k-SMOOTHNESS OF BOUNDED LINEAR OPERATORS AND RELATED TOPICS

Subhrajit Dey

(Index No.: 6/20/Math./26)

THIS THESIS IS SUBMITTED IN PARTIAL FULFILMENT OF THE

REQUIREMENTS FOR THE AWARD OF THE DEGREE OF

DOCTOR OF PHILOSOPHY IN SCIENCE

DEPARTMENT OF MATHEMATICS

JADAVPUR UNIVERSITY

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যাদবপুর বিশ্ববিদ্যালয়

FACULTY OF SCIENCE
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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "k−**SMOOTHNESS OF BOUNDED LINEAR OPERATORS AND RELATED TOPICS**" submitted by **Subhrajit Dey** who got his name registered on 14/09/2020 (**Index No.: 6/20/Math./26**) for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon his own research work under the supervision of **Prof. Kallol Paul**, Department of Mathematics, Jadavpur University, Kolkata 700032, India and that neither this thesis nor any part of it has been submitted for any degree/diploma or any other academic award anywhere before.

PROF. KALLOL PAUL
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K Paul 19.11.22

(Prof. Kallol Paul)
(Signature of the Supervisor
and date with official seal)

*Dedicated to my mother **Mrs. Mita Dey***
*and my father **Mr. Bharat Chandra Dey***

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Subhrajit Dey

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Abstract

The study of $k-$smoothness of operators between Banach spaces is relatively new area of research in the geometry of Banach spaces. The study of $k-$smoothness and Birkhoff-James orthogonality plays an important role in the geometry of Banach spaces. One of the most interesting aspects of Birkhoff-James orthogonality is the relation between orthogonality of operators and that of norm attainment set in the ground space. We first characterize $k-$smoothness of an element on the unit sphere of a finite-dimensional polyhedral Banach space and k-smoothness of an operator $T \in \mathbb{L}(\ell_\infty^n, \mathbb{Y})$, where $\mathbb{Y}$ is a two-dimensional Banach space with the additional condition that T attains norm at each extreme point of the unit ball $B_{\ell_\infty^n}$. Then we characterize $k-$smoothness of an operator defined between ℓ_∞^3 and ℓ_1^3. Next we study $k-$smoothness of bounded linear operators defined between infinite-dimensional Hilbert spaces. We also characterize $k-$smoothness of operators on some particular spaces, namely $\mathbb{L}(\mathbb{X}, \ell_\infty^n)$, $\mathbb{L}(\ell_\infty^3, \mathbb{Y})$, where $\mathbb{X}$ is a finite-dimensional Banach space and $\mathbb{Y}$ is a two-dimensional Banach space. Study of $k-$smoothness is deeply related to extreme contractions, the characterization of which is still elusive, in the general setting of Banach spaces. As an application of the study of k-smoothness of operators, we characterize extreme contractions defined between ℓ_∞^3 and $\mathbb{Y}$, where $\mathbb{Y}$ is a two-dimensional polygonal Banach space. Then we obtain extreme contractions defined between finite dimensional polyhedral Banach spaces using $k-$smoothness of operators. We explicitly compute the number of extreme contractions in some special Banach spaces. Next we explore the connections between the numerical radius norm and operator norm under certain conditions. We characterize nu-smoothness of order k for a bounded linear operator defined on a finite-dimensional Banach space. Also we characterize nu-extreme contractions defined on two-dimensional polygonal Banach spaces. Next we obtain the structure of the set of extreme points in the dual of $\mathbb{L}(\mathbb{X})_w$, where $\mathbb{X}$ is a two-dimensional polygonal Banach space. Then we move our attention to the study of Birkhoff-James orthogonality of bounded linear operators. We explore the relation between the orthogonality of bounded linear operators in the space of operators and that of elements in the ground space. We continue exploring the validity of the BŠ (Bhatia-Šemrl) Property in the setting of different Banach spaces. We characterize the space ℓ_∞^3 among all 3-dimensional polyhedral Banach spaces whose unit ball have exactly eight extreme points.

CONTENTS

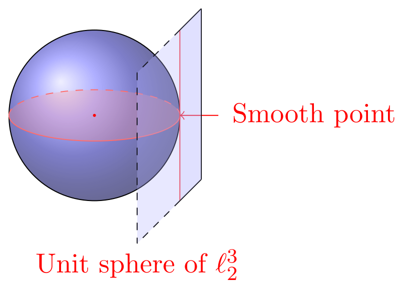
CHAPTER 1

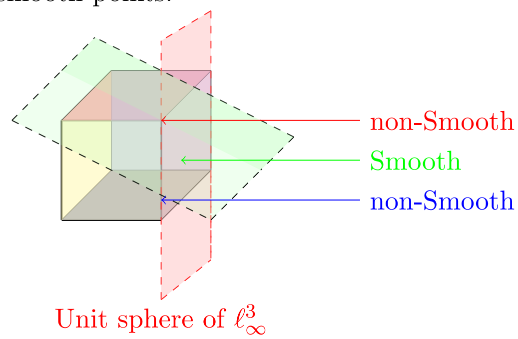
INTRODUCTION

The study of $k-$smoothness is relatively a new area of research in the geometric theory of Banach space. The concept of $k-$smoothness has evolved out of smoothness, the study of which is classical area of research in Banach space theory. Throughout we assume that $\mathbb{X}$, $\mathbb{Y}$ are real Banach spaces and $\mathbb{H}$ is a real Hilbert space. Let $S_{\mathbb{X}}$ and $B_{\mathbb{X}}$ denote the unit sphere and unit ball of the space $\mathbb{X}$, i.e., $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$ and $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}$. By $\mathbb{L}(\mathbb{X}, \mathbb{Y})(\mathbb{K}(\mathbb{X}, \mathbb{Y}))$, we denote the collection of all bounded(compact) linear operators defined between $\mathbb{X}$ and $\mathbb{Y}$. In case $\mathbb{X} = \mathbb{Y}$, we write $\mathbb{L}(\mathbb{X}, \mathbb{Y}) = \mathbb{L}(\mathbb{X})$ and $\mathbb{K}(\mathbb{X}, \mathbb{Y}) = \mathbb{K}(\mathbb{X})$. Let $\mathbb{X}^*$ denote the Banach space of all bounded linear functionals on $\mathbb{X}$, which is known as the dual space of $\mathbb{X}$. An element x of the unit sphere $S_{\mathbb{X}}$ is said to be a smooth element if there exists unique linear functional $x^* \in S_{\mathbb{X}^*}$ such that $x^*(x) = 1$. The question that arises naturally is if x is not a smooth element then how many such norm one linear functionals can be there which attain norm at x and out of them how many will be linearly independent. This concept sowed a seed of motivation to study the order of smoothness of an element x if it is not a smooth element. The major part of the thesis focuses on the study of smoothness of a bounded linear operator defined between two Banach or Hilbert spaces. Several Mathematicians have studied $k-$smoothness on various type of Banach spaces. Readers can look at the following papers [14, 24, 32, 67]. Let us first interpret the geometric notions of smoothness and $k-$smoothness.

Definition 1.1 (Smoothness). *An element $x \in S_{\mathbb{X}}$ is said to be a smooth point if there is a unique linear functional $f \in \mathbb{X}^*$ such that $\|f\| = 1$ and $f(x) = \|x\| = 1$. Equivalently, a geometric definition of a smooth point is as follows: An element $x \in S_{\mathbb{X}}$ is said to be a smooth point if there is a unique hyperplane H supporting the unit ball $B_{\mathbb{X}}$ at x. A Banach space is said* *to be smooth if every element of the unit sphere is smoooth.*

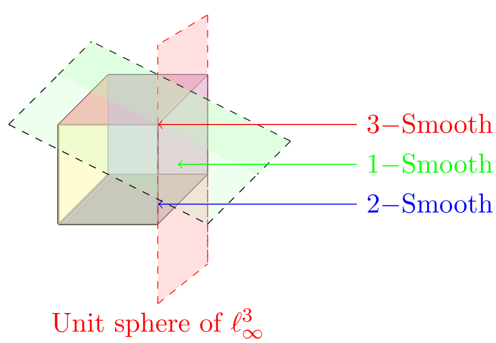
Every element of the unit sphere of $\ell_2^3(\mathbb{R})$ is a smooth element whereas in $\ell_\infty^3(\mathbb{R})$, $(1,0,0)$ is a smooth point but $(1,1,0)$ and $(1,1,1)$ are non-smooth points.





Consider $J(x) = \{x^* \in S_{\mathbb{X}^*} : x^*(x) = 1\}$. It is easy to see that $J(x)$ is a weak*-compact convex subset of $S_{\mathbb{X}^*}$. The notion of smoothness has been generalized by Khalil and Saleh [24] in 2005. They introduced the notion of multi-smoothness or $k-$smoothness depending on the "size" of $J(x)$ as follows:

Definition 1.2 ($k-$smoothness)**.** *[24] Let $\mathbb{X}$ be a Banach space and $x \in S_{\mathbb{X}}$. Then x is said to be $k-$smooth or smooth of order k if $J(x)$ contains exactly k linearly independent elements. In particular, x is $1-$smooth or simply smooth if $J(x)$ is singleton.*



We will study $k-$smoothness of operators defined between finite-dimensional real polyhedral Banach spaces, the definition of which goes as follows.

Definition 1.3. *A polyhedron P is a non-empty compact subset of $\mathbb{X}$ which is the intersection of finitely many closed half-spaces of $\mathbb{X}$, that is, $P = \cap_{i=1}^r M_i$, where M_i are closed half-spaces in $\mathbb{X}$ and $r \in \mathbb{N}$. The dimension $dim(P)$ of the polyhedron P is defined as the dimension of the subspace generated by the differences $v - w$ of vectors $v, w \in P$.*

Definition 1.4. *A polyhedron Q is said to be a face of the polyhedron P if either $Q = P$ or if we can write $Q = P \cap \delta M$, where M is a closed half-space in $\mathbb{X}$ containing P and δM denotes the* *boundary of M. If $dim(Q) = i$, then Q is called an i-face of P. If $dim(P) = n$, then $(n-1)$-faces of P are called facets of P and 1-faces of P are called edges of P.*

Let S be a convex subset of the space $\mathbb{X}$. An element $x \in S$ is said to be an extreme point of the set S if whenever $x = (1-t)y + tz$ for some $t \in (0,1)$ and $y, z \in S$ then $x = y = z$. The collection of all extreme points of S is denoted by $Ext\,(S)$ or $Ext\,S$. The space $\mathbb{X}$ is said to be strictly convex if $Ext\,(B_{\mathbb{X}}) = S_{\mathbb{X}}$. An operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is said to be an extreme contraction if T is an extreme point of the unit sphere of $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. The extreme contractions for operators defined on Hilbert spaces is well-known, they are the maximal partial isometries [13]. However the same for operators defined between Banach spaces is yet to be completely understood, even for finite-dimensional spaces it is still elusive. Here we will try to apply the notion of $k-$smoothness to study the extreme contractions for operators defined between Banach spaces. We need the following well-known results.

Theorem 1.1. *[68, Theorem 2.1] Let $\mathbb{X}, \mathbb{Y}$ be Banach spaces over the field $\mathbb{K}(=\mathbb{R}\, or\, \mathbb{C})$, then $Ext\ B_{\mathbb{K}(\mathbb{X},\mathbb{Y})^*} = \{x^{**} \otimes y^* \in \mathbb{K}(\mathbb{X}, \mathbb{Y})^* : x^{**} \in Ext\ B_{\mathbb{X}^{**}}, y^* \in Ext\ B_{\mathbb{Y}^*}\}$, where $x^{**} \otimes y^* : \mathbb{K}(\mathbb{X}, \mathbb{Y}) \to \mathbb{K},\ (x^{**} \otimes y^*)(T) = x^{**}(T^* y^*)$ for every $T \in \mathbb{K}(\mathbb{X}, \mathbb{Y})$.*

Theorem 1.2. *[68, Corollary 2.2] Let $\mathbb{X}$ be a reflexive Banach space over the field $\mathbb{K}(=\mathbb{R}\, or\, \mathbb{C})$, then $Ext\ B_{\mathbb{K}(\mathbb{X},\mathbb{Y})^*} = \{y^* \otimes x \in \mathbb{K}(\mathbb{X}, \mathbb{Y})^* : x \in Ext\ B_{\mathbb{X}}, y^* \in Ext\ B_{\mathbb{Y}^*}\}$, where $y^* \otimes x : \mathbb{K}(\mathbb{X}, \mathbb{Y}) \to \mathbb{K},\ (y^* \otimes x)(T) = y^*(Tx)$ for every $T \in \mathbb{K}(\mathbb{X}, \mathbb{Y})$.*

Motivated by the work of Lindenstrauss and Perles in [33], the following two definitions were introduced recently in [51, 58], to study extreme contractions.

Definition 1.5. *[58] Let $\mathbb{X}, \mathbb{Y}$ be Banach spaces. We say that the pair $(\mathbb{X}, \mathbb{Y})$ has L-P (abbreviated from Lindenstrauss-Perles) property if $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ is an extreme contraction if and only if $T(Ext(B_{\mathbb{X}})) \subseteq Ext(B_{\mathbb{Y}})$.*

Definition 1.6. *[51] Let $\mathbb{X}, \mathbb{Y}$ be Banach spaces. We say that the pair $(\mathbb{X}, \mathbb{Y})$ has weak L-P property if for each extreme contraction $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$, $T(Ext(B_{\mathbb{X}})) \cap Ext(B_{\mathbb{Y}}) \neq \emptyset$.*

In due course of time we show that some of the results obtained in [58, 51] follows easily from the results obtained by us. The notion of M-ideal plays a very important role in our scheme of things.

Definition 1.7. *Let $\mathbb{X}$ be a Banach space and V is a closed subspace of $\mathbb{X}$. The subspace V is said to be an M-ideal in $\mathbb{X}$ if $\mathbb{X}^* = V^* \oplus_1 V^{\perp}$, where $V^{\perp} = \{x^* \in \mathbb{X}^* : V \subseteq ker(x^*)\}$ and if $x^* = x_1^* + x_2^*$ is the unique decomposition of x^* then $\|x^*\| = \|x_1^*\| + \|x_2^*\|$.*

The role of norm attainment set [60] is essential in our study. For an operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, the norm attainment set, denoted by M_T, is defined as

$$M_T = \{x \in S_{\mathbb{X}}, \|Tx\| = \|T\|\}.$$

Again the structure of norm attainment set is well-known if the operators are defined between Hilbert spaces and the same is still unknown and intriguing for Banach spaces. To study $k-$smoothness of an operator T we have to investigate how many norm one linearly independednt linear functionals are there which attain its norm at T. In this connection the following result describing the $Ext\, J(T)$ under some assumptions on the space and operator is very useful in our study.

Lemma 1.1. *[68, Lemma 3.1] Suppose that $\mathbb{X}$ is a reflexive Banach space. Suppose that $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ is an $M-$ideal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y}), \|T\| = 1$ and $dist(T, \mathbb{K}(\mathbb{X}, \mathbb{Y})) < 1$. Then $M_T \cap Ext(B_{\mathbb{X}}) \neq \emptyset$ and*

$$Ext\ J(T) = \{y^* \otimes x \in \mathbb{K}(\mathbb{X}, \mathbb{Y})^* : x \in M_T \cap Ext(B_{\mathbb{X}}), y^* \in Ext\ J(Tx)\},$$

where $y^* \otimes x : \mathbb{K}(\mathbb{X}, \mathbb{Y}) \to \mathbb{R}$ is defined by $y^* \otimes x(S) = y^*(Sx)$ for every $S \in \mathbb{K}(\mathbb{X}, \mathbb{Y})$.

The numerical range of a bounded linear operator T on a complex Hilbert space $\mathbb{H}$, denoted by $W(T)$, is defined as $W(T) = \{\langle Tx, x\rangle : x \in S_{\mathbb{H}}\}$. The numerical radius of a bounded linear operator T, to be denoted by $w(T)$, is defined as $w(T) = sup\{|\langle Tx, x\rangle| : x \in S_{\mathbb{H}}\}$. If $\mathbb{H}$ is a complex Hilbert space then $w(T)$ defines a norm on $\mathbb{L}(\mathbb{H})$. The natural generalization of numerical radius for the Banach space $\mathbb{X}$ is as follows:

$$w(T) = \sup\{|x^*(Tx)| : x^* \in S_{\mathbb{X}^*}, x \in S_{\mathbb{X}}, x^*(x) = 1\}.$$

The numerical radius $w(T)$ does not always define a norm on $\mathbb{L}(\mathbb{X})$. We consider only those real finite-dimensional Banach spaces $\mathbb{X}$ such that numerical radius defines a norm on $\mathbb{L}(\mathbb{X})$ and use the symbol $\mathbb{L}(\mathbb{X})_w$ to denote the space of bounded linear operators endowed with the numerical radius norm. Motivated by the notion of smooth operator of order k or $k-$smooth operator, we generalize the notion of nu-smooth operator in the following way.

Definition 1.8. *Let $\mathbb{X}$ be a Banach space. A non-zero operator $T \in \mathbb{L}(\mathbb{X})_w$ is said to be nu-smooth of order k if there exist exactly k linearly independent elements $f_1, f_2, \ldots, f_k \in J_w(T)$, where $J_w(T) = \{f \in S_{\mathbb{L}(\mathbb{X})_w^*} : f(T) = w(T)\}$. In other words, T is said to be nu-smooth of order k if*

$$k = \dim\ span\ J_w(T) = \dim\ span\ Ext\, J_w(T).$$

We study nu-smooth operators of order k for some special Banach spaces. We also study nu-extreme contractions for some special Banach spaces. Note that an operator $T \in \mathbb{L}(\mathbb{X})_w$ is said to be nu-extreme contraction if T is an extreme point of the unit sphere of $\mathbb{L}(\mathbb{X})_w$. Last two chapters of the thesis deals with the study of some important properties of Birkhoff-James orthogonality [3, 20, 21] for bounded linear operators. The notion of Birkhoff-James orthogonality in a Banach space is well-known and is used extensively in the study of the geometry of Banach spaces. For $x, y \in \mathbb{X}$, x is said to be orthogonal to y in the sense of Birkhoff-James, written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$. Similarly, for $T, A \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, T is said to be Birkhoff-James orthogonal to A, written as $T \perp_B A$, if $\|T + \lambda A\| \geq \|T\|$ for all $\lambda \in \mathbb{R}$. For the n-dimensional Euclidean space $\mathbb{E}^n$, Bhatia and Šemrl [2] and Paul [49] independently proved that for $T, A \in \mathbb{L}(\mathbb{E}^n)$, $T \perp_B A$ if and only if there exists $x \in S_{\mathbb{E}^n}$ such that $\|Tx\| = \|T\|$ and $Tx \perp_B Ax$. Note that the sufficient part of the above theorem is true whenever the domain space and the co-domain space are any normed linear spaces of any dimension, i.e., if there exists $x \in M_T$ such that $Tx \perp_B Ax$ then $T \perp_B A$. On the other hand, the necessary part of the said theorem is not true in general Banach spaces, even if $\dim(\mathbb{X})$ is finite [31, 63, 62]. Sain and Paul [63] proved that if T is a linear operator on a finite-dimensional Banach space $\mathbb{X}$ with $M_T = D \cup (-D)$, where D is a connected subset of $S_{\mathbb{X}}$ then $T \perp_B A$ imples that there exists $x \in M_T$ such that $Tx \perp_B Ax$. An operator T is said to satisfy BŠ (Bhatia-Šemrl) Property [62] if for an operator A, $T \perp_B A$ implies that there exists $x \in M_T$ such that $Tx \perp_B Ax$. The following theorem characterises BŠ Property when the space is of dimension 2.

Theorem 1.3. *[62, Th. 2.4] A linear operator T on a 2-dimensional Banach space $\mathbb{X}$ satisfies the Bhatia-Šemrl Property if and only if T attains its norm only on $D \cup (-D)$, where D is a non-empty connected subset of $S_{\mathbb{X}}$.*

The validity of the above result remains unknown, when the dimension of $\mathbb{X}$ is strictly greater than 2. The following conjecture remains open to the best of our knowledge.

Conjecture 1.1. *[62, Conj. 2.5] A linear operator T on a finite-dimensional Banach space $\mathbb{X}$ satisfies the Bhatia-Šemrl Property if and only if $M_T = D \cup (-D)$, where D is a connected subset of $S_{\mathbb{X}}$.*

We observe that the sufficient part of the above conjecture is true, but the validity of the necessary part remains unknown. In view of this, the authors [62] defined the Bhatia-Šemrl (BŠ) Property of a bounded linear operator $T \in \mathbb{L}(\mathbb{X})$. It is possible to extend the definition of the BŠ Property in a more general way, without giving the restriction that the domain space and the co-domain space are identical. We now state the following definition of the BŠ Property in more general way.

Definition 1.9. *[50] Let $\mathbb{X}, \mathbb{Y}$ be Banach spaces and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$. We say that T satisfies the Bhatia-Šemrl (BŠ) Property if for any $A \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, $T \perp_B A$ implies that there exists $x \in M_T$ such that $Tx \perp_B Ax$.*

Next we introduce the definition of BŠ pair which plays a crucial role in the whole scheme of things.

Definition 1.10. *Let $\mathbb{X}, \mathbb{Y}$ be Banach spaces. We say that the pair $(\mathbb{X}, \mathbb{Y})$ is a BŠ pair if for every $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, T satisfies the BŠ Property if and only if $M_T = D \cup (-D)$, where D is a non-empty connected subset of $S_{\mathbb{X}}$.*

We focus on the study of BŠ Property and BŠ pair of spaces in our last two chapters. We note that the study further indicates that the conjecture 1.1 most likely to be true. We next give a brief outline of the thesis.

1.1 Outline of the thesis

The thesis consists of eight chapters including the introductory one. In the introductory chapter we provide a brief history of $k-$smoothness of elements and operators and also the history of BŠ Property in the context of Birkhoff-James orthogonality of bounded linear operators. We mention some definitions and notations to be used throughout the thesis.

In Chapter 2, we characterize $k-$smoothness of an element on the unit sphere of a finite-dimensional polyhedral Banach space. Then we study $k-$smoothness of an operator $T \in \mathbb{L}(\ell_\infty^n, \mathbb{Y})$, where $\mathbb{Y}$ is a two-dimensional Banach space with the additional condition that $M_T \subseteq B_{\ell_\infty^n}$. We also characterize $k-$smoothness of an operator $T \in \mathbb{L}(\ell_\infty^3, \ell_1^3)$.

In Chapter 3, we investigate $k-$smoothness of bounded linear operators defined between arbitrary Hilbert spaces. We then study the problem in the setting of both finite and infinite-dimensional Banach spaces. We also characterize $k-$smoothness of operators on some particular spaces. As an application, we characterize extreme contractions on $\mathbb{L}(\ell_\infty^3, \mathbb{Y})$, where $\mathbb{Y}$ is a two-dimensional polygonal Banach space.

In Chapter 4, we characterize extreme contractions defined between finite dimensional polyhedral Banach spaces using $k-$smoothness of operators. Using this we explicitly compute the number of extreme contractions in some special Banach spaces. Our approach in this paper in studying extreme contractions lead to the improvement and generalization of previously known results.

In Chapter 5, we completely characterize the $k-$smoothness of bounded linear operators defined on ℓ_∞^3 the proof of which is given explicitly.

In Chapter 6, we explore the connections between the numerical radius norm and operator norm under certain conditions. Then we characterize nu-smoothness of order k for a bounded linear operator defined on a finite-dimensional Banach space. Also we characterize nu-extreme contractions defined on two-dimensional polygonal Banach spaces. Finally we obtain the structure of the set of extreme points in the dual of $\mathbb{L}(\mathbb{X})_w$, where $\mathbb{X}$ is a two-dimensional polygonal Banach space.

Chapter 7 and 8 deals with the study of some important properties of Birkhoff-James orthogonality of bounded linear operators. In Chapter 7, we explore the relation of Birkhoff-James orthogonality between the elements in operator space and ground space. In this context, we introduce the notion of Property P_n for a Banach space and illustrate its connection with orthogonality of a bounded linear operator between Banach spaces. We further study Property P_n for various polyhedral Banach spaces. In Chapter 8, we study operators satisfying Bhatia-Šemrl(BŠ) Property. We show that $(\ell_1^n, \mathbb{Y})$ is a BŠ pair for any normed linear space $\mathbb{Y}$ and also obtain that $(\ell_\infty^3, \ell_\infty^3)$ is a BŠ pair. Finally, we characterize the space ℓ_∞^3 among all 3-dimensional polyhedral Banach spaces whose unit ball have exactly eight extreme points.

Before we end this section we would like to mention that in the beginning of each of the following chapters, we provide a brief motivation and for the convenience of the reader we provide the relevant notations and terminology to keep each chapter independent.

CHAPTER 2

STUDY OF $K-$SMOOTHNESS ON FINITE-DIMENSIONAL POLYHEDRAL BANACH SPACES

2.1 Introduction

The study of $k-$smoothness plays an important role to identify the structure of the unit ball of a Banach space. The papers [14, 15, 24, 32] contain the study of $k-$smooth points of many of the Banach spaces. There are several papers including [14, 24, 32, 38, 39, 36, 67] that contain the study of $k-$smoothness of operators on different spaces. In [36], authors have obtained a relation between $k-$smoothness and extreme points of the unit ball of a polyhedral Banach space. The purpose of this chapter is to characterize the order of smoothness of an element on the unit sphere of a finite-dimensional polyhedral Banach space. We also study $k-$smoothness of an operator defined between polyhedral Banach spaces. Let us first fix the notation and terminology.

Letters $\mathbb{X}$, $\mathbb{Y}$ denote Banach spaces. Throughout the chapter we assume the Banach spaces

Content of this chapter is based on the following paper:
S. Dey, A. Mal and K. Paul; $k-$smoothness on polyhedral Banach spaces, **Colloq. Math.**, 169 (2022), no. 1, 25-37.

to be real. We denote the unit ball and the unit sphere of $\mathbb{X}$ respectively by $B_{\mathbb{X}}$ and $S_{\mathbb{X}}$, i.e., $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}, S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$. Let $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ denote the space of all bounded linear operators between $\mathbb{X}$ and $\mathbb{Y}$. For $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, M_T denotes the collection of all unit vectors of $\mathbb{X}$ at which T attains its norm, i.e., $M_T = \{x \in S_{\mathbb{X}} : \|Tx\| = \|T\|\}$. For a set A, the cardinality of A is denoted by $|A|$. The dual space of $\mathbb{X}$ is denoted by $\mathbb{X}^*$. An element $x \in S_{\mathbb{X}}$ is said to be an extreme point of the convex set $B_{\mathbb{X}}$ if and only if $x = (1-t)y+tz$ for some $y, z \in B_{\mathbb{X}}$ and $t \in (0, 1)$ implies that $y = z = x$. For $x, y \in \mathbb{X}$, let $L[x, y] = \{tx + (1-t)y : 0 \leq t \leq 1\}$ and $L(x, y) = \{tx + (1-t)y : 0 < t < 1\}$. The set of all extreme points of $B_{\mathbb{X}}$ is denoted by $Ext(B_{\mathbb{X}})$. An element $x^* \in S_{\mathbb{X}^*}$ is said to be a supporting linear functional of $x \in S_{\mathbb{X}}$, if $x^*(x) = 1$. For a unit vector x, let $J(x)$ denote the set of all supporting linear functionals of x, i.e., $J(x) = \{x^* \in S_{\mathbb{X}^*} : x^*(x) = 1\}$. The set $J(x)$ for $x \in S_{\mathbb{X}}$ plays a significant role to study the $k-$smoothness. By the Hahn-Banach Theorem, it is easy to verify that $J(x) \neq \emptyset$, for all $x \in S_{\mathbb{X}}$. We would like to mention that $J(x)$ is a weak*-compact convex subset of $S_{\mathbb{X}^*}$. A unit vector x is said to be a smooth point if $J(x)$ is singleton. $\mathbb{X}$ is said to be a smooth Banach space if every unit vector of $\mathbb{X}$ is smooth. The set of all extreme points of $J(x)$ is denoted by $Ext\ J(x)$, where $x \in S_{\mathbb{X}}$. In 2005, Khalil and Saleh [24] defined $k-$smooth points as follows: An element $x \in S_{\mathbb{X}}$ is said to be $k-$smooth or the order of smoothness of x is k, if $J(x)$ contains exactly k linearly independent supporting linear functionals of x. In other words, x is $k-$smooth, if $\dim\ span\ J(x) = k$. Moreover, from [32, Prop. 2.1], we get that x is $k-$smooth, if $k = \dim\ span\ Ext\ J(x)$. Similarly, for $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ with $\|T\| = 1$, $J(T) = \{F \in \mathbb{L}(\mathbb{X}, \mathbb{Y})^* : \|F\| = 1, F(T) = 1\}$ and T is said to be $k-$smooth operator, if $k = \dim\ span\ J(T) = \dim\ span\ Ext\ J(T)$. Observe that, $1-$smooth points of $S_{\mathbb{X}}$ are the smooth points of $S_{\mathbb{X}}$. The spaces that we are dealing with in this chapter are mostly finite-dimensional polyhedral Banach spaces. A finite-dimensional Banach space $\mathbb{X}$ is said to be *polyhedral* if the unit ball $B_{\mathbb{X}}$ of $\mathbb{X}$ contains only finitely many extreme points. Equivalently, a finite-dimensional Banach space $\mathbb{X}$ is a *polyhedral Banach space*, if $B_{\mathbb{X}}$ is a polyhedron. In particular, a two-dimensional polyhedral Banach space is said to be a *polygonal Banach space*.

For a convex set C, $\text{int}_r(C)$ denotes the relative interior of the set C, i.e., $x \in \text{int}_r(C)$ if there exists $\epsilon > 0$ such that $B(x, \epsilon) \cap \text{affine}(C) \subseteq C$, where $\text{affine}(C)$ is the intersection of all affine sets containing C and an affine set is defined as the translation of a vector subspace. A non-empty convex subset F of C is said to be a face of C, if for $x, y \in C$ and $t \in (0, 1)$, $(1-t)x + ty \in F \Rightarrow x, y \in F$.

In this chapter, we first prove that a point on the relative interior of an i-face of the unit ball of an n-dimensional polyhedral Banach space is $(n-i)-$smooth. In [39], the authors completely characterized the $k-$smoothness of an operator defined between two Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, where $\dim(\mathbb{X}) = \dim(\mathbb{Y}) = 2$ and in [38], the authors characterized the $k-$smoothness of a bounded linear operator defined between ℓ_∞^3 and a two-dimensional Banach space. We continue our study in this direction and characterize the $k-$smoothness of a bounded linear operator defined between ℓ_∞^n and a two-dimensional Banach space with the assumption that the linear operator attains its norm at all the extreme points of the unit ball of ℓ_∞^n. Then we characterize $k-$smoothness of a bounded linear operator defined between ℓ_∞^3 and ℓ_1^3.

We state the following lemma [68, Lemma 3.1], characterizing $Ext\ J(T)$, which will be used often. For simplicity we state the lemma for finite-dimensional Banach spaces.

Lemma 2.1. *[68, Lemma 3.1] Suppose that $\mathbb{X}, \mathbb{Y}$ are finite-dimensional Banach spaces. Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ and $\|T\| = 1$ Then*

$$Ext\ J(T) = \{y^* \otimes x \in \mathbb{L}(\mathbb{X}, \mathbb{Y})^* : x \in M_T \cap Ext(B_{\mathbb{X}}), y^* \in Ext\ J(Tx)\},$$

where $y^* \otimes x : \mathbb{L}(\mathbb{X}, \mathbb{Y}) \to \mathbb{R}$ is defined by $y^* \otimes x(S) = y^*(Sx)$ for every $S \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$.

2.2 $k-$smooth points of polyhedral Banach spaces and operators spaces

We begin this section with a relation between the order of smoothness of a unit vector x in a polyhedral Banach space and the dimension of the face F such that x is in the relative interior of F.

