathcal{S})$ such that $A \leq A^2XA^2$ and so $A \subseteq A^2XA^2$. Thus for $a \in S$ there exists $x \in X \subseteq S$ such that $a = a^2xa^2$. Hence S is completely regular. □

Theorem 4.3.14. *A ternary semigroup S is intra-regular if and only if the ordered power ternary semigroup $\mathcal{P}(\mathcal{S})$ is intra-regular.*

Proof. First suppose that S be an intra-regular ternary semigroup. Let $A \in \mathcal{P}(\mathcal{S})$. Then for each $a \in A$ there exists $x_a, y_a \in S$ such that $a = x_aa^3y_a$. Let $X = \{x_a : a \in A\}$ and $Y = \{y_a : a \in A\}$. Then $X, Y \in \mathcal{P}(\mathcal{S})$ such that $A \subseteq XA^3Y$ i.e. $A \leq XA^3Y$. Thus $A \in (\mathcal{P}(\mathcal{S})A^3\mathcal{P}(\mathcal{S})]$ for all $A \in \mathcal{P}(\mathcal{S}))$. Hence $\mathcal{P}(\mathcal{S})$ is intra-regular.

Conversely, suppose that $\mathcal{P}(\mathcal{S})$ be an intra-regular ordered power ternary semigroup. Let $a \in S$. Then for $A = \{a\} \in \mathcal{P}(\mathcal{S})$ there exists $X, Y \in \mathcal{P}(\mathcal{S})$ such that $A \leq XA^3Y$ and so $A \subseteq XA^3Y$. Thus for $a \in S$ there exists $x, y \in X \subseteq S$ such that $a = xa^3y$. Hence S is intra-regular. $\square$

Theorem 4.3.15. *A ternary semigroup S is a ternary group if and only if the ordered power ternary semigroup $\mathcal{P}(\mathcal{S})$ is simple.*

Proof. First, let us consider S be a ternary group. Let $A, B, C \in \mathcal{P}(\mathcal{S})$. Then for each $a \in A,\ b \in B,\ c \in C$ there exists unique $x,\ y,\ z \in S$ such that $abx = c, ayb = c,\ and\ zab = c$. Let $X = \{x_{a,b,c} : a \in A,\ b \in B,\ c \in C\}$, $Y = \{y_{a,b,c} : a \in A,\ b \in B,\ c \in C\}$, $Z = \{z_{a,b,c} : a \in A,\ b \in B,\ c \in C\}$. Then $X, Y, Z \in \mathcal{P}(\mathcal{S})$ such that $C \subseteq ABX$, $C \subseteq AYB$ and $C \subseteq ZAB$. Let I be a left ideal of $\mathcal{P}(\mathcal{S})$. Then $I \subseteq \mathcal{P}(\mathcal{S})$ such that $\mathcal{P}(\mathcal{S})\mathcal{P}(\mathcal{S})I \subseteq I$ and $(I] = I$. Suppose that $U, V \in \mathcal{P}(\mathcal{S})$ and $W \in I$ i.e. $U, V, W \subseteq S$. Then there exists $X^* \in \mathcal{P}(\mathcal{S})$ such that $U \leq X^*VW$. Thus $U \subseteq X^*VW \in \mathcal{P}(\mathcal{S})\mathcal{P}(\mathcal{S})I \subseteq I$. Hence $\mathcal{P}(\mathcal{S}) \subseteq I$. Thus $\mathcal{P}(\mathcal{S})$ has no proper left ideal and so $\mathcal{P}(\mathcal{S})$ is left simple. In the similar way we can show that $\mathcal{P}(\mathcal{S})$ is right simple and lateral simple.

Conversely, let $\mathcal{P}(\mathcal{S})$ be simple ordered ternary semigroup. Let $a, b, c \in S$. Then $A = \{a\},\ B = \{b\},\ C = \{c\} \in \mathcal{P}(\mathcal{S})$. Now $(\mathcal{P}(\mathcal{S})AB]$ is a left ideal in $\mathcal{P}(\mathcal{S})$. Since $\mathcal{P}(\mathcal{S})$ is left simple by Theorem 1.4.7, we have $(\mathcal{P}(\mathcal{S})AB] = \mathcal{P}(\mathcal{S})$. Thus $C \in \mathcal{P}(\mathcal{S}) = (\mathcal{P}(\mathcal{S})AB]$ and so $C \leq ZAB$ for some $Z \in \mathcal{P}(\mathcal{S})$. Again $(AB\mathcal{P}(\mathcal{S})]$ is a right ideal in $\mathcal{P}(\mathcal{S})$. Since $\mathcal{P}(\mathcal{S})$ is right simple by Theorem 1.4.7, we have $(AB\mathcal{P}(\mathcal{S})] = \mathcal{P}(\mathcal{S})$. Thus $C \in \mathcal{P}(\mathcal{S}) = (AB\mathcal{P}(\mathcal{S})]$ and so $C \leq ABX$ for some $X \in \mathcal{P}(\mathcal{S})$. Since $X \in \mathcal{P}(\mathcal{S})$ then there exists $X^* \in \mathcal{P}(\mathcal{S})$ such that $X \leq X^*AB$ and so $C \leq ABX \leq ABX^*AB \in A\mathcal{P}(\mathcal{S})B$. Thus $C \in (A\mathcal{P}(\mathcal{S})B]$ and so $C \leq AYB$ for some $Y \in \mathcal{P}(\mathcal{S})$. Hence for $A, B, C \in \mathcal{P}(\mathcal{S})$ there exists $X, Y, Z \in \mathcal{P}(\mathcal{S})$ such that $C \leq ABX$, $C \leq AYB$ and $C \leq ZAB$ i.e. $C \subseteq ABX$, $C \subseteq AYB$ and $C \subseteq ZAB$. Hence for $a, b, c \in S$ there exists $x, y, z \in S$ such that $abx = c$, $ayb = c$ and $zab = c$. Hence S is a ternary group. $\square$

Theorem 4.3.16. *Let S be an ordered ternary semigroup. Then S is simple if and only if $a \in (bcSbc]$ for all $a, b, c \in S$.*

Proof. First, let us consider S be a simple ordered ternary semigroup. For $a, b, c \in S$ there exists $x \in S$ such that $a \leq xbc$. Again for $b, c, x \in S$ there exists $y \in S$ such that $x \leq ybc$ which implies $a \leq bcx \leq bcybc$. Thus $a \in (bcSbc]$.

Conversely, $a \in (bcSbc]$ implies $a \leq bcsbc$, $s \in S$ implies $a \leq bcs_1$ where $s_1 = sbc \in S$. In similar way we can show that $a \leq bs_2c$, $a \leq s_3bc$ for some $s_2, s_3 \in S$. Hence S is simple ordered ternary semigroup. □

Theorem 4.3.17. *A ternary semigroup S is a ternary group if and only if the ordered power ternary semigroup $A \in (BC\mathcal{P}(\mathcal{S})BC]$ for all $A, B, C \in \mathcal{P}(\mathcal{S})$.*

Definition 4.3.18. *An ordered ternary semigroup $(S, ., \leq)$ is called a semilattice ordered ternary semigroup if $a \vee b$ exists in the poset $(S, \leq)$ for every $a, b \in S$. Then for all $a, b, c, d \in S$ the following holds :*
(i) $ab(c \vee d) = abc \vee abd$
(ii) $(a \vee b)cd = acd \vee bcd$.

Let T be a ternary semigroup and $P^*(T)$ be the set of all finite subsetes of T. For $A, B, C \in P^*(T)$ the ternary multiplication '.' defined by $ABC = \{abc : a \in A, b \in B, c \in C\}$ and partial order relation '$\leq$' defined by '$A \leq B$ if and only if $A \subseteq B$". Then $P^*(T)$ is a semillatice ordered ternary semigroup with respect to the multiplication '.' and partial order relation '$\leq$'.

Theorem 4.3.19. *Let T be a ternary semigroup, S be a semilattice ordered ternary semigroup and $\phi : T \longrightarrow S$ be a ternary semigroup homomorphism. Then there is an ordered semigroup ternary semigroup homomorphism $g : P^*(T) \longrightarrow S$ such that the $g \circ f = \phi$ where $f : T \longrightarrow P^*(T)$ is defined by $f(x) = \{x\}$.*

Proof. Let us define a mapping $g : P^*(T) \longrightarrow S$ by $g(A) = \vee_{a \in A}\phi(a)$ for all $A \in P^*(T)$.

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "\phi"'] & P^*(T) \arrow[dl, "g"] \\ S & \end{tikzcd}$$

Now for $A, B, C \in P^*(T)$, we have

$$\begin{aligned} g(ABC) &= \vee_{a\in A, b\in B, c\in C}\phi(abc) \\ &= \vee_{a\in A, b\in B, c\in C}\phi(a)\phi(b)\phi(c) \\ &= (\vee_{a\in A}\phi(a))(\vee_{b\in B}\phi(b))(\vee_{c\in C}\phi(c)) \\ &= g(A)g(B)g(C) \end{aligned}$$

Again for all $A, B \in P^*(T)$, if $A \leq B$ then we have $g(A) = \vee_{a\in A}\phi(a) \leq \vee_{a\in B}\phi(B) = g(B)$. Thus g is an ordered ternary homomorphism from $P^*(T)$ to S. Now, $(g \circ f)(x) = g(f(x)) = g(\{x\}) = \vee_{a\in\{x\}}\phi(a) = \phi(x)$. $\square$

Lattice Structures In Ternary Semigroup Of Mappings

Chapter-5

Chapter 5

Lattice structures in ternary semigroup of mappings

5.1 Introduction

The topic in this chapter focuses on ternary semigroups of mappings. A. Chronowski [19] introduced the notion of ternary semigroup of mappings which are a natural generalization of semigroup of mappings and these algebraic structures are used for constructing the natural examples of ternary algebras, which are the counterpart of binary algebras. Some properties of ternary semigroup of homomorphism, ternary semigroup of lattice homomorphism, ternary semigroups of linear mappings are studied by A. Chronowski [21], [22], [24], [23]. The set S(X) of all mappings of a given non-empty set X into itself, where the binary operation is the usual composition of mappings, forms an important class of semigroup which is commonly known as semigroup of mappings. A. Chronowski studied about n-ary semigroup of mappings and in particular, ternary semigroup of mappings, embedding theorem, classical Green's equivalence relation, structure of ternary semigroup of linear mappings, ternary semigroup of matrices etc. The corresponding results for semigroup of mappings have been intensively studied by several authors. Also S. Kar, I. Dutta

[42] studied the notion of various structures of ternary semigroup of mappings. G. Birkhoff [6] disscussed about lattice theory.

In the first section, we give a characterization of the structures of lattices in ternary semigroup of mappings. In remaining sections, we discuss the problem that if $X \cong X'$ and $Y \cong Y'$, then the corresponding ternary semigroups of mappings $T[X, Y]$ and $T[X', Y']$ are isomorphic. The converse statement is not valid. We also derive simple conditions under which the converse is also true. We also introduce a partial order relation in the ternary semigroup of mappings $T[X, Y]$. We also study the notion of ternary semigroup of isotone mappings $O[X, Y]$. Further we present the characterization of regular, intra-regular and idempotent ordered ternary semigroup in $O[X, Y]$.

5.2 Ternary semigroup of mappings $T[X, Y]$

In this section, we are going to define a partial order in $T[X, Y]$ and after that we study different types of lattice structure of $T[X, Y]$. Throughout this paper, $T[X, Y]$ will denote a ternary semigroup of mappings of sets X and Y.

Let X and Y be non-empty sets. By $T(X, Y)$ and $T(Y, X)$ we denote the set of all mappings of the set X into the set Y and the set of all mappings of the set Y into the set X i.e. $T(X, Y) = \{p : p \text{ is a mapping from } X \text{ to } Y\}$ and $T(Y, X) = \{q : q \text{ is a mapping from } Y \text{ to } X\}$. Now consider the set $T[X, Y] = T(X, Y) \times T(Y, X)$.
Define the ternary operation $\cdot : T[X, Y] \times T[X, Y] \times T[X, Y] \longrightarrow T[X, Y]$ as follows:

$$(p_1, q_1) \cdot (p_2, q_2) \cdot (p_3, q_3) = (p_1q_2p_3, q_1p_2q_3) \ \forall (p_1, q_1), (p_2, q_2), (p_3, q_3) \in T[X, Y]$$

The ternary operation defined above is associative. Then $(T[X, Y], \cdot)$ is a ternary semigroup. The ternary semigroup is called the "Ternary Semigroup of Mappings" of sets X and Y.
If $X \cap Y = \{\}$, then $(T[X, Y], \cdot)$ is called the disjoint ternary semigroup of mappings of sets X and Y.

The ternary semigroups $(T[X,Y],\cdot)$ and $(T[Y,X],\cdot)$ are isomorphic for the sets X and Y.

Throughout this section, $T[X,Y]$ will denote a ternary semigroup of mappings of sets X and Y.

5.3 Partial order on $T[X,Y]$

In this section, we are going to define a partial order in $T[X,Y]$ and after that we study different types of lattice structure of $T[X,Y]$.

Let $(X,\leq_X)$ and $(Y,\leq_Y)$ be two posets with respect to the partial orders $\leq_X$ and $\leq_Y$ respectively. Let us define a partial order '$\leq$' on $T[X,Y]$. For all (p_1,q_1), $(p_2,q_2)\in T[X,Y]$ the partial order $\leq$ defined as follows:

$$\text{“}(p_1,q_1)\leq(p_2,q_2)\ if\ and\ only\ if\ p_1(x)\leq_Y p_2(x)\ and\ q_1(y)\leq_X q_2(y)$$
$$for\ all\ x\in X\ and\ y\in Y\text{”}.$$

Thus $(T[X,Y],\leq)$ becomes a poset with respect to the partial order " $\leq$ ".

Theorem 5.3.1. *The ternary semigroup of mappings $(T[X,Y],\leq)$ of X and Y is a lattice if and only if $(X,\leq_X)$ and $(Y,\leq_Y)$ are two lattices.*

Proof. Let $(X,\leq_X)$ and $(Y,\leq_Y)$ be two lattices under the partial order $\leq_X$ and $\leq_Y$ respectively. Let (p_1,q_1), $(p_2,q_2)\in T[X,Y]$. Let $p^*:X\longrightarrow Y$ and $q^*:Y\longrightarrow X$ be any two functions such that $p^*(x)=p_1(x)\wedge p_2(x)$ and $q^*(y)=q_1(y)\wedge q_2(y)$ for all $x\in X$ and $y\in Y$. We have to show that $(p^*,q^*)=\text{Inf}\{(p_1,q_1),\ (p_2,q_2)\}$. Now $p^*(x)=p_1(x)\wedge p_2(x)\Longrightarrow p^*(x)\leq p_1(x),\ p_2(x)$ and $q^*(y)=q_1(y)\wedge q_2(y)\Longrightarrow q^*(y)\leq q_1(y),\ q_2(y)$. Thus (p^*,q^*) is a lower bound of (p_1,q_1) and (p_2,q_2). We show that (p^*,q^*) is the greatest lower bound of $\{(p_1,q_1),\ (p_2,q_2)\}$. If not, then there exists an another lower bound (p,q) of $\{(p_1,q_1),(p_2,q_2)\}$ in $T[X,Y]$ such that $(p^*,q^*)\leq(p,q)\Longrightarrow p^*(x)\leq_Y p(x)$ and $q^*(y)\leq_X q(y)$ for all $x\in X$ and $y\in Y$. Again since (p,q) is a lower bound of (p_1,q_1) and (p_2,q_2), then $(p,q)\leq(p_1,q_1)$

and $(p,q) \leq (p_2,q_2)$. This implies that $p(x) \leq_Y p_1(x)$ and $p(x) \leq_Y p_2(x)$ for all $x \in X$. Thus $p(x) \leq_Y p_1(x) \wedge p_2(x) = p^*(x)$. Hence $p(x) = p^*(x)$. Similarly, we can prove that $q(y) = q^*(y)$. Thus $(p,q) = (p^*,q^*)$ and hence (p^*,q^*) is the Inf$\{(p_1,q_1),$ $(p_2,q_2)\}$.

Similarly, we can show that there exists $(p^{'},q^{'}) \in T[X,Y]$ such that $(p_1,q_1) \vee (p_2,q_2)$ $= (p^{'},q^{'})$ where $p^{'}(x) = p_1(x) \vee p_2(x)$ and $q^{'}(x) = q_1(y) \vee q_2(y)$ for all $x \in X$ and $y \in Y$ will be the Sup$\{(p_1,q_1), (p_2,q_2)\}$. Hence $T[X,Y]$ is a lattice.

Conversely, let $T[X,Y]$ be a lattice. Let $x_1,x_2 \in X$ and $y_1,y_2 \in Y$. Let us define two mappings $q_1,q_2 : Y \longrightarrow X$ such that $q_1(y) = x_1$ and $q_2(y) = x_2$ for all $y \in Y$. Again let us define two mappings $p_1,p_2 : X \longrightarrow Y$ such that $p_1(x) = y_1$ and $p_2(x) = y_2$ for all $x \in X$. Since (p_1,q_1), $(p_2,q_2) \in T[X,Y]$ and $T[X,Y]$ is a lattice then we have (p^*,q^*) in $T[X,Y]$ such that $(p_1,q_1) \wedge (p_2,q_2) = (p^*,q^*)$. Thus we have,

$(p^*,q^*) \leq (p_1,q_1) \Longrightarrow p^*(x) \leq_Y p_1(x) = y_1$ and $q^*(y) \leq_X q_1(y) = x_1$.

$(p^*,q^*) \leq (p_2,q_2) \Longrightarrow p^*(x) \leq_Y p_2(x) = y_2$ and $q^*(y) \leq_X q_2(y) = x_2$.

Thus $q^*(y)$ is a lower bound of x_1 and x_2. Let us assume $q^*(y)$ be not the greatest lower bound of x_1 and x_2. Then we found an another lower bound x^o such that $q^*(y) \leq_X x^o$. Let us define a mapping $q^o : Y \longrightarrow X$ such that $q^o(y) = x^o$ for all $y \in Y$. Thus $q^*(y) \leq_X q^o(y)$ for all $y \in Y$. Again since x^o is a lower bound of x_1 and x_2 then $x^o \leq_X x_1$ and $x^o \leq_X x_2 \Longrightarrow q^o(y) \leq_X q_1(y)$ and $q^o(y) \leq_X q_2(y)$ for all $y \in Y$.

Similarly, $p^*(x)$ is a lower bound of y_1 and y_2. We have to show $p^*(x)$ be the greatest lower bound of y_1 and y_2. If not, then we found an another lower bound y^o such that $p^*(x) \leq_Y y^o$. Let us define a mapping $p^o : X \longrightarrow Y$ such that $p^o(x) = y^o$ for all $x \in X$. Thus $p^*(x) \leq_Y p^o(x)$ for all $x \in X$. Again since y^o is a lower bound of y_1 and y_2 then $y^o \leq_Y y_1$ and $y^o \leq_Y y_2 \Longrightarrow p^o(x) \leq_Y p_1(x)$ and $p^o(x) \leq_Y p_2(x)$ for all $x \in X$. Thus $(p^o,q^o) \leq (p_1,q_1)$ and $(p^o,q^o) \leq (p_2,q_2) \Longrightarrow (p^o,q^o) \leq (p_1,q_1) \wedge (p_2,q_2) =$ (p^*,q^*). Again from $q^*(y) \leq_X x^o$ and $p^*(x) \leq_Y y^o$ we get $(p^*,q^*) \leq (p^o,q^o)$. Thus $(p^o,q^o) = (p^*,q^*)$. Hence $p^o(x) = p^*(x)$ for all $x \in X$ and $q^o(y) = q^*(y)$ for all $y \in Y$ and so $y^o = p^*(x)$ and $x^o = q^*(y)$. Thus $q^*(y) = $ Inf$\{x_1,x_2\}$ and $p^*(x) = $ Inf$\{y_1,y_2\}$.

Proceeding in the same manner we can show that Sup$\{x_1, x_2\}$ and Sup$\{y_1, y_2\}$ exist in X and Y respectively. Thus X and Y are lattices. $\square$

Theorem 5.3.2. *The ternary semigroup of mappings $(T[X,Y], \leq)$ of X and Y is a complete lattice if and only if the posets $(X, \leq_X)$ and $(Y, \leq_Y)$ are two complete lattices.*

Proof. Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be two complete lattices under the partial order $\leq_X$ and $\leq_Y$ respectively. By Theorem 5.3.1 $T[X,Y]$ is a lattice. Let $A = \{(p_\alpha, q_\alpha) : \alpha \in I\}$ be any non empty subset of $T[X,Y]$, I being an index set. Since X is a complete lattice and $\{q_\alpha(y) : y \in Y, \alpha \in I\}$ is a subset of X, then $\{q_\alpha(y) : y \in Y, \alpha \in I\}$ has Inf and Sup in X. Thus $\underset{\alpha\in I}{\wedge} q_\alpha(y)$ and $\underset{\alpha\in I}{\vee} q_\alpha(y)$ exists in X. Similarly, $\underset{\alpha\in I}{\wedge} p_\alpha(x)$ and $\underset{\alpha\in I}{\vee} p_\alpha(x)$ exists in Y. Define $\underset{\alpha\in I}{\wedge}(p_\alpha, q_\alpha) = (p^*, q^*)$ where $p^*(x) = \underset{\alpha\in I}{\wedge} p_\alpha(x)$ and $q^*(y) = \underset{\alpha\in I}{\wedge} q_\alpha(y)$ for all $x \in X$ and $y \in Y$. Now we have to show that (p^*, q^*) be the Inf$A$. If not then there is an another lower bound (p^o, q^o) of A such that $(p^*, q^*) \leq (p^o, q^o)$. Hence $\underset{\alpha\in I}{\wedge} p_\alpha(x) \leq_Y p^o(x)$ and $\underset{\alpha\in I}{\wedge} q_\alpha(y) \leq_X q^o(y)$ for all $x \in X$ and $y \in Y$, which is a contradiction. Thus (p^*, q^*) is the Inf$A$. Similarly, we can show that Sup$A$ exists in $T[X,Y]$. Thus $T[X,Y]$ is a complete lattice.

Conversely, let $T[X,Y]$ be a complete lattice. By Theorem 5.3.1, we can say that X and Y are also lattices. Let $A = \{x_i : i \in I\}$ and $B = \{y_i : i \in I\}$ are non-empty arbitrary sets of X and Y respectively, I being an index set. Let us define two mappings $q_i : Y \longrightarrow X$ such that $q_i(y) = x_i$ and $p_i : X \longrightarrow Y$ such that $p_i(x) = y_i$ for all $x \in X$ and $y \in Y$, where $i \in I$. Since $(p_i, q_i) \in T[X,Y]$ and $T[X,Y]$ is a complete lattice then we have (p^*, q^*) in $T[X,Y]$ such that $(p^*, q^*) = \underset{i\in I}{\wedge}(p_i, q_i)$. Thus $(p^*, q^*) \leq (p_i, q_i)$, $i \in I \Longrightarrow p^*(x) \leq_Y p_i(x) = y_i,\ q^*(y) \leq_X q_i(y) = x_i,\ i \in I$. Thus $q^*(y)$ is a lower bound of $\{x_i : i \in I\} = A$. We show that $q^*(y)$ is the greatest lower bound of A. Let us assume that $q^*(y)$ be not the greatest lower bound of A. Then there exists another lower bound x^o such that $q^*(y) \leq_X x^o$. Let us define a mapping $q^o : Y \longrightarrow X$ such that $q^o(y) = x^o$ for all $y \in Y$.