Theorem 2.1. *Let $\mathbb{X}$ be an n-dimensional polyhedral Banach space. Let F be an i-face of $B_{\mathbb{X}}$. Let $x \in int_r(F)$. Then x is $(n-i)-$smooth.*

Proof. Let $f \in Ext\ J(x)$. Since $x \in \text{int}_r(F)$, we have for all $y \in F$, $f(y) = 1$. Therefore,

$$F \subseteq \cap_{f \in Ext\ J(x)} \{y \in S_{\mathbb{X}} : f(y) = 1\} = A \text{ (say)}.$$

Clearly, A is a face of $B_{\mathbb{X}}$. If possible, suppose that $F \subsetneqq A$. Then there exists $z \in A \setminus F$. Now, $x \in F \subseteq A$ and $z \in A \Rightarrow tx + (1-t)z \in A$ for all $t \in [0, 1]$, since A is a face. Using convexity argument of norm, it is easy to observe that $\|x + \lambda(z - x)\| \geq \|x\|$ for all scalars λ. Moreover, we have $\|x + (z - x)\| = \|z\| = 1$. If possible, let $\|x + \lambda_0(z - x)\| = 1$ for some $\lambda_0 < 0$. Then

$$x = tz + (1-t)\{x + \lambda_0(z - x)\}, \quad \text{where } t = \frac{-\lambda_0}{1 - \lambda_0} \in (0, 1).$$

Since F is a face and $x \in F$, we get $z \in F$, a contradiction. Thus $\|x + \lambda(z - x)\| > 1$ for all $\lambda < 0$. Let $Y = span\{x, z\}$. Then $\dim(Y) = 2$ and Y is a polygonal Banach space. Using $\|x + (z - x)\| = 1$ and $\|x + \lambda(z - x)\| > 1$ for all $\lambda < 0$, it is easy to observe that $x \in Ext(B_Y)$. Thus, by [39, Th. 3.5], we get x is $2-$smooth in Y. Let h, g be two linearly independent elements of $Ext(B_{Y^*})$ such that $h(x) = g(x) = 1$. If $h(z) = g(z) = 1$, then $h(x - z) = g(x - z) = 0 \Rightarrow \ker(h) = \ker(g) \Rightarrow h = \lambda g$, for some scalar λ, a contradiction. Without loss of generality, suppose that $g(z) \neq 1$. By [66, Lemma 1.2, Page 168], there exists $g_1 \in Ext(B_{\mathbb{X}^*})$ such that $g_1|_Y = g$. Now, $g_1(x) = 1$ and $g_1 \in Ext(B_{\mathbb{X}^*}) \Rightarrow g_1 \in Ext\ J(x)$. Thus, $g_1(z) \neq 1$ contradicts that $z \in A$. Therefore, $A \setminus F = \emptyset \Rightarrow A = F$, i.e.,

$$F = \cap_{f \in Ext\ J(x)} \{y \in S_{\mathbb{X}} : f(y) = 1\} = \cap_{f \in Ext\ J(x)} (x + \ker(f)) \cap S_{\mathbb{X}}.$$

This implies that $i = \dim(F) = \dim(\cap_{f \in Ext\ J(x)} ker(f))$. Now, let x be $k-$smooth. Let $\{f_1, f_2, \ldots, f_k\}$ be the set of all linearly independent vectors of $Ext\ J(x)$. Then

$$\begin{aligned}
& f \in Ext\ J(x) \Rightarrow f = \sum_{j=1}^{k} a_j f_j,\ (a_j \in \mathbb{R}) \\
\Rightarrow\ & \cap_{j=1}^{k} \ker(f_j) \subseteq \ker(f) \\
\Rightarrow\ & \cap_{j=1}^{k} \ker(f_j) \subseteq \cap_{f \in Ext\ J(x)} ker(f) \subseteq \cap_{j=1}^{k} \ker(f_j) \\
\Rightarrow\ & \cap_{f \in Ext\ J(x)} \ker(f) = \cap_{j=1}^{k} \ker(f_j) \\
\Rightarrow\ & i = \dim(\cap_{j=1}^{k} \ker(f_j)) = n - k \\
\Rightarrow\ & k = n - i.
\end{aligned}$$

This completes the proof of the theorem. $\square$

Remark 2.2. *Note that, if $\mathbb{X}$ is an n-dimensional polyhedral Banach space and F is a facet of $B_{\mathbb{X}}$, then from Theorem 2.1, we get for each $x \in int_r(F)$, x is smooth. On the other hand, if F is a 0-face, i.e., $F = \{x\}$, then x is $n-$smooth. In this case, clearly x is an extreme point of $B_{\mathbb{X}}$. It is worth mentioning that Theorem 2.1 generalizes [39, Th. 3.5].*

Now, we focus on the space of all operators defined between some particular polyhedral Banach spaces. First we study $k-$smoothness of an operator defined between ℓ_∞^n and an arbitrary two-dimensional Banach space. To do so we need the following two lemmas.

Lemma 2.2. *[39, Lemma 2.1] Suppose $\mathbb{X}, \mathbb{Y}$ are finite-dimensional Banach spaces. If $\{x_1, x_2, \ldots, x_m\}$ is a linearly independent subset of $\mathbb{X}$ and $\{y_1^*, y_2^*, \ldots, y_n^*\}$ is a linearly independent subset of $\mathbb{Y}^*$ then $\{y_i^* \otimes x_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a linearly independent subset of $\mathbb{L}(\mathbb{X}, \mathbb{Y})^*$.*

Lemma 2.3. *Let $\mathbb{X} = \ell_\infty^n$ and $\mathbb{Y}$ be a two-dimensional Banach space. Let $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ be such that Rank$(T) = 2$ and $Ext(B_{\mathbb{X}}) \subseteq M_T$. Then the followings hold:*
(i) $T(B_{\mathbb{X}})$ is a convex set with 4 extreme points.
(ii) If $T(B_{\mathbb{X}})$ is the convex hull of $\{\pm z_1, \pm z_2\}$, then either for each $x \in Ext(B_{\mathbb{X}})$, $Tx \in \pm L[z_1, z_2]$ or for each $x \in Ext(B_{\mathbb{X}})$, $Tx \in \pm L[z_1, -z_2]$.

Proof. (i) Follows from [36, Remark 2.13].

(ii) Suppose $e_i = (0, 0, \ldots, 1, 0, \ldots, 0)$ with 1 at the $i-$th coordinate and 0 at the remaining places. Since, Rank$(T) = 2$, $Te_i \neq 0$ for some $i \in \{1, 2, \ldots, n\}$. Without loss of generality, we assume $i = 1$, i.e., $Te_1 \neq 0$. It is well-known that $B_{\mathbb{X}} = conv(K \cup -K)$, where $K = \{(1, u_2, \ldots, u_n) : |u_i| \leq 1, 2 \leq i \leq n\}$. Now, K can be expressed as $K = e_1 + F$, where $e_1 = (1, 0, \ldots, 0)$ and $F = \{(0, u_2, \ldots, u_n) : |u_i| \leq 1, 2 \leq i \leq n\}$. Observe that $x \in Ext(B_{\mathbb{X}})$ if and only if there exists $u \in Ext(F)$ such that either $x = e_1 + u$ or $x = -e_1 + u$. Clearly, $T(B_{\mathbb{X}}) = conv\Big(T(K) \cup T(-K)\Big)$, where $T(K) = T(e_1 + F) = Te_1 + T(F)$ and $T(-K) = T(-e_1 + F) = -Te_1 + T(F)$. Now, $T(F)$ must be a symmetric convex set about origin, since F is so. If $T(F)$ has more than four extreme points then proceeding similarly as in [36, Lemma 2.11], it can be shown that there exists $v \in Ext(B_{\mathbb{X}})$ such that $v \notin M_T$, a contradiction. Thus, $T(F)$ has at most four extreme points.
First suppose $T(F)$ has only two extreme points say $\pm y$, i.e., $T(F)$ is the convex hull of $\{y, -y\}$. Clearly, $T(B_{\mathbb{X}})$ is the convex hull of $\pm Te_1 \pm y$. Now, for each $x \in Ext(F), Tx \in L[y, -y]$. Therefore, for each $z \in Ext(B_{\mathbb{X}})$, $Tz \in \pm L[Te_1 + y, Te_1 - y]$ and hence we are done.
Next, suppose $T(F)$ has four distinct extreme points say $\pm y_1, \pm y_2$. We prove the rest of the lemma in the following two steps.

Step 1. We claim that $T(B_{\mathbb{X}})$ is of the form $conv\{\pm(Te_1 - y_2), \pm(Te_1 + y_1)\}$ or $conv\{\pm(Te_1 - y_1), \pm(Te_1 + y_2)\}$ or $conv\{\pm(Te_1 + y_1), \pm(Te_1 + y_2)\}$ or $conv\{\pm(Te_1 - y_1), \pm(Te_1 - y_2)\}$.

Since $y_1, y_2 \in Ext(T(F))$, there exist $x_1, x_2 \in Ext(F)$ such that $Tx_1 = y_1, Tx_2 = y_2$. Now, $T(F) = conv\{\pm y_1, \pm y_2\}$ gives that $T(B_{\mathbb{X}}) = conv\{\pm Te_1 \pm y_1, \pm Te_1 \pm y_2\}$. Note that $\pm Te_1 \pm y_i = \pm Te_1 \pm Tx_i$, for $i = 1, 2$ and $\pm e_1 \pm x_i \in Ext(B_{\mathbb{X}})$. Therefore, $\| \pm Te_1 \pm y_i\| = 1$, for $i = 1, 2$. Since $\mathbb{Y}$ is two-dimensional and $\{y_1 + y_2, y_1 - y_2\}$ is linearly independent, $Te_1 = a(y_1 + y_2) + b(y_1 - y_2)$, where $a, b \in \mathbb{R}$. We assert that either $a = 0, b \neq 0$ or $a \neq 0, b = 0$. Clearly, a and b can not be simultaneously zero as $Te_1 \neq 0$. If possible, suppose that $a \neq 0, b \neq 0$. If $a > 0, b > 0$, then

$$\begin{aligned} & Te_1 - y_1 \\ = \; & \frac{2a}{2a+2b+1}(Te_1 + y_2) + \frac{2b}{2a+2b+1}(Te_1 - y_2) \\ + \; & \frac{1}{2a+2b+1}(-Te_1 - y_1) \end{aligned}$$

and $\frac{2a}{2a+2b+1}, \frac{2b}{2a+2b+1}, \frac{1}{2a+2b+1} \in (0,1)$. Moreover, we have, $\|Te_1 + y_2\| = \|Te_1 - y_2\| = \|Te_1 + y_1\| = 1$. Using this, it can be easily observed that $\|Te_1 - Tx_1\| = \|Te_1 - y_1\| < 1$, which contradicts that $e_1 - x_1 \in Ext(B_{\mathbb{X}}) \subseteq M_T$. Similarly, considering the other possible cases $a < 0, b < 0$ or $a < 0, b > 0$ or $a > 0, b < 0$, we get a contradiction. This establishes our assertion. Assume that $a = 0, b > 0$.

Then we have

$$Te_1 - y_1 = \frac{2b}{2b+1}(Te_1 - y_2) + \frac{1}{2b+1}(-Te_1 - y_1) \text{ and} \tag{2.1}$$

$$-Te_1 - y_2 = \frac{1}{2b+1}(Te_1 - y_2) + \frac{2b}{2b+1}(-Te_1 - y_1). \tag{2.2}$$

Thus, the only extreme points of $T(B_{\mathbb{X}})$ are $\pm(Te_1 - y_2)$ and $\pm(Te_1 + y_1)$, i.e, $T(B_{\mathbb{X}}) = conv\{\pm(Te_1 - y_2), \pm(Te_1 + y_1)\}$. Similarly considering other cases, we can show that $T(B_{\mathbb{X}})$ is of the form as claimed in Step 1.

Step 2. Claim that either $Tz \in \pm L[Te_1 + y_1, -Te_1 + y_2]$ for each $z \in Ext(B_{\mathbb{X}})$ or $Tz \in \pm L[Te_1 + y_2, -Te_1 + y_1]$ for each $z \in Ext(B_{\mathbb{X}})$ or $Tz \in \pm L[Te_1 + y_1, -Te_1 - y_2]$ for each $z \in Ext(B_{\mathbb{X}})$ or $Tz \in \pm L[Te_1 - y_1, -Te_1 + y_2]$ for each $z \in Ext(B_{\mathbb{X}})$.

Suppose that $T(B_{\mathbb{X}}) = conv\{\pm(Te_1 - y_2), \pm(Te_1 + y_1)\}$. Let $z \in Ext(B_{\mathbb{X}})$ be arbitrary. We show that $Tz \in \pm L[Te_1 + y_1, -Te_1 + y_2]$. Now, there exists $x \in Ext(F)$ such that either $z = x + e_1$ or $z = x - e_1$. First let $z = x + e_1$. Now, if Tx is an interior point of $T(F)$, then $Tz = Tx + Te_1$ will be an interior point of $T(B_{\mathbb{X}})$ and hence $||Tz|| < ||T||$, a contradiction as $z \in Ext(B_{\mathbb{X}})$. Thus, Tx is on the boundary of $T(F)$, i.e., $Tx \in \pm L[y_1, y_2] \cup \pm L[y_1, -y_2]$. If $Tx \in \pm L[y_1, y_2]$, then clearly $Tz = Tx + Te_1 \in \pm L[Te_1 + y_1, -Te_1 + y_2]$ and we are done.

If possible, let $Tx \in L(y_1, -y_2)$, i.e., $Tx = (1-\lambda)y_1 + \lambda(-y_2), 0 < \lambda < 1$. Then by Equation

(2.2), we have

$$\begin{aligned} & -Tx + Te_1 \\ = \; & (1-\lambda)(Te_1 - y_1) + \lambda(Te_1 + y_2) \\ = \; & (1-\lambda)(Te_1 - y_1) + \lambda\Big(\frac{1}{2b+1}(-Te_1 + y_2) + \frac{2b}{2b+1}(Te_1 + y_1)\Big) \\ = \; & (1-\lambda)(Te_1 - y_1) + \frac{\lambda}{2b+1}(-Te_1 + y_2) + \frac{2b\lambda}{2b+1}(Te_1 + y_1). \end{aligned}$$

Since $0 < \lambda < 1$, we have, $1-\lambda, \frac{\lambda}{2b+1}, \frac{2b\lambda}{2b+1} \in (0,1)$. Moreover, we have, $\|Te_1 - y_1\| = \|-Te_1 + y_2\| = \|Te_1 + y_1\| = 1$. Using this, it can be easily observed that $\|-Tx + Te_1\| < \|T\|$, where $-x + e_1 \in Ext(B_{\mathbb{X}})$, a contradiction.

Similarly, if $Tx \in -L(y_1, -y_2)$, then we can show that $\|Tz\| = \|Tx + Te_1\| < \|T\|$, where $z = x + e_1 \in Ext(B_{\mathbb{X}})$, a contradiction. Therefore, we must have $Tx \in \pm L[y_1, y_2]$, i.e., $Tz = Tx + Te_1 \in \pm L[Te_1 + y_1, -Te_1 + y_2]$.

Now if $z = x - e_1$, following the same line of arguments, we can show that $Tz \in \pm L[Te_1 + y_1, -Te_1 + y_2]$.

Considering the other possibilities of $T(B_{\mathbb{X}})$ and proceeding similarly we can establish our claim stated in Step 2. This completes the proof of the lemma. □

The following lemma is needed to prove the desired theorem.

Lemma 2.4. *Any face of ℓ_∞^n having exactly 2^k extreme points contains exactly $k+1$ linearly independent extreme points.*

Now, we are ready to prove our desired result. We completely characterize $k-$smoothness of an operator defined between ℓ_∞^n and any two-dimensional Banach space with the condition that the operator attains its norm at each element of $Ext(B_{\ell_\infty^n})$. We solve the problem in the following two theorems. In the following theorem, we consider the case in which image of each extreme point of $B_{\ell_\infty^n}$ is smooth.

Theorem 2.3. *Let $\mathbb{X} = \ell_\infty^n$ and $\mathbb{Y}$ be a two-dimensional Banach space. Let $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ be such that $Ext(B_{\mathbb{X}}) \subseteq M_T$ and Tx is smooth for all $x \in Ext(B_{\mathbb{X}})$. Then the followings hold:*

(i) If $Rank(T) = 1$, then T is $n-$smooth.

(ii) Let $Rank(T) = 2$. If Tx is an interior point of some line segment of $T(B_{\mathbb{X}})$ for some $x \in Ext(B_{\mathbb{X}})$, then T is $n-$smooth. Otherwise, T is $(2n-2)-$smooth.

Proof. Let us write $Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \ldots, \pm x_{2^{n-1}}\}$, where $\{x_1, x_2, \ldots, x_n\}$ is linearly independent.

(i) Suppose $Rank(T) = 1$. Then $Tx_i = \pm Tx_1$ for all $2 \leq i \leq 2^{n-1}$. Let $J(Tx_1) = \{y^*\}$. Then for any $i \in \{1, 2, \ldots, 2^{n-1}\}$, either $J(Tx_i) = \{y^*\}$ or $J(Tx_i) = \{-y^*\}$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= \textit{dim span } J(T) \\ &= \textit{dim span Ext } J(T) \\ &= \textit{dim span } \{y^* \otimes x_i : 1 \leq i \leq 2^{n-1}\} \\ &= \textit{dim span } \{y^* \otimes x_i : 1 \leq i \leq n\} \\ &= n, \end{aligned}$$

as $\{y^* \otimes x_i : 1 \leq i \leq n\}$ is linearly independent by Lemma 2.2. Hence, T is $n-$smooth.

(ii) Suppose $Rank(T) = 2$. Then by Lemma 2.3, $T(B_{\mathbb{X}})$ is a convex set with four extreme points. Let $\pm y_1, \pm y_2$ be four distinct extreme points of $T(B_{\mathbb{X}})$.

First suppose Tx is an interior point of some line segment of $T(B_{\mathbb{X}})$ for some $x \in Ext(B_{\mathbb{X}})$, i.e., $Tx_j \in L(y_1, y_2)$ for some $1 \leq j \leq 2^{n-1}$. Again by Lemma 2.3, we get $Tx_i \in \pm L[y_1, y_2]$ for all $1 \leq i \leq 2^{n-1}$. Let $J(Tx_j) = \{y^*\}$. Then it is clear that for any $i \in \{1, 2, \ldots, 2^{n-1}\}$, either $J(Tx_i) = \{y^*\}$ or $J(Tx_i) = \{-y^*\}$. Then as in case (i) it is easy to show that T is $n-$smooth.

Next, suppose that Tx is not an interior point of any line segment of $T(B_{\mathbb{X}})$ for any $x \in Ext(B_{\mathbb{X}})$. Then $Tx_i \notin L(\pm y_1, \pm y_2)$ for any $1 \leq i \leq 2^{n-1}$. Thus, $Tx_i \in \{\pm y_1, \pm y_2\}$ for all $1 \leq i \leq 2^{n-1}$. Since, $Rank(T) = 2$, without loss of generality, we may assume $Tx_1 = y_1, Tx_2 = y_2$. Let $J(Tx_1) = \{z_1^*\}$ and $J(Tx_2) = \{z_2^*\}$. Then for any $i \in \{1, 2, \ldots, 2^{n-1}\}$,

$$J(Tx_i) = \ \{z_1^*\} \textit{ or } \{-z_1^*\} \textit{ or } \ \{z_2^*\} \textit{ or } \{-z_2^*\}.$$

Let $S_1 = \{x_i \in Ext(B_{\mathbb{X}}) : Tx_i = Tx_1\}$ and $S_2 = \{x_i \in Ext(B_{\mathbb{X}}) : Tx_i = Tx_2\}$. Thus, we have $S_1 \cap S_2 = \emptyset$, $\pm S_1 \cup \pm S_2 = Ext(B_{\mathbb{X}})$ and $|S_1 \cup S_2| = |S_1| + |S_2| = 2^{n-1}$. Therefore, for any $i \in \{1, 2, \ldots, 2^{n-1}\}$,

$$\begin{aligned} J(Tx_i) &= \{z_1^*\}, \text{ if } x_i \in S_1 \\ &= \{z_2^*\}, \text{ if } x_i \in S_2. \end{aligned}$$

Now, it is clear that S_1 as well as S_2 cannot contain n linearly independent vectors. For otherwise, we get $Rank(T) = 1$, a contradiction. Thus, maximal linearly independent subsets of S_1 and S_2 contain at most $n-1$ elements. Now, we show that $|S_1| = 2^{n-2}$ and $|S_2| = 2^{n-2}$. If possible, let $|S_1| < 2^{n-2}$. Then we must have $|S_2| > 2^{n-2}$. Observe that $conv(S_2)$ is a face of $B_{\mathbb{X}}$. Clearly, $Ext(conv(S_2)) = S_2$, i.e., $|Ext(conv(S_2))| = |S_2| > 2^{n-2}$. Hence, the face $conv(S_2)$

of $B_{\mathbb{X}}$ contains at least 2^{n-1} extreme points and hence by Lemma 2.4, $conv(S_2)$ contains at least n linearly independent extreme points. Thus, S_2 contains at least n linearly independent vectors. This gives that $Rank(T) = 1$, a contradiction. Therefore, $|S_1| \geq 2^{n-2}$. Similarly, $|S_2| \geq 2^{n-2}$. Thus, $|S_1| = |S_2| = 2^{n-2}$, i.e., $|Ext(conv(S_1))| = |Ext(conv(S_2))| = 2^{n-2}$. Now, by Lemma 2.4, S_1 and S_2 has exactly $n-1$ linearly independent vectors. Without loss of generality, suppose that the set of all linearly independent vectors of S_1 and S_2 are $\{u_1, u_2, \ldots, u_{n-1}\}$ and $\{u_n, u_{n+1}, \ldots, u_{2n-2}\}$ respectively. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= \dim \, span \; J(T) \\ &= \dim \, span \; Ext \; J(T) \\ &= \dim \, span \; \{z_1^* \otimes u_i, z_2^* \otimes u_j : u_i \in S_1, u_j \in S_2\} \\ &= \dim \, span \; \{z_1^* \otimes u_i, z_2^* \otimes u_j : 1 \leq i \leq n-1, n \leq j \leq 2n-2\} \\ &= 2n-2, \text{which can be easily verified.} \end{aligned}$$

Therefore, T is $(2n-2)-$smooth. This completes the proof. $\square$

In addition to Theorem 2.3, if we assume that the range space is strictly convex and smooth, then we obtain the following corollary.

Corollary 2.1. *Let $\mathbb{X} = \ell_\infty^n$ and $\mathbb{Y}$ be a two-dimensional strictly convex, smooth Banach space. Let $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ be such that $Ext(B_{\mathbb{X}}) \subseteq M_T$. Then the followings hold:*
(i) If $Rank(T) = 1$, then T is $n-$smooth.
(ii) If $Rank(T) = 2$, then T is $(2n-2)-$smooth.

Proof. (i) follows clearly from Theorem 2.3. We only prove (ii). Let $Rank(T) = 2$. Then by Lemma 2.3, $T(B_{\mathbb{X}})$ is a convex set with four extreme points. Without loss of generality, let $\pm Tx_1, \pm Tx_2$ be four distinct extreme points of $T(B_{\mathbb{X}})$. If possible, suppose that there exists $x \in Ext(B_{\mathbb{X}})$ such that Tx is an interior point of some line segment of $T(B_{\mathbb{X}})$. Suppose that $Tx \in L(Tx_1, Tx_2)$. Since $x \in M_T, \|Tx\| = 1$. Thus, it is clear that $\|y\| = 1$ for all $y \in L[Tx_1, Tx_2]$, i.e., $L[Tx_1, Tx_2] \subseteq S_{\mathbb{Y}}$. This contradicts that $\mathbb{Y}$ is strictly convex. Therefore, there does not exist any $x \in Ext(B_{\mathbb{X}})$ such that Tx is an interior point of some line segment of $T(B_{\mathbb{X}})$. Hence, from Theorem 2.3, we conclude that if $Rank(T) = 2$, then T is $(2n-2)-$smooth. $\square$

In the next theorem, we consider the remaining case in which the image of at least one extreme point of $B_{\ell_\infty^n}$ is not a smooth point. Note that in a two-dimensional Banach space, a non-zero vector is either smooth or $2-$smooth.

Theorem 2.4. *Let $\mathbb{X} = \ell_\infty^n$ and $\mathbb{Y}$ be a two-dimensional Banach space. Let $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ be such that $Ext(B_\mathbb{X}) \subseteq M_T$. Let $S = \{x \in Ext(B_\mathbb{X}) : Tx$ is not smooth$\}$ be non-empty and S_1 be the subset of S containing all linearly independent vectors of S. If $|S_1| = k$, then T is $(n+k)-$smooth.*

Proof. Let $Ext(B_\mathbb{X}) = \{\pm x_1, \pm x_2, \ldots, \pm x_{2^{n-1}}\}$. First suppose that $Rank(T) = 1$. Since $S \neq \emptyset$, assume $x_1 \in S$. Then for all $2 \leq i \leq 2^{n-1}$, $Tx_i = \pm Tx_1$. Let $Ext\ J(Tx_1) = \{z_1^*, z_2^*\}$ for some $z_1^*, z_2^* \in S_{\mathbb{Y}^*}$. Then either $Ext\ J(Tx_i) = \{z_1^*, z_2^*\}$ or $Ext\ J(Tx_i) = \{-z_1^*, -z_2^*\}$ for all $2 \leq i \leq 2^{n-1}$. Clearly, $|S_1| = n$. Now,

$$\begin{aligned}
& \dim\ span\ Ext\ J(T) \\
= & \dim\ span\ \{z_1^* \otimes x_i, z_2^* \otimes x_i : 1 \leq i \leq 2^{n-1}\} \\
= & \dim\ span\ \{z_1^* \otimes x_i, z_2^* \otimes x_i : 1 \leq i \leq n\} \\
= & 2n, \text{ using Lemma 2.2.}
\end{aligned}$$

Thus, in this case T is $2n-$smooth and we are done.

Next, suppose that $Rank(T) = 2$. Then by Lemma 2.3, $T(B_\mathbb{X})$ is a convex set with four extreme points. Without loss of generality, let $\pm y_1, \pm y_2$ be four distinct extreme points of $T(B_\mathbb{X})$. We consider the following two cases:

Case I : $S = Ext(B_\mathbb{X})$.

Clearly, in this case $|S_1| = n$. Let $S_1 = \{x_1, x_2, \ldots, x_n\}$. Observe that if $Tx_i \in L(\pm y_1, \pm y_2)$, for any $1 \leq i \leq 2^{n-1}$, then Tx_i will be smooth, which contradicts that $S = Ext(B_\mathbb{X})$. Thus, $Tx_i \in \{\pm y_1, \pm y_2\}$ for all $1 \leq i \leq 2^{n-1}$. Since $Rank(T) = 2$, $Tx_i = y_1, Tx_j = y_2$ for some $1 \leq i \neq j \leq 2^{n-1}$. Therefore, y_1, y_2 are not smooth. Suppose that $Ext\ J(y_1) = \{z_1^*, z_2^*\}$ and $Ext\ J(y_2) = \{z_3^*, z_4^*\}$. Then for each $1 \leq i \leq 2^{n-1}$, $Ext\ J(Tx_i)$ is either $\{z_1^*, z_2^*\}$ or $\{-z_1^*, -z_2^*\}$ or $\{z_3^*, z_4^*\}$ or $\{-z_3^*, -z_4^*\}$. Observe that $z_3^*, z_4^* \in span\{z_1^*, z_2^*\}$, since $\dim(\mathbb{Y}) = 2$. Hence, for any $x \in \mathbb{X}, z_3^* \otimes x, z_4^* \otimes x \in span\ \{z_1^* \otimes x, z_2^* \otimes x\}$. Therefore,

$$\begin{aligned}
& \dim\ span\ Ext\ J(T) \\
= & \dim\ span\ \{z_1^* \otimes x_i, z_2^* \otimes x_i : 1 \leq i \leq 2^{n-1}\} \\
= & \dim\ span\ \{z_1^* \otimes x_i, z_2^* \otimes x_i : 1 \leq i \leq n\} \\
= & 2n.
\end{aligned}$$

Thus, in this case T is $2n-$smooth and we are done.

Case II : $S \subsetneqq Ext(B_\mathbb{X})$.

Without loss of generality, we may assume that $S_1 = \{x_1, x_2, \ldots, x_k\}$ and $S = \{\pm x_1, \pm x_2, \ldots, \pm x_k,$

$\pm x_{k+1}, \ldots, \pm x_m\}$, $(1 \leq m < 2^{n-1}, 1 \leq k \leq n)$. As in Case I, $Tx_i \in \{\pm y_1, \pm y_2\}$ for all $1 \leq i \leq m$. Clearly, at least one of y_1, y_2 is not smooth. First we assume both of y_1, y_2 are not smooth. Using Lemma 2.3, we get either $Tx_i \in \pm L[y_1, y_2]$ for each $m < i \leq 2^{n-1}$ or $Tx_i \in \pm L[-y_1, y_2]$ for each $m < i \leq 2^{n-1}$. Without loss of generality, assume that $Tx_i \in \pm L[y_1, y_2]$ for each $m < i \leq 2^{n-1}$. Since, y_1, y_2 are not smooth and Tx_i are smooth, $Tx_i \notin \{\pm y_1, \pm y_2\}$ for $m < i \leq 2^{n-2}$. Therefore, $Tx_i \in \pm L(y_1, y_2)$ for each $m < i \leq 2^{n-1}$. Now, it is easy to observe that either $J(Tx_i) = \{y^*\}$ or $J(Tx_i) = \{-y^*\}$ for each $m < i \leq 2^{n-1}$. Suppose $Ext\ J(y_1) = \{y^*, z_1^*\}$ and $Ext\ J(y_2) = \{y^*, z_2^*\}$. Then for each $1 \leq i \leq m$, $Ext\ J(Tx_i)$ is either $\{y^*, z_1^*\}$ or $\{-y^*, -z_1^*\}$ or $\{y^*, z_2^*\}$ or $\{-y^*, -z_2^*\}$. Observe that $z_2^* \in span\{y^*, z_1^*\}$, since $\dim(\mathbb{Y}) = 2$. Hence, for any $x \in \mathbb{X}, z_2^* \otimes x \in span\ \{y^* \otimes x, z_1^* \otimes x\}$. Therefore,

$$\begin{aligned}
& \dim\ span\ Ext\ J(T) \\
= \ & \dim\ span\ \{y^* \otimes x_i, z_1^* \otimes x_j : 1 \leq i \leq 2^{n-1}, 1 \leq j \leq m\} \\
= \ & \dim\ span\ \{y^* \otimes x_i, z_1^* \otimes x_j : 1 \leq i \leq 2^{n-1}, 1 \leq j \leq k\} \\
= \ & n + k, \ \text{(by Lemma 2.2)},
\end{aligned}$$

since $\{x_i : 1 \leq i \leq 2^{n-1}\}$ contains only n linearly independent vectors. Therefore, T is $(n+k)-$smooth.

Now, if we consider exactly one of y_1, y_2 is not smooth, then following same line of arguments, we can prove that T is $(n+k)-$smooth. This completes the proof of the theorem. □

We would like to mention that Theorem 2.3 and Theorem 2.4 improves on [38, Th. 3.10]. The study of $k-$smoothness of an operator defined between ℓ_∞^n and $\mathbb{Y}$ becomes more complicated when $\dim \mathbb{Y} \geq 3$. The rest of the chapter is devoted to the study of $k-$smoothness of an operator defined between two particular spaces ℓ_∞^3 and ℓ_1^3. We denote the extreme points of $B_{\ell_\infty^3}$ by $\pm x_1 = \pm(1,1,1), \pm x_2 = \pm(-1,1,1), \pm x_3 = \pm(-1,-1,1), \pm x_4 = \pm(1,-1,1)$. $|M_T \cap Ext(B_{\ell_\infty^3})|$ plays an important role in determining the order of smoothness of T. Observe that if $|M_T \cap Ext(B_{\ell_\infty^3})| \leq 6$, then the order of smoothness of T can be obtained using [39, Th. 2.2]. Therefore, we only consider the case for which $|M_T \cap Ext(B_{\ell_\infty^3})| = 8$, i.e., $M_T \cap Ext(B_{\ell_\infty^3}) = \{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$. Note that, for $1 \leq i \leq 4$, Tx_i is $k-$smooth, where $k \in \{1,2,3\}$. Suppose $S_k = \{x \in M_T \cap Ext(B_{\ell_\infty^3}) : Tx \ is\ k-smooth\}$, where $k \in \{1,2,3\}$. Clearly, $|S_1| + |S_2| + |S_3| = 8$. In the following theorem, we consider the case when $|S_1| = 8$.

Theorem 2.5. *Let $\mathbb{X} = \ell_\infty^3$ and $\mathbb{Y} = \ell_1^3$. Let $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ be such that $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$. Let $|S_1| = 8$. Then the followings hold:*
(i) If $\pm J(Tx_i) = \pm J(Tx_j)$ for all $x_i, x_j \in S_1$, then T is $3-$smooth.
(ii) Otherwise, T is $4-$smooth.

Proof. (i) Suppose the given condition is satisfied. Let $\pm J(Tx_i) = \{\pm y^*\}$ for $1 \leq i \leq 4$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= \dim \ span \ J(T) \\ &= \dim \ span \ Ext \ J(T) \\ &= \dim \ span \ \{y^* \otimes x_1, y^* \otimes x_2, y^* \otimes x_3, y^* \otimes x_4\} \\ &= \dim \ span \ \{y^* \otimes x_1, y^* \otimes x_2, y^* \otimes x_3\} \\ &= 3, \ (\text{using Lemma 2.2}). \end{aligned}$$

Hence, T is $3-$smooth.