Thus $q^*(y) \leq_X q^o(y)$ for all $y \in Y$ ....(1)

Again x^o is a lower bound of $A = \{x_i : i \in I\}$.
Thus $x^o \leq_X x_i$, $i \in I \Longrightarrow q^o(y) \leq_X q_i(y)$ ....(2)
Again $p^*(x)$ is a lower bound of $\{y_i : i \in I\} = B$. We show that $p^*(x)$ is the greatest lower bound of B. Assume that $p^*(x)$ is not the greatest lower bound of B. Then there exists another lower bound y^o such that $p^*(x) \leq_Y y^o$. Let us define a mapping $p^o : X \longrightarrow Y$ such that $p^o(x) = y^o$ for all $x \in X$.
Thus $p^*(x) \leq_Y p^o(x)$ for all $x \in X$ ....(3)

Again y^o is a lower bound of $B = \{y_i : i \in I\}$.
Thus $y^o \leq_Y y_i$, $i \in I \Longrightarrow p^o(x) \leq_Y p_i(x)$ ....(4)
Thus from (2) and (4) we get $(p^o, q^o) \leq (p_i, q_i)$, $i \in I \Longrightarrow (p^o, q^o) \leq \bigwedge_{i \in I}(p_i, q_i) \Longrightarrow (p^o, q^o) \leq (p^*, q^*)$. Again from (1) and (3) we get $(p^*, q^*) \leq (p^o, q^o)$. Thus $(p^*, q^*) = (p^o, q^o)$. Hence $p^o(x) = p^*(x)$ and $q^o(y) = q^*(y)$ for all $x \in X$ and $y \in Y$. Thus $y^o = p^*(x)$ and $x^o = q^*(y)$. Proceeding in the same manner we can show that Sup$A$ and Sup$B$ exist in X and Y respectively.
Thus X and Y are complete lattices. □

Theorem 5.3.3. *The ternary semigroup of mappings* $(T[X, Y], \leq)$ *of* X *and* Y *is a modular lattice if and only if the posets* $(X, \leq_X)$ *and* $(Y, \leq_Y)$ *are two modular lattices.*

Proof. Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be two modular lattices under the partial order $\leq_X$ and $\leq_Y$ respectively. Let $(p_1, q_1), (p_2, q_2) \in T[X, Y]$ such that $(p_1, q_1) \leq (p_2, q_2)$. Thus $p_1(x) \leq_Y p_2(x)$ and $q_1(y) \leq_X q_2(y)$ for all $x \in X$ and $y \in Y$. Consider $(p_3, q_3) \in T[X, Y]$. Since X and Y are modular lattices we have,
$p_1(x) \leq_Y p_2(x) \Longrightarrow p_1(x) \vee (p_3(x) \wedge p_2(x)) = (p_1(x) \vee p_3(x)) \wedge p_2(x)$ for all $x \in X$ and
$q_1(y) \leq_X q_2(y) \Longrightarrow q_1(y) \vee (q_3(y) \wedge q_2(y)) = (q_1(y) \vee q_3(y)) \wedge q_2(y)$ for all $y \in Y$.

Therefore, we have $(p_1, q_1) \vee \Big((p_3, q_3) \wedge (p_2, q_2)\Big)$

$$= (p_1, q_1) \vee (p^{'}, q^{'}) \text{ where, } p^{'}(x) = p_3(x) \wedge p_2(x) \text{ and }$$
$$q^{'}(y) = q_3(y) \wedge q_2(y)$$
$$= (p^*, q^*) \text{ where, } p^*(x) = p_1(x) \vee \Big(p_3(x) \wedge p_2(x)\Big) \text{ and }$$
$$q^*(y) = q_1(y) \vee \Big(q_3(y) \wedge q_2(y)\Big)$$
$$\text{for all } x \in X \text{ and } y \in Y.$$

Similarly, we have $\Big((p_1, q_1) \vee (p_3, q_3)\Big) \wedge (p_2, q_2)$

$$= (p^{''}, q^{''}) \wedge (p_2, q_2) \text{ where, } p^{''}(x) = p_1(x) \vee p_3(x) \text{ and }$$
$$q^{''}(y) = q_1(y) \vee q_3(y)$$
$$= (p^o, q^o) \text{ where, } p^o(x) = \Big(p_1(x) \vee p_3(x)\Big) \wedge p_2(x) \text{ and }$$
$$q^o(y) = \Big(q_1(y) \vee q_3(y)\Big) \wedge q_2(y)$$
$$\text{for all } x \in X \text{ and } y \in Y.$$

Since X and Y are modular lattices then $p_1(x) \vee \Big(p_3(x) \wedge p_2(x)\Big) = \Big(p_1(x) \vee p_3(x)\Big) \wedge p_2(x)$ and $q_1(y) \vee \Big(q_3(y) \wedge q_2(y)\Big) = \Big(q_1(y) \vee q_3(y)\Big) \wedge q_2(y)$ for all $x \in X$ and $y \in Y$. Thus $(p^*, q^*) = (p^o, q^o)$ i.e. $(p_1, q_1) \vee \Big((p_3, q_3) \wedge (p_2, q_2)\Big) = \Big((p_1, q_1) \vee (p_3, q_3)\Big) \wedge (p_2, q_2)$. Hence $T[X, Y]$ is a modular lattice.

Conversely, suppose $T[X, Y]$ be a modular lattice. Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $x_1 \leq_X x_2$ and $y_1 \leq_Y y_2$. Let us define two mappings $p_1, p_2 : X \longrightarrow Y$ such that $p_1(x) = y_1$ and $p_2(x) = y_2$ for all $x \in X$. Thus $y_1 \leq_Y y_2 \Longrightarrow p_1(x) \leq_Y p_2(x)$ for all $x \in X$. Again let us define two mappings $q_1, q_2 : Y \longrightarrow X$ such that $q_1(y) = x_1$ and $q_2(y) = x_2$ for all $y \in Y$. Thus $y_1 \leq_Y y_2 \Longrightarrow p_1(x) \leq_Y p_2(x)$ for all $x \in X$ and $x_1 \leq_X x_2 \Longrightarrow q_1(y) \leq_X q_2(y)$ for all $y \in Y$. This implies that $(p_1, q_1) \leq (p_2, q_2)$.

Since $T[X,Y]$ is a modular lattice and $(p_1,q_1),(p_2,q_2) \in T[X,Y]$ such that $(p_1,q_1) \leq (p_2,q_2)$ then we have $(p_1,q_1) \vee \Big((p,q) \wedge (p_2,q_2)\Big) = \Big((p_1,q_1) \vee (p,q)\Big) \wedge (p_2,q_2)$. This shows that,

$$\begin{aligned}
(p_1,q_1) \vee (p^*,q^*) = (p^{'},q^{'}) \wedge (p_2,q_2) \text{ where } & p^*(x) = p(x) \wedge p_2(x),\ q^*(y) = q(y) \wedge q_2(y) \\
& \text{and } p^{'}(x) = p_1(x) \vee p(x),\ q^{'}(y) = q_1(y) \vee q(y) \\
& \text{for all } x \in X \text{ and } y \in Y.
\end{aligned}$$

$$\begin{aligned}
\text{i.e. } (p^{**},q^{**}) = (p^{''},q^{''}) \text{ where } & p^{**}(x) = p_1(x) \wedge p^*(x),\ q^{**}(y) = q_1(y) \wedge q^*(y) \\
& \text{and } p^{''}(x) = p^{'}(x) \vee p_2(x),\ q^{''}(y) = q^{'}(y) \vee q_2(y) \\
& \text{for all } x \in X \text{ and } y \in Y.
\end{aligned}$$

$$\begin{aligned}
\text{i.e. } (p^{**},q^{**}) = (p^{''},q^{''}) \text{ where } & p^{**}(x) = p_1(x) \wedge \Big(p(x) \wedge p_2(x)\Big), \\
& q^{**}(y) = q_1(y) \wedge \Big(q(y) \wedge q_2(y)\Big) \\
& \text{and } p^{''}(x) = \Big(p_1(x) \vee p(x)\Big) \vee p_2(x), \\
& q^{''}(y) = \Big(q_1(y) \vee q(y)\Big) \vee q_2(y) \\
& \text{for all } x \in X \text{ and } y \in Y.
\end{aligned}$$

Thus $(p^{**},q^{**}) = (p^{''},q^{''})$ implies that $p^{**}(x) = p^{''}(x)$ for all $x \in X$ and $q^{**}(y) = q^{''}(y)$ for all $y \in Y$.

i.e. $p_1(x) \vee \Big(p(x) \wedge p_2(x)\Big) = \Big(p_1(x) \vee p(x)\Big) \wedge p_2(x)$ for all $x \in X$ and $q_1(y) \vee \Big(q(y) \wedge q_2(y)\Big) = \Big(q_1(y) \vee q(y)\Big) \wedge q_2(y)$ for all $y \in Y$.

i.e. $y_1 \vee \Big(p(x) \wedge y_2\Big) = \Big(y_1 \vee p(x)\Big) \wedge y_2$ for all $x \in X$ and $x_1 \vee \Big(q(y) \wedge x_2\Big) = \Big(x_1 \vee q(y)\Big) \wedge x_2$ for all $y \in Y$.

Therefore, X and Y are modular lattices. $\square$

Theorem 5.3.4. *The ternary semigroup of mappings $(T[X,Y], \leq)$ of X and Y is a distributive lattice if and only if $(X, \leq_X)$ and $(Y, \leq_Y)$ are two distributive lattices.*

Proof. Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be two distributive lattices under the partial order $\leq_X$ and $\leq_Y$ respectively. Let $(p_1,q_1),(p_2,q_2),(p_3,q_3) \in T[X,Y]$.

Therefore we have, $(p_1, q_1) \wedge \Big((p_2, q_2) \vee (p_3, q_3)\Big)$

$$= (p_1, q_1) \vee (p^{'}, q^{'}) \text{ where, } p^{'}(x) = p_2(x) \vee p_3(x) \text{ and } q^{'}(y) = q_2(y) \vee q_3(y) \text{ for all } x \in X \text{ and } y \in Y.$$

$$= (p^*, q^*) \text{ where, } p^*(x) = p_1(x) \wedge \Big(p_2(x) \vee p_3(x)\Big) \text{ and } q^*(y) = q_1(y) \wedge \Big(q_2(y) \vee q_3(y)\Big) \text{ for all } x \in X \text{ and } y \in Y.$$

Again, $\Big((p_1, q_1) \wedge (p_2, q_2)\Big) \vee \Big((p_1, q_1) \wedge (p_3, q_3)\Big)$

$$= (p^{**}, q^{**}) \vee (p^{''}, q^{''}) \text{ where, } p^{**}(x) = p_1(x) \wedge p_2(x), q^{**}(y) = q_1(y) \wedge q_2(y) \text{ and } p^{''}(x) = p_1(x) \wedge p_3(x), q^{''}(y) = q_1(y) \wedge q_3(y) \text{ for all } x \in X \text{ and } y \in Y.$$

$$= (p^o, q^o) \text{ where, } p^o(x) = \Big(p_1(x) \wedge p_2(x)\Big) \vee \Big(p_1(x) \wedge p_3(x)\Big) \text{ and } q^o(y) = \Big(q_1(y) \wedge q_2(y)\Big) \vee \Big(q_1(y) \wedge q_3(y)\Big) \text{ for all } x \in X \text{ and } y \in Y.$$

Since X and Y are distributive lattices, then $p_1(x) \wedge \Big(p_2(x) \vee p_3(x)\Big) = \Big(p_1(x) \wedge p_2(x)\Big) \vee \Big(p_1(x) \wedge p_3(x)\Big)$ and $q_1(y) \wedge \Big(q_2(y) \vee q_3(y)\Big) = \Big(q_1(y) \wedge q_2(y)\Big) \vee \Big(q_1(y) \wedge q_3(y)\Big)$ for all $x \in X$ and $y \in Y$.
Thus $(p^*, q^*) = (p^o, q^o)$ i.e $(p_1, q_1) \wedge \Big((p_2, q_2) \vee (p_3, q_3)\Big) = \Big((p_1, q_1) \wedge (p_2, q_2)\Big) \vee \Big((p_1, q_1) \wedge (p_3, q_3)\Big)$. Hence $T[X, Y]$ is a distributive lattice.

Conversely, suppose $T[X, Y]$ be a distributive lattice. Let $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$. Let p_1, p_2, p_3 be such mappings from X to Y defined by $p_1(x) = y_1$,

$p_2(x) = y_2$ and $p_3(x) = y_3$ for all $x \in X$. Again let q_1, q_2, q_3 be such mappings from Y to X defined by $q_1(y) = x_1$, $q_2(y) = x_2$ and $q_3(y) = x_3$ for all $y \in Y$. Thus $(p_1, q_1), (p_2, q_2), (p_3, q_3) \in T[X, Y]$. Since $T[X, Y]$ is a distributive lattice then we have $(p_1, q_1) \wedge \Big((p_2, q_2) \vee (p_3, q_3)\Big) = \Big((p_1, q_1) \wedge (p_2, q_2)\Big) \vee \Big((p_1, q_1) \wedge (p_3, q_3)\Big)$
$\Longrightarrow (p_1, q_1) \wedge (p^*, q^*) = (p^{'}, q^{'}) \vee (p^o, q^o)$ where,

$$\begin{aligned}
&p^*(x) = p_2(x) \vee p_3(x),\ q^*(y) = q_2(y) \vee q_3(y)\\
&p^{'}(x) = p_1(x) \wedge p_2(x),\ \ q^{'}(y) = q_1(y) \wedge q_2(y)\\
&p^o(x) = p_1(x) \wedge p_3(x),\ \ q^o(y) = q_1(y) \wedge q_3(y)\\
&\text{for all } x \in X \text{ and } y \in Y.
\end{aligned}$$

$\Longrightarrow (p^{**}, q^{**}) = (p^{''}, q^{''})$ where,

$$\begin{aligned}
&p^{**}(x) = p_1(x) \wedge p^*(x),\ q^{**}(y) = q_1(y) \wedge q^*(y)\\
&p^{''}(x) = p^{'}(x) \vee p^o(x),\ q^{''}(y) = q^{'}(y) \vee q^o(y)\\
&\text{for all } x \in X \text{ and } y \in Y.
\end{aligned}$$

$\Longrightarrow (p^{**}, q^{**}) = (p^{''}, q^{''})$ where

$$\begin{aligned}
&p^{**}(x) = p_1(x) \wedge \Big(p_2(x) \vee p_3(x)\Big),\\
&q^{**}(y) = q_1(y) \wedge \Big(q_2(y) \vee q_3(y)\Big)\\
&p^{''}(x) = \Big(p_1(x) \wedge p_2(x)\Big) \vee \Big(p_1(x) \wedge p_3(x)\Big),\\
&q^{''}(y) = \Big(q_1(y) \wedge q_2(y)\Big) \vee \Big(q_1(y) \wedge q_3(y)\Big)\\
&\text{for all } x \in X \text{ and } y \in Y.
\end{aligned}$$

Now, $(p^{**}, q^{**}) = (p^{''}, q^{''})$ implies that $p^{**}(x) = p^{''}(x)$ and $q^{**}(y) = q^{''}(y)$ for all $x \in X$ and $y \in Y$.
i.e. $p_1(x) \wedge \Big(p_2(x) \vee p_3(x)\Big) = \Big(p_1(x) \wedge p_2(x)\Big) \vee \Big(p_1(x) \wedge p_3(x)\Big)$ and $q_1(y) \wedge \Big(q_2(y) \vee q_3(y)\Big) = \Big(q_1(y) \wedge q_2(y)\Big) \vee \Big(q_1(y) \wedge q_3(y)\Big)$ for all $x \in X$ and $y \in Y$.
i.e. $x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3)$ and $y_1 \wedge (y_2 \vee y_3) = (y_1 \wedge y_2) \vee (y_1 \wedge y_3)$.
Thus X and Y are distributive lattices. □

Theorem 5.3.5. *The ternary semigroup of mappings $(T[X, Y], \leq)$ of X and Y is a Boolean lattice if and only if $(X, \leq_X)$ and $(Y, \leq_Y)$ are two Boolean lattices.*

Proof. Let us assume that $(X, \leq_X)$ and $(Y, \leq_Y)$ be two Boolean lattices with 1_X,

1_Y and 0_X, 0_Y be the greatest and least elements of X and Y respectively. Let $(p,q) \in T[X,Y]$. We define two mappings $p^{'} : X \longrightarrow Y$ and $q^{'} : Y \longrightarrow X$ such that $p^{'}(x) = \overline{p(x)}$ and $q^{'}(y) = \overline{q(y)}$ for all $x \in X$ and $y \in Y$, where $\overline{p(x)}$ and $\overline{q(y)}$ denotes the complement of $p(x)$ and $q(y)$ for all $x \in X$ and $y \in Y$ i.e $p(x) \vee \overline{p(x)} = 1_Y$, $p(x) \wedge \overline{p(x)} = 0_Y$ and $q(y) \vee \overline{q(y)} = 1_X$, $q(y) \wedge \overline{q(y)} = 0_X$ for all $x \in X$ and $y \in Y$. If $(p,q) \vee (p^{'},q^{'}) = (p^*,q^*)$ and $(p,q) \wedge (p^{'},q^{'}) = (p^o,q^o)$ then we have to show that (p^*,q^*) and (p^o,q^o) be the greatest and least element of $T[X,Y]$. We proceed by contradiction, let (p^*,q^*) be not the greatest element of $T[X,Y]$. If not, then there exists an element (p^{**},q^{**}) such that $(p^*,q^*) \leq (p^{**},q^{**}) \Longrightarrow p^*(x) \leq_Y p^{**}(x)$ and $q^*(y) \leq_X q^{**}(y)$ for all $x \in X$ and $y \in Y \Longrightarrow p(x) \vee \overline{p(x)} \leq_Y p^{**}(x)$ and $q(y) \vee \overline{q(y)} \leq_X q^{**}(y) \Longrightarrow 1_Y \leq_Y p^{**}(x)$ and $1_X \leq_X q^{**}(y)$, which is a contradiction. In the similar way, let $(p^{'},q^{'})$ be not the least element of $T[X,Y]$. If not then there exists an element $(p^{''},q^{''})$ such that $(p^{''},q^{''}) \leq (p^{'},q^{'}) \Longrightarrow p^{''}(x) \leq_Y p^{'}(x)$ and $q^{''}(y) \leq_X q^{'}(y)$ for all $x \in X$ and $y \in Y \Longrightarrow p^{''}(x) \leq_Y p(x) \wedge \overline{p(x)}$ and $q^{''}(y) \leq_X q(y) \wedge \overline{q(y)} \Longrightarrow p^{''}(x) \leq_Y 0_Y$ and $q^{''}(y) \leq_X 0_X$, a contradiction. So that (p^*,q^*) and $(p^{'},q^{'})$ be the greatest and least elements of $T[X,Y]$ and hence $T[X,Y]$ is a Boolean lattice.

Conversely, let $T[X,Y]$ be a Boolean lattice. By Theorem 5.3.1 X and Y are also lattices. Let $x_1 \in X$ and $y_1 \in Y$. Let us define two mappings $q_1 : Y \longrightarrow X$ such that $q_1(y) = x_1$ and $p_1 : X \longrightarrow Y$ such that $p_1(x) = y_1$ for all $x \in X$ and $y \in Y$. Thus $(p_1,q_1) \in T[X,Y]$. Then there exists $(p_1^{'},q_1^{'}) \in T[X,Y]$ such that $(p_1,q_1) \vee (p_1^{'},q_1^{'}) = 1_{T[X,Y]}$ and $(p_1,q_1) \wedge (p_1^{'},q_1^{'}) = 0_{T[X,Y]}$, where $1_{T[X,Y]}$ and $0_{T[X,Y]}$ be the greatest and least elements of $T[X,Y]$. Let $1_{T[X,Y]} = (p^*,q^*)$ and $0_{T[X,Y]} = (p^o,q^o)$. Thus $(p_1,q_1) \vee (p_1^{'},q_1^{'}) = (p^*,q^*)$ where $p^*(x) = p_1(x) \vee p_1^{'}(x)$, $q^*(y) = q_1(y) \vee q_1^{'}(y)$ and $(p_1,q_1) \wedge (p_1^{'},q_1^{'}) = (p^o,q^o)$ where $p^o(x) = p_1(x) \wedge p_1^{'}(x)$, $q^o(y) = q_1(y) \wedge q_1^{'}(y)$ for all $x \in X$ and $y \in Y$. We have to show that $q^*(y)$ and $p^*(x)$ be the greatest elements of X and Y. Also $q^o(y)$ and $p^o(x)$ be the least elements of X and Y. Let $q^*(y)$ be no the greatest element in X. Then there exists x^{**} such that $q^*(y) \leq_X x^{**}$.

Let us define a mapping $q^{**} : Y \longrightarrow X$ such that $q^{**}(y) = x^{**}$ for all $y \in Y$. Hence $q^*(y) \leq_X q^{**}(y)$ for all $y \in Y$ Now $q^*(y) = q_1(y) \vee q_1^{'}(y)$. Thus, $q_1(y) \vee q_1^{'}(y) \leq_X q^{**}(y) \Longrightarrow q_1(y) \leq_X q^{**}(y)$ and $q_1^{'}(y) \leq_X q^{**}(y)$. ....(1).
Again let $p^*(x)$ be not the greatest element in Y. Then there exists y^{**} such that $p^*(x) \leq_Y y^{**}$. Let us define another mapping $p^{**} : X \longrightarrow Y$ such that $p^{**}(x) = y^{**}$ for all $x \in X$. Thus $p^*(x) \leq_Y p^{**}(x)$ for all $x \in X \Longrightarrow p_1(x) \vee p_1^{'}(x) \leq_Y p^{**}(x) \Longrightarrow p_1(x) \leq_Y p^{**}(x)$ and $p_1^{'}(x) \leq_Y p^{**}(x)$ ....(2).
From (1) and (2) we get $(p_1, q_1) \leq (p^{**}, q^{**}), (p_1^{'}, q_1^{'}) \leq (p^{**}, q^{**}) \Longrightarrow (p_1, q_1) \vee (p_1^{'}, q_1^{'}) \leq (p^{**}, q^{**}) \Longrightarrow (p^*, q^*) \leq (p^{**}, q^{**})$ which is not possible since (p^*, q^*) is the greatest element in $T[X, Y]$. Thus $q^*(y)$ and $p^*(x)$ be the greatest elements of X and Y. Similarly, we can show that $q^o(y)$ and $p^o(x)$ be the least elements of X and Y. Hence X and Y are Boolean lattices. □

5.4 Isomorphism between ternary semigroup of mappings

In this section, we consider the problem of describing isomorphism between ternary semigroups of mappings.

Definition 5.4.1. *[22, Chronowski] Let $(X, \vee, \wedge)$ and $(Y, \vee, \wedge)$ are two lattices. A mapping $f : X \longrightarrow Y$ is said to be a lattice homomorphism if*
(i) $f(x_1 \vee x_2) = f(x_1) \vee f(x_2)$ and
(ii) $f(x_1 \wedge x_2) = f(x_1) \wedge f(x_2)$ $\forall x_1, x_2 \in X$.
Again f is called a lattice isomorphism if f is one-one and onto.

Theorem 5.4.2. *Let $(X, \wedge, \vee)$, $(X^{'}, \wedge, \vee)$, $(Y, \wedge, \vee)$, $(Y^{'}, \wedge, \vee)$ are lattices. If X is isomorphic to $X^{'}$ and Y is isomorphic to $Y^{'}$, then there is a lattice isomorphism from the ternary semigroup of mappings $T[X, Y]$ of X and Y to the ternary semigroup of mappings $T[X^{'}, Y^{'}]$ of $X^{'}$ and $Y^{'}$.*

Proof. Let us suppose that $X \cong X^{'}$ and $Y \cong Y^{'}$. Let $\phi : X \longrightarrow X^{'}$ and $\psi : Y \longrightarrow Y^{'}$ be two lattice isomorphisms. Let us define a mapping f from $T[X,Y]$ onto $T[X^{'},Y^{'}]$ by

$$f(p,q) = (p^{'},q^{'}) \text{ such that } p^{'}(\phi(x)) = \psi(p(x)),\ q^{'}(\psi(y)) = \phi(q(y)).$$

$$\begin{array}{ccc} X & \xrightarrow{\quad\quad\phi\quad\quad} & X^{'} \\ p \downarrow\uparrow q & & p^{'} \downarrow\uparrow q^{'} \\ Y & \xrightarrow[\quad\quad\psi\quad\quad]{} & Y^{'} \end{array}$$

Let $(p,q),(p_1,q_1) \in T[X,Y]$. Since $T[X,Y]$ is a lattice $\sup\{(p,q),(p_1,q_1)\} = (p,q) \vee (p_1,q_1)$ and $\inf\{(p,q),(p_1,q_1)\} = (p,q) \wedge (p_1,q_1)$ exists in $T[X,Y]$.
Thus we have $f((p,q) \vee (p_1,q_1)) = f(p^o,q^o)$ where $p^o(x) = p(x) \vee p_1(x)$, $q^o(y) = q(y) \vee q_1(y)$ for all $x \in X$ and $y \in Y$ and
$f(p,q) \vee f(p_1,q_1) = (p^{'},q^{'}) \vee (p_1^{'},q_1^{'}) = (p^{o^{'}},q^{o^{'}})$ where $p^{o^{'}}(x^{'}) = p^{'}(x^{'}) \vee p_1^{'}(x^{'})$, $q^{o^{'}}(y^{'})$ $= q^{'}(y^{'}) \vee q_1^{'}(y^{'})$ for all $x^{'} \in X^{'}$ and $y^{'} \in Y^{'}$.
We have to show that $f((p,q) \vee (p_1,q_1)) = f(p,q) \vee f(p_1,q_1) \Longrightarrow f(p^o,q^o) = (p^{o^{'}},q^{o^{'}})$ i.e. $p^{o^{'}}(\phi(x)) = \psi(p^o(x))$ and $q^{o^{'}}(\psi(y)) = \phi(q^o(y))$.
Now $f(p,q) = (p^{'},q^{'}) \Longrightarrow p^{'}(\phi(x)) = \psi(p(x))$, $q^{'}(\psi(y)) = \phi(q(y))$ and
$f(p_1,q_1) = (p_1^{'},q_1^{'}) \Longrightarrow p_1^{'}(\phi(x)) = \psi(p_1(x))$, $q_1^{'}(\psi(y)) = \phi(q_1(y))$
Then, $p^{o^{'}}(\phi(x)) = p^{'}(\phi(x)) \vee p_1^{'}(\phi(x)) = \psi(p(x)) \vee \psi(p_1(x)) = \psi(p(x) \vee p_1(x)) = \psi(p^o(x))$ and
$q^{o^{'}}(\psi(y)) = q^{'}(\psi(y)) \vee q_1^{'}(\psi(y)) = \phi(q(y)) \vee \phi(q_1(y)) = \phi(q(y) \vee q_1(y)) = \phi(q^o(y))$.
Thus $f(p^o,q^o) = (p^{o^{'}},q^{o^{'}})$.
Hence $f((p,q) \vee (p_1,q_1)) = f(p,q) \vee f(p_1,q_1)$.

Similarly, it can be shown that $f((p,q) \wedge (p_1,q_1)) = f(p,q) \wedge f(p_1,q_1)$. Therefore, f is a lattice homomorphism. It remains to show that f is one-one and onto. For $(p,q),(p_1,q_1) \in T[X,Y]$, we have

$$
\begin{aligned}
& f(p,q) = f(p_1,q_1) \\
\Longrightarrow\ & (p^{'},q^{'}) = (p_1^{'},q_1^{'}) \\
\Longrightarrow\ & p^{'}(\phi(x)) = p_1^{'}(\phi(x)),\ q^{'}(\psi(y)) = q_1^{'}(\psi(y)) \text{ for all } x \in X \text{ and } y \in Y \\
\Longrightarrow\ & \psi(p(x)) = \psi(p_1(x)),\ \phi(q(y)) = \phi(q_1(y)) \text{ for all } x \in X \text{ and } y \in Y \\
\Longrightarrow\ & p(x) = p_1(x),\ q(y) = q_1(y) \text{ for all } x \in X \text{ and } y \in Y. \\
\Longrightarrow\ & (p,q) = (p_1,q_1)
\end{aligned}
$$

Hence $(p,q) = (p_1,q_1)$ and f is one-one. For the last part, let $(p^{'},q^{'}) \in T[X^{'},Y^{'}]$. Now $p^{'}(\phi(x)) = y^{'} = \psi(y) = \psi(p(x))$ and $q^{'}(\psi(y)) = x^{'} = \phi(x) = \phi(q(y))$. Thus there exists (p,q) in $T[X,Y]$ such that $f(p,q) = (p^{'},q^{'})$. Hence f is onto and so f is a lattice isomorphism. $\square$

Note 5.4.3. *But the converse of the theorem may not be true. Let us give an example below.*

Example 5.4.4. *Assume that X, $X^{'}$, Y, $Y^{'}$ be non-empty sets such that card (X) = card $(Y^{'})$ = n, $n \in \mathbb{N}$ and card $(X^{'})$ = card (Y) = 1. Let us consider the sets $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1\}$, $X^{'} = \{x_1^{'}\}$ and $Y^{'} = \{y_1^{'}, y_2^{'}, ..., y_n^{'}\}$. Define a partially order relation on X by $x_i \leq x_j$ if and only if $i < j$ and also define a partially order relation on $Y^{'}$ by $y_i^{'} \leq y_j^{'}$ if and only if $i < j$. Assume that $X^{'}$ and Y are trivially ordered sets. Now $T(X,Y) = \{p_1\}$ where $p_1(x) = y_1$ for all $x \in X$, $T(Y,X) = \{q_1, q_2, ..., q_n\}$ where $q_1(y) = x_1$, $q_2(y) = x_2, ...,$ $q_n(y) = x_n$ for all $y \in Y$ and so $T[X,Y] = T(X,Y) \times T(Y,X) = \{(p_1,q_1), (p_1,q_2), ..., (p_1,q_n)\}$.*
Again, $T(X^{'},Y^{'}) = \{p_1^{'}, p_2^{'}, ..., p_n^{'}\}$ where $p_1^{'}(x^{'}) = y_1^{'}$, $p_2^{'}(x^{'}) = y_2^{'}, ...,$ $p_n^{'}(x^{'}) = y_n^{'}$ for all $x^{'} \in X^{'}$, $T(Y^{'},X^{'}) = \{q_1^{'}\}$ where $q_1^{'}(y^{'}) = x_1^{'}$ for all $y^{'} \in Y^{'}$ and so $T[X^{'},Y^{'}] = T(X^{'},Y^{'}) \times T(Y^{'},X^{'}) = \{(p_1^{'},q_1^{'}), (p_2^{'},q_1^{'}), ..., (p_n^{'},q_1^{'})\}$.

Let us define a mapping $f : T[X,Y] \longrightarrow T[X^{'},Y^{'}]$ by $f(p_i,q_j) = (p_j^{'},q_i^{'})$. Then *it can be easily shown that f is an isomorphism from $T[X,Y]$ to $T[X^{'},Y^{'}]$. Since card $(T[X,Y])$ = card $(T[X^{'},Y^{'}])$, then there exists a bijection from $T[X,Y]$ to $T[X^{'},Y^{'}]$. Now $f((p_1,q_1)\wedge(p_1,q_2)\wedge \dots\dots\dots\dots\dots\dots \wedge(p_1,q_n)) = f(p^*,q^*)$ where $p^*(x) = p_1(x)\wedge p_1(x)\wedge p_1(x)\wedge \dots\dots\dots\dots\dots\dots \wedge p_1(x) = p_1(x)$ and $q^*(y) = q_1(y)\wedge q_2(y)\wedge q_3(y)\wedge \dots\dots\dots\dots\dots\dots \wedge q_n(y) = q_1(y)$. Therefore $f((p_1,q_1)\wedge(p_1,q_2)\wedge \dots\dots\dots\dots\dots\dots \wedge(p_1,q_n)) = f(p_1,q_1) = (p_1^{'},q_1^{'})$.*

Again $f(p_1,q_1)\wedge f(p_1,q_2)\wedge \dots\dots\dots\dots\dots\dots \wedge f(p_1,q_n) = (p_1^{'},q_1^{'})\wedge(p_2^{'},q_1^{'})\wedge \dots\dots\dots\dots\dots\dots \wedge (p_n^{'},q_1^{'}) = (p_1^{'},q_1^{'})$. Thus f is a homomorphism and so $T[X,Y] \cong T[X^{'},Y^{'}]$. But X is not isomorphic to $X^{'}$ and Y is not isomorphic to $Y^{'}$ since card $(X) \neq$ card $(X^{'})$ and card $(Y) \neq$ card $(Y^{'})$.

Note 5.4.5. *If card (X) = card $(X^{'})$ and card (Y) = card $(Y^{'})$ and f is an isomorphism from $T[X,Y]$ to $T[X^{'},Y^{'}]$ such that f takes the pair of constant maps to a pair of constant maps, then X is isomorphic to $X^{'}$ and Y is isomorphic to $Y^{'}$. Let us consider p_y, q_x be constant maps such that $p_y(x) = y$ and $q_x(y) = x$ for all $x \in X, y \in Y$. Let us define two mappings $\phi : X \longrightarrow X^{'}$ and $\psi : Y \longrightarrow Y^{'}$ by $\phi(x) = x^{'}$ and $\psi(y) = y^{'}$ such that $f(p_y,q_x) = (p^{'}_{y^{'}},q^{'}_{x^{'}})$. Let (p_y,q_x), $(p_{y_1},q_{x_1}) \in T[X,Y]$. Since f is an isomorphism, we have*

$$
\begin{aligned}
& f((p_y,q_x)\vee(p_{y_1},q_{x_1})) = f(p_y,q_x)\vee f(p_{y_1},q_{x_1}) \\
\Longrightarrow\ & f(p_y\vee p_{y_1}, q_x\vee q_{x_1}) = (p^{'}_{y^{'}},q^{'}_{x^{'}})\vee(p^{'}_{y_1^{'}},q^{'}_{x_1^{'}}) \\
\Longrightarrow\ & f(p_y\vee p_{y_1}, q_x\vee q_{x_1}) = (p^{'}_{y^{'}}\vee p^{'}_{y_1^{'}}, q^{'}_{x^{'}}\vee q^{'}_{x_1^{'}}) \\
\Longrightarrow\ & f(p_{y\vee y_1}, q_{x\vee x_1}) = (p^{'}_{y^{'}\vee y_1^{'}}, q^{'}_{x^{'}\vee x_1^{'}})
\end{aligned}
$$

Thus $\phi(x\vee x_1) = x^{'}\vee x_1^{'} = \phi(x)\vee\phi(x_1)$ and $\psi(y\vee y_1) = y^{'}\vee y_1^{'} = \psi(y)\vee\psi(y_1)$.

Definition 5.4.6. *A mapping $f : X \longrightarrow Y$ is called a isotone of X into Y if $x_1 \leq_X x_2 \Longrightarrow f(x_1) \leq_Y f(x_2)$ for all $x_1,x_2 \in X$.*

Theorem 5.4.7. *Let X and Y be posets and $O(X,Y)$ be the set of all isotone mappings from X to Y. Put $O[X,Y] = O(X,Y)\times O(Y,X) \subset T[X,Y]$. Then $O[X,Y]$*

is an ordered ternary semigroup with respect to the ternary operation and partial order defined in $T[X,Y]$.

Proof. First, suppose that $(p_1, q_1), (p_2, q_2) \in O[X, Y]$ such that $(p_1, q_1) \leq (p_2, q_2) \Longrightarrow$ $p_1(x) \leq_Y p_2(x)$, $q_1(y) \leq_X q_2(y)$ for all $x \in X$ and $y \in Y$ ....(1)
Since (p_3, q_3), (p_4, q_4) are elements of $O[X, Y]$, we have $(p_1, q_1)(p_3, q_3)(p_4, q_4) = (p_1q_3p_4, q_1p_3q_4)$ and $(p_2, q_2)(p_3, q_3)(p_4, q_4) = (p_2q_3p_4, q_2p_3q_4)$. Now

$$(p_1q_3p_4)(x) = (p_1q_3)(p_4(x)) = (p_1q_3)(y) = (p_1)(q_3(y)) = p_1(x^{'}) \text{ and}$$
$$(p_2q_3p_4)(x) = (p_2q_3)(p_4(x)) = (p_2q_3)(y) = (p_2)(q_3(y)) = p_2(x^{'})$$
$$\text{where } p_4(x) = y \text{ and } q_3(y) = x^{'}$$

Thus from (1) we have $p_1(x^{'}) \leq_Y p_2(x^{'}) \Longrightarrow p_1q_3p_4(x) \leq_Y p_2q_3p_4(x)$ for all $x \in X$. In the similar manner we can show that, $(q_1p_3q_4)(y) \leq_Y (q_2p_3q_4)(y)$ for all $y \in Y$. Hence $(p_1q_3p_4, q_1p_3q_4) \leq (p_2q_3p_4, q_2p_3q_4)$ which implies that $(p_1, q_1)(p_3, q_3)(p_4, q_4) \leq (p_2, q_2)(p_3, q_3)(p_4, q_4)$.

Again, we have $(p_3, q_3)(p_4, q_4)(p_1, q_1) = (p_3q_4p_1, q_3p_4q_1)$ and $(p_3, q_3)(p_4, q_4)(p_2, q_2) = (p_3q_4p_2, q_3p_4q_2)$. Now $p_1(x) \leq_Y p_2(x) \Longrightarrow (q_4p_1)(x) \leq_X (q_4p_2)(x)$ [since q_4 is an isotone map] and $(q_4p_1)(x) \leq_X (q_4p_2)(x) \Longrightarrow (p_3q_4p_1)(x) \leq_Y (p_3q_4p_2)(x)$ [since p_3 is an isotone map] for all $x \in X$. Similarly, $(q_3p_4q_1)(y) \leq (q_3p_4q_2)(y)$ for all $y \in Y$. Thus $(p_3q_4p_1, q_3p_4q_1) \leq (p_3q_4p_2, q_3p_4q_2)$ implies that $(p_3, q_3)(p_4, q_4)(p_1, q_1)$ $\leq (p_3, q_3)(p_4, q_4)(p_2, q_2)$.

For the last part, $(p_3, q_3)(p_1, q_1)(p_4, q_4) = (p_3q_1p_4, q_3p_1q_4)$ and $(p_3, q_3)(p_2, q_2)(p_4, q_4)$ $= (p_3q_2p_4, q_3p_2q_4)$. Let $p_4(x) = y$ for some $y \in Y$ and since Y is a poset $y \leq_Y y$ i.e. $p_4(x) \leq_Y p_4(x) \Longrightarrow (q_1p_4)(x) \leq_Y (q_1p_4)(x)$ [since q_1 is an isotone map] $\Longrightarrow (p_3q_1p_4)(x) \leq_Y (p_3q_1p_4)(x)$ [since p_3 is an isotone map]. Thus $(p_3q_1p_4, q_3p_1q_4)$ $\leq (p_3q_2p_4, q_3p_2q_4)$ i.e. $(p_3, q_3)(p_1, q_1)(p_4, q_4) \leq (p_3, q_3)(p_2, q_2)(p_4, q_4)$.
Hence $O[X, Y]$ is an ordered ternary semigroup. □

The ordered ternary semigroup $O[X, Y]$ is called the ternary semigroup of isotone mappings of sets X and Y.

Theorem 5.4.8. *The ternary semigroup of isotone mappings $O[X, Y]$ is a regular ordered ternary semigroup.*

Proof. In [23, Corollary 28.16] we have seen that $T[X, Y]$ is regular ternary semigroup. Now proceeding in the same way we can show that $O[X, Y]$ is also regular ternary semigroup i.e. for every $(p, q) \in O[X, Y]$ there exists $(p^{'}, q^{'}) \in O[X, Y]$ such that $(p, q)(p^{'}, q^{'})(p, q) = (p, q)$. Since $O[X, Y]$ is a partially ordered ternary semigroup then $(p, q) \leq (p, q) = (p, q)(p^{'}, q^{'})(p, q)$. Thus $O[X, Y]$ is a regular ordered ternary semigroup. □

Theorem 5.4.9. *The ternary semigroup of isotone mappings $O[X, Y]$ is an intra-regular ordered ternary semigroup.*

Proof. Let $(p, q) \in O[X, Y]$. Let us define two mappings $p_1 : Im(q) \longrightarrow Y$ and $q_1 : Im(p) \longrightarrow X$ such that $p_1(x) = y$ for all $x \in Im(q)$ and $q_1(y) = x$ for all $y \in Im(p)$. Now $Dom(p_1) = Im(q)$ and $Dom(q_1) = Im(p)$. Let $(p^{'}, q^{'}) \in O[X, Y]$ such that $p^{'}|_{Im(q)} = p_1$ and $q^{'}|_{Im(p)} = q_1$. Hence $pq'p'q'p = p$ and $qp'q'p'q = q$. Thus for every $(p, q) \in O[X, Y]$ there exists $(p^{'}, q^{'}) \in O[X, Y]$ such that $(p, q)(p^{'}, q^{'})(p^{'}, q^{'})(p^{'}, q^{'})(p, q)$ $= (p, q)$ i.e. $(p, q)(p^{'}, q^{'})^3(p, q) = (p, q)$. Since $O[X, Y]$ is a partially ordered ternary semigroup then $(p, q) \leq (p, q) = (p, q)(p^{'}, q^{'})^3(p, q)$. Thus $O[X, Y]$ is an intra-regular ordered ternary semigroup. □

Theorem 5.4.10. *The ternary semigroup of isotone mappings $O[X, Y]$ is an idempotent ordered ternary semigroup if either card$(X) = 1$ or card$(Y) = 1$.*

Proof. Let $(p, q) \in O[X, Y]$. Let card$(X) = 1$. Then Imq contains exactly one element. Let the element be x_1. Thus $q(y) = x_1$ for all $y \in Y$. Let $p(x_1) = y_1$. Then $(qpq)(y) = (qp)(q(y)) = (qp)(x_1) = q(p(x_1)) = q(y_1) = x_1 = q(y)$ and $(pqp)(x) = (pqp)(x_1) = (pq)(p(x_1)) = (pq)(y_1) = p(q(y_1)) = p(x_1) = p(x)$. Hence $(pqp, qpq) = (p, q)$. Since $O[X, Y]$ is a partially ordered ternary semigroup then $(p, q) \leq (p, q) = (pqp, qpq) = (p, q)(p, q)(p, q)$. Thus $O[X, Y]$ is an idempotent ordered ternary semigroup. Similarly, if card$(Y) = 1$ we can prove that $O[X, Y]$ is an idempotent ordered ternary semigroup. □

Definition 5.4.11. *Let $(X, ., \leq_X)$ and $(Y, ., \leq_Y)$ are ordered ternary semigroups. A mapping $f : X \longrightarrow Y$ is said to be an ordered ternary homomorphism if (i) $f(x_1x_2x_3) = f(x_1)f(x_2)f(x_3)$*
(ii) $x_1 \leq_X x_2 \Longrightarrow f(x_1) \leq_Y f(x_2)$ for all $x_1, x_2, x_3 \in X$.
Again, f is called an ordered ternary isomorphism if f is one-one and onto.

Theorem 5.4.12. *Let X, $X^{'}$, Y, $Y^{'}$ are posets. If $\phi : X \longrightarrow X^{'}$ and $\psi : Y \longrightarrow Y^{'}$ are isomorphisms, then there exists an ordered ternary isomorphism f from $O[X, Y]$ to $O[X^{'}, Y^{'}]$.*

Proof. The proof is similar to Theorem 5.4.2. □

The converse statement of the Theorem 5.4.12 is not true. By using Example 5.4.4 we have reached the conclusion.

On Right Chain Ordered Ternary Semigroups

Chapter-6

Chapter 6

On right chain ordered ternary semigroups

6.1 Introduction

In this chapter, we study the ideal theory of a right chain ordered ternary semigroup S. Brungs and Törner develop the ideal theory for right cones [11] and right holoids [12]. Our main aim to study right chain ordered ternary semigroup in terms of prime ideals, completely prime ideals and prime segment. Prime segments are studied by many authors in [10], [9], [8], [70]. Then we are going to extend the concept of "Hoehnke ideal" of an ordered semigroup in an ordered ternary semigroup. In a semigroup, S H. J. [36] Hoehnke introduced the set $M^0 = \{h \in S : m \notin mhS \; for \; all \; m \in M \backslash 0\}$, where M is an S-system. Then M^0 is an ideal of the semigroup S. Later Miguel Ferrero, Ryszard Mazurek, Alveri Sant' Ana [31] defined the Hoehnke ideal of a semigroup S to be the set $\{h \in S : s \notin shS \; for \; all \; s \in S\backslash 0\}$ and denoted by $H(S)$. Thereafter Thawhat Changphas, Panuwat Luangchaisri, Ryszard Mazurek [18] introduced the Hoehnke ideal of a semigroup S asscociated with a proper right ideal A of S defined by $H_A(S) = \{h \in S : s \notin (shS] \; for \; all \; s \in S\backslash A\}$. In this chapter, we introduce the notion of $\mathcal{H}$-right ideal using Hoehnke ideal.