(ii) Let $\pm J(Tx_i) = \{\pm y_i^*\}$ for $1 \leq i \leq 4$. Since (i) is not satisfied, without loss of generality, we assume $y_1^* \neq \pm y_2^*$, i.e., $\{y_1^*, y_2^*\}$ is linearly independent. Let $y_3^* = ay_1^* + by_2^*$ and $y_4^* = cy_1^* + dy_2^*$, where $a, b, c, d \in \mathbb{R}$. Since $\|y_3^*\| = 1$, a and b cannot be zero simultaneously. Similarly, c and d cannot be zero simultaneously. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= \dim \ span \ J(T) \\ &= \dim \ span \ Ext \ J(T) \\ &= \dim \ span \ \{y_1^* \otimes x_1, y_2^* \otimes x_2, y_3^* \otimes x_3, y_4^* \otimes x_4\}. \end{aligned}$$

Using the relations $x_4 = x_1 - x_2 + x_3$, $y_3^* = ay_1^* + by_2^*$, $y_4^* = cy_1^* + dy_2^*$ and Lemma 2.2, it is easy to verify that $\{y_1^* \otimes x_1, y_2^* \otimes x_2, y_3^* \otimes x_3, y_4^* \otimes x_4\}$ is linearly independent. Therefore, $k = 4$ and T is $4-$smooth. □

Proceeding similarly we can find the $k-$smoothness of operator T for other feasible cases. We skip the details of proof to avoid monotonicity. The following two tables illustrates the various possible cases of $k-$smoothness under different conditions on S_1, S_2 and S_3. The first table contains the cases when $S_3 = \emptyset$, i.e., Tx_i is either $1-$smooth or $2-$smooth but not $3-$smooth.

$\lvert S_1\rvert$	$\lvert S_2\rvert$	$\lvert S_3\rvert$	Further conditions on the operator T	T is $k-$smooth with $k=$
8	0	0	$\pm J(Tx_i)=\pm J(Tx_j)\ \forall\ x_i,x_j\in S_1$	3
			Otherwise	4
6	2	0	$\pm J(Tx_i)=\pm J(Tx_j)\ \forall\ x_i,x_j\in S_1$ and $\pm J(Tx_i)\subseteq \pm Ext\ J(Tx_k),\ \forall x_i\in S_1, x_k\in S_2$	4
			Otherwise	5
4	4	0	$\pm J(Tx_i)=\pm J(Tx_j)\ \forall\ x_i,x_j\in S_1$ and $\pm J(Tx_i)\subseteq \pm Ext\ J(Tx_k),\forall x_i\in S_1,\forall x_k\in S_2$	5
			Otherwise	6
2	6	0	$\lvert\cap_{x_k\in S_2}\pm J(Tx_k)\rvert\geq 2$ and $\pm J(Tx_i)\subseteq \pm Ext\ J(Tx_k),\forall x_i\in S_1,\forall x_k\in S_2$	6
			Otherwise	7
0	8	0	$\lvert\cap_{i=1}^4\pm J(Tx_i)\rvert=4$	6
			Either $\lvert\cap_{i=1}^4\pm J(Tx_i)\rvert=2$ or $\lvert\pm Ext\ J(Tx_i)\cap\pm Ext\ J(Tx_j)\rvert\neq 2$, for $1\leq i\neq j\leq 4$	7
			Otherwise	8

The next table exhibits the cases when $S_3\neq\emptyset$.

$\|S_1\|$	$\|S_2\|$	$\|S_3\|$	Further conditions on the operator T	T is $k-$smooth with $k =$
6	0	2	$\pm J(Tx_i) = \pm J(Tx_j), \forall\ x_i, x_j \in S_1$	5
			Otherwise	6
4	2	2	$\pm J(Tx_i) = \pm J(Tx_j), \forall\ x_i, x_j \in S_1$ and $\pm J(Tx_i) \subseteq \pm Ext\ J(Tx_j), \forall\ x_i \in S_1, x_j \in S_2$	6
			Otherwise	7
2	4	2	-	7
0	6	2	$\| \cap_{x_i \in S_2} \pm Ext\ J(Tx_i)\| = 4$	7
			Otherwise	8
4	0	4	-	7
0	4	4	-	8
0	0	8	-	9

Finally we would like to note that the following possibilities are not feasible: (i) $|S_1| = 2, |S_2| = 2, |S_3| = 4$, (ii) $|S_1| = 2, |S_3| = 6$ and (iii) $|S_2| = 2, |S_3| = 6$.

CHAPTER 3

CHARACTERIZATION OF $K-$SMOOTHNESS OF BOUNDED LINEAR OPERATORS

3.1 Introduction

The problem of characterizing $k-$smooth operators defined between arbitrary Banach or Hilbert spaces is relatively new but an important area of research in the field of geometry of Banach spaces. Characterization of smoothness of bounded linear operators has been studied in [56]. There are several papers including [14, 15, 24, 32, 39, 67] that contain the study of $k-$smoothness of operators on different spaces. In this chapter, our objective is to study the $k-$smoothness of bounded linear operators defined between infinite-dimensional spaces. We first fix the notations and terminologies to be used throughout the chapter.

Let $\mathbb{X}$, $\mathbb{Y}$ denote Banach spaces and $\mathbb{H}$ denote Hilbert space. Throughout the chapter we assume that the spaces are real unless otherwise mentioned. The unit ball and the unit sphere of $\mathbb{X}$ are denoted by $B_{\mathbb{X}}$ and $S_{\mathbb{X}}$ respectively, i.e., $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}, S_{\mathbb{X}} = \{x \in \mathbb{X} :$

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$\|x\| = 1\}$. The space of bounded (compact) linear operators between $\mathbb{X}$ and $\mathbb{Y}$ is denoted by $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ ($\mathbb{K}(\mathbb{X}, \mathbb{Y})$). If $\mathbb{X} = \mathbb{Y}$, then we write $\mathbb{L}(\mathbb{X}, \mathbb{Y}) := \mathbb{L}(\mathbb{X})$ and $\mathbb{K}(\mathbb{X}, \mathbb{Y}) := \mathbb{K}(\mathbb{X})$. $\mathbb{X}^*$ denotes the dual space of $\mathbb{X}$. An element $x \in S_{\mathbb{X}}$ is said to be an extreme point of the convex set $B_{\mathbb{X}}$ if and only if $x = (1-t)y + tz$ for some $y, z \in B_{\mathbb{X}}$ and $t \in (0, 1)$ implies that $y = z = x$. The set of all extreme points of $B_{\mathbb{X}}$ is denoted by $Ext(B_{\mathbb{X}})$. For $x, y \in \mathbb{X}$, let $L[x, y] = \{tx+(1-t)y : 0 \leq t \leq 1\}$ and $L(x, y) = \{tx + (1-t)y : 0 < t < 1\}$. A Banach space $\mathbb{X}$ is said to be a strictly convex Banach space if every element of the unit sphere $S_{\mathbb{X}}$ is an extreme point of the unit ball $B_{\mathbb{X}}$, equivalently, $\mathbb{X}$ is said to be a strictly convex Banach space, if the unit sphere of $\mathbb{X}$ does not contain non-trivial straight line segment. A face E of a convex set C is said to be an edge if for each $z \in E$, there exist extreme points x, y in C such that $z \in L[x, y]$. An element $x^* \in S_{\mathbb{X}^*}$ is said to be a supporting linear functional of $x \in S_{\mathbb{X}}$, if $x^*(x) = 1$. For a unit vector x, let $J(x)$ denote the set of all supporting linear functionals of x, i.e., $J(x) = \{x^* \in S_{\mathbb{X}^*} : x^*(x) = 1\}$. By the Hahn-Banach Theorem, $J(x) \neq \emptyset$, for all $x \in S_{\mathbb{X}}$. We would like to note that $J(x)$ is a weak*-compact convex subset of $S_{\mathbb{X}^*}$. The set of all extreme points of $J(x)$ is denoted by $Ext\ J(x)$, where $x \in S_{\mathbb{X}}$. A unit vector x is said to be a smooth point if $J(x)$ is singleton. $\mathbb{X}$ is said to be a smooth Banach space if every unit vector of $\mathbb{X}$ is smooth.

In 2005, Khalil and Saleh [24] generalized the notion of smoothness and introduced the notion of multi-smoothness or $k-$smoothness depending on the "size" of $J(x)$. An element $x \in S_{\mathbb{X}}$ is said to be $k-$smooth or the order of smoothness of x is k, if $J(x)$ contains exactly k linearly independent supporting linear functionals of x. In other words, x is $k-$smooth, if $\dim\ span\ J(x) = k$. Moreover, from [32, Prop. 2.1], we get that x is $k-$smooth, if $k = \dim\ span\ Ext\ J(x)$. Similarly, $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is said to be $k-$smooth operator, if $k = \dim\ span\ J(T) = \dim\ span\ Ext\ J(T)$. Observe that, $1-$smooth points of $S_{\mathbb{X}}$ are actually the smooth points of $S_{\mathbb{X}}$. In our study, the norm attainment set of an operator plays an important role which will be clear in due time. The norm attainment set of T, denoted as M_T, is defined as the collection of all unit vectors x at which T attains its norm, i.e., $M_T = \{x \in S_{\mathbb{X}} : \|Tx\| = \|T\|\}$. The notion of $k-$smoothness has a nice connection with extreme contraction which will be explored later. An operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is said to be an extreme contraction, if T is an extreme point of the unit ball of $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. A two-dimensional Banach space $\mathbb{X}$ is said to be a polygonal Banach space, if $B_{\mathbb{X}}$ contains only finitely many extreme points. Equivalently, a two-dimensional Banach space $\mathbb{X}$ is a polygonal Banach space, if $B_{\mathbb{X}}$ is a polygon.

From [32, Th. 3.8], we know that there is a large class of Banach spaces which does not contain $k-$smooth point, where $k \in \mathbb{N}$. The papers [14, 15, 24, 32, 39, 67] contain extensive study on multi-smoothness in Banach space and in operator space. In [67, Th. 2.4] Wójcik studied $k-$smoothness of compact operators defined between complex (real) Hilbert spaces. In this chapter, we obtain a complete characterization of $k-$smoothness of bounded linear operators defined between complex (real) Hilbert spaces. We prove that a bounded linear operator T defined on a complex (real) Hilbert space $\mathbb{H}$ is n^2-smooth $(\binom{n+1}{2}-$smooth) if and only if $M_T = S_{H_0}$, where $\dim(H_0) = n$ and $\|T\|_{H_0^{\perp}} < \|T\|$. Moving onto Banach spaces, the complete characterization of $k-$smooth operators defined between arbitrary Banach spaces is still not known, in fact it is elusive even for finite-dimensional Banach spaces. In [39], the authors characterized the $k-$smoothness of a bounded linear operator defined between two-dimensional Banach spaces. In this chapter, we continue our study in this direction and obtain sufficient conditions for $k-$smoothness of bounded linear operators defined between infinite-dimensional Banach spaces. We also obtain a relation between the order of smoothness of the operators T and T^*, where T is defined between finite-dimensional Banach spaces and T^* is the adjoint of T. Using this relation, we characterize the order of smoothness of an operator defined from a finite-dimensional Banach space to ℓ_{∞}^n, $(n \in \mathbb{N})$. We also obtain a characterization of the order of smoothness of $T \in \mathbb{L}(\ell_{\infty}^3, \mathbb{Y})$, where $\mathbb{Y}$ is a two-dimensional Banach space. As an application of this result, we characterize the extreme contractions in the space $\mathbb{L}(\ell_{\infty}^3, \mathbb{Y})$, where $\mathbb{Y}$ is a two-dimensional polygonal Banach space.

We state the following lemma [68, Lemma 3.1], characterizing $Ext\ J(T)$, which will be used often.

Lemma 3.1. *[68, Lemma 3.1] Suppose that $\mathbb{X}$ is a reflexive Banach space. Suppose that $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ is an $M-$ideal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y}), \|T\| = 1$ and $dist(T, \mathbb{K}(\mathbb{X}, \mathbb{Y})) < 1$. Then $M_T \cap Ext(B_{\mathbb{X}}) \neq \emptyset$ and*

$$Ext\ J(T) = \{y^* \otimes x \in \mathbb{K}(\mathbb{X}, \mathbb{Y})^* : x \in M_T \cap Ext(B_{\mathbb{X}}), y^* \in Ext\ J(Tx)\},$$

where $y^* \otimes x : \mathbb{K}(\mathbb{X}, \mathbb{Y}) \to \mathbb{R}$ is defined by $y^* \otimes x(S) = y^*(Sx)$ for every $S \in \mathbb{K}(\mathbb{X}, \mathbb{Y})$.

We end this section with the following definition: A subspace M of a Banach space $\mathbb{X}$ is said to be an $M-$ideal if there exists a projection P on $\mathbb{X}^*$ such that $P(\mathbb{X}^*) = \{x^* \in \mathbb{X}^* : x^*(m) = 0\ \forall\ m \in M\}$ and for all $x^* \in \mathbb{X}^*$,

$$\|x^*\| = \|P(x^*)\| + \|x^* - P(x^*)\|.$$

It is well known that for a Hilbert space $\mathbb{H}$, $\mathbb{K}(\mathbb{H})$ is an $M-$ideal in $\mathbb{L}(\mathbb{H})$ and for each $1 < p < \infty$, $\mathbb{K}(\ell_p)$ is an $M-$ideal in $\mathbb{L}(\ell_p)$. Interested readers are referred to [16] for more information in this topic.

3.2 $k-$smoothness of operators defined between Hilbert spaces

We begin this section with the study of $k-$smooth operators defined between arbitrary Hilbert spaces. We use the notion of Birkhoff-James orthogonality to prove the theorem. Recall that, for $x, y \in \mathbb{X}$, x is said to be Birkhoff-James orthogonal [3, 20] to y, written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for each scalar λ. Similarly, for $T, A \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, we say that $T \perp_B A$ if $\|T + \lambda A\| \geq \|T\|$ for each scalar λ.

Theorem 3.1. *Let $\mathbb{H}_1, \mathbb{H}_2$ be Hilbert spaces. Let $T \in S_{\mathbb{L}(\mathbb{H}_1,\mathbb{H}_2)}$ be such that $M_T = S_{H_0}$, where H_0 is a finite-dimensional subspace of $\mathbb{H}_1$ with $\dim(H_0) = n$ and $\|T\|_{H_0^\perp} < 1$. Then T is n^2-smooth, if $\mathbb{H}_1, \mathbb{H}_2$ are considered to be complex and it is $\binom{n+1}{2}-$smooth, if the spaces are considered to be real.*

Proof. We claim that $dist(T, \mathbb{K}(\mathbb{H}_1, \mathbb{H}_2)) < 1$. If possible, suppose that this is not true. Then for every $S \in \mathbb{K}(\mathbb{H}_1, \mathbb{H}_2)$, $dist(T, \text{Span}\{S\}) \geq dist(T, \mathbb{K}(\mathbb{H}_1, \mathbb{H}_2)) \geq 1$, i.e., for every scalar λ, $\|T - \lambda S\| \geq dist(T, \text{Span}\{S\}) \geq 1$. Therefore, $T \perp_B S$. Define $S : \mathbb{H} \to \mathbb{H}$ by $Sx = Tx$ whenever $x \in H_0$ and $Sx = 0$, whenever $x \in H_0^\perp$. Then clearly, $S \in \mathbb{K}(\mathbb{H}_1, \mathbb{H}_2)$, since H_0 is finite-dimensional. Hence, $T \perp_B S$. By [46, Th. 3.1], there exists $x \in M_T = S_{H_0}$ such that $Tx \perp Sx$, i.e., $Tx \perp Tx$, a contradiction. This establishes our claim. We note that $\mathbb{K}(\mathbb{H}_1, \mathbb{H}_2)$ is an $M-$ideal in $\mathbb{L}(\mathbb{H}_1, \mathbb{H}_2)$ and $dist(T, \mathbb{K}(\mathbb{H}_1, \mathbb{H}_2)) < 1$, so by Lemma 3.1, $Ext\ J(T) = \{y^* \otimes x : x \in M_T, y^* \in Ext\ J(Tx)\}$, where $y^* \otimes x(S) = y^*(Sx) = \langle Sx, Tx\rangle$ for every $S \in \mathbb{L}(\mathbb{H}_1, \mathbb{H}_2)$. So we can write $Ext\ J(T) = \{x \otimes Tx : x \in M_T\}$, where $x \otimes Tx(S) = \langle Sx, Tx\rangle$ for every $S \in \mathbb{L}(\mathbb{H}_1, \mathbb{H}_2)$.

Suppose that $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of H_0. Assume that $\mathbb{H}_1, \mathbb{H}_2$ are complex Hilbert spaces. Then

$$\begin{aligned}
& \dim\ span\ Ext\ J(T) \\
= & \dim\ span\ \{x \otimes Tx : x \in M_T\} \\
= & \dim\ span\ \Big\{ \sum_{i,j=1}^{n} a_i \overline{a_j} e_i \otimes Te_j : \sum_{i=1}^{n} |a_i|^2 = 1 \Big\} \\
= & \dim\ span\ \{e_i \otimes Te_j : 1 \leq i, j \leq n\} \\
= & n^2.
\end{aligned}$$

Next assume that the Hilbert spaces are real, then

$$\begin{aligned}
& \dim \ span \ Ext \ J(T) \\
= & \dim \ span \left\{ \sum_{i,j=1}^{n} a_i a_j e_i \otimes Te_j : \sum_{i=1}^{n} |a_i|^2 = 1 \right\} \\
= & \binom{n+1}{2}.
\end{aligned}$$

Thus, T is n^2-smooth, if $\mathbb{H}_1, \mathbb{H}_2$ are considered to be complex and it is $\binom{n+1}{2}-$smooth, if the spaces are considered to be real. □

Noting that for a compact operator T, $dist(T, \mathbb{K}(\mathbb{H}_1, \mathbb{H}_2)) = 0 < 1$, we get the following corollary which provides a sufficient condition for the $k-$smoothness of a compact operator [67, Th. 2.4] defined between Hilbert spaces.

Corollary 3.1. *Let $\mathbb{H}_1, \mathbb{H}_2$ be Hilbert spaces. Let $T \in S_{\mathbb{K}(\mathbb{H}_1,\mathbb{H}_2)}$ be such that $M_T = S_{H_0}$, where H_0 is a finite-dimensional subspace of $\mathbb{H}_1$ with $\dim(H_0) = n$. Then T is n^2-smooth, if $\mathbb{H}_1, \mathbb{H}_2$ are considered to be complex and it is $\binom{n+1}{2}-$smooth, if the spaces are considered to be real.*

Next we show that the conditions mentioned in Theorem 3.1 are necessary for $k-$smoothness. To do so, we need the following theorem on Birkhoff-James orthogonality for complex Hilbert spaces, in case of real Hilbert spaces an analogous theorem can be obtained from [40, Th. 3.2].

Theorem 3.2. *Let $\mathbb{H}_1, \mathbb{H}_2$ be complex Hilbert spaces. Let $T \in S_{\mathbb{L}(\mathbb{H}_1,\mathbb{H}_2)}$ be such that $dist(T, \mathbb{K}(\mathbb{H}_1, \mathbb{H}_2)) < 1$. Then for $A \in \mathbb{L}(\mathbb{H}_1, \mathbb{H}_2)$, $T \perp_B A$ if and only if there exists $x \in M_T$ such that $Tx \perp Ax$.*

Proof. If there exists $x \in M_T$ such that $Tx \perp Ax$, then it is easy to observe that $T \perp_B A$. Conversely, suppose that $T \perp_B A$. Then by [66, Th. 1.1, Page 170], there exists $\lambda_i \geq 0, f_i \in Ext \ J(T)$ for $1 \leq i \leq 3$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $(\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3)(A) = 0$. By Lemma 3.1, there exists $x_i \in M_T, y_i^* \in Ext \ J(Tx_i)$ for $1 \leq i \leq 3$ such that $f_i = y_i^* \otimes x_i$. Since Tx_i is smooth, it is easy to observe that $y_i^* \otimes x_i(A) = \langle Ax_i, Tx_i \rangle$. Thus,

$$\begin{aligned}
\sum_{i=1}^{3} \lambda_i f_i(A) &= 0 \\
\Rightarrow \sum_{i=1}^{3} \lambda_i y_i^* \otimes x_i(A) &= 0
\end{aligned}$$

$$\Rightarrow \sum_{i=1}^{3} \lambda_i \langle Ax_i, Tx_i \rangle = 0. \tag{3.1}$$

Consider the set $W = \{\langle Ax, Tx\rangle : x \in S_{H_0}\} = \{\langle Ax, Tx\rangle : x \in M_T\}$. Since H_0 is a subspace of $\mathbb{H}_1$, following the idea of [8, Th. 1], it can be easily verified that the set W is convex. Now, it follows from (3.1) that 0 is in the convex hull of W, i.e., $0 \in W$. Therefore, there exists $x \in M_T$ such that $\langle Ax, Tx\rangle = 0$ and so $Tx \perp Ax$. This completes the proof of the theorem. □

Now, we are ready to prove our desired theorem.

Theorem 3.3. *Let $\mathbb{H}$ be a separable complex Hilbert space and $T \in S_{\mathbb{L}(\mathbb{H})}$. Then T is n^2-smooth if and only if $M_T = S_{H_0}$, where H_0 is a finite-dimensional subspace of $\mathbb{H}$ with $\dim(H_0) = n$ and $\|T\|_{H_0^\perp} < 1$.*
In case the Hilbert space is real then the result still holds good with n^2-smoothness replaced by $\binom{n+1}{2}-$smoothness.

Proof. First suppose that $\mathbb{H}$ is a complex Hilbert space. The sufficient part of the theorem follows from Theorem 3.1. We only prove the necessary part. Suppose that T is n^2-smooth. Since $\mathbb{H}$ is separable, by [32, Th. 3.8], $\mathbb{L}(\mathbb{H})/\mathbb{K}(\mathbb{H})$ has no operator whose order of smoothness is finite. Hence, by [32, Remark 3.7], $dist(T, \mathbb{K}(\mathbb{H})) < 1$. Thus, by Lemma 3.1, $M_T \neq \emptyset$. From [63, Th. 2.2], we get $M_T = S_{H_0}$ for some subspace H_0 of $\mathbb{H}$. Now, let $A \in \mathbb{L}(\mathbb{H})$ be such that $T \perp_B A$. Then using Theorem 3.2, we get $x \in M_T$ such that $Tx \perp Ax$. Thus, by [46, Th. 3.1], H_0 is a finite-dimensional subspace of $\mathbb{H}$ and $\|T\|_{H_0^\perp} < 1$. Let $\dim(H_0) = k$. Then from Theorem 3.1, we get T is k^2-smooth. Therefore, $k^2 = n^2$, i.e., $k = n$. Thus, $\dim(H_0) = n$.
The proof of the theorem for a real Hilbert space follows similarly using [40, Th. 3.2]. □

3.3 $k-$smoothness of operators defined between Banach spaces

In this section, we study $k-$smoothness of operators defined between Banach spaces. We begin with the following simple lemma, the proof of which is given for the sake of completeness.

Lemma 3.2. *Suppose $\mathbb{X}, \mathbb{Y}$ are Banach spaces. If $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1^*, y_2^*, \ldots, y_n^*\}$ are linearly independent subsets of $\mathbb{X}$ and $\mathbb{Y}^*$ respectively, then $\{y_i^* \otimes x_j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ is a linearly independent subset of $\mathbb{L}(\mathbb{X}, \mathbb{Y})^*$.*

Proof. Let c_{ij} be scalars such that

$$\sum_{1\leq i\leq n, 1\leq j\leq m} c_{ij} y_i^* \otimes x_j = 0. \tag{3.2}$$

Choose $y \in \mathbb{Y}, \phi \in \mathbb{X}^*$. Define $S \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ by $Sx = \phi(x)y$ for all $x \in \mathbb{X}$. Now, from (3.2) we get,

$$\begin{aligned}
& \sum_{1\le i\le n, 1\le j\le m} c_{ij} y_i^* \otimes x_j(S) = 0 \\
\Rightarrow\ & \sum_{1\le i\le n, 1\le j\le m} c_{ij} y_i^* S(x_j) = 0 \\
\Rightarrow\ & \sum_{1\le i\le n, 1\le j\le m} c_{ij} \phi(x_j) y_i^*(y) = 0 \\
\Rightarrow\ & \phi\Bigg(\sum_{1\le i\le n, 1\le j\le m} c_{ij} x_j y_i^*(y) \Bigg) = 0 \\
\Rightarrow\ & \sum_{1\le i\le n, 1\le j\le m} c_{ij} x_j y_i^*(y) = 0, \ \text{(since } \phi \in \mathbb{X}^* \text{ is arbitrary)} \\
\Rightarrow\ & \sum_{1\le i\le n} c_{ij} y_i^*(y) = 0 \text{ for all } 1 \le j \le m \\
\Rightarrow\ & \sum_{1\le i\le n} c_{ij} y_i^* = 0 \text{ (since } y \in \mathbb{Y} \text{ is arbitrary)} \\
\Rightarrow\ & c_{ij} = 0 \text{ for all } 1 \le j \le m, 1 \le i \le n.
\end{aligned}$$

Thus, $\{y_i^* \otimes x_j : 1 \le i \le n, 1 \le j \le m\}$ is a linearly independent subset of $\mathbb{L}(\mathbb{X}, \mathbb{Y})^*$. ☐

We are in a position to prove the following theorem which gives a sufficient condition for $k-$smoothness of operators defined between infinite-dimensional Banach spaces, which improves on [39, Th. 2.2].

Theorem 3.4. *Suppose $\mathbb{X}$ is a reflexive Banach space and $\mathbb{Y}$ is an arbitrary Banach space. Let $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ be an $M-$ideal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. Suppose that $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ is such that $dist(T, \mathbb{K}(\mathbb{X}, \mathbb{Y})) < 1$ and $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \ldots, \pm x_r\}$, where $\{x_1, x_2, \ldots, x_r\}$ is linearly independent in $\mathbb{X}$. Let Tx_i be m_i-smooth for each $1 \le i \le r$. Then T is $k-$smooth, where $m_1 + m_2 + \ldots + m_r = k$.*

Proof. Since Tx_i is m_i-smooth, we have, $m_i = \dim\ span\ Ext\ J(Tx_i)$, for each $1 \le i \le r$. Let $\{y_{ij}^* \in Ext\ J(Tx_i) : 1 \le j \le m_i\}$ be a basis of $span\ Ext\ J(Tx_i)$ for each $1 \le i \le r$. Using similar arguments as in Lemma 3.2, it can be shown that $\{y_{ij}^* \otimes x_i : 1 \le i \le r, 1 \le j \le m_i\}$ is linearly independent. Now, using Lemma 3.1, we get,

$$\begin{aligned}
& span\ Ext\ J(T) \\
= \ & span\ \{y_{ij}^* \otimes x_i : y_{ij}^* \in Ext\ J(Tx_i), 1 \le i \le r\} \\
= \ & span\ \{y_{ij}^* \otimes x_i : 1 \le i \le r, 1 \le j \le m_i\}.
\end{aligned}$$

Therefore, $\dim\ span\ Ext\ J(T) = m_1 + m_2 + \ldots + m_r$. Thus, T is $k-$smooth, where $k = m_1 + m_2 + \ldots + m_r$. This completes the proof of the theorem. ☐

The following corollary now easily follows from Theorem 3.4. Once again, we recall that $\mathbb{K}(\ell_p)$ is an $M-$ideal in $\mathbb{L}(\ell_p)$, where $1 < p < \infty$.

Corollary 3.2. *Let $T \in S_{\mathbb{L}(\ell_p)}$, where $1 < p < \infty$. Suppose that $dist(T, \mathbb{K}(\ell_p)) < 1$ and $M_T = \{\pm x_1, \pm x_2, \ldots, \pm x_k\}$, where $\{x_1, x_2, \ldots, x_k\}$ is linearly independent in ℓ_p. Then T is $k-$smooth.*

Proof. For $1 < p < \infty$, ℓ_p is strictly convex, smooth space. Hence, $x_i \in Ext(B_{\ell_p})$ and Tx_i is smooth for each $1 \leq i \leq k$. Thus, by Theorem 3.4, T is $k-$smooth. □

Now, we exhibit an easy example to show that the converse of Corollary 3.2 is not true, i.e., there exists $k-$smooth operator $T \in \mathbb{L}(\ell_p)$ such that M_T is not of the form $\{\pm x_1, \pm x_2, \ldots, \pm x_k\}$, where $\{x_1, x_2, \ldots, x_k\}$ is linearly independent in ℓ_p.

Example 3.5. *Consider the operator $T \in \mathbb{L}(\ell_p)$, $(1 < p < \infty)$ defined by $T(\sum_i a_i e_i) = a_1 e_1 + a_2 e_2$, where $\{e_i : i \in \mathbb{N}\}$ is the canonical basis of ℓ_p. Then $M_T = span\{e_1, e_2\}$. Let $x = ae_1 + be_2$, where $a, b \neq 0$. Then $x \in M_T$ and so by Lemma 3.1, $e_1^* \otimes e_1, e_2^* \otimes e_2, x^* \otimes x \in Ext\ J(T)$. Now, it is easy to show that $\{e_1^* \otimes e_1, e_2^* \otimes e_2, x^* \otimes x\}$ is linearly independent. Therefore, T is $k-$smooth, where $k \geq 3$. But M_T does not contain 3 linearly independent vectors.*

Remark 3.6. *Example 3.5 illustrates the fact that one part of the Theorem [24, Th. 2.3], namely (i) $\Rightarrow$ (ii), is not correct and Theorem 3.4 improves on the other part of the same theorem.*

Next, we study the $k-$smoothness of a bounded linear operator T for which $M_T \cap Ext(B_{\mathbb{X}})$ may contain linearly dependent vectors.

Theorem 3.7. *Let $\mathbb{X}$ be a reflexive Banach space and $\mathbb{Y}$ be a finite-dimensional Banach space with $\dim(\mathbb{Y}) = m$. Let $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ be such that $\{x_1, x_2, \ldots, x_r\} \subseteq M_T \cap Ext(B_{\mathbb{X}}) \subseteq span\{x_1, x_2, \ldots, x_r\}$, where $\{x_1, x_2, \ldots, x_r\}$ is linearly independent. Suppose Tx_i is $m-$smooth for $i = 1, 2, \ldots, r$. Then T is $mr-$smooth.*

Proof. For each $1 \leq i \leq r$, Tx_i is $m-$smooth. Suppose $\{y_{ij}^* : 1 \leq j \leq m\}$ is a linearly independent subset of $Ext\ J(Tx_i)$ for each $1 \leq i \leq r$. We first show that dim $span\ Ext\ J(T) \leq mr$. Since $\mathbb{Y}$ is finite-dimensional, $\mathbb{L}(\mathbb{X}, \mathbb{Y}) = \mathbb{K}(\mathbb{X}, \mathbb{Y})$. Hence, T is compact operator and $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ is trivially an $M-$ideal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. Thus, by Lemma 3.1,

$$Ext\ J(T) = \{y^* \otimes x \in \mathbb{K}(\mathbb{X}, \mathbb{Y})^* : x \in M_T \cap Ext(B_{\mathbb{X}}), y^* \in Ext\ J(Tx)\}.$$

Let $x \in M_T \cap Ext(B_{\mathbb{X}})$. Then there exist scalars $c_i, i = 1, 2, ..., r$ such that $x = c_1x_1 + c_2x_2 + ... + c_rx_r$. Now, let $y^* \in Ext\ J(Tx)$. Since $\{y^*_{1j} : 1 \leq j \leq m\}$ is linearly independent, and hence forms a basis of $\mathbb{Y}^*$, there exist scalars d_j $(1 \leq j \leq m)$ such that $y^* = \sum_{1\leq j\leq m} d_j y^*_{1j}$. Thus,

$$\begin{aligned} y^* \otimes x &= y^* \otimes (c_1x_1 + c_2x_2 + ... + c_rx_r) \\ &= \Big(\sum_{1\leq j\leq m} d_j y^*_{1j}\Big) \otimes \Big(\sum_{1\leq i\leq r} c_i x_i\Big) \\ &= \sum_{1\leq i\leq r, 1\leq j\leq m} c_i d_j y^*_{1j} \otimes x_i \\ &\in span\{y^*_{1j} \otimes x_i : 1 \leq i \leq r, 1 \leq j \leq m\}. \end{aligned}$$

Since $x \in M_T \cap Ext(B_{\mathbb{X}})$ and $y^* \in Ext\ J(Tx)$ are arbitrary, we have $Ext\ J(T) \subseteq span\{y^*_{1j} \otimes x_i : 1 \leq i \leq r, 1 \leq j \leq m\}$. Now,

$$\begin{aligned} & \dim\ span\ Ext\ J(T) \\ \leq\ & \dim\ span\ \{y^*_{1j} \otimes x_i : 1 \leq i \leq r, 1 \leq j \leq m\} \\ =\ & mr,\ \text{(by Lemma 3.2)}. \end{aligned}$$

Therefore, T is $k-$smooth, where $k \leq mr$. Now, $y^*_{ij} \otimes x_i \in Ext\ J(T)$ for all $1 \leq i \leq r, 1 \leq j \leq m$. Using similar arguments as in Lemma 3.2, it can be shown that $\{y^*_{ij} \otimes x_i : 1 \leq i \leq r, 1 \leq j \leq m\}$ is linearly independent subset of $Ext\ J(T)$. Thus, $\dim\ span\ Ext\ J(T) \geq mr$, i.e., $k \geq mr$. Therefore, $k = mr$ and T is $mr-$smooth. This completes the proof of the theorem. □

To illustrate the usefulness of Theorem 3.7 we cite the following example for which $k-$smooth -ness of the operator can not be obtained using Theorem 3.4 or [39, Th. 2.2] but can be obtained using above theorem.