The concept of $\mathcal{H}$-right ideal is very helpful idea to construct semiprime right ideals in right chain ordered ternary semigroup.

The definition of right chain ordered ternary semigroups is as follows:

Definition 6.1.1. *A right chain ordered ternary semigroup is an ordered ternary semigroup $(S, ., \leq)$ in which right ideals forms a chain by inclusion. In other words, if I, J are right ideals of S then either $I \subseteq J$ or $J \subseteq I$.*

- *In the similar way we can define left chain ordered ternary semigroup and lateral chain ordered ternary semigroup.*
- *The ordered ternary semigroup S is a chain ordered ternary semigroup if it is a right, left and lateral chain ordered ternary semigroup.*

In this chapter, we consider that any ordered ternary semigroup contains the zero element. Therefore, there exists an element '0' in the ordered ternary semigroup S such that $ab0 = a0b = 0ab = 0$ for all $a, b \in S$.

Throughout this chapter, S denotes a right chain ordered ternary semigroup containing '0' which is the zero element of S.

Example 6.1.2. *Let $S = \{0, a, b, c\}$, where '0' is the zero element in S. The ternary multiplication in S defined as follows*

$$xyz = \begin{cases} x & \text{if } y \neq 0 \text{ and } z \neq 0 \\ y & \text{if } y = 0 \\ z & \text{if } z = 0 \end{cases}$$

for all $x, y, z \in S$. Then S with the ternary multiplication forms a ternary semigroup. Let '$\leq$' be a partial order on S defined by

$$\leq := \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c), (0, 0), (0, a), (0, b), (0, c)\}$$

Now $(S, ., \leq)$ is an ordered ternary semigroup with respect to the partial order '$\leq$'. The right ideals of $(S, ., \leq)$ are $\{0\}$, $\{0, a\}$, $\{0, a, b\}$, $\{0, a, b, c\}$ which are comparable.

Thus the right ideals of S form a chain. Hence S is a right chain ordered ternary semigroup.

6.2 $\mathcal{H}$-ideals in right chain ordered ternary semigroup

In this section, we are going to define the $\mathcal{H}$-ideal of ordered ternary semigroup.

Next we have the proposition which we will often use in this chapter.

Proposition 6.2.1. *Let S be an ordered ternary semigroup and I be an ideal of S. Then I is completely prime if and only if I is prime and completely semiprime.*

Proof. From the definitions of prime ideal, completely prime ideal and completely semiprime ideal of an ordered ternary semigroup S it can be easily seen that if an ideal I of S is completely prime then I is prime and completely semiprime.

To prove the reverse side, assume that I is prime and completely semiprime ideal of S. Let $abc \in I$ for any $a, b, c \in S$. Now $(bca)^3 = (bca)(bca)(bca) = bc(abc)(abc)a \in SSIIS \subseteq IIS \subseteq SIS \subseteq I$ and $(cab)^3 = (cab)(cab)(cab) = c(abc)(abc)ab \in SIISS \subseteq SII \subseteq SIS \subseteq I$. Since I is completely semiprime ideal we have $bca \in I$ and $cab \in I$. Then $I(a)I(b)I(c) \subseteq I$ for any $a, b, c \in S$. Since I is prime ideal then $I(a) \subseteq I$ or $I(b) \subseteq I$ or $I(c) \subseteq I$ and hence $a \in I$ or $b \in I$ or $c \in I$. Therefore, I is completely prime ideal of S. □

Definition 6.2.2. *Let $(S, ., \leq)$ be an ordered ternary semigroup. For any proper right ideal A of S we define the $\mathcal{H}$-right ideal of S associated with A to be the set*

$$H(S_A) = \{h \in S : s \notin (s^2hSS] \text{ for all } s \in S \backslash A\}$$

Theorem 6.2.3. *Let $(S, ., \leq)$ be an ordered ternary semigroup and A be a proper right ideal of S. Then for any right ideal I of S, $I \subseteq H(S_A)$ if and only if $s \notin (s^2I]$ for all $s \in S \backslash A$.*

Proof. We continue by contraposition. Suppose that $I \nsubseteq H(S_A)$, then for some $i \in I$ and $s \in S \backslash A$ we have $s \in (s^2 iSS] \subseteq (s^2 ISS] \subseteq (s^2 I]$. Thus we get, if $s \notin (s^2 I]$ for all $s \in S \backslash A$, then $I \subseteq H(S_A)$.

To proof the reverse implication, let $s \in (s^2 I]$ for some $s \in S \backslash A$. Then there exists some $i \in I$ such that $s \leq s^2 i$. Then $s^2 i \leq s^2 isi$ and so $s \leq s^2 isi \in s^2 iSS$. Hence $s \in (s^2 iSS]$, which shows that $i \notin H(S_A)$. Therefore, $I \nsubseteq H(S_A)$ and the proof is done. □

Theorem 6.2.4. *Let $(S, ., \leq)$ be an ordered ternary semigroup and A be a proper right ideal of S. Then the followings hold:*
(1) *$H(S_A)$ is a semiprime right ideal of S.*
(2) *If A is a proper ideal of S, then $A \subseteq H(S_A)$.*

Proof. (1) Suppose $(S, ., \leq)$ be an ordered ternary semigroup and A be a proper right ideal of S. We have to show that $H(S_A)$ is a semiprime right ideal of S. First, we show that $H(S_A)$ is a right ideal of S. Since $0 \in S$, then $0 \in A$. For any $s \in S$ we have, $(s^2 0SS] = (0SS] = (0] \subseteq (A] = A$. Thus $s \notin (s^2 0SS]$ for all $s \in S \backslash A$. Hence $0 \in H(S_A)$, which shows that $H(S_A)$ is nonempty. Next we have to show that $H(S_A)$ is closed under right ternary multiplication i.e. $H(S_A)SS \subseteq H(S_A)$. Let us suppose $H(S_A)SS \nsubseteq H(S_A)$. Then for some $h \in H(S_A)$ and $s_1 s_2 \in S$ we have $hs_1 s_2 \notin H(S_A)$. By definition of $H(S_A)$ there exists $s \in S \backslash A$ such that $s \in (s^2 hs_1 s_2 SS] \subseteq (s^2 hSS]$. Since $h \in H(S_A)$, $s \in A$ which is a contradiction. So, $hs_1 s_2 \in H(S_A)$. Hence $H(S_A)SS \subseteq H(S_A)$. To complete the proof that $(H(S_A)] \subseteq H(S_A)$, let $x \in (H(S_A)]$. Then $x \leq h$ for some $h \in H(S_A)$. For any $s \in S \backslash A$ we have $s^2 x \leq s^2 h$. Hence by Proposition 1.4.3 we have $(s^2 xSS] \subseteq (s^2 hSS]$. Since $h \in H(S_A)$, $s \notin (s^2 hSS]$ and so also $s \notin (s^2 xSS]$ which implies that $x \in H(S_A)$. Therefore, $(H(S_A)] \subseteq H(S_A)$. Hence $H(S_A)$ is a right ideal of S.

The rest of the proof is complete by showing that $H(S_A)$ is a semiprime right ideal of S. Let I be a right ideal of S such that $I^3 \subseteq H(S_A)$. If $I \nsubseteq H(S_A)$ then by Theorem 6.2.3 there exists $s \in S \backslash A$ such that $s \in (s^2 I]$. Now, $s^2 I \subseteq$

$(s^2I]sI \subseteq (s^2I](sI] \subseteq (s^2IsI]$. Thus $s \in (s^2I] \subseteq ((s^2IsI]] = (s^2IsI]$. Again $s^2I \in (s^2IsI]sI \subseteq (s^2IsIsI] \subseteq (s^2I^3]$. Thus $s \in (s^2I^3] \subseteq (s^2H(S_A)]$ which is a contradiction. Hence $I \subseteq H(S_A)$ and $H(S_A)$ is semiprime right ideal of S.
(2) Next we are going to proof the last part of the theorem. Let us assume that, A be an ideal of S. Then $(s^2A] \subseteq (SSA] \subseteq (A] = A$ and $(sAS] \subseteq (SAS] \subseteq (A] = A$ for any $s \in S$. So, $s \notin (s^2A]$ for all $s \in S\backslash A$. Hence by Theorem 6.2.3 we get that $A \subseteq H(S_A)$.
This completes our proof. □

Let A be a non-empty subset of an ordered ternary semigroup S. By A^{2n-1} we mean the set of all products $a_1a_2...a_{2n-1}$ where $a_1, a_2, ..., a_{2n-1}$ are all elements of S and $n \in \mathbb{N}$. i.e.

$$A^{2n-1} = \{a_1a_2a_3....a_{2n-1} : a_1, a_2, ..., a_{2n-1} \in A\}$$

Definition 6.2.5. *Let A be an ideal of ordered ternary semigroup $(S, ., \leq)$.*
(a) An ideal I of S is said to be A-nilpotent if $I^{2n-1} \subseteq A$ for some $n \in \mathbb{N}$.
(b) An element $t \in S$ is said to be A-nilpotent $t^{2n-1} \in A$ for some $n \in \mathbb{N}$.

Proposition 6.2.6. *Let A be a proper right ideal of a right chain ordered ternary semigroup $(S, ., \leq)$. Then the followings hold:*
(1) If I is an ideal of S such that $I \subseteq H(S_A)$ and I is not A-nilpotent, then $\bigcap_{n\in\mathbb{N}}(I^{2n-1}]$ is a completely semiprime right ideal of S.
(2) If $t \in S$ such that $t \in H(S_A)$ and t is not A-nilpotent, then $\bigcap_{n\in\mathbb{N}}(t^{2n}S]$ is a semiprime right ideal of S.

Proof. (1) Assume that I is an ideal of S. Then for all $n \in \mathbb{N}$, $(I^{2n-1}]$ is a right ideal and by Proposition 1.4.2, $\bigcap_{n\in\mathbb{N}}(I^{2n-1}]$ is a right ideal of S. Let a be an arbitrary element of S such that $a^3 \in \bigcap_{n\in\mathbb{N}}(I^{2n-1}]$ but $a \notin \bigcap_{n\in\mathbb{N}}(I^{2n-1}]$. Then $a \notin (I^{2m-1}]$ for some $m \in \mathbb{N}$. Now $R(a) = (a \cup aSS]$ is a right ideal containing a of S and since S is right chain ordered ternary semigroup we have $(I^{2m-1}] \subseteq (a \cup aSS]$.

Since $a^3 \in \bigcap_{n\in\mathbb{N}}(I^{2n-1}]$, then $a^3 \in (I^{2n-1}]$ for all $n \in \mathbb{N}$. Now $6m-2 \in \mathbb{N}$ since $m \in \mathbb{N}$. Then we have $a^3 \in (I^{2(6m-2)-1}] = (I^{12m-5}] = (I^{2m-1}I^{10m-4}] \subseteq ((a \cup aSS]I^{10m-4}] \subseteq (aI^{10m-4}] = (aI^{2m-1}I^{8m-3}] \subseteq (a(a \cup aSS]I^{8m-3}] \subseteq (a^2I^{8m-3}] = (a^2I^{2m-1}I^{6m-2}] \subseteq (a^2(a \cup aSS]I^{6m-2}] \subseteq (a^3I^{6m-2}] = (a^3I^{2m-1}I^{4m-1}] \subseteq (a^3(a \cup aSS]I^{4m-1}] \subseteq (a^4I^{4m-1}] = (a^4I^{2m-1}I^{2m}] \subseteq (a^4(a\cup aSS]I^{2m}] \subseteq (a^5I^{2m}] = (a^5I^{2m-1}I] \subseteq (a^5(a \cup aSS]I] \subseteq (a^6I] = ((a^3)^2I]$. Since $I \subseteq H(S_A)$, then $a^3 \subseteq A$ and so $(I^{2(6m-2)-1}] \subseteq ((a^3)^2I] \subseteq (A^2I] \subseteq (ASS] \subseteq (A] = A$. Hence $(I^{2(6m-2)-1}] \subseteq A$. Taking $m' = 6m-2$ we have $(I^{2m'-1}] \subseteq A$ for some $m' \in \mathbb{N}$ implies that $I^{2m'-1} \subseteq A$ that for some $m' \in \mathbb{N}$ which contradicts the fact that I is not A-nilpotent. Hence $a \in (I^{2n-1}]$ for all $n \in \mathbb{N}$ and so $a \in \bigcap_{n\in\mathbb{N}}(I^{2n-1}]$. Therefore, $\bigcap_{n\in\mathbb{N}}(I^{2n-1}]$ is a completely semiprime right ideal of S.

(2) Let us assume that $t \in S$ such that $t \in H(S_A)$ and t is not A-nilpotent, i.e. $t^{2n-1} \notin A$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, $(t^{2n}S]$ is a right ideal of S, then by Proposition 1.4.2 we have $\bigcap_{n\in\mathbb{N}}(t^{2n}S]$ is a right ideal of S. Let J be a right ideal of S such that $J^3 \subseteq \bigcap_{n\in\mathbb{N}}(t^{2n}S]$ but $J \nsubseteq \bigcap_{n\in\mathbb{N}}(t^{2n}S]$. Thus $J \nsubseteq (t^{2m}S]$ for some $m \in \mathbb{N}$. Since S is a right chain ordered ternary semigroup we must have $(t^{2m}S] \subseteq J$ where $m \in \mathbb{N}$. Now $t \in S \Longrightarrow t^{6m-3} \in S$. Hence

$$\begin{aligned}
t^{6m-3} &= t^{(2m-2)+(2m-2)+(2m-2)+3} \\
&= t^{(2m-2)}.t.t^{(2m-2)}.t.t^{(2m-2)}.t \\
&= t^{2(m-1)}.t.t^{2(m-1)}.t.t^{2(m-1)}.t \\
&\in J^3 \\
&\subseteq (t^{2(6m-2)}S] \\
&= (t^{2(6m-3)}t^2S] \\
&\subseteq (t^{2(6m-3)}(t^2S]] \\
&= ((t^{(6m-3)})^2(t^2S]]
\end{aligned}$$

Since $(t^2S]$ is a right ideal of S and $(t^2S] \subseteq H(S_A)$, by Theorem 6.2.3 we have

$t^{6m-3} \in A$ which implies that $t^{2(3m-1)-1} \in A$. Taking $3m-1 = m^{'}$ we get $t^{2m^{'}-1} \in A$ for some $m^{'} \in \mathbb{N}$ which contradicts the fact that t is not A-nilpotent element. Therefore, $J \subseteq \bigcap_{n\in\mathbb{N}}(t^{2n}S]$ and so $\bigcap_{n\in\mathbb{N}}(t^{2n}S]$ is a semiprime right ideal of S. $\square$

Next we have the following corollary which generalizes this proposition.

Corollary 6.2.7. *Let I be an ideal (resp. right ideal) of a right chain ordered ternary semigroup $(S, ., \leq)$ such that $(I^{2n-1}] \neq (I^{2n+1}]$ for any $n \in \mathbb{N}$. Then $\bigcap_{n\in\mathbb{N}}(I^{2n-1}]$ is a completely semiprime ideal (resp. right prime ideal) of S.*

Proof. Let I be an ideal of a right chain ordered ternary semigroup $(S, ., \leq)$. Then $(I^{2n-1}]$ are ideals of S for any $n \in \mathbb{N}$. Then by Proposition 1.4.2, $\bigcap_{n\in\mathbb{N}}(I^{2n-1}]$ are ideals of S. Let $A = \bigcap_{n\in\mathbb{N}}(I^{2n-1}]$. Let $s \in S \backslash A$, then $s \notin (I^{2m-1}]$ for some $m \in \mathbb{N}$. If $s \in (s^2I]$ then $(s^2I] \subseteq ((s^2I]sI] \subseteq (s^2IsI]$. So, $s \in (s^2IsI]$ then $(s^2I] \subseteq ((s^2IsI]sI] \subseteq (s^2IsIsI] \subseteq (s^2ISISI] \subseteq (s^2I^3]$ and continuing in this way we obatin $s \in (s^2I^{2m-1}] \subseteq (I^{2m-1}]$, a contradiction. Hence $s \notin (s^2I]$, then by Theorem 6.2.3 we say that $I \subseteq H(S_A)$. Next we have to show that I is not A-nilpotent. Suppose that $I^{2k-1} \subseteq A$ for some $k \in \mathbb{N}$. Then $(I^{2k-1}] \subseteq (A] = A$. So, $(I^{2k-1}] \subseteq A = \bigcap_{n\in\mathbb{N}}(I^{2k-1}] \subseteq (I^{2k+1}] \subseteq (I^{2k-1}]$ which implies that $(I^{2k-1}] = (I^{2k+1}]$, a contradiction. Therefore, I is not A-nilpotent. By Proposition 6.2.6 we can say that $\bigcap_{n\in\mathbb{N}}(I^{2n-1}]$ is a completely prime ideal of S. $\square$

Similarly, we can prove the corollary for right ideal.

6.3 Prime, semiprime, completely prime and completely semiprime ideals of right chain ordered ternary semigroups

The next proposition shows that for any right chain ordered ternary semigroup semiprime and prime right ideals are equivalent. Also on the other hand completely semiprime and completely prime ideals are equivalent.

Proposition 6.3.1. *If $(S, ., \leq)$ is a right chain ordered ternary semigroup, then we have the followings :*
(1) A right ideal I of S is semiprime ideal if and only if I is prime ideal.
(2) An ideal (resp. right ideal) I of S is completely semiprime ideal (resp. right ideal) if and only if I is completely prime ideal (resp. right ideal).

Proof. (1) It is obvious that if I is prime then I is semiprime. Asssume that I is a semiprime right ideal of S. Let A, B, C be right ideals of S such that $ABC \subseteq I$. Since S is right chain oredred ternary semigroup we must have $A \subseteq B$ or $B \subseteq A$, $B \subseteq C$ or $C \subseteq B$, $C \subseteq A$ or $A \subseteq C$. Then $A^3 \subseteq ABB \subseteq ABC \subseteq I$ which implies that $A \subseteq I$. Similarly, we have $B \subseteq I$ and $C \subseteq I$. Thus I is prime ideal of S.

(2) Let I be a completely semiprime ideal of S. Consider $a, b, c \in S$ such that $abc \in I$. For $a, b, c \in S$ $R(a), R(b), R(c)$ are the right ideals of S generated by a, b, c respectively. Now $a^3 \in R(a)R(a)R(a)$. Since S is right chain oredred ternary semigroup we have $a^3 \in R(a)R(b)R(c)$ i.e. $a^3 \in (a \cup aSS](b \cup bSS](c \cup cSS] \subseteq (abc \cup abcSS \cup abSSc \cup abSScSS \cup aSSbc \cup aSSbcSS \cup aSSbSSc \cup aSSbSScSS] \subseteq I$. Since I is completely semiprime ideal of S, we have $a \in I$. Similarly, $b \in I$ and $c \in I$. This I is completely prime ideal of S. The converse part is obvious.
Similarly, we can proof that A right ideal I of S is completely semiprime right ideal if and only if I is completely prime right ideal. □

Now we define *associated prime right ideal* in right chain ordered ternary semigroup.

Definition 6.3.2. *Let $(S, ., \leq)$ be an ordered ternary semigroup. For any proper ideal A of S we define the associated prime right ideal of S to be the set*

$$P(S_A) = \{p \in S : xyp \; for \; some \; x, y \in S \backslash A\}$$

Proposition 6.3.3. *Let A be proper ideal of a right chain ordered ternary semigroup $(S, ., \leq)$. Then $P(S_A)$ is a completely prime right ideals of S conatining A.*

Proof. Since A is a proper ideal of S. Hence there exists an element x in S such that $x \notin A$ i.e. $x \in S \backslash A$. Now for any $a \in A$ we have $x^2 a \in SSA \subseteq A$ it follows that $a \in P(S_A)$. Thus $A \subseteq P(S_A)$

Now we have to show that $P(S_A)$ is a right ideal of S i.e. $P(S_A)SS \subseteq P(S_A)$. Let $y \in P(S_A)SS$. Then $y = ps_1s_2$ for some $p \in P(S_A)$ and $s_1, s_2 \in S$. Hence there exist $u, v \in S \backslash A$ such that $uvp \in A$. Thus $(uvp)s_1s_2 \in ASS \subseteq A$. So, $uv(ps_1s_2) = (uvp)s_1s_2 \in A$ which implies that $ps_1s_2 \in P(S_A)$ which proves that $P(S_A)SS \subseteq P(S_A)$. Now we have to show that $(P(S_A)] = P(S_A)$. Let $q \in (P(S_A)]$ then there exists $p \in P(S_A)$ such that $q \leq p$ and $zwp \in A$ for some $z, w \in S \backslash A$. For $z, w \in S \backslash A$ we have $zwq \leq zwp \in A$. Thus $zwq \in (A] = A$. Hence $q \in P(S_A)$. Thus $(P(S_A)] \subseteq P(S_A)$ and so $(P(S_A)] = P(S_A)$. Therefore, $P(S_A)$ is right ideal.

To complete the proof it remains to show that the right ideal $P(S_A)$ is completely prime. Let $a, b, c \in S$ such that $abc \in P(S_A)$. So there exist $r, t \in S \backslash A$ such that $rt(abc) \in A \Longrightarrow (rta)bc \in A$. Now we have the following two cases.

Case 1: If $rta \in A$, then $a \in P(S_A)$.

Case 2: If $rta \notin A$ i.e. $rta \in S \backslash A$, then we have two posibilities either $b \in A$ or $b \notin A$.

- If $b \in A$ then $rtb \in SSA \subseteq A$. Thus $b \in P(S_A)$.
- If $b \notin A$. Then $(rta)bc \in A$ implies that $c \in A$ [since $rta \notin A$].