Example 3.8. *Let $\mathbb{X} = \ell^4_\infty$ and $\mathbb{Y}$ be a two-dimensional Banach space such that $B_{\mathbb{Y}}$ is the convex hull of $\{\pm(2, 1), \pm(2, -1)\}$. Define $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ by $T(x, y, z, w) = (y + w, x)$. Then*

$$M_T \cap Ext(B_{\mathbb{X}}) = \{\pm(1, 1, 1, 1), \pm(1, 1, -1, 1), \pm(-1, 1, 1, 1), \pm(1, -1, 1, -1)\}.$$

Clearly, $\{(1, 1, 1, 1),(1, 1, -1, 1),(-1, 1, 1, 1),(1, -1, 1, -1)\}$ is linearly dependent. Therefore, the order of smoothness of T cannot be obtained from Theorem 3.4 or [39, Th. 2.2]. Observe that, $M_T \cap Ext(B_{\mathbb{X}}) \subseteq$ span $\{\pm(1, 1, -1, 1), \pm(-1, 1, 1, 1), \pm(1, -1, 1, -1)\}$, where $\{(1, 1, -1, 1), (-1, 1, 1, 1),(1, -1, 1, -1)\}$ is linearly independent. Moreover, $T(1, 1, -1, 1)$, $T(-1, 1, 1, 1)$, $T(1, -1, 1, -1)$ are $2-$smooth. Therefore, using Theorem 3.7 we get, T is $6-$smooth.

We now turn our attention to the study of $k-$smoothness of operators in the setting of special Banach spaces. We characterize the $k-$smoothness of an operator defined from a finite-dimensional Banach space to $\ell_\infty^n,$ $(n \in \mathbb{N})$. To do so we need [39, Cor. 2.3] and the following proposition which gives a nice relation between the order of smoothness of an operator and its adjoint.

Proposition 3.1. *Let $\mathbb{X}, \mathbb{Y}$ be finite-dimensional Banach spaces. Let $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$. Then T is $k-$smooth if and only if T^* is $k-$smooth.*

Proof. We first show that $Ext\ J(T) = Ext\ J(T^*)$. Let $y^* \otimes x \in Ext\ J(T)$. Then $x \in M_T \cap Ext(B_{\mathbb{X}})$ and $y^* \in Ext\ J(Tx)$. Now,

$$y^* \otimes x(T) = 1 \Rightarrow y^*(Tx) = 1 \Rightarrow x(T^*y^*) = 1 \Rightarrow \|T^*y^*\| = 1.$$

Thus, $y^* \in M_{T^*}$ and $x \in J(T^*y^*)$. Moreover, $x \in Ext(B_{\mathbb{X}})$ and $Ext\ J(Tx) \subseteq Ext(B_{\mathbb{Y}^*})$. Therefore, $x \in Ext\ J(T^*y^*)$ and $y^* \in M_{T^*} \cap Ext(B_{\mathbb{Y}^*})$ and so $y^* \otimes x \in Ext\ J(T^*)$. Hence, $Ext\ J(T) \subseteq Ext\ J(T^*)$. Now, replacing T by T^*, we get $Ext\ J(T^*) \subseteq Ext\ J(T)$. Thus, $Ext\ J(T) = Ext\ J(T^*)$, i.e., dim $span\ Ext\ J(T) =$ dim $span\ Ext\ J(T^*)$. Therefore, T is $k-$smooth if and only if T^* is $k-$smooth. $\square$

Corollary 3.3. *Let $\mathbb{X}$ be a finite-dimensional Banach space. Let $T \in S_{\mathbb{L}(\mathbb{X},\ell_\infty^n)}$. Then T is $k-$smooth if and only if $M_{T^*} \cap Ext(B_{\ell_1^n}) = \{\pm e_1, \pm e_2, \ldots \pm e_r\}$ for some $1 \leq r \leq n,$ T^*e_i is m_i-smooth for each $1 \leq i \leq r$ and $m_1 + m_2 + \ldots + m_r = k$.*

Proof. From Proposition 3.1, T is $k-$smooth if and only if T^* is $k-$smooth. Now, $T^* \in S_{\mathbb{L}(\ell_1^n,\mathbb{X}^*)}$. Moreover, from [39, Cor. 2.3], we get that T^* is $k-$smooth if and only if $M_{T^*} \cap Ext(B_{\ell_1^n}) = \{\pm e_1, \pm e_2, \ldots \pm e_r\}$ for some $1 \leq r \leq n,$ T^*e_i is m_i-smooth for each $1 \leq i \leq r$ and $m_1 + m_2 + \ldots + m_r = k$. This completes the proof of the corollary. $\square$

We next determine the order of smoothness of $T \in \mathbb{L}(\ell_\infty^3, \mathbb{Y})$, where $\mathbb{Y}$ is an arbitrary two-dimensional Banach space, depending on $|M_T \cap Ext(B_{\ell_\infty^3})|$. Observe that if $|M_T \cap Ext(B_{\ell_\infty^3})| \leq 6$, then the order of smoothness of T can be obtained using Theorem 3.4. Therefore, we only consider the case for which $|M_T \cap Ext(B_{\ell_\infty^3})| = 8$, i.e., $M_T \cap Ext(B_{\ell_\infty^3}) = \{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$, where $x_1 = (1,1,1), x_2 = (-1,1,1), x_3 = (-1,-1,1), x_4 = (1,-1,1)$. Note that, for each $1 \leq i \leq 4,$ Tx_i is either smooth or $2-$smooth.

Theorem 3.9. *Let $\mathbb{X} = \ell_\infty^3$ and $\mathbb{Y}$ be two-dimensional Banach space. Let $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ be such that $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$. Suppose $S_1 = \{x_i : 1 \leq i \leq 4,\ Tx_i$ is smooth$\}$. Then the following hold:*

(I) *Let $|S_1| = 4$.*

(a) If either $Rank(T) = 1$ or for each $i, j (1 \leq i \neq j \leq 4)$ either Tx_i, Tx_j or $Tx_i, -Tx_j$ belong to the same straight line contained in $S_{\mathbb{Y}}$, then T is $3-$smooth.
(b) Otherwise, T is $4-$smooth.
***(II)** If $|S_1| = 3$, then T is $4-$smooth.*
***(III)** If $|S_1| = 2$, then T is $5-$smooth.*
***(IV)** If $|S_1| < 2$, then $S_1 = \emptyset$ and T is $6-$smooth.*

Proof. Clearly, T is $k-$smooth for $1 \leq k \leq 6$, since $\dim(\mathbb{X}) = 3$ and $\dim(\mathbb{Y}) = 2$.

(I) Let $|S_1| = 4$. Then Tx_i is smooth for $1 \leq i \leq 4$.

(a) If the given condition is satisfied, then it is clear that there exists $y^* \in S_{\mathbb{Y}^*}$ such that for all $i (1 \leq i \leq 4)$, $J(Tx_i) = \{y^*\}$ or $\{-y^*\}$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= \dim \ span \ J(T) \\ &= \dim \ span \ Ext \ J(T) \\ &= \dim \ span \ \{y^* \otimes x_1, y^* \otimes x_2, y^* \otimes x_3, y^* \otimes x_4\} \\ &= \dim \ span \ \{y^* \otimes x_1, y^* \otimes x_2, y^* \otimes x_3\} \\ &= 3, \ (\text{by Lemma 3.2}). \end{aligned}$$

Hence, T is $3-$smooth.

(b) Suppose the condition (a) is not satisfied. Thus, there exist $1 \leq i, j \leq 4$ such that $Tx_i \neq \pm Tx_j$ and neither Tx_i, Tx_j nor $Tx_i, -Tx_j$ belong to the same straight line contained in $S_{\mathbb{Y}}$. Without loss of generality, we assume $i = 1, \ j = 2$. Let $J(Tx_i) = \{y_i^*\}, (1 \leq i \leq 4)$. Then it is easy to observe that $\{y_1^*, y_2^*\}$ is linearly independent. Let $y_3^* = ay_1^* + by_2^*$ and $y_4^* = cy_1^* + dy_2^*$, where $a, b, c, d \in \mathbb{R}$. Since $\|y_3^*\| = 1$, a and b cannot be zero simultaneously. Similarly, c and d cannot be zero simultaneously. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= \dim \ span \ J(T) \\ &= \dim \ span \ Ext \ J(T) \\ &= \dim \ span \ \{y_1^* \otimes x_1, y_2^* \otimes x_2, y_3^* \otimes x_3, y_4^* \otimes x_4\}. \end{aligned}$$

We show that $\{y_1^* \otimes x_1, y_2^* \otimes x_2, y_3^* \otimes x_3, y_4^* \otimes x_4\}$ is linearly independent. Let $c_i (1 \leq i \leq 4) \in \mathbb{R}$ be such that

$$c_1 y_1^* \otimes x_1 + c_2 y_2^* \otimes x_2 + c_3 y_3^* \otimes x_3 + c_4 y_4^* \otimes x_4 = 0.$$

Then
$c_1 y_1^* \otimes x_1 + c_2 y_2^* \otimes x_2 + c_3(ay_1^* + by_2^*) \otimes x_3 + c_4(cy_1^* + dy_2^*) \otimes (x_1 - x_2 + x_3) = 0.$

$\Rightarrow (c_1+c_4c)y_1^*\otimes x_1+(c_2-c_4d)y_2^*\otimes x_2+(c_3a+c_4c)y_1^*\otimes x_3+(c_3b+c_4d)y_2^*\otimes x_3-c_4cy_1^*\otimes x_2+c_4dy_2^*\otimes x_1=0.$

Since $\{x_1,x_2,x_3\}$ is linearly independent subset of $\mathbb{X}$ and $\{y_1^*,y_2^*\}$ is linearly independent subset of $\mathbb{Y}^*$, from Lemma 3.2, we get that $\{y_1^*\otimes x_1, y_2^*\otimes x_1, y_1^*\otimes x_2, y_2^*\otimes x_2, y_1^*\otimes x_3, y_2^*\otimes x_3\}$ is linearly independent subset of $\mathbb{L}(\mathbb{X},\mathbb{Y})^*$. Therefore,

$$c_1+c_4c=0,\ c_2-c_4d=0,\ c_3a+c_4c=0,\ c_3b+c_4d=0,\ c_4c=0,\ c_4d=0.$$

Now, solving these equations, we obtain $c_i=0$ for all $1\leq i\leq 4$. Hence, $\{y_1^*\otimes x_1, y_2^*\otimes x_2, y_3^*\otimes x_3, y_4^*\otimes x_4\}$ is linearly independent. Thus, $k=4$ and T is $4-$smooth.

(II) Without loss of generality, assume that $S_1=\{x_2,x_3,x_4\}$, i.e., Tx_2,Tx_3,Tx_4 are smooth points of $S_{\mathbb{Y}}$ and Tx_1 is $2-$smooth point of $S_{\mathbb{Y}}$. Clearly, $Rank(T)=2$. Now, by [36, Lemma 2.11], $T(B_{\mathbb{X}})$ is a polygon with 4 extreme points. Since Tx_1 is a $2-$smooth point of $S_{\mathbb{Y}}$, by [24, Th. 4.1], $\pm Tx_1$ must be two extreme points of $B_{\mathbb{Y}}$. Since $\|T\|=1,\ T(B_{\mathbb{X}})\subseteq B_{\mathbb{Y}}$, i.e., $Ext(B_{\mathbb{Y}})\cap T(B_{\mathbb{X}})\subseteq Ext(T(B_{\mathbb{X}}))$. Therefore, $\pm Tx_1\in Ext(T(B_{\mathbb{X}}))$. Suppose that the other two extreme points of the polygon $T(B_{\mathbb{X}})$ are $\pm Tx_2$. Then $L[Tx_2,-Tx_1]$ is an edge of the polygon $T(B_{\mathbb{X}})$. Now, $L[x_2,-x_1]\cap L[x_3,-x_4]=\{(-1,0,0)\}$. Therefore, $T(-1,0,0)\in L[Tx_2,-Tx_1]\cap L[Tx_3,-Tx_4]$. This implies that $Tx_3,-Tx_4\in L[Tx_2,-Tx_1]$. Since Tx_3,Tx_4 are smooth points and Tx_1 is $2-$smooth point, $Tx_3\neq -Tx_1$ and $-Tx_4\neq -Tx_1$. Now, $Rank(T)=2$ implies that either $Tx_3\neq Tx_2$ or $-Tx_4\neq Tx_2$. Without loss of generality, let us assume that $Tx_3\neq Tx_2$. Then $Tx_3\in L(Tx_2,-Tx_1)$. Since $\|Tx_3\|=1$, $L[Tx_2,-Tx_1]\subseteq S_{\mathbb{Y}}$. Now, let $J(Tx_3)=\{y^*\}$. Then it is easy to see that $J(Tx_2)=\{y^*\}$ and $J(Tx_4)=\{-y^*\}$. Since Tx_1 is $2-$smooth, it is clear that $Ext\ J(Tx_1)=\{y_1^*,y_2^*\}$, where $\{y_1^*,y_2^*\}$ is a linearly independent subset of $\mathbb{Y}^*$. Let $y^*=ay_1^*+by_2^*$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= \dim\ span\ J(T)\\ &= \dim\ span\ Ext\ J(T)\\ &= \dim\ span\ \{y_1^*\otimes x_1, y_2^*\otimes x_1, y^*\otimes x_2, y^*\otimes x_3, y^*\otimes x_4\}\\ &= \dim\ span\ \{y_1^*\otimes x_1, y_2^*\otimes x_1, y^*\otimes x_2, y^*\otimes x_3\}. \end{aligned}$$

Now, using Lemma 3.2 it can be observed that $\{y_1^*\otimes x_1, y_2^*\otimes x_1, y^*\otimes x_2, y^*\otimes x_3\}$ is a linearly independent subset of $\mathbb{L}(\mathbb{X},\mathbb{Y})^*$. Hence, $k=4$ and T is $4-$smooth.

(III) Without loss of generality, we assume $S_1=\{x_3,x_4\}$. Then $\pm Tx_3,\pm Tx_4$ are smooth points of $S_{\mathbb{Y}}$ and $\pm Tx_1,\pm Tx_2$ are $2-$smooth points of $S_{\mathbb{Y}}$. Clearly, $Rank(T)=2$. By [24, Th.

4.1], $\pm Tx_1, \pm Tx_2 \in Ext(B_{\mathbb{Y}})$. Since $\|T\| = 1, T(B_{\mathbb{X}}) \subseteq B_{\mathbb{Y}}$. This gives that $Ext(B_{\mathbb{Y}}) \cap T(B_{\mathbb{X}}) \subseteq Ext(T(B_{\mathbb{X}}))$. Thus, we get, $\pm Tx_1, \pm Tx_2 \in Ext(T(B_{\mathbb{X}}))$. If possible, suppose that $Tx_1 = -Tx_2$. Then $x_4 = x_1 - x_2 + x_3$ implies that $Tx_1 = \frac{Tx_4 - Tx_3}{2}$. Since $Tx_1 \in Ext(B_{\mathbb{Y}})$, we must have $Tx_1 = Tx_4 = -Tx_3$. Thus, $Rank(T) = 1$, a contradiction. Therefore, $Tx_1 \neq -Tx_2$. First assume that $Tx_1 = Tx_2$. Then from $x_4 = x_1 - x_2 + x_3$, we get $Tx_3 = Tx_4$. Let $J(Tx_3) = J(Tx_4) = \{y^*\}$ and $Ext\ J(Tx_1) = Ext\ J(Tx_2) = \{y_1^*, y_2^*\}$. Then it is easy to see that T is $5-$smooth. Now, assume that $Tx_1 \neq Tx_2$. Then $\pm Tx_1, \pm Tx_2$ are 4 distinct extreme points of the polygon $T(B_{\mathbb{X}})$ and $L[Tx_2, -Tx_1]$ is an edge of $T(B_{\mathbb{X}})$. Now, as in **(II)**, it can be shown that $Tx_3, -Tx_4 \in L[Tx_2, -Tx_1]$. Since $Tx_3, -Tx_4$ are smooth points, $Tx_3, -Tx_4 \in L(Tx_2, -Tx_1)$. From $\|Tx_3\| = 1$, we can show that $L[Tx_2, -Tx_1] \subseteq S_{\mathbb{Y}}$. Let $J(Tx_4) = \{y^*\}$. Then $J(Tx_3) = \{-y^*\}$. Let $Ext\ J(Tx_1) = \{y_1^*, y_2^*\}$ and $Ext\ J(Tx_2) = \{y_3^*, y_4^*\}$. Clearly, $\{y_1^*, y_2^*\}$ and $\{y_3^*, y_4^*\}$ are linearly independent. Now, if T is $k-$smooth, then

$$
\begin{aligned}
k &= \dim\ span\ J(T) \\
&= \dim\ span\ Ext\ J(T) \\
&= \dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_3^* \otimes x_2, y_4^* \otimes x_2, y^* \otimes x_3, y^* \otimes x_4\} \\
&= \dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_3^* \otimes x_2, y_4^* \otimes x_2, y^* \otimes x_3\} \\
&= \dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y^* \otimes x_3\} \\
&= 5, \text{by simple calculation.}
\end{aligned}
$$

Hence, T is $5-$smooth.

(IV) Since $|S_1| < 2$, at least 3 points of $Tx_i,\ 1 \leq i \leq 4$ are $2-$smooth. Without loss of generality, suppose that Tx_1, Tx_2, Tx_3 are $2-$smooth. If $Rank(T) = 1$, then it is easy to see that Tx_4 is $2-$smooth. Suppose $Rank(T) = 2$. Then by [36, Lemma 2.11], $T(B_{\mathbb{X}})$ is a polygon with 4 extreme points. First let $Ext\ (T(B_{\mathbb{X}})) = \{\pm Tx_1, \pm Tx_2\}$. Then using similar arguments as in (**III**) we can show that $Tx_3,\ -Tx_4 \in L[Tx_2, -Tx_1]$. Since Tx_3 is $2-$smooth, we must have either $Tx_3 = Tx_2$ or $Tx_3 = -Tx_1$. If $Tx_3 = Tx_2$, then from $x_4 = x_1 - x_2 + x_3$, we get $Tx_4 = Tx_1$. If $Tx_3 = -Tx_1$, then similarly, we get $Tx_4 = -Tx_2$. In each case, Tx_4 is $2-$smooth. Similarly, considering other cases, we can conclude that if $|S_1| < 2$, then Tx_i are $2-$smooth for all $1 \leq i \leq 4$, i.e., $S_1 = \emptyset$. Using Theorem 3.7, we can now say that T is $6-$smooth. This completes the proof of the theorem. $\square$

In Theorem 3.9, if we further assume that $\mathbb{Y}$ is a two-dimensional strictly convex, smooth Banach space, then we obtain the following corollary.

Corollary 3.4. *Let $\mathbb{X} = \ell_\infty^3$ and $\mathbb{Y}$ be a two-dimensional strictly convex, smooth Banach space. Let $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ and $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$. Then the following hold:*
(i) If $Rank(T) = 1$ then T is $3-$smooth.
(ii) If $Rank(T) = 2$, then T is $4-$smooth.

Proof. Observe that, since $\mathbb{Y}$ is strictly convex, $S_{\mathbb{Y}}$ does not contain non-trivial straight line segment. Now, since $\mathbb{Y}$ is smooth, Tx_i is smooth for all $1 \leq i \leq 4$. Thus, the corollary follows from case **(I)** of Theorem 3.9. $\square$

As an immediate application of Theorem 3.9, we can characterize the extreme contractions defined from ℓ_∞^3 to arbitrary two-dimensional polygonal Banach space.

Theorem 3.10. *Let $\mathbb{X} = \ell_\infty^3$ and $\mathbb{Y}$ be a two-dimensional polygonal Banach space. Let $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$. Then T is an extreme contraction if and only if $|M_T \cap Ext(B_{\mathbb{X}})| \geq 6$ and $T(M_T \cap Ext(B_{\mathbb{X}})) \subseteq Ext(B_{\mathbb{Y}})$.*

Proof. First let T be an extreme contraction. Then by [36, Th. 2.2], T is $6-$smooth. From [58, Th. 2.2], we get $span(M_T \cap Ext(B_{\mathbb{X}})) = \mathbb{X}$, i.e., $|M_T \cap Ext(B_{\mathbb{X}})| \geq 6$. Let $|M_T \cap Ext(B_{\mathbb{X}})| = 6$. Then $M_T \cap Ext(B_{\mathbb{X}})$ is of the form $\{\pm x_1, \pm x_2, \pm x_3\}$, where $\{x_1, x_2, x_3\}$ is linearly independent. Now, from Theorem 3.4 it is clear that Tx_i is $2-$smooth for each $1 \leq i \leq 3$. Therefore, by [24, Th. 4.1], $Tx_i \in Ext(B_{\mathbb{Y}})$ for all $1 \leq i \leq 3$. Now, suppose that $|M_T \cap Ext(B_{\mathbb{X}})| = 8$. Then from Theorem 3.9, we can conclude that for all $x \in M_T \cap Ext(B_{\mathbb{X}})$, Tx is $2-$smooth, i.e., $Tx \in Ext(B_{\mathbb{Y}})$.
Conversely, suppose that $|M_T \cap Ext(B_{\mathbb{X}})| \geq 6$ and $T(M_T \cap Ext(B_{\mathbb{X}})) \subseteq Ext(B_{\mathbb{Y}})$. If $|M_T \cap Ext(B_{\mathbb{X}})| = 6$, then from Theorem 3.4, we get, T is $6-$smooth. Hence, by [36, Th. 2.2], T is an extreme contraction. If $|M_T \cap Ext(B_{\mathbb{X}})| = 8$, then from Theorem 3.9, we get T is $6-$smooth. Thus, again by [36, Th. 2.2], T is an extreme contraction. This completes the proof of the theorem. $\square$

We end this article with the following question:

Question 3.11. *Suppose $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces and $T \in S_{L(\mathbb{X},\mathbb{Y})}$, then what are the necessary and sufficient conditions for T to be multi-smooth point of finite order? One can consider the case $\mathbb{X} = \ell_\infty^n, \mathbb{Y} = \ell_1^n, (n \geq 3)$. There are many more cases where the question is still unanswered.*

CHAPTER 4

STUDY OF EXTREME CONTRACTIONS THROUGH $K-$SMOOTHNESS OF OPERATORS

4.1 Introduction

The study of extreme contractions and smoothness of operators between Banach spaces are two classical and fertile areas of research in Banach space theory. While the characterization of extreme contractions defined between Hilbert spaces is well known [11, 23, 44, 66], the characterization of the same is still elusive, in the general setting of Banach spaces. There are several papers including [1, 6, 7, 12, 19, 25, 30, 29, 33, 51, 54, 58, 64, 65], that deal with the study of extreme contractions of operators defined between some special Banach spaces. The purpose of this chapter is to study extreme contractions between polyhedral Banach spaces and explore interesting connections between the order of smoothness of an operator and extreme contraction. In particular, we generalize and improve on the results obtained in [51] in an elegent way. Before proceeding further, we first establish the notations and terminologies.

Content of this chapter is based on the following paper:
A. Mal, K. Paul and **S. Dey**; Characterization of extreme contractions through $k-$smoothness of operators, **Linear Multilinear Algebra**, (2021), DOI:10.1080/03081087.2021.1913086

We denote the Banach spaces by the letters $\mathbb{X}$ and $\mathbb{Y}$. Throughout the chapter, we assume that the Banach spaces are real. $|A|$ denotes the cardinality of a set A. An element x of a convex set A is said to be an extreme point of A, if $x = ty + (1-t)z$ for some $t \in (0,1)$ and $y, z \in A$ implies that $x = y = z$. The set of all extreme points of a convex set A is denoted by $Ext(A)$. The unit ball and the unit sphere of $\mathbb{X}$ are denoted by $B_{\mathbb{X}}$ and $S_{\mathbb{X}}$ respectively, that is, $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$. $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ denotes the space of all bounded linear operators defined from $\mathbb{X}$ to $\mathbb{Y}$ endowed with the usual operator norm. M_T denotes the set of all unit vectors at which T attains its norm, that is, $M_T = \{x \in S_{\mathbb{X}} : \|Tx\| = \|T\|\}$. For $x_1, x_2 \in \mathbb{X}$, $L[x_1, x_2], L(x_1, x_2)$ and $L[x_1, x_2[$ represent the following sets:

$$L[x_1, x_2] = \{tx_1 + (1-t)x_2 : 0 \leq t \leq 1\},$$

$$L(x_1, x_2) = \{tx_1 + (1-t)x_2 : 0 < t < 1\} \text{ and}$$

$$L[x_1, x_2[= L[x_1, x_2] \cup \{tx_2 - (t-1)x_1 : t > 1\}.$$

$\mathbb{X}^*$ denotes the dual space of $\mathbb{X}$. A bounded linear functional $x^* \in S_{\mathbb{X}^*}$ is said to be a supporting linear functional of a non-zero vector $x \in \mathbb{X}$, if $x^*(x) = \|x\|$. For $x \in S_{\mathbb{X}}$, the set of all supporting linear functionals of x is denoted by $J(x)$, that is, $J(x) = \{x^* \in S_{\mathbb{X}^*} : x^*(x) = 1\}$. Note that, $J(x)$ is a non-empty, weak*-compact, convex subset of $S_{\mathbb{X}^*}$. $x \in S_{\mathbb{X}}$ is said to be a smooth point if $J(x)$ is singleton. $x \in S_{\mathbb{X}}$ is said to be a $k-$smooth point [24] or the order of smoothness of x is said to be k, if $J(x)$ contains exactly k linearly independent functionals, that is, $k = \dim \ span \ J(x)$. From [32, Prop. 2.1], we get, if x is $k-$smooth, then $k = \dim \ span \ Ext \ J(x)$. Likewise an operator $T \in S_{\mathbb{L}(\mathbb{X}, \mathbb{Y})}$ is said to be $k-$smooth operator, if $k = \dim \ span \ J(T) = \dim \ span \ Ext \ J(T)$. For more information on $k-$smoothness in Banach space, the readers may go through [14, 15, 24, 32, 38, 39, 68]. An operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is said to be an extreme contraction, if T is an extreme point of the unit ball of $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. Observe that, if T is an extreme contraction, then $\|T\| = 1$. Recall that, a finite-dimensional Banach space is said to be a polyhedral Banach space, if the unit ball contains only finitely many extreme points. Equivalently, a finite-dimensional Banach space is said to be polyhedral if $B_{\mathbb{X}}$ is a polyhedron. In particular, a two-dimensional polyhedral Banach space is said to be a polygonal Banach space. Note that by [27, Th. 2.11] a finite-dimensional Banach space is polyhedral if and only if its dual is polyhedral. Motivated by the work of Lindenstrauss and Perles in [33], the following two definitions have been introduced recently in [51, 58], to study extreme contractions.

In this chapter, we first obtain a characterization of extreme contractions defined between finite-dimensional polyhedral Banach spaces in terms of $k-$smoothness of the operators. As an immediate application of this result, we characterize the extreme contractions defined between two-dimensional polygonal Banach spaces. Next, we obtain a sufficient condition for a pair of finite-dimensional polyhedral Banach spaces to satisfy weak L-P property. This result generalizes [51, Th. 2.1] and also improves on [51, Th. 2.5]. Then we show that the sufficient condition for a pair $(\mathbb{X}, \mathbb{Y})$ to satisfy weak L-P property, given in Theorem 4.3 of this chapter, is also a necessary condition, if $\mathbb{X}$ is two-dimensional polygonal Banach space and $\mathbb{Y} = \ell_\infty^2$. However, by exhibiting proper examples, we show that this is not true in general. As a final result of this chapter, we explicitly compute the exact number of extreme contractions defined on $\mathbb{X}$, where $S_{\mathbb{X}}$ is a regular hexagon. All the results obtained here, highlight the pivotal role of the order of smoothness of an operator in the study of extreme contractions defined between finite-dimensional polyhedral Banach spaces.

We will use [68, Lemma 3.1] in describing the structure of $Ext\ J(T)$. For simplicity we state the lemma for finite-dimensional Banach spaces.

Lemma 4.1. *[68, Lemma 3.1] Suppose that $\mathbb{X}, \mathbb{Y}$ are finite-dimensional Banach spaces. Let $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$. Then $M_T \cap Ext(B_{\mathbb{X}}) \neq \emptyset$ and*

$$Ext\ J(T) = \{y^* \otimes x \in \mathbb{L}(\mathbb{X}, \mathbb{Y})^* : x \in M_T \cap Ext(B_{\mathbb{X}}), y^* \in Ext\ J(Tx)\},$$

where $y^* \otimes x : \mathbb{L}(\mathbb{X}, \mathbb{Y}) \to \mathbb{R}$ is defined by $y^* \otimes x(S) = y^*(Sx)$ for every $S \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$.

4.2 Role of $k-$smoothness to extreme contractions

We begin this section with the characterization of exposed points of the unit ball of a finite-dimensional polyhedral Banach space, which clearly follows from [39, Th. 3.5] and [68, Th. 4.2]. Recall that an element $x \in S_{\mathbb{X}}$ is said to be an exposed point of the unit ball $B_{\mathbb{X}}$, if there exists a supporting linear functional x^* of x such that x^* attains norm only at $\pm x$. We also observe that in a finite-dimensional polyhedral Banach space, a point is an extreme point of the unit ball if and only if it is an exposed point of the same. We write these observations in the form of the following proposition.

Proposition 4.1. *Let $\mathbb{X}$ be a polyhedral Banach space of dimension n. Let $x \in S_{\mathbb{X}}$. Then the following are equivalent:*
(a) x is an exposed point of $B_{\mathbb{X}}$.

(b) x is an extreme point of $B_{\mathbb{X}}$.

(c) x is $n-$smooth.

In the next theorem, we prove a characterization of extreme contractions defined between finite-dimensional polyhedral Banach spaces in terms of $k-$smoothness.

Theorem 4.1. *Let $\mathbb{X}, \mathbb{Y}$ be polyhedral Banach spaces such that $\dim(\mathbb{X}) = n$ and $\dim(\mathbb{Y}) = m$. Then $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ is an extreme contraction if and only if T is $mn-$smooth.*

Proof. Since $\mathbb{X}, \mathbb{Y}$ are finite-dimensional Banach spaces, from [28, Th. 1], we get

$$Ext(B_{\mathbb{L}(\mathbb{X},\mathbb{Y})^*}) = Ext(B_{\mathbb{Y}^*}) \otimes Ext(B_{\mathbb{X}}).$$

Since $Ext(B_{\mathbb{Y}^*})$ and $Ext(B_{\mathbb{X}})$ are finite sets, there are only finitely many extreme points in the unit ball of $\mathbb{L}(\mathbb{X},\mathbb{Y})^*$. Therefore, $\mathbb{L}(\mathbb{X},\mathbb{Y})^*$ is a polyhedral Banach space. Moreover, $\mathbb{L}(\mathbb{X},\mathbb{Y})^*$ is a finite-dimensional Banach space. Therefore, $\mathbb{L}(\mathbb{X},\mathbb{Y})$ is also a finite-dimensional polyhedral Banach space. Now, $\dim(\mathbb{L}(\mathbb{X},\mathbb{Y})) = mn$. Hence, from Proposition 4.1, we can conclude that $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ is an extreme contraction if and only if T is $mn-$smooth. □

Using Theorem 4.1, we can now characterize the extreme contractions between two-dimensional polygonal Banach spaces.

Theorem 4.2. *Let $\mathbb{X}, \mathbb{Y}$ be polygonal Banach spaces such that $\dim(\mathbb{X}) = \dim(\mathbb{Y}) = 2$. Let $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$. Then T is an extreme contraction if and only if either of the following holds:*

(i) $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2\}$ and $Tx_1, Tx_2 \in Ext(B_{\mathbb{Y}})$.