Therefore, $P(S_A)$ is a completely prime right ideal of S containing A. □

Proposition 6.3.4. *Let A be a prime right ideal of a right chain ordered ternary semigroup $(S, ., \leq)$. Then for any proper ideal I of S we have, either $I \subseteq A$ or $P(S_A) \subseteq I$.*

Proof. Let S be an ordered ternary semigroup and A be a prime right ideal of S. Let I be a proper ideal of S such that $P(S_A) \nsubseteq I$. Thus there exists an element $p \in P(S_A)$ such that $p \notin I$. Since $p \in P(S_A)$ there exists $x, y \in S \backslash A$ such that $xyp \in A$. Also $p \notin I \Longrightarrow (p \cup pSS] \nsubseteq I$. Since S is a right chain ordered ternary semigroup we have $I \subseteq (p \cup pSS]$. Let us assume that $A \subseteq I$. Now for $y \in S \backslash A$,

$(SSyI^2]SS \subseteq (SSyI^2](S](S] \subseteq (SSyI^2SS] \subseteq (SSyI^2]$. Thus $(SSyI^2]$ is a right ideal of S. If $y \in (SSyI^2] \subseteq (SSSI^2] \subseteq (I] = I$. Then $S\backslash A \subseteq I$. Thus $(S\backslash A) \cup A \subseteq I$ and so $I \subseteq S$, which is a contradiction. Thus $I \subseteq A$. If $y \notin (SSyI^2]$. Since $(SSyI^2]$ is a right ideal of S and S is a right chain ordered ternary semigroup we have $(SSyI^2] \subseteq (y \cup ySS]$. Now,

$$\begin{aligned}
(x \cup xSS](y \cup ySS]I^3 &= xyI^3 \cup xySSI^3 \cup xSSyI^3 \cup xSSySSI^3 \\
&\subseteq xyI^3 \cup xyI^3 \cup xSSyI^3 \cup xSSyI^3 \\
&= xyI^3 \cup xSSyI^3 \\
&\subseteq xyI \cup x(y \cup ySS]I \\
&= xyI \cup xyI \cup xySSI \\
&\subseteq xyI \\
&\subseteq xy(p \cup pSS] \\
&= (xyp \cup xypSS] \\
&\in (A \cup ASS] \\
&\subseteq (A] = A
\end{aligned}$$

Since $x, y \notin A$ and A is a prime right ideal of S, we have $I^3 \subseteq A$ and so $I \subseteq A$. $\square$

Lemma 6.3.5. *If A is a proper ideal of a right chain ordered ternary semigroup $(S, ., \leq)$ such that $A = (A^3]$, then $A = (s^{2n}A]$ for any $s \in S\backslash A$ and $n \in \mathbb{N}$.*

Proof. Let A be a proper ideal of a right chain ordered ternary semigroup and $s \in S\backslash A$. i.e. $s \notin A$. Thus $R(s) = (s \cup sSS] \nsubseteq A$. Since S is a right chain ordered ternary semigroup we must have $A \subseteq (s \cup sSS]$. Hence $A = (A^3] = (AAA] \subseteq ((s \cup sSS]AA] \subseteq (sAA \cup sSSAA] \subseteq (sAA \cup sAA] \subseteq (sAA] \subseteq (s(s \cup sAA]A] \subseteq (s^2A \cup s^2SSA] \subseteq (s^2A \cup s^2A] \subseteq (s^2A] \subseteq (SSA] \subseteq (A] = A$. Thus we get $A = (s^2A]$. So, it is true for $n = 1$. Assume that it is true for $n = k$. Hence $A = (s^{2k}A] = (s^{2k}(s^2A]] = (s^{2k+1}A] = (s^{2(k+1)}A]$. Hence the result is true for $n = k + 1$. Thus the result follows by induction. □

Next we define *exceptional prime ideal* and *minimal ideal* of a right chain ordered ternary semigroup.

Definition 6.3.6. *Let $(S, ., \leq)$ be a right chain ordered ternary semigroup. An exceptional prime ideal Q of S is an ideal which is prime but not completely prime ideal of S. Similarly, exceptional right, left and lateral prime ideals are defined.*

Definition 6.3.7. *If $I \subset J$ are ideals (resp. right ideals, left ideals and lateral ideals) of a right chain ordered ternary semigroup S such that there are no further ideals (resp. right ideals, left ideals and lateral ideals) properly between I and J, then we say that J is minimal over I.*

Proposition 6.3.8. *let Q be an exceptional prime ideal of a right chain ordered ternary semigroup $(S, ., \leq)$. Then there exists a unique ideal R of S such that $Q \subset R$ and R is minimal over Q. Moreover, $R = (R^3]$ i.e. R is an idempotent ideal of S.*

Proof. Let $R = \bigcap_{Q \subseteq I} I$, i.e. R denote the intersection of all ideals I of S such that $Q \subseteq I$ which implies that $Q \subseteq R$. Since Q is exceptional prime ideal, then Q is prime ideal of S. Then by Proposition 6.3.4 we have either $P(S_Q) \subseteq I$ or $I \subseteq Q$, for any such ideal I of S. Since $Q \subseteq I$, then $I \nsubseteq Q$ for any such ideal I. Then $P(S_Q) \subseteq I$ for any ideal I such that $Q \subseteq I$. Thus $P(S_Q) \subseteq \bigcap_{Q \subseteq I} I = R$. By Proposition 6.3.3 $P(S_Q)$ is completely prime ideal of S containing Q. Thus $Q \subseteq P(S_Q)$. But since Q is exceptional prime ideal $Q \neq P(S_Q)$ and so $Q \subset P(S_Q)$. Hence $Q \subset R$. By Proposition 1.4.2, $R = \bigcap_{Q \subseteq I} I$ is an ideal of S and R is the smallest ideal containing Q. Thus we can say that R is minimal over Q.
For the second part, let $R \neq (R^3]$. Then $(R^3] \subseteq (SSR] \subseteq (R] = R$. So, $R^3 \subseteq R$. Since R is minimal over Q, $R^3 \subseteq Q$ and since Q is a prime ideal of S, $R^3 \subset Q \Longrightarrow R \subseteq Q$, which contradicts the fact that $Q \subset R$.
Therefore, $R = (R^3]$ i.e. R is an idempotent ideal of S. □

Proposition 6.3.9. *let Q be an exceptional prime ideal of a right chain ordered ternary semigroup $(S, ., \leq)$ and R be the unique ideal of S such that R is minimal over Q and $R = (R^3]$. Then there exists an element $a \in R\backslash Q$ such that $Q \subseteq \bigcap_{n\in\mathbb{N}}(a^{2n}S]$. In particular, if $a \in R\backslash Q$ then a is not Q-nilpotent.*

Proof. Let us consider the set $X = \{x \in S : \bigcap_{n\in\mathbb{N}}(x^{2n}R] \subseteq Q\}$, where $R = (R^3]$ i.e. R is the unique idempotent ideal minimal over Q.
Let $q \in Q$. Then $(q^2R] \subseteq (q^2S] \subseteq (QSS] \subseteq (Q] = Q$ and $(q^4R] = (q^2(q^2R]] \subseteq (q^2Q] \subseteq (SSQ] \subseteq (Q] = Q$. Again $(q^6R] = (q^2(q^4R]] \subseteq (q^2Q] \subseteq (SSQ] \subseteq (Q] = Q$ and continuing in this way we get $(q^{2n}R] \subseteq Q$ for all $n \in \mathbb{N}$. Thus $Q \subseteq X$ and thus the set X is non empty.
We claim that $X \subseteq R$. If $s \notin R$, then $s \in S\backslash R$. Since R is an idempotent ideal of S i.e. $R = (R^3]$, then by Lemma 6.3.5 we can say that $R \nsubseteq Q \Longrightarrow (s^{2n}R] = R$ for all $n \in \mathbb{N}$. So, $\bigcap_{n\in\mathbb{N}}(s^{2n}R] \nsubseteq Q$ which implies that $s \notin X$. By contraposition we have $X \subseteq R$ which proves our claim. Since Q is exceptional prime ideal of S, then Q is prime but not completely prime and by Proposition 6.3.1 it is not completely semiprime. Then there exists $y \in S$ such that $y^3 \in Q$ but $y \notin Q$, i.e. $y \in S\backslash Q$. If $y \in (x^2yRR]$ for some $x \in X$. Then $(y] \subseteq ((x^2yRR]] = x^2yRR \subseteq (x^2yRR] \subseteq (x^2(x^2yRR]RR] = (x^4yRRRR] \subseteq (x^4yRR] \subseteq (x^4(x^2yRR]RR] = (x^6yRRRR] \subseteq (x^6yRR] \subseteq .... \subseteq (x^8yRR] \subseteq .... \subseteq (x^{10}yRR] \subseteq ....(x^{12}yRR] \subseteq ....$

Continuing in this way we obtain $(y] \subseteq (x^{2m}yRR]$ for arbitrary $m \in \mathbb{N}$. So, $(y] \subseteq (x^{2m}(x^2yRR]RR] = (x^{2m+2}yRRRR] \subseteq (x^{2m+2}yRR] = (x^{2(m+1)}yRR]$ for $m + 1 \in \mathbb{N}$. Thus it is true for all $n \in \mathbb{N}$. Hence $(y] \subseteq \bigcap_{n\in\mathbb{N}}(x^{2n}yRR]$ for all $n \in \mathbb{N}$. Now $y \in \bigcap_{n\in\mathbb{N}}(x^{2n}yRR] \subseteq \bigcap_{n\in\mathbb{N}}(x^{2n}R] \subseteq Q$, which is a contradiction. So, $y \notin (x^2yRR]$. Hence $(y] \nsubseteq (x^2yRR] \Longrightarrow (y] \cup (ySS] \nsubseteq (x^2yRR] \Longrightarrow (y \cup ySS] \nsubseteq (x^2yRR]$. Since S is right chain ordered ternary semigroup we have $(x^2yRR] \subseteq (y \cup ySS] \Longrightarrow (X^2yRR] \subseteq (y \cup ySS]$.
If $X = R$, then $(R^2yRR] \subseteq (y \cup ySS]$. Thus we have $((y \cup ySS]RR]^3$ is a right ideal of S. Then $((y \cup ySS]RR]^3 = (yRR \cup ySSRR]^3 \subseteq (yRR]^3 = (yRR](yRR](yRR] \subseteq (yRRyRRyRR] \subseteq (y(y \cup ySS]yRR] = (y^3RR \cup y^2SSyRR] \subseteq (y^2SRR \cup y^2SSSRR]$

$\subseteq (y^2SSR \cup y^2SSSSR] \subseteq (y^2R]$. Let $Y = ((y \cup ySS]RR]^3$. Then Y is also a right ideal of S. Now we have,

$$
\begin{aligned}
Y^3 &\subseteq (y^2R]^3 \\
&= (y^2R](y^2R](y^2R] \\
&\subseteq (y^2Ry^2Ry^2R] \\
&= (y^2(R]^3y^2(R]^3y^2R] \\
&= (y^2RRRy^2RRRy^2R] \\
&\subseteq (y^2RRSSySSRSSR] \\
&\subseteq (y^2RRSSySSRSSR] \\
&\subseteq (y^2RRyRR] \\
&\subseteq (y^2(y \cup ySS]] \\
&= (y^3 \cup y^3SS] \\
&\subseteq (Q] = Q.
\end{aligned}
$$

Since Q is a prime ideal we have $Y \subseteq Q$ i.e. $((y \cup ySS]RR]^3 \subseteq Q \implies ((y \cup ySS]RR] \subseteq Q \implies (y \cup ySS]RR \subseteq Q$. Again since Q is a prime ideal we have either $(y \cup ySS] \subseteq Q$ or $R \subseteq Q$. But $R \subseteq Q$ contradicts the fact that $Q \subset R$. Again $(y \cup ySS] \subseteq Q \implies y \in Q$ which is also a contradiction. Thus $X \neq R$ and we must have $X \subset R$.

To complete the proof, take any $a \in R \backslash X \subseteq R \backslash Q$, then $a \in R$ but $a \notin X$. So, $\bigcap_{n \in \mathbb{N}}(a^{2n}R] \nsubseteq Q$. Since $(a^{2n}R]$ is a right ideal of S for all $n \in \mathbb{N}$ by Proposition 1.4.2, $\bigcap_{n \in \mathbb{N}}(a^{2n}R]$ is also a right ideal of S. Since S is a right chain ordered ternary semigroup , we must have $Q \subseteq \bigcap_{n \in \mathbb{N}}(a^{2n}R] \subseteq \bigcap_{n \in \mathbb{N}}(a^{2n}S]$.
Again $\bigcap_{n \in \mathbb{N}}(a^{2n}R] \nsubseteq Q \implies (a^{2n}R] \nsubseteq Q$ for all $n \in \mathbb{N} \implies a^{2n}a \notin Q$ for all $n \in \mathbb{N} \implies a^{2n+1} \notin Q$ for all $n \in \mathbb{N}$. Also $a \notin X$ and $Q \subseteq X \implies a \in Q$. Thus $a \notin Q$ and $a^{2n+1} \notin Q$ implies that $a^{2n-1} \notin Q$ for all $n \in \mathbb{N}$. Hence for any element $a \in R \backslash X \subseteq R \backslash Q$, we have a is not Q-nilpotent.

This completes the proof. □

6.4 Prime right segments of right chain ordered ternary semigroup

Following [31], we define a prime (resp. prime left, prime right, prime lateral) segment of a right chain ordered ternary semigroup $(S, ., \leq)$.

Definition 6.4.1. *Let $(S, ., \leq)$ be a right chain ordered ternary semigroup.*
(a) A prime segment of S is a pair (P_1, P_2) of completely prime ideals of S such that $P_1 \subset P_2$ and there are no further completely prime ideal of S exists between P_1 and P_2.
(b) A prime right segment of S is a pair (R_1, R_2) of completely prime right ideals of S such that $R_1 \subset R_2$ and there are no further completely prime right ideal of S exists between R_1 and R_2.
(c) A prime left segment of S is a pair (L_1, L_2) of completely prime left ideals of S such that $L_1 \subset L_2$ and there are no further completely prime left ideal of S exists between L_1 and L_2.
(d) A prime lateral segment of S is a pair (M_1, M_2) of completely prime lateral ideals of S such that $M_1 \subset M_2$ and there are no further completely prime lateral ideal of S exists between M_1 and M_2.

Next we show that for a prime right segment of a right chain ordered ternary semigroup four different posibilities may happen.

Definition 6.4.2. *Let $(S, ., \leq)$ be a right chain ordered ternary semigroup, and let (R_1, R_2) be a right prime segment of S. The right prime segment is called simple if there are no further right ideals of S between R_1 and R_2. i.e. there are no further ideals of S between R_1 and R_2.*

Definition 6.4.3. *Let $(S, ., \leq)$ be a right chain ordered ternary semigroup, and let (R_1, R_2) be a right prime segment of S. The right prime segment is called* *archimedean if for every $a \in R_2 \backslash R_1$ there exists a right ideal $I \subseteq R_2$ of S such that $a \in I$ and $\bigcap_{n \in \mathbb{N}} (I^{2n-1}] = R_1$.*

Definition 6.4.4. *Let $(S, ., \leq)$ be a right chain ordered ternary semigroup, and let (R_1, R_2) be a right prime segment of S. The right prime segment is called exceptional if there exists a prime right ideal Q of S with $R_1 \subset Q \subset R_2$.*

Definition 6.4.5. *Let $(S, ., \leq)$ be a right chain ordered ternary semigroup, and let (R_1, R_2) be a right prime segment of S. The right prime segment is called supplementary if there exists a right ideal D of S such that $R_1 \subset D \subset R_2$ and D is minimal over R_1.*

Next we have the following theorem for prime right segments:

Theorem 6.4.6. *Let $(S, ., \leq)$ be a right chain ordered ternary semigroup, and (R_1, R_2) be a prime right segment of S. Then the prime right segment (R_1, R_2) is either simple or archimedean or exceptional or supplementary.*

Proof. Let (R_1, R_2) be a prime right segment of a right chain ordered ternary semigroup S. Then prime right segment (R_1, R_2) is either simple or not simple. If (R_1, R_2) is simple, then our aim is done. Let us suppose that the prime right segment is not simple. Then there exists an ideal I of S such that $R_1 \subset I \subset R_2$.
Case 1: First assume that $R_2 \nsubseteq H(S_I)$. Then $H(S_I) \subset R_2$, since S is a right chain ordered semigroup. So, by Theorem 6.2.4 we have $H(S_I)$ is a semiprime right ideal of S and $I \subseteq H(S_I)$. Since S is a right chain ordered ternary semigroup by Corollary 6.3.1, $H(S_I)$ is a prime right ideal of S and $R_1 \subset I \subseteq H(S_I) \subset R_2$. So, $H(S_I)$ is a prime right ideal lying properly between R_1 and R_2. Thus the prime right segment is exceptional in this case.
Case 2: Next we assume that $R_2 \subseteq H(S_I)$. Here we have two cases.

Subcase 2a: First consider the case where the prime right segment (R_1, R_2)

contains the ideal I of S such that $(I^{2m-1}] = (I^{2m+1}]$ for some $m \in \mathbb{N}$. Then

$$(I^{2m+1}] = (I^{2m-1+2}] = (I^{2m-1}I^2] = ((I^{2m-1}]I^2] = ((I^{2m+1}]I^2] = (I^{2m+3}]$$
$$(I^{2m+3}] = (I^{2m-1+4}] = (I^{2m-1}I^4] = ((I^{2m-1}]I^4] = ((I^{2m+1}]I^4] = (I^{2m+5}]$$
$$(I^{2m+5}] = (I^{2m-1+6}] = (I^{2m-1}I^6] = ((I^{2m-1}]I^6] = ((I^{2m+1}]I^6] = (I^{2m+7}]$$

....

So continuing in this way we get $(I^{2m-1}] = (I^{2m+1}] = (I^{2m+3}] = \ldots\ldots\ldots = (I^{2m+(2n-1)}]$ for all $n \in \mathbb{N}$ i.e. $(I^{2m-1}] = (I^{2m-1+2}] = (I^{2m-1+4}] = \ldots\ldots\ldots\ldots = (I^{2m-1+2n}]$ for all $n \in \mathbb{N}$. So, $(I^{2m-1}] = (I^{2m-1+2k}]$ for all $k \in \mathbb{N}$. Let $D = (I^{2m-1}]$. Then D is a right ideal of S. Thus we have $D = (I^{2m-1}] = (I^{(2m-1)(2n-1)}] = ((I^{2m-1}]^{(2n-1)}] = (D^{2n-1}]$ for all $n \in \mathbb{N}$ and $D = (I^{2m-1}] \subseteq (I] = I \subseteq R_2$. Now if $D = (I^{2m-1}] \subseteq R_1$, since R_1 is completely prime hence prime we would get $I \subseteq R_1$, which is a contradiction. Thus $D \nsubseteq R_1$ and so $R_1 \subset D$, since S is a right chain ordered ternary semigroup. Next we show that D is minimal over R_1. Let us suppose that D is not minimal over R_1, then there exists a right ideal A of S such that $R_1 \subset A \subset D$. Then $R_1 \subset A \subset R_2$. Again $A \subset D$ implies that $D \nsubseteq A$ and so $D^{2n-1} \nsubseteq A$ for all $n \in \mathbb{N}$. Thus D is not A-nilpotent. Hence by Proposition 6.2.6, $\bigcap_{n\in\mathbb{N}}(D^{2n-1}] = D$ is a completely prime right ideal of S and so D is a completely prime right ideal of S which contradicts the fact that (R_1, R_2) is prime right segment. Hence D is minimal over R_1 and thus the prime right segment is supplementary in this case.

<u>Subcase 2b</u>: Next consider the case where $(I^{2n-1}] \neq (I^{2n+1}]$ for all $n \in \mathbb{N}$. Since $R_1 \subset I$ then $I \nsubseteq R_1$. Now for all $n \in \mathbb{N}$, $I^{2n-1} \subseteq R_1 \Longrightarrow I \subseteq R_1$ since R_1 is completely prime right ideal of S. So, $I^{2n-1} \nsubseteq R_1$. Thus we have $R_1 \subseteq I^{2n-1}$ for all $n \in \mathbb{N}$, since S is a right chain ordered ternary semigroup and thus $R_1 \subseteq (I^{2n-1}] \subset R_2$ for all $n \in \mathbb{N} \Longrightarrow R_1 \subseteq \bigcap_{n\in\mathbb{N}}(I^{2n-1}] \subset R_2$. By Corollary 6.2.7, the ideal $\bigcap_{n\in\mathbb{N}}(I^{2n-1}]$ is completely prime right ideal of S. Since (R_1, R_2) is a prime right segment, there are no further completely prime right ideals between R_1 and R_2. Hence $\bigcap_{n\in\mathbb{N}}(I^{2n-1}] = R_1$.

Let us consider the set $\mathcal{I}$ be the collection of all ideals I_k of S such that $R_1 \subset I_k \subset R_2$, where $k \in \Lambda$ and Λ is an index set.

$$\mathcal{I} = \{I_k \subseteq S : \ I_k \ is \ an \ ideal \ of \ S \ and \ R_1 \subset I_k \subset R_2, \ k \in \Lambda\}$$

Then for any ideal $I_k \in \mathcal{I}$, we have $\bigcap_{n\in\mathbb{N}}({I_k}^{2n-1}] = R_1$. Let us suppose $X = \bigcup_{I\in\mathcal{I}} I$ then by Proposition 1.4.2 X is also a right ideal of S. If $X = R_2$ then $a \in R_2 \backslash R_1 = R_2 \backslash X$ there is an ideal I_k such that $a \in I_k$ where $I_k \subseteq \bigcup_{I_k\in\mathcal{I}} I = X = R_2$ and $\bigcap_{n\in\mathbb{N}}({I_k}^{2n-1}] = R_1$. Thus in this case the prime right segment (R_1, R_2) is archimedean.

If $X \neq R_2$ then $X \subseteq R_2$. Then we have the following two cases:

Subcase 2b(i): $(R_2^3] \neq R_2$. Then $(R_2^3] \subset R_2$. If $(R_2^3] \subseteq R_1$, then since R_1 is completely prime right ideals of S we have $R_2 \subset R_1$ which is a contradiction. Thus $(R_2^3] \nsubseteq R_1$. Since S is a right chain ordered ternary semigroup, $R_1 \subset (R_2^3]$ and so $R_1 \subset (R_2^3] \subset R_2$. Thus $(R_2^3] \subset \mathcal{I}$. Then $\bigcap_{n\in\mathbb{N}}((R_2^3]^{2n-1}] = R_1 \Longrightarrow \bigcap_{n\in\mathbb{N}}(R_2^{3(2n-1)}] = R_1$. Again $R_1 \subset R_2 \Longrightarrow R_2 \nsubseteq R_1$. Thus for any $n \in \mathbb{N}$, $(R_2^{2n-1}] \subseteq R_1$ implies that $R_2 \subseteq R_1$, since R_1 is completely prime right ideal. Hence $(R_2^{2n-1}] \nsubseteq R_1$ for all $n \in \mathbb{N}$. Thus $R_1 \subseteq \bigcap_{n\in\mathbb{N}}(R_2^{2n-1}] \subseteq \bigcap_{n\in\mathbb{N}}(R_2^{3(2n-1)}] = R_1$. Hence $R_1 = \bigcap_{n\in\mathbb{N}}(R_2^{2n-1}]$. Thus for every $a \in R_2 \backslash R_1$ we have $\bigcap_{n\in\mathbb{N}}(R_2^{2n-1}] = R_1$. Therefore, the prime segment is archimedean in this case.