(ii) $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3\}$ and $|\{x_i : Tx_i \in Ext(B_{\mathbb{Y}}), 1 \leq i \leq 3\}| \geq 2$.

(iii) $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3\}$, $Tx_1 \in Ext(B_{\mathbb{Y}})$, $Tx_2, Tx_3 \notin Ext(B_{\mathbb{Y}})$ and there exist edges F, G of $B_{\mathbb{Y}}$ such that $Tx_2 \in F, Tx_3 \in G$ and $F \neq \pm G$.

(iv) $|M_T \cap Ext(B_{\mathbb{X}})| \geq 8$ and there exists $x \in M_T \cap Ext(B_{\mathbb{X}})$ such that $Tx \in Ext(B_{\mathbb{Y}})$.

(v) $|M_T \cap Ext(B_{\mathbb{X}})| \geq 8$ and for each $x \in M_T \cap Ext(B_{\mathbb{X}})$, $Tx \notin Ext(B_{\mathbb{Y}})$. Moreover, there exist $x_i \in M_T \cap Ext(B_{\mathbb{X}})$ and $y_i^* \in Ext\ J(Tx_i)$ for $1 \leq i \leq 4$ such that $x_2 = ax_1 + bx_3, x_4 = cx_1 + dx_3, y_2^* = \alpha_1 y_1^* + \alpha_2 y_3^*, y_4^* = \beta_1 y_1^* + \beta_2 y_3^*$ and $\beta_1\alpha_2 ad - \beta_2\alpha_1 bc \neq 0$.

Proof. From Theorem 4.1, we can say that T is an extreme contraction if and only if T is $4-$smooth. If T is an extreme contraction, then from [58, Th. 2.2], we get, $span(M_T \cap Ext(B_{\mathbb{X}})) = \mathbb{X}$, that is, $|M_T \cap Ext(B_{\mathbb{X}})| \geq 4$. Hence, we only assume that $|M_T \cap Ext(B_{\mathbb{X}})| \geq 4$.

First let $|M_T \cap Ext(B_{\mathbb{X}})| = 4$. In this case, we show that T is an extreme contraction if and only if (i) holds. Let $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2\}$ for some $x_1, x_2 \in S_{\mathbb{X}}$. Clearly, $\{x_1, x_2\}$ is linearly independent. Therefore, from [39, Th. 2.2], we can conclude that T is extreme contraction, that is, T is $4-$smooth if and only if Tx_1 and Tx_2 are $2-$smooth. Thus, by Proposition 4.1, $Tx_1, Tx_2 \in Ext(B_{\mathbb{Y}})$. Therefore, if $|M_T \cap Ext(B_{\mathbb{X}})| = 4$, then T is an extreme contraction if and only if (i) holds.

Now, let $|M_T \cap Ext(B_{\mathbb{X}})| = 6$. In this case, we show that T is an extreme contraction if and only if either (ii) or (iii) holds. Let $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3\}$ for some $x_1, x_2, x_3 \in S_{\mathbb{X}}$. In this case, from [39, Th. 3.1], renaming vectors x_1, x_2 and x_3 if necessary, we get, T is an extreme contraction, that is, T is $4-$smooth if and only if Tx_1 is non-smooth, Tx_2, Tx_3 are not interior point of same line segment of $S_{\mathbb{Y}}$ and $Tx_2, -Tx_3$ are not interior point of same line segment of $S_{\mathbb{Y}}$. Since Tx_1 is non-smooth, Tx_1 is $2-$smooth. Hence, $Tx_1 \in Ext(B_{\mathbb{Y}})$. Now, if at least one of $Tx_2, Tx_3 \in Ext(B_{\mathbb{Y}})$, then (ii) holds. Suppose $Tx_2, Tx_3 \notin Ext(B_{\mathbb{Y}})$. Then Tx_2, Tx_3 are interior points of line segments of $S_{\mathbb{Y}}$. So there exist edges F, G of $B_{\mathbb{Y}}$ such that $Tx_2 \in F, Tx_3 \in G$. Since Tx_2, Tx_3 are not interior point of same line segment of $S_{\mathbb{Y}}$ and $Tx_2, -Tx_3$ are not interior point of same line segment of $S_{\mathbb{Y}}$, we get that $F \neq \pm G$. Thus, (iii) holds. Therefore, if $|M_T \cap Ext(B_{\mathbb{X}})| = 6$, then T is an extreme contraction if and only if either (ii) or (iii) holds.

Next, let $|M_T \cap Ext(B_{\mathbb{X}})| \geq 8$. Then from [39, Th. 3.3], we can easily conclude that T is an extreme contraction if and only if either (iv) or (v) holds. □

We now turn our attention to the study of weak L-P property of a pair of Banach spaces. Note that, if $\mathbb{X}$ is a reflexive Banach space, then each functional $f \in \mathbb{X}^*$ attains its norm at an extreme point of $B_{\mathbb{X}}$. As a consequence, the pair $(\mathbb{X}, \mathbb{R})$ satisfies weak L-P property. The following lemma will be used in the next theorem.

Lemma 4.2. *Let $\mathbb{X}, \mathbb{Y}$ be Banach spaces. Suppose $\emptyset \neq E_1 \subseteq \mathbb{X}$, $\emptyset \neq E_2 \subseteq \mathbb{Y}^*$ and $B_i \subseteq E_i$ is a basis of span E_i for $i = 1, 2$. Then $\{y^* \otimes x : x \in B_1, y^* \in B_2\}$ is a basis of span $\{y^* \otimes x : x \in E_1, y^* \in E_2\}$.*

Proof. We first show that the set $\{y^* \otimes x : x \in B_1, y^* \in B_2\}$ is linearly independent. Suppose $\{y_i^* \otimes x_j : x_j \in B_1, y_i^* \in B_2, 1 \leq i \leq n, 1 \leq j \leq m\}$ is a finite subset of $\{y^* \otimes x : x \in B_1, y^* \in B_2\}$. Let c_{ij} be scalars such that

$$\sum_{1 \leq i \leq n, 1 \leq j \leq m} c_{ij} y_i^* \otimes x_j = 0. \tag{4.1}$$

Choose $y \in \mathbb{Y}, \phi \in \mathbb{X}^*$. Define $S \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ by $Sx = \phi(x)y$ for all $x \in \mathbb{X}$. Now, from (4.1) we get,

$$
\begin{aligned}
& \sum_{1\leq i\leq n, 1\leq j\leq m} c_{ij} y_i^* \otimes x_j(S) = 0 \\
\Rightarrow \quad & \sum_{1\leq i\leq n, 1\leq j\leq m} c_{ij} y_i^* S(x_j) = 0 \\
\Rightarrow \quad & \sum_{1\leq i\leq n, 1\leq j\leq m} c_{ij} \phi(x_j) y_i^*(y) = 0 \\
\Rightarrow \quad & \phi\left(\sum_{1\leq i\leq n, 1\leq j\leq m} c_{ij} x_j y_i^*(y)\right) = 0 \\
\Rightarrow \quad & \sum_{1\leq i\leq n, 1\leq j\leq m} c_{ij} x_j y_i^*(y) = 0, \text{ (since } \phi \in \mathbb{X}^* \text{ is arbitrary)} \\
\Rightarrow \quad & \sum_{1\leq i\leq n} c_{ij} y_i^*(y) = 0 \text{ for all } 1 \leq j \leq m \\
\Rightarrow \quad & \sum_{1\leq i\leq n} c_{ij} y_i^* = 0 \text{ (since } y \in \mathbb{Y} \text{ is arbitrary)} \\
\Rightarrow \quad & c_{ij} = 0 \text{ for all } 1 \leq j \leq m, 1 \leq i \leq n.
\end{aligned}
$$

Thus, $\{y_i^* \otimes x_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is linearly independent and hence the set $\{y^* \otimes x : x \in B_1, y^* \in B_2\}$ is linearly independent.

Now, let $y^* \in E_2$ and $x \in E_1$. Then there exists scalars a_i, b_j and $x_j \in B_1, y_i^* \in B_2$ for $1 \leq i \leq n, 1 \leq j \leq m$ such that $y^* = \sum_{i=1}^n a_i y_i^*, x = \sum_{j=1}^m b_j x_j$. Thus,

$$y^* \otimes x = \sum_{1\leq i\leq n, 1\leq j\leq m} a_i b_j y_i^* \otimes x_j.$$

This shows that $y^* \otimes x \in span\ \{y^* \otimes x : x \in B_1, y^* \in B_2\}$. Therefore, $\{y^* \otimes x : x \in B_1, y^* \in B_2\}$ is a basis of $span\ \{y^* \otimes x : x \in E_1, y^* \in E_2\}$. This completes the proof of the lemma. $\square$

In the next theorem, we obtain a sufficient condition for a pair of finite-dimensional polyhedral Banach spaces to satisfy weak L-P property.

Theorem 4.3. *Let $\mathbb{X}, \mathbb{Y}$ be polyhedral Banach spaces and $\dim(\mathbb{X}) = n, \dim(\mathbb{Y}) = m$. Let $|Ext(B_{\mathbb{X}})| = 2(n+p)$. If $mp < n+p$, then the pair $(\mathbb{X}, \mathbb{Y})$ satisfies weak L-P property.*

Proof. Let $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ be an extreme contraction. We show that there exists $x \in M_T \cap Ext(B_{\mathbb{X}})$ such that $Tx \in Ext(B_{\mathbb{Y}})$. From [58, Th. 2.2], we get $span(M_T \cap Ext(B_{\mathbb{X}})) = \mathbb{X}$, that is, $M_T \cap Ext(B_{\mathbb{X}})$ contains at least n linearly independent elements. Let $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_i : 1 \leq i \leq r\}$ such that $\{x_1, x_2, \ldots, x_n\}$ is linearly independent. Then $r \leq n+p$. If possible, suppose that $Tx_i \notin Ext(B_{\mathbb{Y}})$, for any i, $1 \leq i \leq r$. Then by Proposition 4.1, Tx_i is not $m-$smooth for each $1 \leq i \leq r$. Let Tx_i be k_i-smooth for all $1 \leq i \leq r$. Then $k_i \leq (m-1)$ for all $1 \leq i \leq r$. Clearly, $k_i = \dim\ span\ Ext\ J(Tx_i)$. Let $\{y_{ij}^* \in Ext\ J(Tx_i) : 1 \leq j \leq k_i\}$ be a basis of $span\ Ext\ J(Tx_i)$ for each $1 \leq i \leq r$. Let

$$W_i = span\ \{y^* \otimes x_i : y^* \in Ext\ J(Tx_i)\} \text{ for each } 1 \leq i \leq r.$$

It now follows from Lemma 4.2 that $B_i = \{y_{ij}^* \otimes x_i : 1 \leq j \leq k_i\}$ is a basis of W_i and so $\dim(W_i) = k_i$ for each $1 \leq i \leq r$. Now, let T be $k-$smooth. Then

$$\begin{aligned} k &= \dim\ span\ J(T) \\ &= \dim\ span\ Ext\ J(T) \\ &= \dim\ span\{y^* \otimes x_i : y^* \in Ext\ J(Tx_i), 1 \leq i \leq r\} \\ &= dim\ W, \text{ where,} \\ W &= span\ \{y^* \otimes x_i : y^* \in Ext\ J(Tx_i), 1 \leq i \leq r\}. \end{aligned}$$

Clearly, $W \subseteq W_1 + W_2 + \ldots + W_r$. Therefore,

$$k = \dim(W) \leq \dim(\sum_{i=1}^{r} W_i) \leq \sum_{i=1}^{r} \dim(W_i) = \sum_{i=1}^{r} k_i \leq (m-1)r \leq (m-1)(n+p).$$

Now, $mp < n+p$ implies that $(m-1)(n+p) < mn$, and thus, $k < mn$. Therefore, from Theorem 4.1, we conclude that T is not an extreme contraction. This is a contradiction. Thus, there exists $1 \leq i \leq r$ such that $Tx_i \in Ext(B_{\mathbb{Y}})$. Hence, the pair $(\mathbb{X}, \mathbb{Y})$ satisfies weak L-P property. This completes the proof of the theorem. $\square$

The following corollary now follows easily from Theorem 4.3.

Corollary 4.1. *Let $\mathbb{X}, \mathbb{Y}$ be polyhedral Banach spaces such that $\dim(\mathbb{X}) = n, \dim(\mathbb{Y}) = m$. Let $|Ext(B_{\mathbb{X}})| = 2n+2$ and $m \leq n$. Then the pair $(\mathbb{X}, \mathbb{Y})$ satisfies weak L-P property.*

Remark 4.4. *(i) In [51, Th. 2.1], Ray et al. proved that if $\mathbb{X}$ is an n-dimensional polyhedral Banach space with exactly $(2n+2)$ extreme points and $m \leq n$, then the pair $(\mathbb{X}, \ell_\infty^n)$ satisfies weak L-P property. Clearly, Corollary 4.1 improves on [51, Th. 2.1].*
(ii) In [51, Th. 2.5], Ray et al. proved that if $\mathbb{X}$ is an n-dimensional polyhedral Banach space with exactly $(2n+2)$ extreme points and $m(m-1) \leq n$, then the pair $(\mathbb{X}, \ell_1^m)$ satisfies weak L-P property. Observe that if $m > 1$ and $m(m-1) \leq n$, then $m \leq m(m-1) \leq n$. Therefore, for $m > 1$, Corollary 4.1 improves on [51, Th. 2.5].
(iii) Our Theorem 4.3 unifies Theorems [51, Th. 2.1] and [51, Th. 2.5] and holds for the pair $(\mathbb{X}, \mathbb{Y})$ with $\mathbb{Y}$ as m-dimensional polyhedral Banach space instead of the special Banach spaces ℓ_∞^m ([51, Th. 2.1]) and ℓ_1^m ([51, Th. 2.5]).

The natural question that arises now is whether the sufficient condition given in Theorem 4.3 for a pair of Banach spaces to satisfy weak L-P property is also a necessary condition or not. In the next theorem, we show that the condition is both necessary and sufficient for the pair $(\mathbb{X}, \ell_\infty^2)$, where $\mathbb{X}$ is a two-dimensional polygonal Banach space.

Theorem 4.5. *Let $\mathbb{X}$ be a two-dimensional polygonal Banach space. Then the pair $(\mathbb{X}, \ell_\infty^2)$ satisfies weak L-P property if and only if $|Ext(B_{\mathbb{X}})| \leq 6$.*

Proof. Let $|Ext(B_{\mathbb{X}})| \leq 6$. Then from Theorem 4.3, we conclude that the pair $(\mathbb{X}, \ell_\infty^2)$ satisfies weak L-P property. Conversely, suppose that $|Ext(B_{\mathbb{X}})| \geq 8$. We show that the pair $(\mathbb{X}, \ell_\infty^2)$ does not satisfy weak L-P property. Clearly, $\mathbb{X}^*$ is a two-dimensional polygonal Banach space such that $|Ext(B_{\mathbb{X}^*})| \geq 8$. So we can choose $\{x_1, x_2, x_3, x_4\} \subseteq Ext(B_{\mathbb{X}^*})$ such that $L[x_1, x_3]$ and $L[x_1, -x_3]$ are not edges of $B_{\mathbb{X}^*}$. Now, define $T \in \mathbb{L}(\ell_1^2, \mathbb{X}^*)$ by $Te_1 = x_1$ and $Te_2 = x_3$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Then $M_T = \{\pm e_1, \pm e_2\}$. Since x_1, x_3 are extreme points of $B_{\mathbb{X}^*}$ and $\mathbb{X}^*$ is polygonal Banach space, by Proposition 4.1, x_1, x_3 are $2-$smooth points. Thus, by [39, Th. 2.2], T is $4-$smooth. Hence, by Theorem 4.1, T is an extreme contraction. It is easy to observe that $T^* : \mathbb{X} \to \ell_\infty^2$ is also an extreme contraction. We claim that $T^*(Ext(B_{\mathbb{X}})) \cap Ext(B_{\ell_\infty^2}) = \emptyset$. If possible, suppose that there exists $u \in Ext(B_{\mathbb{X}})$ such that $T^*u \in Ext(B_{\ell_\infty^2})$. Then T^*u is $2-$smooth. Clearly, $Ext\ J(T^*u) \subset Ext(B_{\ell_1^2}) = \{\pm e_1, \pm e_2\}$ and so without loss of generality, we may assume that $Ext\ J(T^*u) = \{e_1, e_2\}$. Thus,

$$
\begin{aligned}
& e_1(T^*u) = e_2(T^*u) = 1 \\
\Rightarrow\quad & u(Te_1) = u(Te_2) = 1 \\
\Rightarrow\quad & u(x_1) = u(x_3) = 1 \\
\Rightarrow\quad & u(tx_1 + (1-t)x_3) = 1 \text{ for all } t \in [0, 1] \\
\Rightarrow\quad & \|tx_1 + (1-t)x_3\| = 1 \text{ for all } t \in [0, 1].
\end{aligned}
$$

Hence, $L[x_1, x_3]$ is an edge of $B_{\mathbb{X}^*}$, a contradiction. Therefore, $T^*(Ext(B_{\mathbb{X}})) \cap Ext(B_{\ell_\infty^2}) = \emptyset$. Thus, if $|Ext(B_{\mathbb{X}})| \geq 8$, then the pair $(\mathbb{X}, \ell_\infty^2)$ does not satisfy weak L-P property. This completes the proof of the theorem. □

Remark 4.6. *If $(\mathbb{X}, \mathbb{Y})$ and $(\mathbb{X}_0, \mathbb{Y}_0)$ are pair of Banach spaces such that $\mathbb{X}$ is isometrically isomorphic to $\mathbb{X}_0$ and $\mathbb{Y}$ is isometrically isomorphic to $\mathbb{Y}_0$, it is clear that $(\mathbb{X}, \mathbb{Y})$ satisfies L-P property (or weak L-P property) if and only if the same happens with $(\mathbb{X}_0, \mathbb{Y}_0)$. Thus, Theorem 4.5 is also true if ℓ_∞^2 is replaced by a two-dimensional Banach space $\mathbb{Y}$ such that $|Ext(B_{\mathbb{Y}})| = 4$.*

In the next theorem, we show that, given a two-dimensional polygonal Banach space $\mathbb{X}$ with $|Ext(B_{\mathbb{X}})| \geq 8$, there exists a two-dimensional polygonal Banach space $\mathbb{Y}$ with $|Ext(B_{\mathbb{Y}})| \geq 6$, such that the pair $(\mathbb{X}, \mathbb{Y})$ does not satisfy weak L-P property.

Theorem 4.7. *Let $\mathbb{X}$ be a two-dimensional polygonal Banach space such that $|Ext(B_{\mathbb{X}})| \geq 8$. Then for each $n \in \mathbb{N}\setminus\{1,2\}$, there exists a two-dimensional polygonal Banach space $\mathbb{Y}$ such that $|Ext(B_{\mathbb{Y}})| = 2n$ and the pair $(\mathbb{X}, \mathbb{Y})$ does not satisfy weak L-P property.*

Proof. Suppose $\{x_1, x_2, x_3, x_4\} \subseteq Ext(B_{\mathbb{X}})$ such that $x_i \neq \pm x_j$ for all $1 \leq i, j \leq 4$ with $i \neq j$ and $L[x_i, x_{i+1}]$ is an edge of $B_{\mathbb{X}}$ for each $1 \leq i \leq 3$. Let $L[x_1, x_2[\ \cap L[x_4, x_3[= \{y_1\}$ and $L[x_2, x_1[\ \cap L[-x_3, -x_4[= \{y_2\}$. Let $z_1 = \frac{y_1+x_2}{2}$ and $z_{n-1} = \frac{y_1+x_3}{2}$. Now, for each $2 \leq i \leq n-2$, we can easily choose vectors $z_i = a_i z_1 + b_i z_{n-1}$, where $a_i, b_i > 0$ such that the convex hull of $\{\pm y_2, \pm z_j : 1 \leq j \leq n-1\}$ is a symmetric convex set with $2n$ extreme points $\{\pm y_2, \pm z_j : 1 \leq j \leq n-1\}$. Let $\mathbb{Y}$ be the Banach space such that $B_{\mathbb{Y}}$ is the convex hull of $\{\pm y_2, \pm z_j : 1 \leq j \leq n-1\}$. Then $|Ext(B_{\mathbb{Y}})| = 2n$. It is clear that x_i $(1 \leq i \leq 4)$ are smooth points of $\mathbb{Y}$. Now, consider the operator $I : \mathbb{X} \to \mathbb{Y}$ defined by $Ix = x$ for all $x \in \mathbb{X}$. Then $M_I \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$. Observe that, $x_1, x_2 \in L[z_1, y_2]$ and $x_3, x_4 \in L[-y_2, z_{n-1}]$. So there exist $f, g \in S_{\mathbb{Y}^*}$ such that $J(x_1) = J(x_2) = \{f\}$ and $J(x_3) = J(x_4) = \{g\}$. Let k be the order of smoothness of I, where $1 \leq k \leq 4$. Then

$$\begin{aligned} k &= \dim\ span\ J(I) \\ &= \dim\ span\ Ext\ J(I) \\ &= \dim\ span\ \{f \otimes x_1, f \otimes x_2, g \otimes x_3, g \otimes x_4\} \\ &= 4. \end{aligned}$$

Therefore, I is $4-$smooth. Hence, by Theorem 4.1, I is an extreme contraction. Now, observe that if $x \in Ext(B_{\mathbb{X}}) \setminus M_I$, then $\|Ix\| < \|I\| = 1$, that is, $x \notin Ext(B_{\mathbb{Y}})$. If $x \in M_I \cap Ext(B_{\mathbb{X}})$, then Ix is smooth point of $B_{\mathbb{Y}}$. Therefore, $I(Ext(B_{\mathbb{X}})) \cap Ext(B_{\mathbb{Y}}) = \emptyset$. Thus, the pair $(\mathbb{X}, \mathbb{Y})$ does not satisfy weak L-P property. This completes the proof of the theorem. $\square$

Although the above theorem indicates that the condition stated in Theorem 4.3 may be necessary for arbitrary two-dimensional polygonal Banach spaces $\mathbb{X}, \mathbb{Y}$ to satisfy weak L-P property, the answer is still not known in its full generality. However, if $\dim(\mathbb{X}) > 2$, then the condition is not necessary. We exhibit polyhedral Banach spaces $\mathbb{X}, \mathbb{Y}$ that satisfy weak L-P property but $\dim(\mathbb{X}) = n, \dim(\mathbb{Y}) = m, |Ext(B_{\mathbb{X}})| = 2(n+p)$ and $mp \geq n+p$ hold. To do so we need the following two lemmas.

Lemma 4.3. *Let $\mathbb{X} = \ell_\infty^3$ and $\mathbb{Y}$ be a two-dimensional Banach space. Let $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ be such that $Rank(T) = 2$ and $Ext(B_{\mathbb{X}}) \subseteq M_T$. Then $T(B_{\mathbb{X}})$ is a convex set with 4 extreme points.*

Proof. Let us consider the facets $\pm G_1$ of $B_{\mathbb{X}}$, where $G_1 = \{(1, y, z) : |y|, |z| \leq 1\}$. Now, G_1 can be expressed as

$$G_1 = x + F,$$

where $x = (1, 0, 0)$ and $F = \{(0, y, z) : |y|, |z| \leq 1\}$. Then

$$-G_1 = -x + F.$$

It is clear that $B_{\mathbb{X}}$ is the convex hull of the sets $G_1 = x+F$ and $-G_1 = -x+F$. Therefore, $T(B_{\mathbb{X}})$ is the convex hull of $T(x+F) = Tx+T(F)$ and $T(-x+F) = -Tx+T(F)$. Here, F is the convex hull of $\{\pm(0, 1, 1), \pm(0, 1, -1)\}$, and thus, $T(F)$ is the convex hull of $\{\pm T(0, 1, 1), \pm T(0, 1, -1)\}$. Hence, $T(F)$ must be a symmetric set having at most four extreme points. If $T(F)$ has two extreme points say $\pm z$, then $Tx \neq 0$ and the extreme points of $T(B_{\mathbb{X}})$ are $\pm Tx \pm z$ and we are done.

Now, let $T(F)$ be a symmetric set having exactly four distinct extreme points $\pm T(0, 1, 1)$ and $\pm T(0, 1, -1)$. We denote $T(0, 1, 1)$ and $T(0, 1, -1)$ by y_1 and y_2 respectively. Clearly, $T(B_{\mathbb{X}})$ is the convex hull of $\{\pm Tx \pm y_1, \pm Tx \pm y_2\}$. Since $\mathbb{Y}$ is two-dimensional and $\{y_1 + y_2, y_1 - y_2\}$ is linearly independent, we have, $Tx = a(y_1 + y_2) + b(y_1 - y_2)$, where $a, b \in \mathbb{R}$. We claim that either $a = 0$ or $b = 0$. If possible, let $a \neq 0, b \neq 0$. First assume that $a > 0, b > 0$. Then

$$Tx - y_1 = \frac{2a}{2a + 2b + 1}(Tx + y_2) + \frac{2b}{2a + 2b + 1}(Tx - y_2) + \frac{1}{2a + 2b + 1}(-Tx - y_1).$$

Since $a > 0, b > 0$, we have, $\frac{2a}{2a+2b+1}, \frac{2b}{2a+2b+1}, \frac{1}{2a+2b+1} \in (0, 1)$. Moreover, we have, $\|Tx + y_2\| = \|Tx - y_2\| = \|Tx + y_1\| = 1$. Since the vectors $Tx + y_2, Tx - y_2, -Tx - y_1$ are not collinear, we get, $\|Tx - y_1\| < 1$, that is, $\|T(1, -1, -1)\| < 1$, which contradicts that $Ext(B_{\mathbb{X}}) \subseteq M_T$. Similarly, considering $a < 0, b < 0$, or $a < 0, b > 0$ or $a > 0, b < 0$, we get a contradiction. This proves our claim. Now, the possible alternatives are as follows:

(i) $a = b = 0$. (ii) $a = 0, b \neq 0$, (iii) $a \neq 0, b = 0$.

The proof would end with the discussion of these alternatives.

(i) Let $a = b = 0$. Then $Tx = 0$. Thus, $T(B_{\mathbb{X}})$ is the convex hull of $\{\pm y_1, \pm y_2\}$. So the extreme points of $T(B_{\mathbb{X}})$ are $\pm y_1, \pm y_2$ and we are done.

(ii) Let $a = 0, b \neq 0$. Then

$$Tx - y_1 = \frac{2b}{2b + 1}(Tx - y_2) + \frac{1}{2b + 1}(-Tx - y_1)$$

and

$$-Tx - y_2 = \frac{1}{2b + 1}(Tx - y_2) + \frac{2b}{2b + 1}(-Tx - y_1).$$

Thus, if $a = 0$ and $b > 0$, then the only extreme points of $T(B_{\mathbb{X}})$ are $\pm(Tx - y_2)$ and $\pm(Tx + y_1)$. We can also write

$$Tx - y_2 = \frac{2b}{2b-1}(Tx - y_1) + \frac{-1}{2b-1}(-Tx - y_2)$$

and

$$-Tx - y_1 = \frac{-1}{2b-1}(Tx - y_1) + \frac{2b}{2b-1}(-Tx - y_2).$$

Thus, if $a = 0$ and $b < 0$, then the only extreme points of $T(B_{\mathbb{X}})$ are $\pm(Tx - y_1)$ and $\pm(Tx + y_2)$. Hence, we are done.

(iii) Let $a \neq 0, b = 0$. Then

$$Tx - y_1 = \frac{2a}{2a+1}(Tx + y_2) + \frac{1}{2a+1}(-Tx - y_1)$$

and

$$-Tx + y_2 = \frac{1}{2a+1}(Tx + y_2) + \frac{2a}{2a+1}(-Tx - y_1).$$

Thus, if $a > 0$ and $b = 0$, then the only extreme points of $T(B_{\mathbb{X}})$ are $\pm(Tx + y_2)$ and $\pm(Tx + y_1)$. We can also write

$$Tx + y_2 = \frac{2a}{2a-1}(Tx - y_1) + \frac{-1}{2a-1}(-Tx + y_2)$$

and

$$-Tx - y_1 = \frac{-1}{2a-1}(Tx - y_1) + \frac{2a}{2a-1}(-Tx + y_2).$$

Thus, if $a < 0$ and $b = 0$, then the only extreme points of $T(B_{\mathbb{X}})$ are $\pm(Tx - y_1)$ and $\pm(Tx - y_2)$. Hence, we are done.

This completes the proof of the lemma.

□

Lemma 4.4. *Let $\mathbb{X} = \ell_\infty^4$ and $\mathbb{Y}$ be a two-dimensional Banach space. Let $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ be such that $Rank(T) = 2$ and $Ext(B_{\mathbb{X}}) \subseteq M_T$. Then $T(B_{\mathbb{X}})$ is a convex set with 4 extreme points.*

Proof. Let us consider the facets $\pm G_1$ of $B_{\mathbb{X}}$, where $G_1 = \{(1, y, z, w) : |y|, |z|, |w| \leq 1\}$. Now, G_1 can be expressed as

$$G_1 = x + F,$$

where $x = (1, 0, 0, 0)$ and $F = \{(0, y, z, w) : |y|, |z|, |w| \leq 1\}$. Then

$$-G_1 = -x + F.$$

Observe that, there exist $x_i \in F (1 \leq i \leq 4)$ such that $\pm x + x_i \in Ext(B_{\mathbb{X}})$ and F is the convex hull of $\{\pm x_i : 1 \leq i \leq 4\}$. It is clear that $B_{\mathbb{X}}$ is the convex hull of the sets $G_1 = x + F$

and $-G_1 = -x + F$. Therefore, $T(B_{\mathbb{X}})$ is the convex hull of $T(x + F) = Tx + T(F)$ and $T(-x+F) = -Tx+T(F)$. Here, F is a symmetric cube about the origin and $T(F)$ has at most eight extreme points.
If $T(F)$ has eight extreme points or $T(F)$ has six extreme points and the two remaining points (say $\pm y_4$, where $y_j = Tx_j$, for $j = 1, 2, 3, 4$) belong to the boundary of $T(F)$, then the operator $T_0 : \ell_\infty^3 \to \mathbb{Y}_0$ given by $T_0(t_1, t_2, t_3) = T(0, t_1, t_2, t_3)$, where $\mathbb{Y}_0$ is the space $\mathbb{Y}$ endowed with a norm whose unit ball is $T(F)$, satisfies the hypothesis but not the thesis of Lemma 4.3. Since this is not possible, $|Ext(T(F))| \leq 6$ and if the equality holds, the remaining two points (say, as before, $\pm y_4$) necessarily belong to the interior of $T(F)$. However, the latter implies that $Tx+y_4$ belongs to the interior of $B_{\mathbb{Y}}$, which contradicts the hypothesis of the lemma. Therefore, $|Ext(T(F))| \leq 4$. From now on, arguing similarly as in Lemma 4.3, we can show that $T(B_{\mathbb{X}})$ has exactly four extreme points. This completes the proof of the lemma. $\square$

Remark 4.8. *Following the same line of arguments we can show that Lemma 4.4 holds for $\mathbb{X} = \ell_\infty^n$, i.e., if $T \in S_{\mathbb{L}(\ell_\infty^n, \mathbb{Y})}$ with $Rank(T) = 2$ and $Ext(B_{\mathbb{X}}) \subseteq M_T$, then $T(B_{\mathbb{X}})$ is a convex set with 4 extreme points.*

Next, we obtain a bound of the order of smoothness of a class of bounded linear operators defined between ℓ_∞^4 and a two-dimensional Banach space.

Theorem 4.9. *Let $\mathbb{X} = \ell_\infty^4$ and $\mathbb{Y}$ be any two-dimensional Banach space. Suppose $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ is such that $Ext(B_{\mathbb{X}}) \subseteq M_T$ and Tx is smooth for all $x \in Ext(B_{\mathbb{X}})$. Then T is $k-$smooth where $k \leq 6$.*

Proof. Let us write $Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \ldots, \pm x_8\}$, where $\{x_1, x_2, x_3, x_4\}$ is linearly independent. Let $S = \{x_1, x_2, \ldots, x_8\}$.
First suppose $Rank(T) = 1$. Then there exists $y^* \in S_{\mathbb{Y}^*}$ such that for any $i \in \{1, 2, \ldots, 8\}$, $J(Tx_i) = \{y^*\}$ or $\{-y^*\}$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= dim\ span\ J(T) \\ &= dim\ span\ Ext\ J(T) \\ &= dim\ span\ \{y^* \otimes x_i : 1 \leq i \leq 8\} \\ &= dim\ span\ \{y^* \otimes x_i : 1 \leq i \leq 4\} \\ &= 4, \end{aligned}$$

as $\{y^* \otimes x_i : 1 \leq i \leq 4\}$ is linearly independent by [39, Lemma 2.1]. Hence T is $4-$smooth.
Let $Rank(T) = 2$. Then by Lemma 4.4, $T(B_{\mathbb{X}})$ is a convex set with four extreme points. Without loss of generality, let $\pm Tx_1, \pm Tx_2$ be four distinct extreme points of $T(B_{\mathbb{X}})$. Suppose

$Tx_i \in L(Tx_1, Tx_2)$ for some $3 \leq i \leq 8$. We claim that for each $3 \leq j \leq 8, Tx_j \in L[Tx_1, Tx_2] \cup L[-Tx_1, -Tx_2]$. Since $\|Tx_i\| = 1, L[Tx_1, Tx_2] \subseteq S_{\mathbb{Y}}$. Let $J(Tx_i) = \{y^*\}$. Then for each $y \in L[Tx_1, Tx_2], y^*(y) = 1$. If possible, let there exist $3 \leq j (\neq i) \leq 8$ such that $Tx_j \in L(Tx_1, -Tx_2)$. Let $J(Tx_j) = \{z^*\}$. Then $y^* \neq \pm z^*$. Now, for all $y \in L[Tx_1, -Tx_2], z^*(y) = 1$. Thus, $y^*, z^* \in J(Tx_1)$, contradicts that Tx_1 is smooth. Therefore, $Tx_j \notin L(Tx_1, -Tx_2)$. Similarly, it can be shown that $Tx_j \notin L(-Tx_1, Tx_2)$. Therefore, for any $i \in \{1, 2, \ldots, 8\}$, $J(Tx_i) = \{y^*\}$ or $\{-y^*\}$. Now, it is easy to observe that T is $4-$smooth.

Now, suppose that $Tx_i \notin L(\pm Tx_1, \pm Tx_2)$ for any $3 \leq i \leq 8$. Then $Tx_i \in \{\pm Tx_1, \pm Tx_2\}$ for all $3 \leq i \leq 8$. Let $J(Tx_1) = \{y_1^*\}$ and $J(Tx_2) = \{y_2^*\}$. Then for any $i \in \{1, 2, \ldots, 8\}$,

$$J(Tx_i) = \quad \{y_1^*\} \text{ or } \{-y_1^*\} \text{ or } \quad \{y_2^*\} \text{ or } \{-y_2^*\}.$$

Thus, there exist two subsets S_1 and S_2 $(S_1 \cap S_2 = \emptyset, S_1 \cup S_2 = S)$ of S such that $T(S_1) = \pm Tx_1$ and $T(S_2) = \pm Tx_2$. Therefore, we have for any $i \in \{1, 2, \ldots, 8\}$,

$$\begin{aligned} J(Tx_i) &= \{y_1^*\} \text{ or } \{-y_1^*\}, \text{ if } x_i \in S_1 \\ &= \{y_2^*\} \text{ or } \{-y_2^*\}, \text{ if } x_i \in S_2. \end{aligned}$$

Now, it is clear that any 4 elements of S_1 as well as S_2 are linearly dependent. Otherwise, if $\{x_{11}, x_{12}, x_{13}, x_{14}\}$ is a linearly independent subset of S_1, then for any $x \in \mathbb{X}$,

$$\begin{aligned} x &= \sum_{i=1}^{4} \lambda_i x_{1i}, \text{ where } \lambda_i \text{ are scalars,} \\ \Rightarrow \quad Tx &= \sum_{i=1}^{4} \lambda_i Tx_{1i} \in span\{Tx_1\}. \end{aligned}$$

Hence, $Rank(T) = 1$, a contradiction. Thus, maximal linearly independent subsets of S_1 and S_2 contain at most 3 elements. Let us write those linearly independent subsets of S_1 and S_2 respectively by $A_1 = \{x_{1i} : 1 \leq i \leq n_1\}$ and $A_2 = \{x_{2i} : 1 \leq i \leq n_2\}$, where $n_1, n_2 \leq 3$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= dim\ span\ J(T) \\ &= dim\ span\ Ext\ J(T) \\ &= dim\ span\ \{y_1^* \otimes x, y_2^* \otimes z : x \in S_1, z \in S_2\} \\ &= dim\ span\ \{y_1^* \otimes x_{1i}, y_2^* \otimes x_{2i} : 1 \leq i \leq n_1, 1 \leq j \leq n_2\} \\ &\leq n_1 + n_2 \leq 6. \end{aligned}$$

Therefore, T is $k-$smooth, where $k \leq 6$. This completes the proof of the theorem. $\square$

Now, we are in a position to show that although the pair $(\ell_\infty^4, \mathbb{Y})$ does not satisfy the condition given in Theorem 4.3, the pair $(\ell_\infty^4, \mathbb{Y})$ satisfies weak L-P property, where $\mathbb{Y}$ is a two-dimensional polygonal Banach space.

Theorem 4.10. *Let $\mathbb{Y}$ be a two-dimensional polygonal Banach space. Then the pair $(\ell_\infty^4, \mathbb{Y})$ satisfies weak L-P property.*

Proof. Let $\mathbb{X} = \ell_\infty^4$. Then $|Ext(B_{\mathbb{X}})| = 16 = 2(4+4)$. Observe that, comparing with Theorem 4.3, here we have $m = 2, n = 4$ and $p = 4$. Thus, $mp < n + p$ is not satisfied. We now show that the pair $(\mathbb{X}, \mathbb{Y})$ satisfies weak L-P property. Let $T \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ be an extreme contraction. First let us assume $|M_T \cap Ext(B_{\mathbb{X}})| = 16$. If $Tx \in Ext(B_{\mathbb{Y}})$ for some $x \in M_T \cap Ext(B_{\mathbb{X}})$, then we are done. If possible, let $Tx \notin Ext(B_{\mathbb{Y}})$ for any $x \in M_T \cap Ext(B_{\mathbb{X}})$. Then from Proposition 4.1, we get Tx is smooth for all $x \in M_T \cap Ext(B_{\mathbb{X}})$. Now, from Theorem 4.9, we see that T is $k-$smooth, where $k \leq 6$. Hence, by Theorem 4.1, T is not an extreme contraction, a contradiction. Thus, $Tx \in Ext(B_{\mathbb{Y}})$ for some $x \in M_T \cap Ext(B_{\mathbb{X}})$. Now, let $|M_T \cap Ext(B_{\mathbb{X}})| = 2q$, where $1 \leq q \leq 7$. Suppose $Tx \notin Ext(B_{\mathbb{Y}})$ for any $x \in M_T \cap Ext(B_{\mathbb{X}})$. Thus, Tx is smooth for all $x \in M_T \cap Ext(B_{\mathbb{X}})$. Let $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \ldots, \pm x_q\}$ and $J(Tx_i) = \{y_i^*\}, 1 \leq i \leq q$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= dim\ span\ J(T) \\ &= dim\ span\ Ext\ J(T) \\ &= dim\ span\ \{y_i^* \otimes x_i : 1 \leq i \leq q\} \\ &\leq q \leq 7. \end{aligned}$$

So, T can not be $8-$smooth and hence not an extreme contraction, a contradiction. Therefore, $Tx \in Ext(B_{\mathbb{Y}})$ for some $x \in M_T \cap Ext(B_{\mathbb{X}})$. Thus, the pair $(\mathbb{X}, \mathbb{Y})$ satisfies weak L-P property. This completes the proof of the theorem. $\square$

As an immediate application of Theorem 4.2 (or Theorem 4.3), we can compute the number of extreme contractions defined on $\mathbb{X}$, where $S_{\mathbb{X}}$ is a regular hexagon.

Theorem 4.11. *Let $\mathbb{X}$ be a two-dimensional polygonal Banach space such that $S_{\mathbb{X}}$ is a regular hexagon. Then $|Ext(B_{\mathbb{L}(\mathbb{X},\mathbb{X})})| = 30$.*

Proof. Without loss of generality, we may assume that the vertices of $S_{\mathbb{X}}$ are $\pm x_1 = \pm(1,0), \pm x_2 = \pm(\frac{1}{2}, \frac{\sqrt{3}}{2}), \pm x_3 = \pm(-\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let $T \in Ext(B_{\mathbb{L}(\mathbb{X},\mathbb{X})})$. Then from Theorem 4.2, we can say that either (i) $|M_T \cap Ext(B_{\mathbb{X}})| = 4$ or (ii) $|M_T \cap Ext(B_{\mathbb{X}})| = 6$.

(i) First consider the case $|M_T \cap Ext(B_{\mathbb{X}})| = 4$. Let $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2\}$. Then by Theorem 4.2, we get $Tx_1, Tx_2 \in Ext(B_{\mathbb{X}})$. Observe that, if x, y are two distinct extreme points of $B_{\mathbb{X}}$, then $\|x - y\| \geq 1$. Now, $x_3 = x_2 - x_1$ and $x_3 \notin M_T$ ensures that Tx_1 and Tx_2 cannot be distinct extreme points of $B_{\mathbb{X}}$. Therefore, $Tx_1 = Tx_2$. Now, there are 6 possibilities for Tx_1. Hence, there are 6 extreme contractions $T \in \mathbb{L}(\mathbb{X}, \mathbb{X})$ such that $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2\}$. Similarly, it can shown that there are 6 extreme contractions $T \in \mathbb{L}(\mathbb{X}, \mathbb{X})$ such that $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_2, \pm x_3\}$. Now, suppose that $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_3\}$. Since $x_2 = x_1 + x_3$ and $\|T\| = 1$, $Tx_1 \neq Tx_3$. Observe that, if x, y are two linearly independent extreme points of $B_{\mathbb{X}}$, then $\|x + y\| \geq 1$. Therefore, taking into account that $Tx_1, Tx_3 \in Ext(B_{\mathbb{X}})$ and $x_2 \notin M_T$, we conclude that Tx_1, Tx_3 are not linearly independent. Hence, $Tx_1 = -Tx_3$. Now, there are 6 possibilities for Tx_1. Thus, there are 6 extreme contractions $T \in \mathbb{L}(\mathbb{X}, \mathbb{X})$ such that $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_3\}$. So we get 18 extreme contractions T such that $|M_T \cap Ext(B_{\mathbb{X}})| = 4$.

(ii) Now, consider the case $|M_T \cap Ext(B_{\mathbb{X}})| = 6$, that is, $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3\}$. We show that in this case, T is an isometry. By Theorem 4.2 (or Theorem 4.3), $Tx_i \in Ext(B_{\mathbb{X}})$ for some $1 \leq i \leq 3$. Without loss of generality, let $Tx_1 \in Ext(B_{\mathbb{X}})$ and $Tx_1 = x_1$. Now, the following cases may hold:

(1) $Tx_3 \in L[x_1, x_2]$,	(2) $Tx_3 \in L[x_2, x_3]$,	(3) $Tx_3 \in L[x_3, -x_1]$,
(4) $Tx_3 \in L[-x_1, -x_2]$,	(5) $Tx_3 \in L[-x_2, -x_3]$,	(6) $Tx_3 \in L[-x_3, x_1]$.

We consider each case separately.

(1) Let $Tx_3 \in L[x_1, x_2]$. Then $Tx_3 = tx_1 + (1-t)x_2$, for some $t \in [0, 1]$. Then $Tx_2 = Tx_1 + Tx_3 = x_1 + tx_1 + (1-t)x_2 = (1+t)x_1 + (1-t)x_2 = \left(\frac{3}{2} + \frac{t}{2}, (1-t)\frac{\sqrt{3}}{2}\right)$. Thus, $\|Tx_2\| > 1$, a contradiction.

(2) Let $Tx_3 \in L[x_2, x_3]$. Then $Tx_3 = tx_2 + (1-t)x_3$, for some $t \in [0, 1]$. Thus, $Tx_2 = tx_1 + x_2 = (t + \frac{1}{2}, \frac{\sqrt{3}}{2})$. Since $\|Tx_2\| = 1$, we must have $t = 0$, that is, $Tx_3 = x_3$.

(3) Let $Tx_3 \in L[x_3, -x_1]$. Then $Tx_3 = tx_3 - (1-t)x_1$ for some $t \in [0, 1]$. Thus, $Tx_2 = tx_2$. Since $\|Tx_2\| = 1$, we have $t = 1$. Thus, $Tx_3 = x_3$.

(4) Let $Tx_3 \in L[-x_1, -x_2]$. Then $Tx_3 = -tx_1 - (1-t)x_2$ for some $t \in [0, 1]$. Thus, $Tx_2 = -(1-t)x_3$. Since $\|Tx_2\| = 1$, we have $t = 0$. Thus, $Tx_3 = -x_2$.

(5) Let $Tx_3 \in L[-x_2, -x_3]$. Then $Tx_3 = -tx_2 - (1-t)x_3$ for some $t \in [0, 1]$. Thus, $Tx_2 = (1-t)x_1 - x_3 = \left(\frac{3}{2} - t, -\frac{\sqrt{3}}{2}\right)$. Since $\|Tx_2\| = 1$, we have $t = 1$. Thus, $Tx_3 = -x_2$.

(6) Let $Tx_3 \in L[-x_3, x_1]$. Then $Tx_3 = tx_1 - (1-t)x_3$ for some $t \in [0, 1]$. Similarly as case (1),

we can show that $\|Tx_2\| > 1$, a contradiction.

Therefore, if $Tx_1 = x_1$, then considering all possibilities for Tx_3, we get that either $Tx_3 = x_3$ or $Tx_3 = -x_2$. In each case, T is an isometry. Clearly, an isometry is an extreme contraction. Now, it is easy to observe that there are 12 isometries on $\mathbb{X}$. Therefore, there are 12 extreme contractions T such that $|M_T \cap Ext(B_{\mathbb{X}})| = 6$.

Combining (i) and (ii), we get total $18 + 12 = 30$ extreme contractions on $\mathbb{X}$. $\square$

CHAPTER 5

COMPLETE CHARACTERIZATION OF $K-$SMOOTHNESS OF OPERATORS DEFINED ON ℓ_∞^3

5.1 Introduction

The problem of characterizing $k-$smooth operators defined between arbitrary Banach spaces is relatively new but an important area of research which plays an important role to identify the structure of the unit ball of a Banach space. There are several papers including [14, 15, 24, 32, 39, 67] that contain the study of $k-$smoothness of operators on different spaces. In this chapter, our objective is to study the $k-$smoothness of bounded linear operators defined on ℓ_∞^3. Let us first fix the notation and terminology.

We denote $\mathbb{X}$ as real Banach space throughout the chapter. The unit sphere and the unit ball of $\mathbb{X}$ respectively denoted by $S_{\mathbb{X}}$ and $B_{\mathbb{X}}$. Let $\mathbb{L}(\mathbb{X})$ denote the space of all bounded linear operators defined on $\mathbb{X}$ endowed with the usual operator norm. For $T \in \mathbb{L}(\mathbb{X})$, M_T denotes the collection of all unit vectors of $\mathbb{X}$ at which T attains its norm, i.e., $M_T = \{x \in S_{\mathbb{X}} : \|Tx\| = \|T\|\}$. For a set A, the cardinality of A is denoted by $|A|$. The dual space of $\mathbb{X}$ is denoted by $\mathbb{X}^*$. An element $x \in S_{\mathbb{X}}$ is said to be an extreme point of the convex set $B_{\mathbb{X}}$ if and only if $x = (1-t)y+tz$ for some $y, z \in B_{\mathbb{X}}$ and $t \in (0,1)$ implies that $y = z = x$. The set of all extreme points of $B_{\mathbb{X}}$ is denoted by $Ext(B_{\mathbb{X}})$. An element $x^* \in S_{\mathbb{X}^*}$ is said to be a supporting linear functional of $x \in S_{\mathbb{X}}$, if $x^*(x) = 1$. For a unit vector x, let $J(x)$ denote the set of all supporting linear functionals of

x, i.e., $J(x) = \{x^* \in S_{\mathbb{X}^*} : x^*(x) = 1\}$. The set $J(x)$ for $x \in S_{\mathbb{X}}$ plays a significant role to study the $k-$smoothness of x. By the Hahn-Banach Theorem, it is easy to verify that $J(x) \neq \emptyset$, for all $x \in S_{\mathbb{X}}$. We would like to mention that $J(x)$ is a weak*-compact convex subset of $S_{\mathbb{X}^*}$. A unit vector x is said to be a smooth point if $J(x)$ is singleton. $\mathbb{X}$ is said to be a smooth Banach space if every unit vector of $\mathbb{X}$ is smooth. The set of all extreme points of $J(x)$ is denoted by $Ext\ J(x)$, where $x \in S_{\mathbb{X}}$.
In 2005, Khalil and Saleh [24] defined $k-$smooth points as follows: An element $x \in S_{\mathbb{X}}$ is said to be $k-$smooth or the order of smoothness of x is k, if $J(x)$ contains exactly k linearly independent supporting linear functionals of x. In other words, x is $k-$smooth, if dim $span\ J(x) = k$. Moreover, from [32, Prop. 2.1], we get that x is $k-$smooth, if $k =$ dim $span\ Ext\ J(x)$. Similarly, for $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ with $\|T\| = 1$, $J(T) = \{F \in \mathbb{L}(\mathbb{X}, \mathbb{Y})^* : \|F\| = 1, F(T) = 1\}$ and T is said to be $k-$smooth operator, if $k =$ dim $span\ J(T) =$ dim $span\ Ext\ J(T)$. Observe that, $1-$smooth points of $S_{\mathbb{X}}$ are the smooth points of $S_{\mathbb{X}}$. For a convex set C, $\text{int}_r(C)$ denotes the relative interior of the set C, i.e., $x \in \text{int}_r(C)$ if there exists $\epsilon > 0$ such that $B(x, \epsilon) \cap \text{affine}(C) \subseteq C$, where affine$(C)$ is the intersection of all affine sets containing C and an affine set is defined as the translation of a vector subspace. A non-empty convex subset F of C is said to be a face of C, if for $x, y \in C$ and $t \in (0, 1)$, $(1-t)x + ty \in F \Rightarrow x, y \in F$.

We state the following lemma [68, Lemma 3.1], characterizing $Ext\ J(T)$, which will be used often. For simplicity we state the lemma for finite-dimensional Banach space $\mathbb{Y} = \mathbb{X}$.

Lemma 5.1. *[68, Lemma 3.1] Suppose that $\mathbb{X}$ is a finite-dimensional Banach space. Let $T \in \mathbb{L}(\mathbb{X})$ and $\|T\| = 1$ Then*

$$Ext\ J(T) = \{y^* \otimes x \in \mathbb{L}(\mathbb{X})^* : x \in M_T \cap Ext(B_{\mathbb{X}}), y^* \in Ext\ J(Tx)\},$$

where $y^* \otimes x : \mathbb{L}(\mathbb{X}) \to \mathbb{R}$ is defined by $y^* \otimes x(S) = y^*(Sx)$ for every $S \in \mathbb{L}(\mathbb{X})$.

We state the following useful lemma which will be used often to prove maximum the theorems of the chapter. We give the statement in more simplified form (considering $\mathbb{Y} = \mathbb{X}$.)

Lemma 5.2. *[39, Lemma 2.1] Suppose $\mathbb{X}$ is a finite-dimensional Banach space. If $\{x_1, x_2, \ldots, x_m\}$ is a linearly independent subset of $\mathbb{X}$ and $\{y_1^*, y_2^*, \ldots, y_n^*\}$ is a linearly independent subset of $\mathbb{X}^*$, then $\{y_i^* \otimes x_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a linearly independent subset of $\mathbb{L}(\mathbb{X})^*$.*

5.2 Complete characterization of $k-$smooth elements of $\mathbb{L}(\ell_\infty^3)$

We denote the extreme points of ℓ_∞^3 by $\pm x_1 = \pm(1,1,1), \pm x_2 = \pm(-1,1,1), \pm x_3 = \pm(-1,-1,1)$, $\pm\, x_4 = \pm(1,-1,1)$. $|M_T \cap Ext(B_{\ell_\infty^3})|$ plays an important role in determining the order of smoothness of T. Observe that if $|M_T \cap Ext(B_{\ell_\infty^3})| \leq 6$, then the order of smoothness of T can be obtained using [39, Th. 2.2]. Therefore, we only consider the case for which $|M_T \cap Ext(B_{\ell_\infty^3})| = 8$, i.e., $M_T \cap Ext(B_{\ell_\infty^3}) = \{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$. We denote the facets $\{(x,y,z) : x = 1\}$, $\{(x,y,z) : y = 1\}$ and $\{(x,y,z) : z = 1\}$ of $S_{\mathbb{X}}$ respectively by F_1, F_2 and F_3. Let us denote the supporting functionals corresponding to the factes F_1, F_2 and F_3 respectively by y_1^*, y_2^* and y_3^*.
Note that, for $1 \leq i \leq 4$, Tx_i is $k-$smooth, where $k \in \{1,2,3\}$. Suppose $S_k = \{x \in M_T \cap Ext(B_{\ell_\infty^3}) : Tx \text{ is } k-smooth\}$, where $k \in \{1,2,3\}$. Clearly, $|S_1| + |S_2| + |S_3| = 8$.
In the following theorem, we consider the case when $|S_1| = 8$.

Theorem 5.1. *Let $\mathbb{X} = \ell_\infty^3$ and let $T \in S_{\mathbb{L}(\mathbb{X})}$ be such that $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$. Let $|S_1| = 8$. Then the following hold:*
(i) If $\pm J(Tx_i) = \pm J(Tx_j)$ for all $x_i, x_j \in S_1$, then T is $3-$smooth.
(ii) Otherwise, T is $4-$smooth.

Proof. (i) Suppose the given condition is satisfied. Let $\pm J(Tx_i) = \{\pm y^*\}$ for $1 \leq i \leq 4$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= \dim\ span\ J(T) \\ &= \dim\ span\ Ext\ J(T) \\ &= \dim\ span\ \{y^* \otimes x_1, y^* \otimes x_2, y^* \otimes x_3, y^* \otimes x_4\} \\ &= \dim\ span\ \{y^* \otimes x_1, y^* \otimes x_2, y^* \otimes x_3\} \\ &= 3, \text{ (using Lemma 5.2)}. \end{aligned}$$

Hence, T is $3-$smooth.
(ii) Let $\pm J(Tx_i) = \{\pm y_i^*\}$ for $1 \leq i \leq 4$. Since (i) is not satisfied, without loss of generality, we assume $y_1^* \neq \pm y_2^*$, i.e., $\{y_1^*, y_2^*\}$ is linearly independent. Let $y_3^* = ay_1^* + by_2^*$ and $y_4^* = cy_1^* + dy_2^*$, where $a, b, c, d \in \mathbb{R}$. Since $\|y_3^*\| = 1$, a and b cannot be zero simultaneously. Similarly, c and d

can not be zero simultaneously. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= \dim \ span\ J(T) \\ &= \dim \ span\ Ext\ J(T) \\ &= \dim \ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_2, y_3^* \otimes x_3, y_4^* \otimes x_4\}. \end{aligned}$$

We show that $\{y_1^* \otimes x_1, y_2^* \otimes x_2, y_3^* \otimes x_3, y_4^* \otimes x_4\}$ is linearly independent. Let $c_i(1 \leq i \leq 4) \in \mathbb{R}$ be such that

$$c_1 y_1^* \otimes x_1 + c_2 y_2^* \otimes x_2 + c_3 y_3^* \otimes x_3 + c_4 y_4^* \otimes x_4 = 0.$$

Then

$$c_1 y_1^* \otimes x_1 + c_2 y_2^* \otimes x_2 + c_3(a y_1^* + b y_2^*) \otimes x_3 + c_4(c y_1^* + d y_2^*) \otimes (x_1 - x_2 + x_3) = 0.$$

$$\Rightarrow (c_1 + c_4 c) y_1^* \otimes x_1 + (c_2 - c_4 d) y_2^* \otimes x_2 + (c_3 a + c_4 c) y_1^* \otimes x_3 + (c_3 b + c_4 d) y_2^* \otimes x_3$$
$$-c_4 c y_1^* \otimes x_2 + c_4 d y_2^* \otimes x_1 = 0.$$

Since $\{x_1, x_2, x_3\}$ is linearly independent subset of $\mathbb{X}$ and $\{y_1^*, y_2^*\}$ is linearly independent subset of $\mathbb{X}^*$, from Lemma 5.2, we get that $\{y_1^* \otimes x_1, y_2^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_1^* \otimes x_3, y_2^* \otimes x_3\}$ is linearly independent subset of $\mathbb{L}(\mathbb{X})^*$. Therefore,

$$c_1 + c_4 c = 0,\ c_2 - c_4 d = 0,\ c_3 a + c_4 c = 0,\ c_3 b + c_4 d = 0,\ c_4 c = 0,\ c_4 d = 0.$$

Now, solving these equations, we obtain $c_i = 0$ for all $1 \leq i \leq 4$. Therefore, $\{y_1^* \otimes x_1, y_2^* \otimes x_2, y_3^* \otimes x_3, y_4^* \otimes x_4\}$ is linearly independent. Hence, $k = 4$ and T is $4-$smooth. □

In the following theorem, we consider the case when $|S_1| = 6$ and $|S_2| = 2$.

Theorem 5.2. *Let $\mathbb{X} = \ell_\infty^3$ and let $T \in S_{\mathbb{L}(\mathbb{X})}$ be such that $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$. Let $|S_1| = 6$ and $|S_2| = 2$. Then the following hold:*
(i) If $\pm J(Tx_i) = \pm J(Tx_j)$ for all $x_i, x_j \in S_1$ and $\pm J(Tx_i) \subseteq \pm Ext\ J(Tx_k)$ for all $x_i \in S_1$ and $x_k \in S_2$, then T is $4-$smooth.
(ii) Otherwise, T is $5-$smooth.

Proof. Without loss of generality, we assume Tx_1 is $2-$smooth and Tx_i are smooth for $2 \leq i \leq 4$.

(i) Let the given condition be satisfied. Without loss of generality, we may assume Tx_1 belongs to the edge $\{(1,1,z) : |z| \leq 1\}$ and Tx_2, Tx_3, Tx_4 belong to the facet F_1. Thus $Ext\ J(Tx_1) = \{y_1^*, y_2^*\}$ and $J(Tx_i) = \{y_1^*\}$ for $i = 2, 3, 4$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= dim\ span\ J(T) \\ &= dim\ span\ Ext\ J(T) \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_1^* \otimes x_2, y_1^* \otimes x_3, y_1^* \otimes x_4\} \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_1^* \otimes x_2, y_1^* \otimes x_3\} \\ &= 4, \text{ (by Lemma 5.2).} \end{aligned}$$

Hence T is $4-$smooth.

(ii) Suppose the condition (i) is not satisfied. Without loss of generality, we may assume Tx_1 belongs to the edge $\{(1,1,z) : |z| \leq 1\}$, Tx_2, Tx_3 belong to the facet F_1 and Tx_4 belongs to the facet F_2. Thus $J(Tx_1) = \{y_1^*, y_2^*\}$, $J(Tx_i) = \{y_1^*\}$ for $i = 2, 3$ and $J(Tx_4) = \{y_2^*\}$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= dim\ span\ J(T) \\ &= dim\ span\ Ext\ J(T) \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_1^* \otimes x_2, y_1^* \otimes x_3, y_2^* \otimes x_4\}. \end{aligned}$$

We now show that $\{y_1^* \otimes x_1, y_2^* \otimes x_1, y_1^* \otimes x_2, y_1^* \otimes x_3, y_2^* \otimes x_4\}$ is linearly independent. Let us consider the relation

$$c_1 y_1^* \otimes x_1 + c_2 y_2^* \otimes x_1 + c_3 y_1^* \otimes x_2 + c_4 y_1^* \otimes x_3 + c_5 y_2^* \otimes x_4 = 0, \text{ where } c_1, c_2, c_3, c_4 \in \mathbb{R}.$$

Then

$$c_1 y_1^* \otimes x_1 + c_2 y_2^* \otimes x_1 + c_3 y_1^* \otimes x_2 + c_4 y_1^* \otimes x_3 + c_5 y_2^* \otimes (x_1 - x_2 + x_3) = 0,$$

i.e.,

$$c_1 y_1^* \otimes x_1 + (c_2 + c_5) y_2^* \otimes x_1 + c_3 y_1^* \otimes x_2 - c_5 y_2^* \otimes x_2 + c_4 y_1^* \otimes x_3 + c_5 y_2^* \otimes x_3 = 0.$$

By Lemma 5.2, $\{y_1^* \otimes x_1, y_2^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_1^* \otimes x_3, y_2^* \otimes x_3\}$ is linearly independent. Therefore, $c_1 = c_2 = c_3 = c_4 = c_5 = 0$. Thus, $k = 5$ and hence T is $5-$smooth. In all the other cases, exactly in a similar manner, we can show that T is $5-$smooth. This completes the proof of the theorem. $\square$

Next, we consider the case when $|S_1| = |S_2| = 4$.

Theorem 5.3. *Let $\mathbb{X} = \ell_\infty^3$ and let $T \in S_{\mathbb{L}(\mathbb{X})}$ be such that $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$. Let $|S_1| = |S_2| = 4$. Then the following hold:*
(i) If $\pm J(Tx_i) = \pm J(Tx_j)$ for all $x_i, x_j \in S_1$ and $\pm J(Tx_i) \subseteq \pm Ext\ J(Tx_k)$ for all $x_i \in S_1$ and $x_k \in S_2$, then T is $5-$smooth.
(ii) Otherwise, T is $6-$smooth.

Proof. Without loss of generality, we assume Tx_1, Tx_2 are $2-$smooth points and Tx_3, Tx_4 are smooth points.

(i) Let the given condition be satisfied, i.e., either Tx_3, Tx_4 or $Tx_3, -Tx_4$ belong to the same facet which is adjacent to both of the edges E_l, where E_l contains Tx_l or $-Tx_l$ for $l = 1, 2$. Without loss of generality, we assume Tx_1 belongs to the edge $\{(1, 1, z) : |z| \leq 1\}$, Tx_2 belongs to the edge $\{(1, y, 1) : |y| \leq 1\}$, Tx_3, Tx_4 belong to the facet F_1. Thus, $Ext\ J(Tx_1) = \{y_1^*, y_2^*\}$, $Ext\ J(Tx_2) = \{y_1^*, y_3^*\}$ and $J(Tx_3) = J(Tx_4) = \{y_1^*\}$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= dim\ span\ J(T) \\ &= dim\ span\ Ext\ J(T) \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_1^* \otimes x_2, y_3^* \otimes x_2, y_1^* \otimes x_3, y_1^* \otimes x_4\} \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_1^* \otimes x_2, y_3^* \otimes x_2, y_1^* \otimes x_3\} \\ &= 5 \text{ (by Lemma 5.2).} \end{aligned}$$

Hence T is $5-$smooth. In all the other cases, similarly we can show that T is $5-$smooth.

(ii) Let the condition (a) be not satisfied. Then one of the following conditions hold:
(1) neither Tx_3, Tx_4 nor $Tx_3, -Tx_4$ belong to the same facet.
(2) either one of the pairs of Tx_3, Tx_4 and $Tx_3, -Tx_4$ belongs to the same facet which is not the common adjacent facet of the edges E_l, where E_l contains Tx_l or $-Tx_l$ for $l = 1, 2$.
Let (1) be true. Without loss of generality, we assume Tx_1, Tx_2 belong to the edge $\{(1, 1, z) : |z| \leq 1\}$, Tx_3 belongs to the facet F_1 and Tx_4 belongs to the facet F_2. Thus, $J(Tx_1) = J(Tx_2) =$

$\{y_1^*, y_2^*\}$, $J(Tx_3) = \{y_1^*\}$ and $J(Tx_4) = \{y_2^*\}$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= dim\ span\ J(T) \\ &= dim\ span\ Ext\ J(T) \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_1^* \otimes x_3, y_2^* \otimes x_4\} \\ &= 6, \text{by simple calculation.} \end{aligned}$$

Hence T is $6-$smooth. If (2) is true, then similarly we can show that T is $6-$smooth. This completes the proof of the theorem.

□

In the following theorem, we show that if $|S_1| = 2$ and $|S_2| = 6$, then T is $6-$smooth.