Subcase 2b(ii): $(R_2^3] = R_2$. We show that the ideal X is prime and our aim is done. Let A be a right ideal of S such that $A^3 \subseteq X \subset R_2$, which implies that $A \subseteq R_2$ since R_2 is completely prime. If $A = R_2$, then $R_2 = (R_2^3] = (A^3] \subseteq (X] = X$, which contradicts the fact that $X \subset R_2$. Hence $R_2 \neq A$ and so $A \subset R_2$. Now $(R_2^3] \subseteq R_1$ imples that $R_2^3 \subseteq R_1 \Longrightarrow R_2 \subseteq R_1$, which is a contradiction. Thus $R_1 \subset (R_2^3] \subseteq (A^3] \subseteq (ASS] \subseteq (A] = A$. Hence $R_1 \subset A \subset R_2$ and so $A \subseteq \bigcup_{I\in\mathcal{I}} I = X$. Thus X is prime right ideal properly lying between R_1 and R_2. Therefore, the prime right segment is exceptional in this case. $\square$

We explain the proof of the above theorem by the following chart:

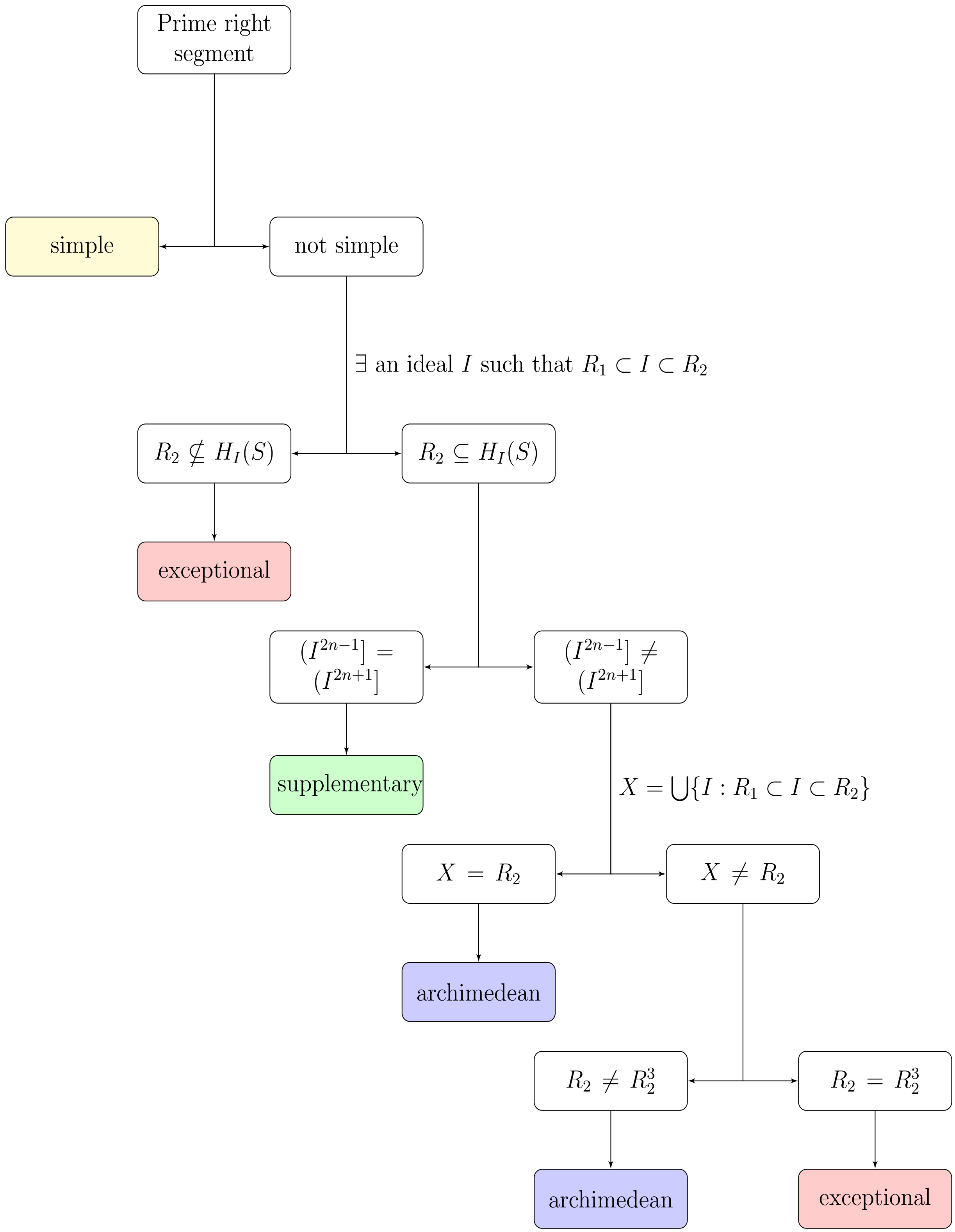


Figure 6.1: Chart

In the next corollary we characterize archimedian prime right segements of right chain ordered ternary semigroups.

Corollary 6.4.7. *Let (R_1, R_2) be a prime right segment of a right chain ordered ternary semigroup $(S, ., \leq)$. Then the following conditions are equivalent:*
(i) The prime right segment (R_1, R_2) is archimedean.
(ii) For any $a \in R_2 \backslash R_1$, $\bigcap_{n\in\mathbb{N}}(a^{2n}S] = R_1$.
(iii) For any $a \in R_2 \backslash R_1$, $(R_2aR_2] \subset (a \cup aSS]$.

Proof. (i) $\Longrightarrow$ (ii) Let us consider the prime right segment (R_1, R_2) is archimedian. Then for every $a \in R_2 \backslash R_1$ there exists an right ideal $I \subseteq R_2$ of S such that $a \in I$ and $\bigcap_{n\in\mathbb{N}}(I^{2n-1}] = R_1$.

$$\begin{aligned}
&\text{Let } x \in \bigcap_{n\in\mathbb{N}}(a^{2n}S] \\
&\Longrightarrow x \in (a^{2n}S] \text{ for all } n \in \mathbb{N} \\
&\Longrightarrow x \in (I^{2n}S] \text{ for all } n \in \mathbb{N} \\
&\Longrightarrow x \in (I^{2n-2}IIS] \text{ for all } n \in \mathbb{N} \\
&\Longrightarrow x \in (I^{2n-2}ISS] \text{ for all } n \in \mathbb{N} \\
&\Longrightarrow x \in (I^{2n-1}] \text{ for all } n \in \mathbb{N} \\
&\Longrightarrow x \in \bigcap_{n\in\mathbb{N}}(I^{2n-1}] = R_1.
\end{aligned}$$

Thus $\bigcap_{n\in\mathbb{N}}(a^{2n}S] \subseteq R_1$. Again, for any $n \in \mathbb{N}$, $(a^{2n}R_2] \subseteq R_1 \Longrightarrow a^{2n}a \in R_1 \Longrightarrow a^{2n+1} \in R_1 \Longrightarrow a \in R_1$ [since R_1 is completely prime]. This is a contradiction. Thus $(a^{2n}R_2] \nsubseteq R_1$ and for any $n \in \mathbb{N}$. Then $R_1 \subseteq (a^{2n}R_2] \subseteq (a^{2n}S]$ for all $n \in \mathbb{N}$. Therefore, $R_1 \subseteq \bigcap_{n\in\mathbb{N}}(a^{2n}S]$ and hence $\bigcap_{n\in\mathbb{N}}(a^{2n}S] = R_1$.
(ii) $\Longrightarrow$ (iii) Let us suppose for any $a \in R_2 \backslash R_1$ and $(a \cup aSS] \subseteq (R_2aR_2]$. Then $a \in (R_2aR_2]$ which implies that $a \leq raq$ for some $r, q \in R_2$. If either r or q in R_1 then $a \leq raq \in R_1SR_1 \subseteq R_1SS \subseteq R_1 \Longrightarrow a \in (R_1] = R_1$, which is a contradiction. Thus $r, q \notin R_1$ and so $r, q \in R_2 \backslash R_1$. Moreover, $a \leq raq$ implies that

$a \leq r(raq)q = r^2aq^2 \leq r^2(raq)q^2 = r^3aq^3 \leq r^3(raq)q^3 = r^4aq^4 \leq r^4(raq)q^4 = ...$ continuing in this way we get $a \leq r^naq^n$ for all $n \in \mathbb{N}$. Thus $a \leq r^{2n}aq^{2n}$ for all $n \in \mathbb{N}$. So, $a \in (r^{2n}SS^{2n}] = (r^{2n}S^{2n+1}] = (r^{2n}S]$ for all $n \in \mathbb{N}$. Thus $a \in \bigcap_{n\in\mathbb{N}}(r^{2n}S]$. By (ii) $\bigcap_{n\in\mathbb{N}}(r^{2n}S] = R_1$. which implies that $a \in R_1$, which is not possible. Hence $(a \cup aSS] \nsubseteq (R_2aR_2]$ and so $(R_2aR_2] \subset (a \cup aSS]$.

(iii) $\Longrightarrow$ (i) Assume that (iii) holds. Then for any $a \in R_2 \backslash R_1$, $(R_2aR_2] \subseteq R_1 \Longrightarrow a^3 \in R_1 \Longrightarrow a \in R_1$ [since R_1 is completely prime ideal of S]. Thus for any $a \in R_2 \backslash R_1$, we have $R_1 \subseteq (R_2aR_2] \subseteq (a \cup aSS] \subseteq (R_2 \cup R_2SS] \subseteq (R_2] = R_2$, and thus the prime segment (R_1, R_2) is not simple.

Suppose the prime segment (R_1, R_2) is exceptional, i.e. there exists a prime right ideal Q of S such that $R_1 \subset Q \subset R_2$. Then by Proposition 6.3.8 there exists an ideal D of S which is minimal over Q. Now (iii) implies that for any $a \in D \backslash Q$ we have $Q \subseteq (R_2aR_2] \subseteq (a \cup aSS] \subseteq D$, which contradicts the fact that R is minimal over Q. Thus the prime right segment (R_1, R_2) is not exceptional.

Next, suppose the prime segment (R_1, R_2) is supplementary. Then there exists a right ideal K of S such that $R_1 \subset K \subset R_2$ and K is minimal over R_1. Then by (iii), for any $a \in K \backslash R_1$ we have $R_1 \subseteq (R_2aR_1] \subseteq (a \cup aSS] \subseteq K$. Then there exists right ideals of S properly lying between K and R_1, which is a contradiction. Hence the prime segment (R_1, R_2) is neither simple, nor exceptional, nor supplementary, and thus by Theorem 6.4.6 it must be archimedean. This completes our proof. $\square$

(n, m, l)-Ideals In Ordered Ternary Semigroup

Chapter-7

Chapter 7

(n, m, l)-ideals in ordered ternary semigroup

7.1 Introduction

In this chapter we have introduced the concept of (n, m, l)-ideal in ordered ternary semigroup and study properties of (n, m, l)-ideal in different classes of ordered ternary semigroups. Let l, m, n be non-negetive odd integers. A ternary subsemigroup A of an ordered ternary semigroup S is called an (n, m, l)-ideal of S if it satisfies the following conditions: (i) $A^n S A^m S A^l \subseteq A$ (ii) $(A] = A$ i.e. for $y \in S$ and $x \in A$, $y \leq x \Rightarrow y \in A$.

Throughout this chapter, S denotes an ordered ternary semigroup.

7.2 Characterization of (n, m, l)-ideal in ordered ternary semigroup

Theorem 7.2.1. *Let S be an ordered ternary semigroup and A be an (n, m, l)-ideal of S. Then for any ternary subsemigroup T of S, $(A \cap T]$ is an (n, m, l)-ideal of T,* *where l, m, n are non-negetive odd integers.*

Proof. Since A is an (n, m, l)-ideal of S, A is a ternary subsemigroup of S. Now $(A \cap T]^3 = (A \cap T](A \cap T](A \cap T] \subseteq (A^3 \cap T^3] \subseteq (A \cap T]$. Again let $x \in (A \cap T]^n T(A \cap T]^m T(A \cap T]^l$. Thus $x = a^n sb^m tc^l$ for some $a, b, c \in (A \cap T] \subseteq (A]$ and $s, t \in T \subseteq S$. Hence $x = a^n sb^m tc^l \in (A]^n S(A]^m S(A]^l \subseteq A^n SA^m SA^l \subseteq A$. On the other hand, $x = a^n sb^m tc^l \in (T]^n T(T]^m T(T]^l \subseteq (T^n TT^m TT^l] \subseteq (T]$. Hence $x \in A \cap (T] \subseteq (A] \cap (T] \subseteq (A \cap T]$. Also, $((A \cap T]] = (A \cap T]$. Hence $(A \cap T]$ is an (n, m, l)-ideal of T. □

Theorem 7.2.2. *The non-empty intersection of any collection of (n, m, l)-ideals of an ordered ternary semigroup S is an (n, m, l)-ideal of S where l, m, n are non-negetive odd integers.*

Proof. Let S be an ordered ternary semigroup and $\{A_\alpha : \alpha \in \Delta\}$ be the collection of (n, m, l)-ideals of S. Suppose $\bigcap\limits_{\alpha\in\Delta} A_\alpha \neq \{\}$. Then $(\bigcap\limits_{\alpha\in\Delta} A_\alpha) \subseteq A_\beta$ for all $\beta \in \Delta$. So $(\bigcap\limits_{\alpha\in\Delta} A_\alpha)^3 = (\bigcap\limits_{\alpha\in\Delta} A_\alpha)(\bigcap\limits_{\alpha\in\Delta} A_\alpha)(\bigcap\limits_{\alpha\in\Delta} A_\alpha) \subseteq A_\beta A_\beta A_\beta = {A_\beta}^3 \subseteq A_\beta$.

Thus $\bigcap\limits_{\alpha\in\Delta} A_\alpha$ is a subsemigroup of S. Again for all $\gamma \in \Delta$ we have,

$(\bigcap\limits_{\alpha\in\Delta} A_\alpha)^n S(\bigcap\limits_{\alpha\in\Delta} A_\alpha)^m S(\bigcap\limits_{\alpha\in\Delta} A_\alpha)^l \subseteq A_\gamma SA_\gamma SA_\gamma \subseteq A_\gamma$.

$\Longrightarrow (\bigcap\limits_{\alpha\in\Delta} A_\alpha)^n S(\bigcap\limits_{\alpha\in\Delta} A_\alpha)^m S(\bigcap\limits_{\alpha\in\Delta} A_\alpha)^l \subseteq \bigcap\limits_{\alpha\in\Delta} A_\alpha$.

Also $(\bigcap\limits_{\alpha\in\Delta} A_\alpha] \subseteq \bigcap\limits_{\alpha\in\Delta} (A_\alpha] = \bigcap\limits_{\alpha\in\Delta} A_\alpha \subseteq (\bigcap\limits_{\alpha\in\Delta} A_\alpha]$.

Therefore, the non-empty intersection $(\bigcap\limits_{\alpha\in\Delta} A_\alpha]$ of any collection $\{A_\alpha : \alpha \in \Delta\}$ of (n, m, l)-ideals of an ordered ternary semigroup S is an (n, m, l)-ideal of S. □

Theorem 7.2.3. *Let $(S, ., \leq)$ be an ordered ternary semigroup and $a \in S$. Then the intersection of all (n, m, l)-ideals of S containing a is an (n, m, l)-ideal of S denoted by $[a]_{(n,m,l)}$ and it is of the form*

$$[a]_{(n,m,l)} = (\bigcup_{i=2k-1, k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^n Sa^m Sa^l]$$

where l, m, n are non-negetive odd integers.

Proof. Suppose $\{A_i : i \in I\}$ be the set of all (n,m,l)-ideals of S containing a. Then $\bigcap_{i\in I} A_i$ is non-empty since $a \in \bigcap_{i\in I} A_i$ and by Theorem 7.2.2 we have $\bigcap_{i\in I} A_i$ is an (n,m,l)-ideal of S. Thus $[a]_{(n,m,l)} = \bigcap_{i\in I} A_i$.

$$\begin{aligned}
\text{Now, } & \big(\big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l}\{a^i\} \cup a^n S a^m S a^l\big]\big)^n S \\
&= \big(\big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l}\{a^i\} \cup a^n S a^m S a^l\big]\big)^{n-1}\big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l}\{a^i\} \cup a^n S a^m S a^l\big] S \\
&\subseteq \big(\big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l}\{a^i\} \cup a^n S a^m S a^l\big]\big)^{n-1}(aS] \\
&\subseteq \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\subseteq \big(\big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l}\{a^i\} \cup a^n S a^m S a^l\big]\big)^{n-n}(a^n S] \\
&= (a^n S]
\end{aligned}$$

Similarly, $\big(\big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l}\{a^i\} \cup a^n S a^m S a^l\big]\big)^m S \subseteq (a^m S]$.

So, $\big(\big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l}\{a^i\} \cup a^n S a^m S a^l\big]\big)^n S\big(\big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l}\{a^i\} \cup a^n S a^m S a^l\big]\big)^m S\big(\big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l}\{a^i\} \cup a^n S a^m S a^l\big]\big)^l \subseteq (a^n S](a^m S](a^l] \subseteq (a^n S a^m S a^l] \subseteq \big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l}\{a^i\} \cup a^n S a^m S a^l\big]$.

Thus we have $\big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l}\{a^i\} \cup a^n S a^m S a^l\big]$ is an (n,m,l)-ideal of S containing a and hence $[a]_{(n,m,l)} \subseteq \big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l}\{a^i\} \cup a^n S a^m S a^l\big]$.

Again $(a^n S a^m S a^l] \subseteq ([a]_{(n,m,l)}{}^n S [a]_{(n,m,l)}{}^m S [a]_{(n,m,l)}{}^l] \subseteq [a]_{(n,m,l)}$.

Therefore, $\big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l}\{a^i\} \cup a^n S a^m S a^l\big] \subseteq [a]_{(n,m,l)}$.

Thus for any element a of S, we have $[a]_{(n,m,l)} = \big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l}\{a^i\} \cup a^n S a^m S a^l\big]$.

□

Theorem 7.2.4. *Let X and Y be two subsets of an ordered ternary semigroup S*

and A be an (n, m, l)-ideal of S, where l, m, n are non-negetive odd integers. Then $(AXY]$, $(XYA]$ and $(YAX]$ are (n, m, l)-ideals of S if $AXY \subseteq A$ or $XYA \subseteq A$ or $YAX \subseteq A$.

Proof. Let us assume $AXY \subseteq A$. Then $(AXY]^3 = (AXY](AXY](AXY] \subseteq (A](A](AXY] \subseteq (A^3XY] \subseteq (AXY]$. Thus $(AXY]$ is a ternary subsemigroup of S. Now,

$$\begin{aligned}
&(AXY]^n S(AXY]^m S(AXY]^l \\
&\subseteq (AXY]^n (S](AXY]^m (S](AXY]^l \\
&\subseteq (A]^n (S](A]^m (S](AXY]^{l-1}(AXY] \\
&\subseteq (A]^n (S](A]^m S(A]^{l-1}(AXY] \\
&= (A^n S A^m S A^l XY] \\
&\subseteq (AXY]
\end{aligned}$$

Hence $(AXY]$ is an (n, m, l)-ideal of S.

Also, $(XYA]^3 = (XYA](XYA](XYA] \subseteq (XYAXYAXYA] \subseteq (XYA^3] \subseteq (XYA]$. Thus $(XYA]$ is also a ternary subsemigroup of S. Now,

$$\begin{aligned}
&(XYA]^n = (XYA](XYA](XYA]........(XYA] \;(\; n \; times \;) \\
&\subseteq (XYAXYAXYA........XYA] \\
&\subseteq (XY \underbrace{AAA........A}_{n \; times}] \\
&= (XYA^n]
\end{aligned}$$

Then $(XYA]^n S(XYA]^m S(XYA]^l \subseteq (XYA^n](S](XYA^m](S](XYA^l] = (XYA^n SXY A^m SXYA^l] \subseteq (XYA^n SSSA^m SSSA^l] \subseteq (XYA^n SA^m SA^l] \subseteq (XYA]$.
Hence $(XYA]$ is also an (n, m, l)-ideal of S.

Again, $(YAX]^3 = (YAX](YAX](YAX] \subseteq (YAXYAXYAX] \subseteq (YA^3X] \subseteq (YAX]$.

On the other hand,

$$
\begin{aligned}
&(YAX]^n S(YAX]^m S(YAX]^l \\
&\subseteq (YA^nX](S](YA^mX](S](YA^lX] \\
&= (YA^nXSYA^mXSYA^lX] \\
&\subseteq (YA^nSSSA^mSSSA^lX] \\
&\subseteq (YA^nSA^mSA^lX] \subseteq (YAX]
\end{aligned}
$$

Hence $(YAX]$ is an (n, m, l)-ideal of S.

Similarly, we can prove the result if one of the conditions $XYA \subseteq A$ or $YAX \subseteq A$ holds. □

Corollary 7.2.5. *Let A be an (n, m, l)-ideal of an ordered ternary semigroup S where l, m, n are non-negetive odd integers and a and b be two arbitrary elements of S. Then $(abA]$, $(Aab]$ and $(aAb]$ are also (n, m, l)-ideals of S.*

Definition 7.2.6. *An ordered ternary semigroup is called (n, m, l)-simple if it does not contain any proper (n, m, l)-ideal, where n, m, l are non-negetive odd integers.*

Lemma 7.2.7. *A ternary semigroup S is $(n, 0, 0)$ (resp. $(0, m, 0)$, $(0, 0, l)$)-simple if and only if $(a^nSS] = S$ (resp. $(Sa^mS] = S$, $(SSa^l] = S$), for all $a \in S$ where n, m, l are non-negetive odd integers.*

Proof. Let A be an $(n, 0, 0)$-ideal of S and $(a^nSS] = S$ for all $a \in S$. Let $x \in S$. Then $x \leq a^nyz$ for some $y, z \in S$. So, $x \leq a^nyz \in A^nSS \subseteq A$ [since A is an $(n, 0, 0)$-ideal of S] $\Longrightarrow x \in (A] = A$. Thus $S \subseteq A$ and hence S is $(n, 0, 0)$-simple.

Conversely, let S is $(n, 0, 0)$-simple and $a \in S$. Then $(a^nSS]$ is an $(n, 0, 0)$-ideal of S. Since S has no proper $(n, 0, 0)$-ideal then $(a^nSS] = S$.

Similar proof for $(0, m, 0)$-simple, $(0, 0, l)$-simple ternary semigroup. □

Theorem 7.2.8. *Let n, m, l be non-negative odd integers. An ordered ternary semigroup does not contain proper (n, m, l)-ideal if and only if it is $(n, 0, 0)$-simple, $(0, m, 0)$-simple and $(0, 0, l)$-simple.*

Proof. Let S is $(n, 0, 0)$-simple, $(0, m, 0)$-simple and $(0, 0, l))$-simple. Thus $(a^nSS] = S$, $(Sa^mS] = S$ and $(SSa^l] = S$ for all $a \in S$. Let A be an (n, m, l)-ideal in S. Let $x \in A$. Then $S = (x^nSS] = (x^n(Sx^mS]S] = (x^nSx^mSS] = (x^nSx^mS(SSx^l]] = (x^nSx^mSSSx^l] \subseteq (x^nSx^mSx^l] \subseteq (A^nSA^mSA^l] \subseteq (A] = A$. Thus S has no proper (n, m, l)-ideal.