Theorem 5.4. *Let $\mathbb{X} = \ell_\infty^3$ and let $T \in S_{\mathbb{L}(\mathbb{X})}$ be such that $M_T \cap Ext(B_\mathbb{X}) = \{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$. Let $|S_1| = 2, |S_2| = 6$. Then T is $6-$smooth.*

Proof. Without loss of generality, we assume Tx_1, Tx_2, Tx_3 are $2-$smooth points and Tx_4 is smooth point. If possible, let the edges $E_l, l = 1, 2, 3$ have no common adjacent facet, where E_l contains Tx_l or $-Tx_l$.
Without loss let us assume $Tx_1 = (\pm1, \pm1, a)$, $Tx_2 = (\pm1, a', \pm1)$, $Tx_3 = (a'', \pm1, \pm1)$, where $|a|, |a'|, |a''| < 1$. Then

$$\begin{aligned} & Tx_4 \\ = \ & Tx_1 - Tx_2 + Tx_3 \\ = \ & (\pm1, \pm1, a) - (\pm1, a', \pm1) + (a'', \pm1, \pm1), \end{aligned}$$

which shows thar either $\|Tx_4\| > 1$ or $\|Tx_4\| < 1$, a contradiction. Therefore, the edges $E_l, l = 1, 2, 3$ have common adjacent facet, where E_l contains Tx_l or $-Tx_l$.
If possible let Tx_4 does not belong to the common adjacent facet of $E_l, l = 1, 2, 3$. Without loss let $Tx_1 = (\pm1, \pm1, a)$, $Tx_2 = (\pm1, \pm1, a')$, $Tx_3 = (\pm1, b, \pm1)$, where $|a|, |a'|, |b| < 1$. Then

$$\begin{aligned} & Tx_4 \\ = \ & Tx_1 - Tx_2 + Tx_3 \\ = \ & (\pm1, \pm1, a) - (\pm1, \pm1, a') + (\pm1, b, \pm1) \\ = \ & (\pm1, b, a - a' \pm 1), \end{aligned}$$

otherwise $\|Tx_4\| > 1$, which shows that either Tx_4 or $-Tx_4$ must belong to the common adjacent facet of $E_l, l = 1, 2, 3$.

Without loss of generality we assume Tx_1, Tx_2 belong to the edge $\{(1,1,z) : |z| \leq 1\}$, Tx_3 belongs to the edge $\{(1,y,1) : |y| \leq 1\}$, Tx_4 belongs to the facet F_1. Thus, $Ext\ J(Tx_1) = Ext\ J(Tx_2) = \{y_1^*, y_2^*\}$, $Ext\ J(Tx_3) = \{y_1^*, y_3^*\}$ and $J(Tx_4) = \{y_1^*\}$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= dim\ span\ J(T) \\ &= dim\ span\ Ext\ J(T) \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_1^* \otimes x_3, y_3^* \otimes x_3, y_1^* \otimes x_4\} \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_1^* \otimes x_3, y_3^* \otimes x_3\} \\ &= 6, \text{ (by Lemma 5.2)}. \end{aligned}$$

Hence T is $6-$smooth. In all the other cases, similarly we can show that T is $6-$smooth. □

Theorem 5.5. *Let $\mathbb{X} = \ell_\infty^3$ and let $T \in S_{\mathbb{L}(\mathbb{X})}$ be such that $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$. Let $|S_2| = 8$. Then the following hold:*
(i) If $\left| \cap_{i=1}^4 \pm Ext\ J(Tx_i) \right| = 4$, then T is $6-$smooth.
(ii) Otherwise, T is $7-$smooth.

Proof. (i) Suppose the given condition is satisfied. Without loss of generality we assume Tx_1 belongs to the edge $\{(1,1,z) : |z| \leq 1\}$, , Tx_2 belongs to the edge $\{(-1,1,z) : |z| \leq 1\}$, Tx_3 belongs to the edge $\{(-1,-1,z) : |z| \leq 1\}$ and Tx_4 belongs to the edge $\{(1,-1,z) : |z| \leq 1\}$. Thus, $Ext\ J(Tx_1) = \{y_1^*, y_2^*\}$, $Ext\ J(Tx_2) = \{y_2^*, -y_1^*\}$, $Ext\ J(Tx_3) = \{-y_1^*, -y_2^*\}$ and $Ext\ J(Tx_4) = \{y_1^*, -y_2^*\}$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= dim\ span\ J(T) \\ &= dim\ span\ Ext\ J(T) \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_2^* \otimes x_2, y_1^* \otimes x_2, y_1^* \otimes x_3, y_2^* \otimes x_3, y_1^* \otimes x_4, \\ &\qquad y_2^* \otimes x_4\} \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_1^* \otimes x_3, y_2^* \otimes x_3\} \\ &= 6 \text{ (by Lemma 5.2)}. \end{aligned}$$

Hence, T is $6-$smooth. In all the other cases, similarly we can show that T is $6-$smooth.

(ii) Suppose the condition of (a) is not satisfied. We first show that the edges $E_l, l = 1,2,3$ have a common adjacent facet, where E_l contains Tx_l or $-Tx_l$. If possible, let the edges $E_l, l = 1,2,3$ have no common adjacent facet, where E_l contains Tx_l or $-Tx_l$. Let $Tx_1 = (\pm 1, \pm 1, a)$,

$Tx_2 = (\pm 1, a', \pm 1)$, $Tx_3 = (a'', \pm 1, \pm 1)$, where $|a|, |a'|, |a''| < 1$. Then

$$\begin{aligned} & Tx_4 \\ = \ & Tx_1 - Tx_2 + Tx_3 \\ = \ & (\pm 1, \pm 1, a) - (\pm 1, a', \pm 1) + (a'', \pm 1, \pm 1), \end{aligned}$$

which shows that either $\|Tx_4\| > 1$ or $\|Tx_4\| < 1$, a contradiction. Now, let exactly four points of the $2-$smooth points belong to parallel edges. Let $Tx_1 = (\pm 1, \pm 1, a)$, $Tx_2 = (\pm 1, \pm 1, a')$, $Tx_3 = (\pm 1, b, \pm 1)$, where $|a|, |a'|, |b| < 1$. Then

$$\begin{aligned} & Tx_4 \\ = \ & Tx_1 - Tx_2 + Tx_3 \\ = \ & (\pm 1, \pm 1, a) - (\pm 1, \pm 1, a') + (\pm 1, b, \pm 1) \\ = \ & (\pm 1, b, a - a' \pm 1), \end{aligned}$$

otherwise $||Tx_4|| > 1$. So we must have $a = a'$ as Tx_4 is $2-$smooth and then $\{(x, y, z) : x = 1\}$ is the common facet which is adjacent to the edges containing Tx_i, or $-Tx_i$, $i = 1, 2, 3, 4$. The other cases are similar. Without loss of generality, we assume Tx_1, Tx_2 belong to the edge $\{(1, 1, z) : |z| < 1\}$, Tx_3 belongs to the edge $\{(x, 1, 1) : |x| < 1\}$, Tx_4 belongs to the edge $\{(x, -1, 1) : |x| < 1\}$. Therefore, F_2 is the required common facet. Thus, $Ext\ J(Tx_1) = J(Tx_2) = \{y_1^*, y_2^*\}$, $Ext\ J(Tx_3) = \{y_2^*, y_3^*\}$ and $Ext\ J(Tx_4) = \{-y_2^*, y_3^*\}$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= dim\ span\ J(T) \\ &= dim\ span\ Ext\ J(T) \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_2^* \otimes x_3, y_3^* \otimes x_3, -y_2^* \otimes x_4, \\ & \quad y_3^* \otimes x_4\} \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_2^* \otimes x_3, y_3^* \otimes x_3, y_3^* \otimes x_4\} \\ &= 7, \text{ (by simple calculation)}. \end{aligned}$$

Hence T is $7-$smooth. In all the other cases, similarly we can show that T is $7-$smooth. This completes the proof of the theorem. □

Now, we turn our attention to the case $S_3 \neq \emptyset$. In Theorem 5.6 we assume $|S_3| = 2, S_2 = \emptyset$ and in Theorem 5.7, we consider $S_3 = 2, S_2 \neq \emptyset$.

Theorem 5.6. *Let $\mathbb{X} = \ell_\infty^3$ and let $T \in S_{\mathbb{L}(\mathbb{X})}$ be such that $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$. Let $|S_1| = 6, |S_3| = 2$. Then the following hold:*
(i) If $\pm J(Tx_i) = \pm J(Tx_j)$ for all $x_i, x_j \in S_1$, then T is $5-$smooth.
(ii) Otherwise, T is $6-$smooth.

Proof. Without loss of generality we assume $\pm Tx_1$ are $3-$smooth.
(i) Suppose the given condition is satisfied. Without loss of generality, we may assume $Tx_1 = (1,1,1)$ and Tx_2, Tx_3, Tx_4 belong to the facet F_1. Thus, $Ext\ J(Tx_1) = \{y_1^*, y_2^*, y_3^*\}$ and $J(Tx_i) = \{y_1^*\}$ for $i = 2,3,4$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= dim\ span\ J(T) \\ &= dim\ span\ Ext\ J(T) \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_3^* \otimes x_1, y_1^* \otimes x_2, y_1^* \otimes x_3, y_1^* \otimes x_4\} \\ &= dim\ span\ \{y^* \otimes x_1, y_2^* \otimes x_1, y_1^* \otimes x_1, y_1^* \otimes x_2, y_1^* \otimes x_3\} \\ &= 5 \text{ (by Lemma 5.2).} \end{aligned}$$

Hence T is $5-$smooth. In all the other cases, similarly we can show that T is $5-$smooth.
(ii) Suppose the condition (a) is not satisfied. Without loss of generality, we may assume $Tx_1 = (1,1,1)$, and Tx_2, Tx_3 belong to the facet F_1 and Tx_4 belongs to the facet F_2. Thus, $J(Tx_1) = \{y_1^*, y_2^*, y_3^*\}$ and $J(Tx_2) = J(Tx_3) = \{y_1^*\}$ and $J(Tx_4) = \{y_2^*\}$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= dim\ span\ J(T) \\ &= dim\ span\ Ext\ J(T) \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_3^* \otimes x_1, y_1^* \otimes x_2, y_1^* \otimes x_3, y_2^* \otimes x_4\} \\ &= 6, \text{ (by simple calculation).} \end{aligned}$$

Hence T is $6-$smooth. In all the other cases, similarly we can show that T is $6-$smooth.

□

Theorem 5.7. *Let $\mathbb{X} = \ell_\infty^3$ and let $T \in S_{\mathbb{L}(\mathbb{X})}$ be such that $M_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$. Let $|S_3| = 2, S_2 \neq \emptyset$. Then the following hold:*
(I) *If $|S_2| = 2$, then T is $6-$smooth.*
(II) *If $|S_2| = 4$, then T is $7-$smooth.*
(III) *If $|S_2| = 6$, then T is $7-$smooth.*

Proof. Without loss of generality we assume Tx_1 is $3-$smooth. So, $Tx_1 = (\pm1, \pm1, \pm1)$.

(**I**) Let $|S_2| = 2$. Without loss of generality, we assume Tx_2 is $2-$smooth and $Tx_2 = (\pm1, \pm1, c)$, where $|c| < 1$. Then Tx_i are smooth for $3 \leq i \leq 4$. If possible, suppose that $Tx_3 = (l, m, \pm1)$, where $|l|, |m| < 1$. Then

$$\begin{aligned}
& Tx_4 \\
= \; & Tx_1 - Tx_2 + Tx_3 \\
= \; & (\pm1, \pm1, \pm1) - (\pm1, \pm1, c) + (l, m, \pm1) \\
= \; & (l, m, c),
\end{aligned}$$

otherwise $\|Tx_4\| > 1$. So $\|Tx_4\| < 1$, which is a contradiction. Therefore, either $Tx_3 = (\pm1, a, b)$, where $|a|, |b| < 1$ or $Tx_3 = (p, \pm1, q)$, where $|p|, |q| < 1$. Without loss of generality, let $Tx_3 = (\pm1, a, b)$, where $|a|, |b| < 1$. Then

$$\begin{aligned}
& Tx_4 \\
= \; & Tx_1 - Tx_2 + Tx_3 \\
= \; & (\pm1, \pm1, \pm1) - (\pm1, \pm1, c) + (\pm1, a, b) \\
= \; & (\pm1, a, \pm1 - c + b),
\end{aligned}$$

otherwise $\|Tx_4\| > 1$. If $b = c$, then Tx_4 will be $2-$smooth, a contradiction. Therefore, $b \neq c$ and Tx_3, Tx_4 belong to the same facet which is one of the adjacent facet of the edge containing Tx_2. For simplicity, suppose $Tx_1 = (1, 1, 1)$, Tx_2 belongs to the edge $\{(1, 1, z) : |z| \leq 1\}$ and Tx_3, Tx_4 belong to the facet F_1. Thus, we have $Ext\ J(Tx_1) = \{y_1^*, y_2^*, y_3^*\}$, $Ext\ J(Tx_2) = \{y_1^*, y_2^*\}$ and $J(Tx_3) = J(Tx_4) = \{y_1^*\}$. Now, if T is $k-$smooth, then

$$\begin{aligned}
k &= dim\ span\ J(T) \\
&= dim\ span\ Ext\ J(T) \\
&= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_3^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_1^* \otimes x_3, y_1^* \otimes x_4\} \\
&= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_3^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_1^* \otimes x_3\} \\
&= 6 \text{ (by Lemma 5.2)}.
\end{aligned}$$

Hence T is $6-$smooth. Considering other cases, we can similarly show that T is $6-$smooth.

(**II**) Let $|S_2| = 4$. Without loss of generality, we assume Tx_2, Tx_3 are $2-$smooth and Tx_4

is smooth. Without loss we may assume $Tx_2 = (\pm 1, \pm 1, c), |c| < 1$. If possible, suppose that $Tx_3 = (\pm 1, \pm 1, c')$, where $|c'| < 1$. Then

$$\begin{aligned}
& Tx_4 \\
= \; & Tx_1 - Tx_2 + Tx_3 \\
= \; & (\pm 1, \pm 1, \pm 1) - (\pm 1, \pm 1, c) + (\pm 1, \pm 1, c') \\
= \; & (\pm 1, \pm 1, \pm 1 - c + c'),
\end{aligned}$$

otherwise $\|Tx_4\| > 1$. So Tx_4 is at least $2-$smooth, which is a contradiction. Therefore, either $Tx_3 = (\pm 1, b, \pm 1)$, where $|b| < 1$ or $Tx_3 = (a, \pm 1, \pm 1)$, where $|a| < 1$. Let $Tx_3 = (\pm 1, b, \pm 1)$, where $|b| < 1$. Then

$$\begin{aligned}
& Tx_4 \\
= \; & Tx_1 - Tx_2 + Tx_3 \\
= \; & (\pm 1, \pm 1, \pm 1) - (\pm 1, \pm 1, c) + (\pm 1, b, \pm 1) \\
= \; & (\pm 1, b, -c),
\end{aligned}$$

otherwise $\|Tx_4\| > 1$. So Tx_4 or $-Tx_4$ belongs to the facet which is the common adjacent facet to the edges containing Tx_2, Tx_3 or $Tx_2, -Tx_3$. Similarly, considering $Tx_3 = (a, \pm 1, \pm 1)$, where $|a| < 1$, we can show that either Tx_4 or $-Tx_4$ belongs to the facet which is the common adjacent facet to the edges containing Tx_2, Tx_3 or $Tx_2, -Tx_3$. For simplicity, let $Tx_1 = (1, 1, 1)$, Tx_2 belongs to the edge $\{(1, 1, z) : |z| \leq 1\}$, Tx_3 belongs to the edge $\{(1, y, 1) : |y| \leq 1\}$ and Tx_4 belongs to the facet $\{(x, y, z) : x = 1\}$. Thus, we have $Ext\ J(Tx_1) = \{y_1^*, y_2^*, y_3^*\}$, $Ext\ J(Tx_2) = \{y_1^*, y_2^*\}$, $Ext\ J(Tx_3) = \{y_1^*, y_3^*\}$ and $J(Tx_4) = \{y_1^*\}$. Now, if T is $k-$smooth, then

$$\begin{aligned}
k &= dim\ span\ J(T) \\
&= dim\ span\ Ext\ J(T) \\
&= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_3^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_1^* \otimes x_3, y_3^* \otimes x_3, \\
&\quad y_1^* \otimes x_4\} \\
&= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_3^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_1^* \otimes x_3, y_3^* \otimes x_3\} \\
&= 7 \text{ (by Lemma 5.2)}.
\end{aligned}$$

Hence T is $7-$smooth. Considering other cases, similarly we can show that T is $7-$smooth.

(**III**) Let $|S_2| = 6$. Then Tx_2, Tx_3, Tx_4 are $2-$smooth. Following the same argument used in (**II**), we can conclude that all $2-$smooth points $Tx_i, i = 2, 3, 4$ belong to the parallel edges, that is, if $Tx_2 = (\pm1, \pm1, c), |c| < 1$, then $Tx_3 = (\pm1, \pm1, c')$, where $|c'| < 1$ and $Tx_4 = (\pm1, \pm1, \pm1 - c + c')$. For simplicity, let $Tx_1 = (1, 1, 1)$, Tx_2, Tx_3, Tx_4 belong to the edge $\{(1, 1, z) : |z| \leq 1\}$. Thus, we have $Ext\ J(Tx_1) = \{y_1^*, y_2^*, y_3^*\}$, $Ext\ J(Tx_2) = Ext\ J(Tx_3) = Ext\ J(Tx_4) = \{y_1^*, y_2^*\}$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= dim\ span\ J(T) \\ &= dim\ span\ Ext\ J(T) \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_3^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_1^* \otimes x_3, y_2^* \otimes x_3, \\ &\quad y_1^* \otimes x_4, y_2^* \otimes x_4\} \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_3^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_1^* \otimes x_3, y_2^* \otimes x_3\} \\ &= 7 \text{ (by Lemma 5.2).} \end{aligned}$$

Hence T is $7-$smooth. Considering other cases, similarly we can show that T is $7-$smooth. This completes the proof of the theorem. ☐

To completely determine the order of smoothness of $T \in \mathbb{L}(\mathbb{X})$, the only remaining case to study is $|S_3| \geq 4$. Now, we consider this case.

Theorem 5.8. *Let $\mathbb{X} = \ell_\infty^3$ and let $T \in S_{\mathbb{L}(\mathbb{X})}$ be such that $M_T \cap Ext(B_\mathbb{X}) = \{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$. Let $|S_3| \geq 4$. Then the following hold:*
(I) *If $|S_3| = 4$ and $S_2 = \emptyset$, then T is $7-$smooth.*
(II) *If $|S_3| = 4$ and $S_2 \neq \emptyset$, then T is $8-$smooth.*
(III) *If $|S_3| > 4$ then $|S_3| = 8$ and T is $9-$smooth.*

Proof. (**I**) Let $|S_3| = 4$ and $S_2 = \emptyset$. Then $|S_1| = 4$. Without loss of generality, we assume Tx_1, Tx_2 are $3-$smooth and Tx_3, Tx_4 are smooth. So, $Tx_1 = Tx_2 = (\pm1, \pm1, \pm1)$. First assume that $Tx_3 = (\pm1, a, b)$, where $|a|, |b| < 1$. Then

$$\begin{aligned} & Tx_4 \\ =\ & Tx_1 - Tx_2 + Tx_3 \\ =\ & (\pm1, \pm1, \pm1) - (\pm1, \pm1, \pm1) + (\pm1, a, b) \\ =\ & (\pm1, a, b), \end{aligned}$$

otherwise $||Tx_4|| > 1$. So Tx_3, Tx_4 belong to the facet F_1 or $-F_1$. Similarly, $Tx_3 = (a, \pm1, b)$, where $|a| < 1, |b| < 1$, implies that Tx_3, Tx_4 belong to the facet F_2 or $-F_2$ and $Tx_3 = (a, b, \pm1)$, where $|a| < 1, |b| < 1$, implies that Tx_3, Tx_4 belong to the facet F_3 or $-F_3$. For simplicity, we assume that $Tx_1 = Tx_2 = (1,1,1)$, Tx_3, Tx_4 belong to $\{(x,y,z) : x = 1\}$. Thus, we have $Ext\ J(Tx_1) = Ext\ J(Tx_2) = \{y_1^*, y_2^*, y_3^*\}$ and $J(Tx_3) = J(Tx_4) = \{y_1^*\}$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= dim\ span\ J(T) \\ &= dim\ span\ Ext\ J(T) \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_3^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_3^* \otimes x_2, y_1^* \otimes x_3, \\ &\quad y_1^* \otimes x_4\} \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_3^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_3^* \otimes x_2, y_1^* \otimes x_3\} \\ &= 7 \text{ (by Lemma 5.2)}. \end{aligned}$$

Hence T is $7-$smooth. Similarly, considering other cases, we can show that T is $7-$smooth.

(II) Let $|S_3| = 4$ and $S_2 \neq \emptyset$. Without loss of generality, we assume Tx_1, Tx_2 are $3-$smooth. So, $Tx_1 = Tx_2 = (\pm1, \pm1, \pm1)$. Suppose Tx_3 is $2-$smooth and $Tx_3 = (\pm1, \pm1, a)$, where $|a|, < 1$. Then

$$\begin{aligned} & Tx_4 \\ = \ & Tx_1 - Tx_2 + Tx_3 \\ = \ & (\pm1, \pm1, \pm1) - (\pm1, \pm1, \pm1) + (\pm1, \pm1, a) \\ = \ & (\pm1, \pm1, a), \end{aligned}$$

otherwise $||Tx_4|| > 1$. So Tx_4 must be $2-$smooth and Tx_3, Tx_4 belong to the parallel edges. Similarly, considering other cases, we can show that Tx_4 must be $2-$smooth and Tx_3, Tx_4 belong to the parallel edges. Now, for simplicity, assume that $Tx_1 = Tx_2 = (1,1,1)$, Tx_3, Tx_4 belongs to the edge $\{(1,1,z) : |z| \leq 1\}$. Thus, we have $Ext\ J(Tx_1) = Ext\ J(Tx_2) = \{y_1^*, y_2^*, y_3^*\}$

and $J(Tx_3) = J(Tx_4) = \{y_1^*, y_2^*\}$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= dim\ span\ J(T) \\ &= dim\ span\ Ext\ J(T) \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_3^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_3^* \otimes x_2, y_1^* \otimes x_3, \\ &\quad y_2^* \otimes x_3, y_1^* \otimes x_4, y_2^* \otimes x_4\} \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_3^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_3^* \otimes x_2, y_1^* \otimes x_3, \\ &\quad y_2^* \otimes x_3\} \\ &= 8 \text{ (by Lemma 5.2).} \end{aligned}$$

Hence T is $8-$smooth. Similarly, considering other cases we can show that T is $8-$smooth.

(III) Let $|S_3| > 4$. Suppose Tx_1, Tx_2, Tx_3 are $3-$smooth. So, we must have $Tx_1 = Tx_2 = Tx_3 = (\pm1, \pm1, \pm1)$. Then

$$\begin{aligned} & Tx_4 \\ = \ & Tx_1 - Tx_2 + Tx_3 \\ = \ & (\pm1, \pm1, \pm1) - (\pm1, \pm1, \pm1) + (\pm1, \pm1, \pm1) \\ = \ & (\pm1, \pm1, \pm1), \end{aligned}$$

otherwise $||Tx_4|| > 1$. Thus, Tx_4 is also $3-$smooth and $|S_3| = 8$. Then we have $\pm Ext\ J(Tx_1) \pm = Ext\ J(Tx_2) = \pm Ext\ J(Tx_3) = \pm Ext\ J(Tx_4) = \{\pm y_1^*, \pm y_2^*, \pm y_3^*\}$. Now, if T is $k-$smooth, then

$$\begin{aligned} k &= dim\ span\ J(T) \\ &= dim\ span\ Ext\ J(T) \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_3^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_3^* \otimes x_2, y_1^* \otimes x_3, \\ &\quad y_2^* \otimes x_3, y_3^* \otimes x_3, y_1^* \otimes x_4, y_2^* \otimes x_4, y_3^* \otimes x_4\} \\ &= dim\ span\ \{y_1^* \otimes x_1, y_2^* \otimes x_1, y_3^* \otimes x_1, y_1^* \otimes x_2, y_2^* \otimes x_2, y_3^* \otimes x_2, y_1^* \otimes x_3, \\ &\quad y_2^* \otimes x_3, y_3^* \otimes x_3\} \\ &= 9 \text{ (by Lemma 5.2).} \end{aligned}$$

Hence T is $9-$smooth. In all the other cases, similarly we can show that T is $9-$smooth. This completes the proof of the theorem.

□

CHAPTER 6

NUMERICAL RADIUS NORM AND NU-SMOOTHNESS OF ORDER K

6.1 Introduction

The purpose of this chapter is to study a generalized notion of smoothness and extreme contraction in the space of bounded linear operators endowed with numerical radius norm. We also obtain necessary and sufficient conditions on operators having equal operator norm and numerical radius norm. Let us first fix the notations and terminologies.

Let $\mathbb{H}$ and $\mathbb{X}$ denote respectively Hilbert space and Banach space over the field $\mathbb{R}$. Suppose $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$ respectively denote the unit ball and the unit sphere of $\mathbb{X}$. $\mathbb{X}^*$ is the dual space of $\mathbb{X}$. For a set A, the cardinality of A is denoted by $|A|$. An element $x \in S_{\mathbb{X}}$ is said to be an extreme point of the convex set $B_{\mathbb{X}}$ if and only if $x = (1-t)y + tz$ for some $y, z \in B_{\mathbb{X}}$ and $t \in (0,1)$ implies that $y = z = x$. The set of all extreme points of $B_{\mathbb{X}}$ is denoted by $Ext(B_{\mathbb{X}})$. For $x, y \in \mathbb{X}$, let $L[x,y] = \{tx + (1-t)y : 0 \leq t \leq 1\}$ and $L(x,y) = \{tx + (1-t)y : 0 < t < 1\}$. An element $x^* \in S_{\mathbb{X}^*}$ is said to be a supporting linear functional of $x \in S_{\mathbb{X}}$, if $x^*(x) = 1$. For a unit vector x, let $J(x) = \{x^* \in S_{\mathbb{X}^*} : x^*(x) = 1\}$. The set $J(x)$ for $x \in S_{\mathbb{X}}$ plays a significant role in our study. By the Hahn-Banach Theorem, it is easy to verify that $J(x) \neq \emptyset$, for all $x \in S_{\mathbb{X}}$. We would like to mention that $J(x)$ is a weak*-compact, convex subset of $S_{\mathbb{X}^*}$. A unit vector x is said to be a smooth point if $J(x)$ is singleton. $\mathbb{X}$ is said to be a smooth Banach space if every unit vector of $\mathbb{X}$ is smooth.

Let $\mathbb{L}(\mathbb{H})$ and $\mathbb{L}(\mathbb{X})$ denote the set of all bounded linear operators on $\mathbb{H}$ and $\mathbb{X}$ respectively, endowed with the usual operator norm. For $T \in \mathbb{L}(\mathbb{H})$, the numerical range and numerical radius of T, respectively denoted by $W(T)$ and $w(T)$ are defined as

$$W(T) = \{\langle Tx, x\rangle : x \in S_{\mathbb{H}}\}.$$

$$w(T) = \sup\{|\langle Tx, x\rangle| : x \in S_{\mathbb{H}}\}.$$

Readers can look into [41]. The natural generalization of numerical radius of an operator T on a Banach space $\mathbb{X}$ is as follows:

$$w(T) = \sup\{|x^*(Tx)| : x^* \in S_{\mathbb{X}^*}, x \in S_{\mathbb{X}}, x^*(x) = 1\}.$$

Numerical radius always defines a norm on the corresponding space if the underlying scalar field is complex. However, there are some real Banach spaces $\mathbb{X}$ such that numerical radius defines a norm on $\mathbb{L}(\mathbb{X})$. In this context, readers may follow [5, 4, 43] for further details. Here we consider only those real finite-dimensional Banach spaces $\mathbb{X}$ such that numerical radius defines a norm on $\mathbb{L}(\mathbb{X})$ and throughout the chapter we use the symbol $\mathbb{L}(\mathbb{X})_w$ to denote the space of bounded linear operators endowed with the numerical radius norm. For a bounded linear operator T on $\mathbb{X}$, its norm attainment set M_T and numerical radius attainment set V_T are defined as follows:

$$M_T = \{x \in S_{\mathbb{X}} : \|Tx\| = \|T\|\}.$$

$$V_T = \{x \in S_{\mathbb{X}} : \ \exists\ x^* \in J(x) \text{ such that } |x^*(Tx)| = w(T)\}.$$

Note that both the sets M_T and V_T are non-empty when $\mathbb{X}$ is finite-dimensional. We refers the reader [55] for numerical radius attainment set. Suppose that

$$J(T) = \{f : S_{\mathbb{L}(\mathbb{X})^*} : \ f(T) = \|T\|\} \text{ and}$$

$$J_w(T) = \{f : S_{\mathbb{L}(\mathbb{X})^*_w} : \ f(T) = w(T)\}.$$

A non-zero operator T is said to be a smooth operator if $J(T)$ is singleton and T is said to be $k-$smooth operator if $k = \dim\ span\ J(T)$. Following [53], we call smooth operators of $\mathbb{L}(\mathbb{X})_w$ as nu-smooth operators. Clearly, for any non-zero $T \in \mathbb{L}(\mathbb{X})_w$, T is nu-smooth if and only if $J_w(T)$ is singleton. Motivated by the notion of smooth operator of order k or k-smooth operator, we generalize the notion of nu-smooth operator in the following way.

Definition 6.1. *Let $\mathbb{X}$ be a Banach space. A non-zero operator $T \in L(\mathbb{X})_w$ is said to be nu-smooth of order k if there exist exactly k linearly independent elements $f_1, f_2, \ldots, f_k \in J_w(T)$.*

In other words, T is said to be nu-smooth of order k if

$$k = \dim\ span\ J_w(T) = \dim\ span\ Ext\ J_w(T).$$

Recall that an operator $T \in S_{\mathbb{L}(\mathbb{X})}$ is called an extreme contraction if T is an extreme point of $B_{\mathbb{L}(\mathbb{X})}$. Similarly, we say that $T \in S_{\mathbb{L}(\mathbb{X})_w}$ is an nu-extreme contraction if T is an extreme point of $B_{\mathbb{L}(\mathbb{X})_w}$. For $x^* \in \mathbb{X}^*$ and $x \in \mathbb{X}$, the symbol $x^* \otimes x$ denotes a linear functional on the space of operators defined as $(x^* \otimes x)(S) = x^*(Sx)$ for every operator S on $\mathbb{X}$.

After this introductory part, the chapter has three sections. In Section 6.2, we characterize those operators which have equal operator norm and numerical radius norm. Then in Section 6.3, we find the structure of *span Ext* $J_w(T)$ for a bounded linear operator T on a finite-dimensional Banach space. With the help of this structure, we obtain the order of nu-smoothness of some class of operators. Moreover, we completely characterize the nu-smooth operators of order k on two-dimensional Banach spaces. We devote Section 6.4 to study the nu-extreme contraction on a Banach space. Furthermore, we completely characterize the set of extreme points of $B_{\mathbb{L}(\mathbb{X})^*_w}$, where $\mathbb{X}$ a two-dimensional polygonal Banach space. To serve our purpose, we need the following definition.

Definition 6.2. *Let $\mathbb{X}$ be a Banach space. An element $x \in S_{\mathbb{X}}$ is said to be nu-smooth of order k with respect to T if there exist exactly k linearly independent elements $x_1^*, x_2^*, \ldots, x_k^* \in J(x)$ such that $|x_i^*(Tx_i)| = w(T)$, i.e., x is said to be nu-smooth of order k with respect to T, if $k = \dim\ span\ J_w(Tx)$, where $J_w(Tx) = \{x^* \in J(x) : |x^*(Tx)| = w(T)\}$.*

6.2 Operators with equal operator norm and numerical radius norm

We begin this section with an easy observation.

Proposition 6.1. *Suppose $\mathbb{X}$ is a Banach space. Let $T \in \mathbb{L}(\mathbb{X})$ be such that $\pm J(Tx) \cap \pm J(x) \neq \emptyset$ for some $x \in M_T$. Then $\|T\| = w(T)$.*

Proof. Let $x \in M_T$ be such that $\pm J(Tx) \cap \pm J(x) \neq \emptyset$. As $x \in M_T$, Tx belongs to the boundary of the sphere centered at 0 of radius $\|T\|$. Let $x^* \in \pm J(Tx) \cap \pm J(x)$. Then $|x^*(Tx)| = \|Tx\| = \|T\|$. Thus by the definition of $w(T)$, we have $\|T\| = w(T)$. □

We would like to mention the following remark.

Remark 6.1. *If $\mathbb{X}$ is a finite-dimensional strictly convex Banach space then it is easy to observe that $\pm J(Tx_0) \cap \pm J(x_0) \neq \emptyset$ for some $x_0 \in M_T$ implies that $Tx_0 = x_0$ and hence clearly $\|T\| = w(T)$.*

Next, we prove the converse of Proposition 6.1, when $\mathbb{X}$ is finite-dimensional.