Conversely, S does not contain any proper (n, m, l)-ideal. Let B be a $(n, 0, 0)$-ideal of S. Then $B^nSB^mSB^l \subseteq B^nSS^mSS^l \subseteq B^nSS \subseteq B$. Thus B is an (n, m, l)-ideal in S and so $B = S$. Hence S is $(n, 0, 0)$-simple. Similarly, we can prove that S is $(0, m, 0)$-simple and $(0, 0, l)$-simple. □

Definition 7.2.9. *An ordered ternary semigroup S is called a ternary group like ordered ternary semigroup if for all $a, b, c \in S$ there are $x, y, z \in S$ such that $a \leq xbc$, $a \leq byc$ and $a \leq bcz$.*

Theorem 7.2.10. *An ordered ternary semigroup S is a ternary group like ordered ternary semigroup if and only if it contains no proper (n, m, l)-ideals, where n, m, l are non-negetive odd integers.*

Proof. Let S be a group like ordered ternary semigroup and let A be an (n, m, l)-ideal of S. Let $a \in A$ and $b, c \in S$. Then $a^n \in A^n \subseteq A \subseteq S$. Thus $b \leq a^nxc \in a^nSS$ for some $x \in S$. So, $b \in (a^nSS]$. Hence $S \subseteq (a^nSS]$. On the other hand, $(a^nSS] \subseteq (S^nSS] \subseteq (S] \subseteq S$. So, $S = (a^nSS]$. Therefore, S is $(n, 0, 0)$-simple. Similarly, we can prove S is $(0, m, 0)$-simple and $(0, 0, l)$-simple. Then by Theorem 7.2.8 the ordered ternary semigroup conatins no proper (n, m, l)-ideal.

Conversely, S conatins no proper (n, m, l)-ideal. Let $b, c \in S$. Then by Corollary 7.2.5 we have $(bcS]$, $(Sbc]$ and $(bSc]$ are (n, m, l)-ideals of S. Hence, $S = (bcS]$, $S = (Sbc]$ and $S = (bSc]$. Hence for all $a, b, c \in S$ we have $a \leq bcx$, $a \leq ybc$ and $a \leq bzc$ for some $x, y, z \in S$. Therefore, S is a ternary group like ordered ternary semigroup. □

Theorem 7.2.11. *Let S be an ordered ternary semigroup and the ordered ternary subsemigroups of S satiesfies the descending chain condition. If S has at least one* *proper (n, m, l)-ideal where $n > 1$, $m > 1$ and $l > 1$ then S has either a proper $(1, p, q)$-ideal or $(p, 1, q)$-ideal or $(p, q, 1)$-ideal, where l, m, n are non-negetive odd integers.*

Proof. Let n_1 be the smallest positive integer such that (n_1, m, l)-ideal exists, m_1 be the smallest positive integer such that (n, m_1, l)-ideal exists and l_1 be the smallest positive integer such that (n, m, l_1)-ideal exists. We show that either $n_1 \leq m, l$ or $m_1 \leq n, l$ or $l_1 \leq n, m$ holds. If $n_1 > m, l$ and $m_1 > n, l$ holds then $n_1 \leq n$ and $m < n_1 \leq n < m_1$ contradicts the minimality of m_1. Similarly, $m_1 > n, l$ and $l_1 > n, m$ contradicts the minimality of l_1 and $n_1 > m, l$ and $l_1 > m, n$ contradicts the minimality of n_1. Let $1 < n_1 \leq m, l$ and A is a proper (n_1, m, l)-ideal of S. Let $A^{n_1}SA^mSA^l \subseteq A$ and $B_{i+1} = {B_i}^{n_1}S{B_i}^mS{B_i}^l$ where $i = 1, 2, 3, \ldots\ldots$ Thus $B_{i+1} \subseteq A$. Since B_i satiesfies the descending chain condition of subsemigroup of S, then there exists a positive integer j such that $B_j = B_{j+k}$ for all $k \geq 1$ i.e. $B_j = {B_j}^{n_1}S{B_j}^mS{B_j}^l$. If we take $B = B_j$, then $B = B^{n_1}SB^mSB^l$. Therefore,

$$
\begin{aligned}
& B = B^{n_1}SB^mSB^l \\
\Longrightarrow\; & B^{n_1}SB^mSB^pSB^mSB^l = BSB^mSB^l \\
\Longrightarrow\; & B^{n_1}SB^mSB^lB^{-n_1}B^{n_1}SB^mSB^l = BSB^mSB^l \\
\Longrightarrow\; & B^{n_1}SB^mSB^{l-n_1}(B^{n_1}SB^mSB^l) = BSB^mSB^l \\
\Longrightarrow\; & B^{n_1}SB^mSB^{l-n_1}B = BSB^mSB^l \\
\Longrightarrow\; & B^{n_1}SB^mSB^{l-n_1+1} = BSB^mSB^l \\
\Longrightarrow\; & B^{n_1}SB^mSB^{l-n_1+1}B^{n_1-1} = BSB^mSB^lB^{n_1-1} \\
\Longrightarrow\; & B^{n_1}SB^mSB^l = BSB^mSB^{l+n_1-1} \\
\Longrightarrow\; & B = BSB^mSB^{l+n_1-1}
\end{aligned}
$$

Thus B is a $(1, m, l + n_1 - 1)$-ideal. Taking $p = m$, $q = l + n_1 - 1$ we can say that S has a proper $(1, p, q)$-ideal in S.

In the similar way we can say that, if $m_1 \leq n, l$ holds then S has a proper $(p, 1, q)$-

ideal and if $l_1 \leq n, m$ holds then S has a proper $(p, q, 1)$-ideal. $\square$

7.3 (n, m, l)-ideals in (n, m, l)-regular ordered ternary semigroup

Definition 7.3.1. *Let S be an ordered ternary semigroup and n, m, l are non-negetive odd integers.*

- *An (n, m, l)-ideal A of S is called quasi-prime if $(A_1A_2A_3] \subseteq A \implies A_1 \subseteq A$ or $A_2 \subseteq A$ or $A_3 \subseteq A$, where A_1, A_2, A_3 are (n, m, l)-ideals S.*
- *An (n, m, l)-ideal A of S is called quasi-semiprime if $(A_1^3] \subseteq A \implies A_1$, where A_1, A_2, A_3 are (n, m, l)-ideals S.*
- *An (n, m, l)-ideal A of S is called strongly quasi-prime if $(A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2] \subseteq A \implies A_1 \subseteq A$ or $A_2 \subseteq A$ or $A_3 \subseteq A$ where A_1, A_2, A_3 are (n, m, l)-ideals S.*

Note that strongly quasi-prime ideals are quasi-prime ideals. Also quasi-prime ideals are quasi-semiprime ideals of S.

Definition 7.3.2. *Let S be an ordered ternary semigroup and n, m, l are non-negetive odd integers.*

- *An (n, m, l)-ideal A of S is called irreducible if $A_1 \cap A_2 \cap A_3 = A \Rightarrow A_1 = A$ or $A_2 = A$ or $A_3 = A$ where A_1, A_2, A_3 are (n, m, l)-ideals S.*
- *An (n, m, l)-ideal A of S is called strongly irreducible if $A_1 \cap A_2 \cap A_3 \subseteq A \Rightarrow A_1 \subseteq A$ or $A_2 \subseteq A$ or $A_3 \subseteq A$ where A_1, A_2, A_3 are (n, m, l)-ideals S.*

Strongly irreducible (n, m, l)-ideal $\implies$ irreducible (n, m, l)-ideal

Theorem 7.3.3. *The non-empty intersection of any collection of quasi-semiprime (n, m, l)-ideals of an ordered ternary semigroup S is a quasi-semiprime (n, m, l)-ideal of S.*

Theorem 7.3.4. *Let A be an (n, m, l)-ideal of an ordered ternary semigroup S. If A is strongly irreducible and quasi-semiprime, then A is strongly quasi-prime.*

Proof. Let A be a strongly irreducible and quasi-semiprime (n, m, l)-ideal S and suppose $(A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2] \subseteq A$ where A_1, A_2, A_3 are (n, m, l)-ideals S. Now $(A_1 \cap A_2 \cap A_3)^3 \subseteq A_1A_2A_3$. Thus $(A_1 \cap A_2 \cap A_3)^3 \subseteq A_1A_2A_3 \cap A_2A_3A_1 \cap A_3A_1A_2 \subseteq (A_1A_2A_3 \cap A_2A_3A_1 \cap A_3A_1A_2] \subseteq (A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2] \subseteq A$.

If $A_1 \cap A_2 \cap A_3 = \{\}$. Then $A_1 \cap A_2 \cap A_3 \subseteq A$. If $A_1 \cap A_2 \cap A_3 \neq \{\}$ then $A_1 \cap A_2 \cap A_3$ is an (n, m, l)-ideal of S. Since A is quasi-semiprime, $(A_1 \cap A_2 \cap A_3)^3 \subseteq A \Longrightarrow A_1 \cap A_2 \cap A_3 \subseteq A$. Again A is strongly irreducible, hence $A_1 \cap A_2 \cap A_3 \subseteq A \Longrightarrow A_1 \subseteq A$ or $A_2 \subseteq A$ or A_3. Therefore, A is strongly quasi-prime (n, m, l)-ideal of an ordered ternary semigroup S. $\square$

Definition 7.3.5. *An ordered ternary semigroup is called (n, m, l)-regular if $a \in (a^nSa^mSa^l]$ for all $a \in S$, where n, m, l are non-negetive odd integers.*

Theorem 7.3.6. *Let S be an ordered ternary semigroup. Then S is (n, m, l)-regular if and only if $[a]_{(n,m,l)} = (a^nSa^mSa^l]$ for all $a \in S$.*

Proof. Let S be an (n, m, l)-regular ordered ternary semigroup. Let $x \in [a]_{(n,m,l)}$ for some $a \in S$. Thus $x \in (\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^nSa^mSa^l] = (\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{a^i\}] \cup (a^nSa^mSa^l]$.
Therefore, we have either $x \in (a^nSa^mSa^l]$ or $x \in (\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} a^i]$.
If $x \in (a^nSa^mSa^l]$, then our proof is done.
If $x \in (\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} a^i]$, then $x \in (a^j]$ where $2k - 1(k \in \mathbb{N}) \leq j \leq n + m + l$. Thus $x \in ((a^nSa^mSa^l]^j] \subseteq ((a^nSa^mSa^l]] \subseteq (a^nSa^mSa^l]$. Again $(a^nSa^mSa^l] \subseteq [a]_{(n,m,l)}$. Therefore, $[a]_{(n,m,l)} = (a^nSa^mSa^l]$.

Conversely, suppose for all $a \in S$, we have $[a]_{(n,m,l)} = (a^nSa^mSa^l]$. Since $[a]_{(n,m,l)}$ contains a, $a \in (a^nSa^mSa^l]$. Therefore, S is an (n, m, l)-regular ordered ternary semigroup. $\square$

Lemma 7.3.7. *Let A be an (n, m, l)-ideal of an ordered ternary semigroup $(S, ., \leq)$*

where n, m, l are non-negetive odd integers and B be a non-empty subset of A. Then $(B] = (B]_A$.

Proof. It is obvuious that $(B]_A \subseteq (B]$. Let $x \in (B]$. Then $x \in S$ such that $x \leq b$ for some $b \in B \subseteq A$. Thus $x \in (A] = A$. Hence $x \in A$ such that $x \leq b \in B$ implies that $x \in (B]_A$ and so $(B] \subseteq (B]_A$. Therefore, $(B] = (B]_A$. □

Theorem 7.3.8. *Let A be an (n, m, l)-ideal of an ordered ternary semigroup $(S, ., \leq)$ where n, m, l are non-negetive odd integers and B be a non-empty subset of A. Then*

$$\big(([B_A]_{(n,m,l)})^n S([B_A]_{(n,m,l)})^m S([B_A]_{(n,m,l)})^l\big] = (B^n SB^m SB^l]$$

where $[B_A]_{(n,m,l)}$ defined by $[B_A]_{(n,m,l)} = (\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{B^i\} \cup B^n AB^m AB^l]_A$.

Proof. Let B be a non-empty subset of an (n, m, l)-ideal A. Then $B \subseteq (B] = (B]_A \subseteq (\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{B^i\} \cup B^n SB^m SB^l]_A = [B_A]_{(n,m,l)}$. Therefore $(B^n AB^m AB^l] \subseteq \big(([B_A]_{(n,m,l)})^n S([B_A]_{(n,m,l)})^m S([B_A]_{(n,m,l)})^l\big]$. Now let $x \in \big(([B_A]_{(n,m,l)})^n S([B_A]_{(n,m,l)})^m S([B_A]_{(n,m,l)})^l\big]$. Then we have $x \leq {b_1}^n y {b_2}^m z {b_3}^l$ for some $y, z \in S$ and $b_1, b_2, b_3 \in [B_A]_{(n,m,l)} = (\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{B^i\} \cup B^n AB^m AB^l]_A \subseteq \big((\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{B^i\} \cup B^n AB^m AB^l]_A\big] = \big((\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{B^i\} \cup B^n AB^m AB^l]\big] = \bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{B^i\} \cup B^n AB^m AB^l$ [since we have, $\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{B^i\} \cup B^n AB^m AB^l \subseteq A$]. The following cases may arise:

Case 1: If $b_1, b_2, b_3 \in \bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{B^i\}$, then we have $b_1 \in B^r, b_2 \in B^s, b_3 \in B^t$ for some $r, s, t \in \{1, 3, 5, ..., n + m + l\}$. Then ${b_1}^n y {b_2}^m z {b_3}^l \in (B^r)^n S(B^s)^m S(B^t)^l = B^{rn} SB^{sm} SB^{tl} \subseteq B^n SB^m SB^l$ and hence $x \in (B^n SB^m SB^l]$.

Case 2: If $b_1, b_2 \in \bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{B^i\}$ and $b_3 \in B^n AB^m AB^l \subseteq B^n SB^m SB^l$, then we have $b_1 \in B^r, b_2 \in B^s$ for some $r, s \in \{1, 3, 5, ..., n + m + l\}$. Then ${b_1}^n S {b_2}^m S {b_3}^l \in (B^r)^n S(B^s)^m S(B^n SB^m SB^l)^l \subseteq B^n SB^m SB^l$. Thus $x \in (B^n SB^m SB^l]$.

Case 3: $b_1, b_3 \in \bigcup\limits_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{B^i\}$ and $b_2 \in B^nAB^mAB^l$. Proof is similar to Case 2.

Case 4: $b_2, b_3 \in \bigcup\limits_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{B^i\}$ and $b_1 \in B^nAB^mAB^l$. Proof is similar to Case 2.

Case 5: If $b_1 \in \bigcup\limits_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{B^i\}$ and $b_2, b_3 \in B^nAB^mAB^l$. Then $b_1 \in B^r$ for some $r \in \{1, 3, 5, ..., n+m+l\}$. Then ${b_1}^nS{b_2}^mS{b_3}^l \in (B^r)^nS(B^nAB^mAB^l)^mS(B^nAB^mAB^l)^l \subseteq (B^r)^nS(B^nSB^mSB^l)^mS(B^nSB^mSB^l)^l \subseteq B^nSB^mSB^l$. Thus $x \in (B^nSB^mSB^l]$.

Case 6: If $b_2 \in \bigcup\limits_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{B^i\}$ and $b_1, b_3 \in B^nAB^mAB^l$. The proof is similar to case 5.

Case 7: If $b_3 \in \bigcup\limits_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{B^i\}$ and $b_1, b_2 \in B^nAB^mAB^l$. The proof is similar to case 5.

Case 8: If $b_1, b_2, b_3 \in B^nAB^mAB^l \subseteq B^nSB^mSB^l$. Then we have ${b_1}^nS{b_2}^mS{b_3}^l \in (B^nSB^mSB^l)^nS(B^nSB^mSB^l)^mS(B^nSB^mSB^l)^l \subseteq B^nSB^mSB^l$. Thus $x \in (B^nSB^mSB^l]$.

So, in all cases we have $x \in (B^nSB^mSB^l]$.

Hence $\Big(([B_A]_{(n,m,l)})^nS([B_A]_{(n,m,l)})^mS([B_A]_{(n,m,l)})^l\Big] \subseteq (B^nSB^mSB^l]$.

Therefore, $\Big(([B_A]_{(n,m,l)})^nS([B_A]_{(n,m,l)})^mS([B_A]_{(n,m,l)})^l\Big] = (B^nSB^mSB^l]$. □

Theorem 7.3.9. *Let $(S, ., \leq)$ be an ordered ternary semigroup and A be an (n, m, l)-ideal of S. Then every (n, m, l)-ideal of A is an (n, m, l)-ideal of S if and only if for each non-empty subset B of A, $(B^nSB^mSB^l] \subseteq [B_A]_{(n,m,l)}$.*

Proof. Let A be an (n, m, l)-ideal of an ordered ternary semigroup $(S, ., \leq)$ and every (n, m, l)-ideal of A is an (n, m, l)-ideal S. Let B be a non empty subset of A. Now $[B_A]_{(n,m,l)} = (\bigcup\limits_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{B^i\} \cup B^nSB^mSB^l]_A$ is an (n, m, l)-ideal of A and hence an (n, m, l)-ideal of S. Thus $B^nSB^mSB^l \subseteq (B^nSB^mSB^l]\Big(([B_A]_{(n,m,l)})^mS([B_A]_{(n,m,l)})^lS$ $([B_A]_{(n,m,l)})^n\Big] \subseteq ([B_A]_{(n,m,l)}] \subseteq [B_A]_{(n,m,l)}$.

For the converse part let us assume that for some non-empty subset B of A, $B^nSB^mSB^l \subseteq [B_A]_{(n,m,l)}$. Let I be an (n, m, l)-ideal of A. Thus $I \subseteq A$.

Now we have,

$$\begin{aligned}
& I^n S I^m S I^l \\
& \subseteq [I_A]_{(n,m,l)} \\
& = (\bigcup_{i=2k-1, k\in\mathbb{N}}^{n+m+l} \{I^i\} \cup I^n S I^m S I^l]_A \\
& = (I \cup I^3 \cup \ldots.. \cup I^{n+m+l} \cup I^n S I^m S I^l]_A \\
& \subseteq (I]_A = I
\end{aligned}$$

Hence I is an (n, m, l)-ideal of S. □

Theorem 7.3.10. *Let $(S, ., \leq)$ be an ordered ternary semigroup. Let $R_{(n,0,0)}$, $M_{(0,m,0)}$ and $L_{(0,0,l)}$ are the set of all $(n, 0, 0)$-ideals, $(0, m, 0)$-ideals and $(0, 0, l)$-ideals respectively. Then the following statements hold:*
(i) S is $(n, 0, 0)$-regular if and only if $(R^n SS] = R$ for all $R \in R_{(n,0,0)}$
(ii) S is $(0, m, 0)$-regular if and only if $(SM^m S] = M$ for all $M \in M_{(0,m,0)}$
(iii) S is $(0, 0, l)$-regular if and only if $(SSL^l] = L$ for all $L \in L_{(0,0,l)}$
(iv) S is (n, m, l)-regular if and only if $(R^n M^m L^l] = R \cap M \cap L$ for all $R \in R_{(n,0,0)}$, $M \in M_{(0,m,0)}$ and $L \in L_{(0,0,l)}$.

Proof. (i) Let S be $(n, 0, 0)$-regular ordered ternary semigroup. Let $a \in R \subseteq S$. Thus $a \in (a^n SS]$ and so $a \leq a^n xy$ for some $x, y \in S$. Thus $a \in (R^n SS]$. Again $(R^n SS] \subseteq (R] = R$. Therefore, $(R^n SS] = R$.
Conversely, let $(R^n SS] = R$ for all $R \in R_{(n,0,0)}$. Let a be an arbitrary element in S. Now $[a]_{(n,0,0)} \in R_{(n,0,0)}$. Now $[a]_{(n,0,0)} = (([a]_{(n,0,0)})^n SS] \subseteq (a^n SS]$. Hence $a \in (a^n SS]$ and S is $(n, 0, 0)$-regular.
Similar proof for (ii) and (iii).
(iv) Let S be an (n, m, l)-regular ordered ternary semigroup. Let $a \in R \cap M \cap L$ where $R \in R_{(n,0,0)}$, $M \in M_{(0,m,0)}$ and $L \in L_{(0,0,l)}$. Since $a \in S$, $a \leq a^n x a^m y a^l$ for some $x, y \in S$. Now $a \leq a^n x a^m y a^l = a^{n-1} a x a^m y a^l \leq a^{n-1}(a^n x a^m y a^l) x a^m y a^l =$

$a^{2n-1}xa^m ya^l xa^m ya^l = a^n a^{n-1} xa^m ya^l xa^m ya^l \in R^n S^{n-1} SM^m SS^l SS^m SL^l = R^n S^n M^m S^{m+l+3} L^l \subseteq R^n SM^m SL^l \subseteq R^n M^m L^l$ [since M is a $(0, m, 0)$-ideal, $SM^m S \subseteq M$]. Thus $a \in (R^n M^m L^l]$. Again $(R^n M^m L^l] \subseteq (S^n S^m L^l] \subseteq (SSL^l] \subseteq L$. Similarly $(R^n M^m L^l] \subseteq M$ and $(R^n M^m L^l] \subseteq R$. Thus $(R^n M^m L^l] \subseteq R \cap M \cap L \subseteq (R \cap M \cap L]$ and hence $(R^n M^m L^l] = (R \cap M \cap L]$.

Conversely, let $R \cap M \cap L = (R^n M^m L^l]$. Now $a \in [a]_{(n,0,0)}$. Again $[a]_{(n,0,0)} \in R_{(n,0,0)}$, $[a]_{(0,m,0)} \in M_{(0,m,0)}$, $[a]_{(0,0,l)} \in L_{(0,0,l)}$. Then $[a]_{(n,0,0)} \cap [a]_{(0,m,0)} \cap [a]_{(0,0,l)} = \big(([a]_{(n,0,0)})^n ([a]_{(0,m,0)})^m ([a]_{(0,0,l)})^l\big] \subseteq \big(([a]_{(n,0,0)}]^n S^m S^l\big] \subseteq (a^n SS]$. Thus $[a]_{(n,0,0)} \subseteq (a^n SS]$. Also $(a^n SS]$ is an $(n, 0, 0)$-ideal of S. Similary $[a]_{(0,m,0)} \subseteq (Sa^m S]$, $[a]_{(0,0,l)} \subseteq (SSa^l]$ and $(Sa^m S]$, $(SSa^l]$ are $(0, m, 0)$-ideal, $(0, 0, l)$-ideal of S respectively. Now

$$\begin{aligned}
[a]_{(n,0,0)} &= [a]_{(n,0,0)} \cap [a]_{(0,m,0)} \cap [a]_{(0,0,l)} \\
&\subseteq (a^n SS] \cap (Sa^m S] \cap (SSa^l] \\
&= \big((a^n SS]^n (Sa^m S]^m (SSa^l]^l\big] \\
&\subseteq (a^n SS](Sa^m S](SSa^l] \\
&\subseteq (a^n SSSa^m SSSa^l] \\
&\subseteq (a^n Sa^m Sa^l]
\end{aligned}$$

Therefore, $a \in (a^n Sa^m Sa^l]$ and hence S is (n, m, l)-regular ordered ternary semigroup. □

Corollary 7.3.11. *Let S be an ordered ternary semigroup. Then S is (n, m, l)-regular if and only if $[a]_{(n,0,0)} \cap [a]_{(0,m,0)} \cap [a]_{(0,0,l)} = ([a]_{(n,0,0)})^n([a]_{(0,m,0)})^m([a]_{(0,0,l)})^l$ for all $a \in S$.*

Theorem 7.3.12. *Let S be an ordered ternary semigroup. Then S is both (n, m, l)-regular and intra-regular if and only if $(A^3] = A$ for every (n, m, l)-ideal A of S.*

Proof. Let us assume that S be an intra-regular and (n, m, l)-regular ordered ternary semigroup. Let A be an (n, m, l)-ideal of S. Thus A is a ternary subsemigroup of

S. So $A^3 \subseteq A \Rightarrow (A^3] \subseteq (A] = A$. It remains to show that $A \subseteq (A^3]$. The following cases may arise:

Case 1: $n = 1, m = 1, l = 1$

Here A is a $(1,1,1)$-ideal of S, thus $ASASA \subseteq A$. Again S is a $(1,1,1)$-regular and intra-regular ternary semigroup. Thus $A \subseteq (ASASA]$ and $A \subseteq (SA^3S]$. Then we have $A \subseteq (ASASA] \subseteq ((ASASA]S(SA^3S]S(ASASA]] = (ASASASSA^3SSASASA]$ $\subseteq (ASASAASSASASA] \subseteq (AASSASASA] \subseteq (AASS(SA^3S]S(SA^3S]SA] = (AAS$ $SSA^3SSSA^3SSA] \subseteq (AASASAAASSA] \subseteq (AAA(SA^3S]SSA] = (AAASA^3SSSA]$ $\subseteq (AAASASA] \subseteq (AAA] = (A^3]$.

Case 2: $n = 1, m = 1, l > 1$

In this case S is an intra-regular and $(1,1,l)$-regular ordered ternary semigroup and A is a $(1,1,l)$-ideal of S. Thus $ASASA^l \subseteq A$. Since S is $(1,1,l)$-regular $A \subseteq$ $(ASASA^l] = (ASASA^{l-2}A^2] \subseteq (ASASA^{l-2}(ASASA^l](ASASA^l]] \subseteq (ASASA^{l-2}(A]$ $(A]] \subseteq (ASAAA] \subseteq (AS(ASASA^l]AA] = (ASASASA^lAA] \subseteq (AAA] = (A^3]$.

Case 3: $n = 1, m > 1, l = 1$

Thus S is an intra-regular and $(1,m,1)$-regular ordered ternary semigroup and A is a $(1,m,1)$-ideal of S. Therefore, $ASA^mSA \subseteq A$. Thus $A \subseteq (ASA^mSA] =$ $(ASA^{m-2}A^2SA] \subseteq (ASAAASA] \subseteq (AS(ASA^mSA]AAS(ASA^mSA]] = (ASASA^mS$ $AAASASA^mSA] \subseteq (AAA] = (A^3]$.

Case 4: $n > 1, m = 1, l = 1$

Here A is an $(n,1,1)$-ideal of S, thus $A^nSASA \subseteq A$. Again S is a $(n,1,1)$-regular and intra-regular ordered ternary semigroup. Thus $A \subseteq (A^nSASA] = (A^{n-2}A^2SASA] \subseteq$ $(A^3SASA] \subseteq (A^2(A^nSASA]SASA] \subseteq (A^2A^nSASA] \subseteq (A^3]$.

Case 5: $n > 1, m > 1, l = 1$

In this case S is an intra-regular and $(n,m,1)$-regular ordered ternary semigroup. Let A be an $(n,m,1)$-ideal of S i.e $A^nSA^mSA \subseteq A$. Thus $A \subseteq (A^nSA^mSA] =$ $(A^nSA^{m-2}AASA] \subseteq (A^nSAAASA]$
$\subseteq (A^nS(A^nSA^mSA]A(A^nSA^mSA]SA] \subseteq (A^nSA^mSAAA^nSA^mSA] \subseteq (A^3]$.

Case 6: $n = 1, m > 1, l > 1$

Let A be an $(1, m, l)$-ideal of S. In this case S is an intra-regular and $(1, m, l)$-regular ordered ternary semigroup. Thus $A \subseteq (ASA^mSA^l] = (ASA^{m-2}AASA^l] \subseteq (ASAAASA^l] \subseteq (AS(ASA^mSA^l]A(ASA^mSA^l]SA^l] \subseteq (ASA^mSA^lAASA^mSA^l] \subseteq (A^3]$.

Case 7: $n > 1, m = 1, l > 1$

Here A is a $(n, 1, l)$-ideal of S i.e $A^nSASA^l \subseteq A$. Again S is a $(n, 1, l)$-regular and intra-regular ternary semigroup. Thus $A \subseteq (A^nSASA^l] = (A^{n-2}A^2SASA^l] \subseteq (AAASASA^l] \subseteq (AA(A^nSASA^l]SASA^l] \subseteq (AAA^nSASA^l] \subseteq (A^3]$.

Case 8: $n > 1, m > 1, l > 1$

Here A is a (n, m, l)-ideal of S i.e $A^nSA^mSA^l \subseteq A$. Again S is a (n, m, l)-regular and intra-regular ternary semigroup. Thus $A \subseteq (A^nSA^mSA^l] = (A^nSA^mSA^{l-2}AA] \subseteq (A^nSA^mSAAA] \subseteq (A^nSA^mS(A^nSA^mSA^l]SAA] \subseteq (A^nSA^mSA^lAA] \subseteq (A^3]$.

Thus in all cases we have $A \subseteq (A^3]$.

Conversely, let $a \in S$. Then $(\bigcup\limits_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^nSa^mSa^l]$ is an (n, m, l)-ideal of S containing a. Since $(A^3] = A$ for every (n, m, l)-ideal A of S, we have

$$\Big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^nSa^mSa^l\Big]$$
$$= \Big(\Big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^nSa^mSa^l\Big]^3\Big]$$
$$= \Big(\Big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^nSa^mSa^l\Big]\Big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^nSa^mSa^l\Big]\Big(\Big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^nSa^mSa^l\Big]^3\Big]\Big]$$
$$= \Big(\Big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^nSa^mSa^l\Big)\Big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^nSa^mSa^l\Big)\Big(\Big(\bigcup_{i=2k-1,k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^nSa^mSa^l\Big]^3\Big)\Big]$$

$$= \Big(\Big(\bigcup_{i=2k-1, k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^n S a^m S a^l\Big)\Big(\bigcup_{i=2k-1, k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^n S a^m S a^l\Big)\Big(\bigcup_{i=2k-1, k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^n S a^m S a^l\Big)^3\Big]$$

$$= \Big(\Big(\bigcup_{i=2k-1, k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^n S a^m S a^l\Big)^5\Big] = \Big(\Big(\bigcup_{i=2k-1, k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^n S a^m S a^l\Big)^{3+2}\Big]$$

Continuing in the similar way we get,

$$\Big(\bigcup_{i=2k-1, k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^n S a^m S a^l\Big] = \Big(\Big(\bigcup_{i=2k-1, k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^n S a^m S a^l\Big)^{n+m+l+2}\Big].$$

Thus $a \in \Big(\Big(\bigcup_{i=2k-1, k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^n S a^m S a^l\Big)^{n+m+l+2}\Big] \subseteq (a^n S a^m S a^l]$ and also $a \in \Big(\Big(\bigcup_{i=2k-1, k\in\mathbb{N}}^{n+m+l} \{a^i\} \cup a^n S a^m S a^l\Big)^5\Big] \subseteq (Sa^3S]$.

So, S is both (n, m, l)-regular and intra-regular ordered ternary semigroup. □

Lemma 7.3.13. *Let S be an ordered ternary semigroup. Then the followings are equivalent:*

(i) $(A^3] = A$ for every (n, m, l)-ideal A of S.

(ii) $A_1 \cap A_2 \cap A_3 = (A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2]$ for all (n, m, l)-ideal A_1, A_2 and A_3 of S.

(iii) Every (n, m, l)-ideal of S is quasi semiprime.

Proof. (i)$\Longrightarrow$(ii) Suppose A_1, A_2 and A_3 are (n, m, l)-ideal of S.

Case 1: Let us consider the case when $A_1 \cap A_2 \cap A_3 = \{\}$.

So, $(A_1A_2A_3]^n S (A_1A_2A_3]^m S (A_1A_2A_3]^l$

$\subseteq (A_1A_2A_3]^n (S] (A_1A_2A_3]^m (S] (A_1A_2A_3]^l$

$\subseteq ((A_1A_2A_3)(A_1A_2A_3)^{n-1} S (A_1A_2A_3)(A_1A_2A_3)^{m-1} S (A_1A_2A_3)^{l-1}(A_1A_2A_3)$

$\subseteq (A_1SSS^{n-1}SA_1SSS^{m-1}SS^{l-1}A_1A_2A_3]$

$= (A_1S^{n+2}A_1S^{m+l+1}A_1A_2A_3]$

$\subseteq (A_1^n S A_1^m S A_1^l A_2A_3] \subseteq (A_1A_2A_3]$

Also $((A_1A_2A_3]] = (A_1A_2A_3]$. Hence $(A_1A_2A_3]$ is an (n, m, l)-ideal of S. Similarly, $(A_2A_3A_1]$ and $(A_3A_1A_2]$ are also (n, m, l)-ideal of S. Let us assume $(A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2] \neq \{\}$. Then $(A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2]$ is an (n, m, l)-ideal of S. Therefore,

$$\begin{aligned}
&((A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2]) \\
&\subseteq (((A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2])^3] \\
&\subseteq ((A_1A_2A_3](A_2A_3A_1](A_3A_1A_2]] \\
&= (A_1A_2A_3A_2A_3A_1A_3A_1A_2] \\
&\subseteq (A_1SSSA_1SSSA_1] \\
&\subseteq (A_1SA_1SA_1] \\
&\subseteq (A_1^nSA_1^mSA_1^l] \\
&\subseteq (A_1] = A_1
\end{aligned}$$

Similarly, $(A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2] \subseteq A_2$ and $(A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2] \subseteq A_3$. So $(A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2] \subseteq A_1 \cap A_2 \cap A_3 = \{\}$. Thus our assumption is not true. Hence $(A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2] = \{\} = A_1 \cap A_2 \cap A_3$.

Case 2: Let us consider the case when $A_1 \cap A_2 \cap A_3 \neq \{\}$.

Thus $A_1 \cap A_2 \cap A_3$ is an (n, m, l)-ideal of S. This implies that $A_1 \cap A_2 \cap A_3 = ((A_1 \cap A_2 \cap A_3)^3] \subseteq (A_1A_2A_3 \cap A_2A_3A_1 \cap A_3A_1A_2] \subseteq (A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2]$. Hence $(A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2]$ is non empty. By case 1 we have $(A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2] \subseteq A_1 \cap A_2 \cap A_3$. Therefore, $(A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2] = A_1 \cap A_2 \cap A_3$.

(ii)$\Rightarrow$(iii) Let A be an (n, m, l)-ideal of S such that $(A_1^3] \subseteq A$ for some (n, m, l)-ideal A_1 of S. Thus $A_1 \cap A_1 \cap A_1 = (A_1A_1A_1] \cap (A_1A_1A_1] \cap (A_1A_1A_1] = (A_1^3] \subseteq A$. Thus $A_1 \subseteq A$ and hence A is quasi-semiprime (n, m, l)-ideal of S.

(iii)$\Rightarrow$(i) Let A be an (n, m, l)-ideal of S. Thus A is a ternary subsemigroup of S. So, $A^3 \subseteq A \Rightarrow (A^3] \subseteq (A] \subseteq A$. Now $(A^3]^nS(A^3]^mS(A^3]^l \subseteq (A^{3n}SA^{3m}SA^{3l}] =$

$(A^n A^{2n} S A^m A^{2m} S A^l A^l A^l] = (A^n A^{2n} S A^m A^{2m} S A^l A^l A^{l-2} AA] = (A^n S^{2n+1} A^m S^{2m+1} A^l A^{l-2} A^l AA] \subseteq (A^n S^{2n+1} A^m S^{2m+2l-1} A^l AA] \subseteq (A^n S A^m S A^l AA] \subseteq (A^3]$. Again $((A^3]] = (A^3]$. Thus $(A^3]$ an (n, m, l)-ideal of S. Since $A^3 \subseteq (A^3]$ and $(A^3]$ is quasi semiprime $A \subseteq (A^3]$. Thus $(A^3] = A$. □

Corollary 7.3.14. *Let S be an (n, m, l)-regular and intra-regular ordered ternary semigroup. Then an (n, m, l)-ideal A of S is strongly irreducible if and only if A is strongly quasi-prime.*

Proof. Let S be an (n, m, l)-regular and intra-regular ordered ternary semigroup. Then by Theorem 7.3.12 $(A^3] = A$ for every (n, m, l)-ideal A of S. Let us assume that A be an (n, m, l)-ideal of S is strongly quasi-prime and $A_1 \cap A_2 \cap A_3 \subseteq A$ for some (n, m, l)-ideals A_1, A_2, A_3 of S. By Lemma 7.3.13 $A_1 \cap A_2 \cap A_3 = (A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2]$. Thus $(A_1A_2A_3] \cap (A_2A_3A_1] \cap (A_3A_1A_2] \subseteq A$. Since A is strongly quasi-prime we have $A_1 \subseteq A$ or $A_2 \subseteq A$ or $A_3 \subseteq A$. Therefore, A is strongly irreducible.

Conversely, suppose A be a strongly irreducible (n, m, l)-ideal of S. Since S be an (n, m, l)-regular and intra-regular ordered ternary semigroup by Lemma 7.3.13 it follows that A is strongly quasi-prime. □

Remark 7.3.15. *Let S be an (n, m, l)-regular and intra-regular ordered ternary semigroup. Then we have the following result.*
Strongly quasi-prime (n, m, l)-ideal $\Longleftrightarrow$ Strongly irreducible (n, m, l)-ideal.

Lemma 7.3.16. *Let S Let S be an ordered ternary semigroup. Then the following statement are equivalent:*
(i) The set $A = \{A_i : A_i^n S A_i^m S A_i^l \subseteq A_i\}$ is a chain under inclusion.
(ii) Every (n, m, l)-ideal is strongly irreducible and $A_1 \cap A_2 \cap A_3 \neq \{\}$ for all (n, m, l)-ideal A_1, A_2 and A_3 of S.
(iii) Every (n, m, l)-ideal of S is irreducible.

Proof. (i)$\Longrightarrow$(ii) Let us assume the condition (i) holds. Let A be an (n, m, l)-ideal of S such that $A_1 \cap A_2 \cap A_3 \subseteq A$ for some (n, m, l)-ideal A_1, A_2 and A_3 of S. Since $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$ we have $A_1 \cap A_2 \cap A_3 \neq \{\}$.

Case 1: Let $A_1 \subseteq A_2$. Then $A_1 \cap A_2 = A_1$. Again A_1 and A_3 are (n, m, l)-ideal of S. Thus we have either $A_1 \subseteq A_3$ or $A_3 \subseteq A_1$. If $A_1 \subseteq A_3$, then $A_1 = A_1 \cap A_1 = (A_1 \cap A_2) \cap A_1 \subseteq A_1 \cap A_2 \cap A_3 \subseteq A$. If $A_3 \subseteq A_1$, then $A_3 = A_3 \cap A_3 \subseteq A_3 \cap A_1 = A_1 \cap A_2 \cap A_3 \subseteq A$.

Case 2: Let $A_2 \subseteq A_1$. Then $A_1 \cap A_2 = A_2$. Again A_2 and A_3 are (n, m, l)-ideal of S. Thus we have either $A_2 \subseteq A_3$ or $A_3 \subseteq A_2$. If $A_2 \subseteq A_3$, then $A_2 = A_2 \cap A_2 = (A_1 \cap A_2) \cap A_2 \subseteq (A_1 \cap A_2) \cap A_3 = A_1 \cap A_2 \cap A_3 \subseteq A$. If $A_3 \subseteq A_2$, then $A_3 = A_3 \cap A_3 \subseteq A_2 \cap A_3 = A_1 \cap A_2 \cap A_3 \subseteq A$.

So in both cases we have $A_1 \subseteq A$, $A_2 \subseteq A$ or $A_3 \subseteq A$. Hence A is strongly irreducible (n, m, l)-ideal of S.

(ii)$\Longrightarrow$(iii) This proof is straightforward.

(iii)$\Longrightarrow$(i) Let us assume A be an (n, m, l)-ideal of S which is irreducible and $A_1 \cap A_2 \cap A_3 \neq \{\}$ for all (n, m, l)-ideal A_1, A_2 and A_3 of S. Thus $A_1 \cap A_2 \cap A_3$ is an (n, m, l)-ideal of S. Let $A_1 \cap A_2 \cap A_3 = A$. This implies $A_1 = A$, $A_2 = A$ or $A_3 = A$. If $A_1 = A$, then $A_1 = A_1 \cap A_2 \cap A_3$. Thus $A_1 \subseteq A_2 \cap A_3 \Rightarrow A_1 \subseteq A_2$ or $A_1 \subseteq A_3$. Simililarly if $A_2 = A$, then $A_2 \subseteq A_1$ or $A_2 \subseteq A_3$ and if $A_3 = A$, then $A_3 \subseteq A_1$ or $A_3 \subseteq A_2$. Thus for A_1, A_2 we have either $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$. Similarly for A_2, A_3 and A_3, A_1 we have either $A_2 \subseteq A_3$ or $A_3 \subseteq A_2$ and $A_3 \subseteq A_1$ or $A_1 \subseteq A_3$. Therefore, the set of all (n, m, l)-ideal of S is a chain. $\square$

We conclude this chapter with following theorem:

Theorem 7.3.17. *Let $(S, . \leq)$ be an ordered ternary semigroup. Then every (n, m, l)-ideal of S is strongly quasi-prime and $A_1 \cap A_2 \cap A_3 \neq \{\}$ for all (n, m, l)-ideal of S if and only if S is (n, m, l)-regular, intra-regular and the set of all (n, m, l)-ideals is a chain.*

Proof. Let every (n, m, l)-ideal of S is strongly quasi-prime and $A_1 \cap A_2 \cap A_3 \neq \{\}$. Then $(A^3] = A$ for all (n, m, l)-ideal A of S (By Lemma 7.3.13). Hence by Lemma

7.3.12 S is both intra-regular and (n, m, l)-regular. Thus by Corollary 7.3.14 S is strongly irreducible.

Conversely, S is (n, m, l)-regular, intra-regular ordered ternary semigroup and the set of all (n, m, l)-ideals is a chain. By Lemma 7.3.16 every (n, m, l)-ideal of S is strongly quasi-prime and $A_1 \cap A_2 \cap A_3 \neq \{\}$ for all (n, m, l)-ideals A_1, A_2, A_3 of S. By Corollary 7.3.14 S is strongly quasi-prime. $\square$

Conclusion and Future Scope

Conclusion and Future Scope

Analysis of Contributions

In this research work, the goal was to evaluate different classes of ternary semigroup. Also, the objective of the present investigation was to study ordered ternary semigroup. We mainly discussed the various types of regularity in ordered ternary semigroup. Ideal theory is a key concept in ternary semigroup. We charaterized ordered ternary semigroup by using different types of ideals. Also bi-ideal, quasi-ideal, prime ideal, completely prime ideal, semiprime ideal, completely semiprime ideal plays a major role to study regularities in ordered ternary semigroup. We discuss the connection between a semigroup cover of a ternary semigroup and the corrresponding ternary semigroup. Furthermore we find relation between the ordered power ternary semigroup and ternary semigroup. Then we introduced the notion of lattice structures in ternary semigroup of mappings. Afterthat, we also introduced the notion of right chain ordered ternary semigroup. Finally, we established the concept of (n, m, l)-ideal in an ordered ternary semigroup.

Scope for further research

The present study lays the groundwork for future research on different algebraic structures. This would be a fruitful area for further work. The knowledge gained in this study can inspire other scholars for future studies. We have already started to work on different classes of ternary semigroups and investigated many problems in ternary semigroup and ordered ternary semigroup. There are still several questions to be answered.

In chapter 3, we study different classes on the semigroup cover of a ternary semigroup and also discuss the isomorphism problem. One of the most significant conclusions that emerge from this study is that if two semigroup S_1 and S_2 are isomorphic then the associated semigroup cover $Q(S_1)$ and $Q(S_2)$ are isomorphic. But the converse is not true. We have shown by an example. Afterthat we give some few classes where the converse is true. So, there will be a scope to find some other classes in which the converse statement also holds.

Ternary semigroup of mappings denoted by $T[X, Y]$ is another significant topic which we discussed in chapter 5. We study lattice structures in $T[X, Y]$. There is enough oppourtunity to study some other algebraic properties in ternary semigroup of mappings $T[X, Y]$.

We introduced the notion of right chain ordered ternary semigroup in chapter 6. This is also a attractive perspective to work with. The methods used for this construction can be applied to another algebraic structutes in other branches of research work.

This thesis has provided a deeper insights for upcoming research.

List Of Publications

(1) Kar, S., Roy, A., and Dutta, I. : On regularities in po-ternary semigroups, Quasigroup and Related Systems 28.1 (2020) 149–158, (10 *pages*).

(2) Kar, S., Roy, A., and Dutta, I. : Ordered power ternary semigroups, Asian-European Journal of Mathematics 15.10, 2250180 (2022) (14 *pages*).
`https://doi.org/10.1142/S1793557122501807`

(3) Kar, S., Roy, A., and Purakait, S. : Lattice structures on ternary semigroup of mappings $T[X,Y]$, Afrika Matematika 33.92 (2022) (10 *pages*).
`https://doi.org/10.1007/s13370-022-01028-2`

(4) Kar, S., Roy, A., and Dutta, I : Semigroup cover of ternary semigroup. (Communicated)

(5) Kar, S., Roy, A. : (n,m,l)-ideals in ordered ternary semigroups. (Communicated)

(6) Kar, S., Roy, A. : On right chain ordered ternary semigroups. (Communicated)

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