Proposition 6.2. *Suppose $\mathbb{X}$ is a finite-dimensional Banach space and $T \in \mathbb{L}(\mathbb{X})$ be such that $||T|| = w(T)$. Then $\pm J(Tx) \cap \pm J(x) \neq \emptyset$ for some $x \in M_T$.*

Proof. Without loss of generality, we may assume that $\|T\| = 1$. Let $w(T) = \|T\|$, i.e., $\sup\{|x^*(Tx)| : x^* \in J(x), x \in S_{\mathbb{X}}\} = ||T||$. Then there exist a sequence $\{x_n\} \subseteq S_{\mathbb{X}}$ and $x_n^* \in J(x_n)$ for each $n \in \mathbb{N}$, such that $|x_n^*(Tx_n)| \to \|T\|$. Since $\mathbb{X}$ is finite-dimensional, $S_{\mathbb{X}}$ and $S_{\mathbb{X}^*}$ are compact. Thus $\{x_n\}$ and $\{x_n^*\}$ have convergent sequences. Without loss of generality, we assume $x_n \to x$ and $x_n^* \to x^*$. Thus $x \in S_{\mathbb{X}}$ and $x^* \in S_{\mathbb{X}^*}$. Now, $|x_n^*(x_n) - x^*(x)| \leq |x_n^*(x_n) - x_n^*(x)| + |x_n^*(x) - x^*(x)| \leq \|x_n - x\| + \|x_n^* - x^*\| \to 0$ and so $x^*(x) = 1$. This implies that $x^* \in J(x)$. Also, $|x_n^*(Tx_n) - x^*(Tx)| \leq |x_n^*(Tx_n) - x_n^*(Tx)| + |x_n^*(Tx) - x^*(Tx)| \leq \|Tx_n - Tx\| + \|x_n^* - x^*\| \to 0$. Then $x_n^*(Tx_n) \to x^*(Tx)$ so that $|x^*(Tx)| = \|T\|$ and hence $|x^*(Tx)| = w(T)$. Now $x^* \in J(x)$ forces $\|Tx\| = \|T\|$ so that $x \in M_T$. Thus $x^* \in \pm J(Tx) \cap \pm J(x)$, where $x \in M_T$. This proves the proposition. $\square$

The following example shows the necessity of the assumption that $\dim(\mathbb{X})$ is finite.

Example 6.2. *Let $\mathbb{X} = \ell_p,\ 1 < p < \infty$ and e_n be the sequence whose n-th coordinate is 1 and all other coordinates are zero. Let $T : \mathbb{X} \to \mathbb{X}$ be defined by*

$$T(x_1, x_2, x_3, x_4, \dots) = (x_2, 0, \frac{2}{3}x_3, \frac{3}{4}x_4, \dots).$$

Then it is easy to observe that T is a linear operator, $\|T\| = 1$ and $M_T = \{\pm e_2\}$. Now to find $\pm J(Te_2) \cap \pm J(e_2)$, let $y^* \in \pm J(Te_2) \cap \pm J(e_2)$. If $y^* = (y_1, y_2, y_3, \dots)$, then $|y^*(e_2)| = 1 \Rightarrow |(y_1, y_2, y_3, \dots)(0, 1, 0, \dots)| = 1 \Rightarrow |y_2| = 1$. Thus $y^* = (0, \pm 1, 0, \dots)$. Again $|y^*(Te_2)| = |y^*(e_1)| = |(0, \pm 1, 0, \dots)(1, 0, 0, \dots)| = 0 \Rightarrow y^* \notin \pm J(Te_2)$. Therefore, $\pm J(Te_2) \cap \pm J(e_2) = \emptyset$. Clearly $\|T\| = w(T)$, which follows from the fact that

$$1 = \|T\| \geq w(T) \geq sup_{n \in \mathbb{N}}\ |e_n^*(Te_n)| = 1.$$

Combining Proposition 6.1 and Proposition 6.2, we get the following characterization of a bounded linear operator on a finite-dimensional Banach space with equal operator norm and numerical radius norm.

Theorem 6.3. *Suppose $\mathbb{X}$ is a finite-dimensional Banach space. Then for any $T \in \mathbb{L}(\mathbb{X})$, $\|T\| = w(T)$ if and only if $\pm J(Tx) \cap \pm J(x) \neq \emptyset$ for some $x \in M_T$.*

The following example illustrates the above theorem.

Example 6.4. *Consider the two-dimensional Banach space $\mathbb{X}$ such that $S_{\mathbb{X}}$ is a regular octagon having vertices $\pm x_1 = \pm(1,0), \pm x_2 = \pm(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \pm x_3 = \pm(0,1),\ \pm x_4 = \pm(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Let $T \in \mathbb{L}(\mathbb{X})$ be defined by $T(1,0) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $T(0,1) = (0,0)$. Here $\pm J(Tx_1) \cap \pm J(x_1) \neq \emptyset$ where $x_1 \in M_T$ and hence by Theorem 6.3, $\|T\| = w(T)$.*

We end this section with the following theorem which gives another sufficient condition for the equality of operator norm and numerical radius norm of a bounded linear operator defined on a Banach space.

Theorem 6.5. *Let $\mathbb{X}$ be any Banach space and $T \in S_{\mathbb{L}(\mathbb{X})}$. If for any $\epsilon > 0$, there exists $x \in S_{\mathbb{X}}$ such that $\|Tx - x\| < \epsilon$, then $\|T\| = w(T)$.*

Proof. For each $n \in \mathbb{N}$, there exists $x_n \in S_{\mathbb{X}}$ satisfying $\|Tx_n - x_n\| < \frac{1}{n}$. Consider the sequence $\{x_n^*\} \in S_{\mathbb{X}^*}$ such that $x_n^*(x_n) = 1$. So we have $|x_n^*(Tx_n - x_n)| \leq \|x_n^*\|\|Tx_n - x_n\| \to 0$. Thus $\lim_{n\to\infty} x_n^*(Tx_n - x_n) = 0 \Rightarrow \lim_{n\to\infty} x_n^*(Tx_n) = \lim_{n\to\infty} x_n^*(x_n) = 1$. Hence $w(T) = \|T\|$. $\square$

Let us give an example to see the role of the above theorem to find the equality of two norms on operator space.

Example 6.6. *Let $\mathbb{X} = \ell_p,\ 1 \leq p < \infty$ and $T \in \mathbb{L}(\mathbb{X})$ be defined by*

$$Te_n = (1 - \frac{1}{n})e_n, n \in \mathbb{N},$$

where e_n is an element of ℓ_p whose n-th coordinate is 1 and all other coordinates are 0 for $n \in \mathbb{N}$. Then it is easy to observe that $\|T\| = 1$ and for any $\epsilon > 0$, there exists $x \in S_{\mathbb{X}}$ such that $\|Tx - x\| < \epsilon$. Thus by above Theorem we conclude that $\|T\| = w(T)$.

6.3 Nu-smoothness of order k in the space of bounded linear operators

We begin this section with a lemma which will be used to obtain the order of nu-smoothness of operators.

Lemma 6.1. *Suppose $\mathbb{X}$ is a finite-dimensional Banach space and let $T \in L(\mathbb{X})_w$. Then*
(i) span $(J_w(Tx) \cap Ext(B_{\mathbb{X}^*})) =$ span $J_w(Tx)$, for each $x \in S_{\mathbb{X}}$.
(ii) span $\{x^* \otimes x : x \in V_T \cap Ext(B_{\mathbb{X}}), x^* \in J_w(Tx) \cap Ext(B_{\mathbb{X}^*})\} =$ span $\{x^* \otimes x : x \in V_T \cap Ext(B_{\mathbb{X}}), x^* \in J_w(Tx)\}$.
(iii) span Ext $J_w(T) =$ span S, where $S = \{x^* \otimes x : x \in Ext(B_{\mathbb{X}}), x^* \in Ext(B_{\mathbb{X}^*}), x^*(x) = 1, |x^*(Tx)| = w(T)\}$.

Proof. (i) Observe that $J_w(Tx) \cap Ext(B_{\mathbb{X}^*}) \neq \emptyset$. Let $x^* \in J_w(Tx)$. First we show that *span* $J_w(Tx) \subseteq$ *span* $(J_w(Tx) \cap Ext(B_{\mathbb{X}^*}))$. If $x^* \in J_w(Tx) \cap Ext(B_{\mathbb{X}^*})$, we are done. Suppose x^* is not an extreme point of $B_{\mathbb{X}^*}$. Then there exist $x_1^*, x_2^*, \ldots, x_n^* \in Ext(B_{\mathbb{X}^*})$ such that $x^* = \sum_{i=1}^n \lambda_i x_i^*$, where $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. Now,

$$1 = x^*(x) = (\sum_{i=1}^n \lambda_i\ x_i^*)(x) = \sum_{i=1}^n \lambda_i\ x_i^*(x) \leq \sum_{i=1}^n \lambda_i = 1.$$

Thus $x_i^*(x) = 1$ for all $1 \leq i \leq n$, i.e., $x_i^* \in J(x)$ for all $1 \leq i \leq n$. Also $w(T) = |x^*(Tx)| = |(\sum_{i=1}^n \lambda_i\ x_i^*)(Tx)| = |\sum_{i=1}^n \lambda_i\ x_i^*(Tx)| \leq \sum_{i=1}^n \lambda_i\ |x_i^*(Tx)| \leq \sum_{i=1}^n \lambda_i w(T) = w(T)$. Thus $|x_i^*(Tx)| = w(T)$ for all $1 \leq i \leq n$, and so $x_i^* \in J_w(Tx) \cap Ext(B_{\mathbb{X}^*})$ for all $1 \leq i \leq n$. Therefore, $x^* \in$ *span* $(J_w(Tx) \cap Ext(B_{\mathbb{X}^*}))$ and hence *span* $J_w(Tx) \subseteq$ *span* $(J_w(Tx) \cap Ext(B_{\mathbb{X}^*}))$. The other inclusion is obvious. This completes the proof.

(ii) The proof is based on analogous arguments used in (i).

(iii) Let $f \in Ext\ J_w(T)$. Since $J_w(T)$ is the extremal subset of $Ext(B_{\mathbb{L}(\mathbb{X})_w^*})$, $f \in Ext(B_{\mathbb{L}(\mathbb{X})_w^*})$. From [34, Th. 2.3] it follows that $f = x^* \otimes x$, where $x \in Ext(B_{\mathbb{X}}), x^* \in Ext(B_{\mathbb{X}^*}), |x^*(x)| = 1$. Also, $|x^*(Tx)| = |(x^* \otimes x)(T)| = |f(T)| = w(T)$. If $x^*(x) = 1$, then $x^* \otimes x \in S$. Let $x^*(x) = -1$. Then $-x \in Ext(B_{\mathbb{X}})$, $x^* \in Ext(B_{\mathbb{X}^*})$, $x^*(-x) = 1$ and also $|x^*(T(-x))| = w(T)$. So, $-f = -(x^* \otimes x) = x^* \otimes (-x) \in S$ and hence $Ext\ J_W(T) \subseteq$ *span* S. Thus *span* $Ext\ J_W(T) \subseteq$ *span* S. For the reverse inclusion, let $x^* \otimes x \in S$, where $x \in Ext(B_{\mathbb{X}}), x^* \in Ext(B_{\mathbb{X}^*})$ with $x^*(x) = 1$ and $|x^*(Tx)| = w(T)$. We show that $x^* \otimes x \in$ *span* $Ext\ J_w(T)$. Now clearly $x^* \otimes x \in \pm J_w(T) \subseteq$ *span* $J_w(T)$. Also *span* $J_w(T) =$ *span* $Ext\ J_w(T)$. Thus $x^* \otimes x \in$ *span* $Ext\ J_w(T)$. This completes the proof of the lemma. $\square$

Using Lemma 6.1, we now obtain the order of nu-smoothness for a class of operators defined on a finite-dimensional Banach space. The idea of the proof is motivated by [39, Th. 2.2]. For the convenience of the reader, we give a sketch of the proof here.

Theorem 6.7. *Suppose $\mathbb{X}$ is a finite-dimensional Banach space. Let $T \in L(\mathbb{X})_w$ be such that $V_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \ldots, \pm x_r\}$, where $\{x_1, x_2, \ldots, x_r\}$ is a linearly independent set in $\mathbb{X}$. Suppose x_i is nu-smooth of order m_i with respect to T for each $1 \leq i \leq r$. Then T is nu-smooth of order k, where $m_1 + m_2 + \ldots + m_r = k$.*

Proof. Let $\dim(\mathbb{X}) = n$. Then $r \leq n$. We extend the linearly independent set $\{x_1, x_2, \ldots, x_r\}$ to a basis $\{x_1, x_2, \ldots, x_n\}$ of the space $\mathbb{X}$, if $r < n$ and in the case $r = n$ the set $\{x_1, x_2, \ldots, x_r\}$ is already a basis. Suppose that T is nu-smooth of order k and each x_i is nu-smooth of order m_i with respect to T. By Lemma 6.1, we have $span\ Ext\ J_w(T) = span\ \{x^* \otimes x : x \in Ext(B_{\mathbb{X}}), x^* \in Ext(B_{\mathbb{X}^*}), x^*(x) = 1, |x^*(Tx)| = w(T)\} = span\ \{x^* \otimes x : x \in V_T \cap Ext(B_{\mathbb{X}}), x^* \in J_w(Tx) \cap Ext(B_{\mathbb{X}^*})\}$. Since, x_i is nu-smooth of order m_i with respect to T, we have, $m_i = \dim\ span\ J_w(Tx_i)$, for each $1 \leq i \leq n$. Let $\{x^*_{ij} \in J_w(Tx_i) : 1 \leq j \leq m_i\}$ be a basis of $span\ J_w(Tx_i)$ for each $1 \leq i \leq n$. Let $W_i = span\ \{x^*_{ij} \otimes x_i : x^*_{ij} \in J_w(Tx_i)\}$ for each $1 \leq i \leq n$. Clearly $B_i = \{x^*_{ij} \otimes x_i : 1 \leq j \leq m_i\}$ is a basis of W_i. So, $\dim(W_i) = m_i, 1 \leq i \leq n$. Now,

$$\begin{aligned}
k &= \dim\ span\ Ext\ J_w(T) \\
&= \dim\ span\ \{x^* \otimes x : x \in V_T \cap Ext(B_{\mathbb{X}}), x^* \in J_w(Tx) \cap Ext(B_{\mathbb{X}^*})\} \\
&= \dim\ span\ \{x^* \otimes x : x \in V_T \cap Ext(B_{\mathbb{X}}), x^* \in J_w(Tx)\} \\
&= \dim\ span\ \{x^*_{ij} \otimes x_i : x^*_{ij} \in J_w(Tx_i), 1 \leq i \leq r\} \\
&= \dim(W),
\end{aligned}$$

where $W = span\ \{x^*_{ij} \otimes x_i : x^*_{ij} \in J_w(Tx_i), 1 \leq i \leq r\}$. Proceeding similarly as in [39, Th. 2.2], we can show that $W = \oplus_{i=1}^r W_i$. Hence,

$$k = \dim(W) = \dim(\oplus_{i=1}^r W_i) = \oplus_{i=1}^r \dim(W_i) = m_1 + m_2 + \cdots + m_r.$$

This completes the proof of the theorem. □

We now exhibit an example to show the role of above theorem to find the order of nu-smoothness of an operator on a finite-dimensional Banach space.

Example 6.8. *Consider the three-dimensional Banach space $\mathbb{X}$ such that $S_{\mathbb{X}}$ is a polyhedron having vertices $\pm x_1, \pm x_2, \ldots, \pm x_6$, where $x_1 = (1, 0, 1), x_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 1)$, $x_3 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}, 1), x_4 = (-1, 0, 1), x_5 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1), x_6 = (\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1)$. Let $T \in \mathbb{L}(\mathbb{X})$ be defined by*

$$T(x, y, z) = (0, -y + \frac{\sqrt{3}}{2}z, -\frac{1}{\sqrt{3}}y + \frac{1}{2}z).$$

Then $Tx_1 = (0, \frac{\sqrt{3}}{2}, \frac{1}{2})$. It is easy to verify that $w(T) = \frac{1}{2}$ and $V_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1\}$. Now,

$J_w(Tx_1) = \{x_1^*, -x_3^*, x_4^*\}$, *where* $x_1^*(x,y,z) = x+\frac{1}{\sqrt{3}}y$, $x_3^*(x,y,z) = -x+\frac{1}{\sqrt{3}}y$ *and* $x_4^*(x,y,z) = z$. *Now,* x_1 *is nu-smooth of order* 3 *with respect to* T. *Therefore, from Theorem 6.7, we conclude that* T *is nu-smooth of order* 3.

Let us now state the following easy lemma which will be used to prove some of the theorems. We omit the proof to avoid monotonicity.

Lemma 6.2. *Suppose* $\mathbb{X}$ *is a finite-dimensional Banach space. If* $\{x_1, x_2, \ldots, x_m\}$ *is a linearly independent subset of* $\mathbb{X}$ *and* $\{x_1^*, x_2^*, \ldots, x_n^*\}$ *is a linearly independent subset of* $\mathbb{X}^*$ *then* $\{x_i^* \otimes x_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ *is a linearly independent subset of* $\mathbb{L}(\mathbb{X})^*$.

In the next theorem, we obtain the order of nu-smoothness of another class of operators defined on a finite-dimensional Banach space. Here, we remove the assumption of Theorem 6.7 that the set $\{x_1, x_2, \ldots, x_m\}$ is linearly independent, where $V_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \ldots, \pm x_m\}$. Instead we assume that $span(V_T \cap Ext(B_{\mathbb{X}})) = \mathbb{X}$.

Theorem 6.9. *Let* $\mathbb{X}$ *be a finite-dimensional Banach space with* $\dim(\mathbb{X}) = n$. *Let* $T \in \mathbb{L}(\mathbb{X})_w$ *be such that* $V_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \ldots, \pm x_m\}$ *where,* $span\{x_1, x_2, \ldots, x_m\} = \mathbb{X}$. *Suppose* x_i *is nu-smooth of order* n *with respect to* T *for* $i = 1, 2, \ldots, m$. *Then* T *is nu-smooth of order* n^2.

Proof. Clearly $m \geq n$. If $m = n$, then $\{x_1, x_2, \ldots, x_m\}$ is linearly independent and by Theorem 6.7 we get that T is nu-smooth of order n^2. Now suppose that $\{x_1, x_2, \ldots, x_m\}$ is linearly dependent. When $m > n$ there exists a subset of $\{x_1, x_2, \ldots, x_m\}$ containing n elements which is a basis of $\mathbb{X}$. Without loss of generality, we may assume the subset as $\{x_1, x_2, \ldots, x_n\}$. Now, as x_i is nu-smooth of order n with respect to T for $i = 1, 2, \ldots, m$, we have dim $span\ J_w(Tx) = n$. Suppose, $\{x_{ij}^* : 1 \leq j \leq n\}$ is a linearly independent subset of $span\ J_w(Tx_i)$ for each $1 \leq i \leq m$. Now by Lemma 6.1,

$$\begin{aligned}
& span\ Ext\ J_w(T) \\
= \ & span\ \{x^* \otimes x : x \in Ext(B_{\mathbb{X}}), x^* \in Ext(B_{\mathbb{X}^*}), x^*(x) = 1, |x^*(Tx)| = w(T)\} \\
= \ & span\ \{x^* \otimes x : x \in V_T \cap Ext(B_{\mathbb{X}}), x^* \in J_w(Tx)\}.
\end{aligned}$$

Let $x \in V_T \cap Ext(B_{\mathbb{X}})$. Then there exist scalars $a_i, i = 1, 2, ..., n$ such that $x = a_1x_1 + a_2x_2 + ... + a_nx_n$. Now, let $x^* \in J_w(Tx)$. Since $\{x_{1j}^* : 1 \leq j \leq n\}$ is a basis of $\mathbb{X}$, $x^* = \sum_{1 \leq j \leq n} b_j x_{1j}^*$ for scalars b_j $(1 \leq j \leq n)$. Therefore,

$$\begin{aligned} x^* \otimes x &= x^* \otimes (a_1x_1 + a_2x_2 + ... + a_nx_n) \\ &= \Big(\sum_{1\leq j\leq n} b_j x_{1j}^* \Big) \otimes \Big(\sum_{1\leq i\leq n} a_i x_i \Big) \\ &= \sum_{1\leq i\leq n, 1\leq j\leq n} a_i b_j x_{1j}^* \otimes x_i \\ &\in span\{x_{1j}^* \otimes x_i : 1 \leq i \leq n, 1 \leq j \leq n\}. \end{aligned}$$

Since $x \in V_T \cap Ext(B_{\mathbb{X}})$ and $x^* \in J_w(Tx)$ are arbitrary, we have $span\ Ext\ J_w(T) = span\{x_{1j}^* \otimes x_i : 1 \leq i, j \leq n\}$. Thus

$$\begin{aligned} & \dim\ span\ Ext\ J_w(T) \\ =\ & \dim\ span\ \{x_{1j}^* \otimes x_i : 1 \leq i, j \leq n\} \\ =\ & n^2, \text{ (by Lemma 6.2).} \end{aligned}$$

Therefore, T is nu-smooth of order n^2. This completes the proof of the theorem. □

The following example shows the applicability of Theorem 6.9 over Theorem 6.7.

Example 6.10. *Consider the two-dimensional polygonal Banach space $\mathbb{X}$ such that $S_{\mathbb{X}}$ is a polygon having vertices $\pm(1,0), \pm(\frac{1}{2}, \frac{\sqrt{3}}{2}), \pm(-\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let $T \in \mathbb{L}(\mathbb{X})$ be defined by*

$$T(1,0) = (0, \frac{\sqrt{3}}{2}),\ T(\frac{1}{2}, \frac{\sqrt{3}}{2}) = (-\frac{3}{4}, \frac{\sqrt{3}}{4}).$$

Then

$$T(-\frac{1}{2}, \frac{\sqrt{3}}{2}) = (-\frac{3}{4}, -\frac{\sqrt{3}}{4}).$$

An easy calculation shows that

$$w(T) = \frac{1}{2},\ \text{and}\ \ V_T \cap Ext(B_{\mathbb{X}}) = \{\pm(1,0), \pm(\frac{1}{2}, \frac{\sqrt{3}}{2}), \pm(-\frac{1}{2}, \frac{\sqrt{3}}{2})\}.$$

Clearly, for each $x \in V_T \cap Ext(B_{\mathbb{X}})$, x is nu-smooth of order 2 with respect to T. Now by Theorem 6.9, we conclude that T is nu-smooth of order 4.

Note that, if $\mathbb{X}$ is a two-dimensional Banach space, $T \in \mathbb{L}(\mathbb{X})_w$ and $|V_T \cap Ext(B_{\mathbb{X}})| \leq 4$, then using Theorem 6.7, we get the order of nu-smoothness of the operator T. Whenever, $|V_T \cap Ext(B_{\mathbb{X}})| > 4$, we have to consider two separate cases, namely $|V_T \cap Ext(B_{\mathbb{X}})| = 6$ and $|V_T \cap Ext(B_{\mathbb{X}})| > 8$ to get the order of nu-smoothness of the operator T. We consider these two cases in the next two theorems. The proofs can be completed proceeding similarly as [39, Th. 3.1 & Th. 3.3]. We only state the theorems here.

Theorem 6.11. *Suppose $\mathbb{X}$ is a two-dimensional Banach space and $T \in \mathbb{L}(\mathbb{X})_w$ is such that $V_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3\}$. Then the following holds:*
(i) If x_i is nu-smooth with respect to T for each $1 \leq i \leq 3$, then T is nu-smooth of order 3.
(ii) Let x_1 be not nu-smooth with respect to T.
(a) If x_2, x_3 are nu-smooth with respect to T and $\pm J_w(Tx_2) = \pm J_w(Tx_3)$, then T is nu-smooth of order 3.
(b) Otherwise, T is nu-smooth of order 4.

Theorem 6.12. *Suppose $\mathbb{X}$ is a two-dimensional Banach space. Let $T \in \mathbb{L}(\mathbb{X})_w$ be such that $|V_T \cap Ext(B_{\mathbb{X}})| \geq 8$. Then the following holds:*
(i) If x is not nu-smooth with respect to T for some $x \in V_T \cap Ext(B_{\mathbb{X}})$, then T is nu-smooth of order 4.
(ii) Suppose x is nu-smooth with respect to T for each $x \in V_T \cap Ext(B_{\mathbb{X}})$. If $x_i \in V_T \cap Ext(B_{\mathbb{X}})$, $x_i^* \in J_w(Tx_i)$ for $i = 1, 2, 3, 4$ are such that $x_2 = \lambda_1 x_1 + \lambda_3 x_3, x_4 = \mu_1 x_1 + \mu_3 x_3$ and $x_2^* = \gamma_1 x_1^* + \gamma_3 x_3^*, x_4^* = \delta_1 x_1^* + \delta_3 x_3^*$ with $\delta_1\gamma_3\lambda_1\mu_3 - \delta_3\gamma_1\lambda_3\mu_1 \neq 0$, then T is nu-smooth of order 4. Otherwise T is nu-smooth of order 3.

We end this section with the following interesting theorem which gives us the relation of the order of nu-smoothness of an operator with its adjoint operator.

Theorem 6.13. *Suppose $\mathbb{X}$ is a finite-dimensional Banach space and $T \in \mathbb{L}(\mathbb{X})_w$. Then T is nu-smooth of order k if and only if T^* is nu-smooth of order k.*

Proof. We first show that *span Ext* $J_w(T) =$ *span Ext* $J_w(T^*)$. Let $f \in Ext\ J_w(T)$. Then $f \in Ext(B_{\mathbb{L}(\mathbb{X})_w^*})$. Now by [34, Th. 2.3], there exist $x \in Ext(B_{\mathbb{X}})$ and $x^* \in Ext(B_{\mathbb{X}^*})$ with $|x^*(x)| = 1$ such that $f = x^* \otimes x$. Let $x^*(x) = 1$. Here,

$$\begin{aligned}
& f(T) = w(T) \\
\Rightarrow\ & (x^* \otimes x)(T) = w(T) \\
\Rightarrow\ & x^*(Tx) = w(T) \\
\Rightarrow\ & (T^*x^*)(x) = w(T) \\
\Rightarrow\ & x(T^*x^*) = w(T^*).
\end{aligned}$$

Now, $x^* \in V_{T^*} \cap Ext(B_{\mathbb{X}^*})$. Also $x^*(x) = 1 \Rightarrow x(x^*) = 1$ and hence $x \in J_w(T^*x^*)$. Thus

$x^* \otimes x \in span\ Ext\ J_w(T^*)$. If $x^*(x) = -1$ then $x^*(-x) = 1$ and

$$\begin{aligned} & |f(T)| = w(T) \\ \Rightarrow\ & |(x^* \otimes (-x))(T)| = w(T) \\ \Rightarrow\ & |x^*(Tx)| = w(T) \\ \Rightarrow\ & |(T^*x^*)(x)| = w(T) \\ \Rightarrow\ & |x(T^*x^*)| = w(T^*) \\ \Rightarrow\ & |-x(T^*x^*)| = w(T^*). \end{aligned}$$

Now, $x^* \in V_{T^*} \cap Ext(B_{\mathbb{X}^*})$. Also $x^*(-x) = 1 \Rightarrow (-x)(x^*) = 1$ and hence $-x \in J_w(T^*x^*)$. Thus $-(x^* \otimes x) = x^* \otimes (-x) \in span\ Ext\ J_w(T^*)$ which implies that $f \in span\ Ext\ J_w(T^*)$. Therefore, $Ext\ J_w(T) \subseteq span\ Ext\ J_w(T^*)$ and hence $span\ Ext\ J_w(T) \subseteq span\ Ext\ J_w(T^*)$. Replacing T by T^* we get $span\ Ext\ J_w(T^*) \subseteq span\ Ext\ J_w(T)$. Thus $span\ Ext\ J_w(T) = span\ Ext\ J_w(T^*)$ and hence

$$\dim\ span\ Ext\ J_w(T) = \dim\ span\ Ext\ J_w(T^*).$$

Therefore, T is nu-smooth of order k if and only if T^* is nu-smooth of order k. $\square$

6.4 Nu-extreme contractions on finite -dimensional polyhedral Banach spaces

We first recall that a finite-dimensional Banach space is said to be a polyhedral Banach space if its unit ball has only finitely many extreme points. In particular, a two-dimensional polyhedral Banach space is said to be a polygonal Banach space. The following proposition is necessary in our study.

Proposition 6.3. *[36, Proposition 2.1] Let $\mathbb{X}$ be a polyhedral Banach space of dimension n. Let $x \in S_{\mathbb{X}}$. Then the following are equivalent:*
(a) x is an exposed point of $B_{\mathbb{X}}$.
(b) x is an extreme point of $B_{\mathbb{X}}$.
(c) x is $n-$smooth.

In the next theorem, we obtain a relation between the order of nu-smoothness and nu-extreme contraction on finite-dimensional polyhedral Banach spaces.

Theorem 6.14. *Let $\mathbb{X}$ be a polyhedral Banach space such that $\dim(\mathbb{X}) = n$. Then $T \in S_{\mathbb{L}(\mathbb{X})_w}$ is an nu-extreme contraction if and only if T is nu-smooth of order n^2.*

Proof. Since $\mathbb{X}$ is finite-dimensional, we have from [34, Th. 2.3],

$$\begin{aligned} Ext(B_{\mathbb{L}(\mathbb{X})^*_w}) &\subseteq \{x^* \otimes x : x \in Ext(B_{\mathbb{X}}), x^* \in Ext(B_{\mathbb{X}^*}), |x^*(x)| = 1\} \\ &\subseteq \{x^* \otimes x : x \in Ext(B_{\mathbb{X}}), x^* \in Ext(B_{\mathbb{X}^*})\} \\ &= Ext(B_{\mathbb{X}^*}) \otimes Ext(B_{\mathbb{X}}). \end{aligned}$$

Since $Ext(B_{\mathbb{X}^*})$ and $Ext(B_{\mathbb{X}})$ are finite sets, there are only finitely many extreme points in the unit ball of $\mathbb{L}(\mathbb{X})^*_w$. Therefore, $\mathbb{L}(\mathbb{X})^*_w$ is a polyhedral Banach space. Moreover, $\mathbb{L}(\mathbb{X})^*_w$ is a finite-dimensional Banach space. Therefore, $\mathbb{L}(\mathbb{X})_w (= \mathbb{L}(\mathbb{X})^{**}_w)$ is also a finite-dimensional polyhedral Banach space. Now, $\dim(\mathbb{L}(\mathbb{X})_w) = n^2$. Hence, from Proposition 6.3, we can conclude that $T \in S_{\mathbb{L}(\mathbb{X})_w}$ is an nu-extreme contraction if and only if T is nu-smooth of order n^2. □

With the help of above theorem we now completely characterize nu-extreme contractions on a two-dimensional polygonal Banach space.

Theorem 6.15. *Let $\mathbb{X}$ be a two-dimensional polygonal Banach space. Let $T \in S_{\mathbb{L}(\mathbb{X})_w}$. Then T is an nu-extreme contraction if and only if either of the following holds:*
(i) $V_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2\}$ and x_1, x_2 are not nu-smooth with respect to T.
(ii) $V_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3\}$ and
$|\{x_i : x_i$ is not nu-smooth with respect to $T\}| \geq 2$.
(iii) $V_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3\}$, x_1 is not nu-smooth with respect to T, x_2, x_3 are nu-smooth with respect to T, and $J_w(Tx_2) \neq \pm J_w(Tx_3)$.
(iv) $|V_T \cap Ext(B_{\mathbb{X}})| \geq 8$ and there exists $x \in V_T \cap Ext(B_{\mathbb{X}})$ such that x is not nu-smooth with respect to T.
(v) $|V_T \cap Ext(B_{\mathbb{X}})| \geq 8$ and x is nu-smooth with respect to T for each $x \in V_T \cap Ext(B_{\mathbb{X}})$. Moreover, there exist $x_i \in V_T \cap Ext(B_{\mathbb{X}})$, $x_i^* \in J_w(Tx_i)$ for $i = 1, 2, 3, 4$ such that $x_2 = \lambda_1 x_1 + \lambda_3 x_3, x_4 = \mu_1 x_1 + \mu_3 x_3$ and $x_2^* = \gamma_1 x_1^* + \gamma_3 x_3^*, x_4^* = \delta_1 x_1^* + \delta_3 x_3^*$ with $\delta_1\gamma_3\lambda_1\mu_3 - \delta_3\gamma_1\lambda_3\mu_1 \neq 0$.

Proof. From Theorem 6.14, we find that T is an nu-extreme contraction if and only if T is nu-smooth of order 4. Observe that, if $|V_T \cap Ext(B_{\mathbb{X}})| < 4$, then from Theorem 6.7, we can conclude that T is not nu-smooth of order 4. Therefore, if T is nu-smooth of order 4, then $|V_T \cap Ext(B_{\mathbb{X}})| \geq 4$. Hence, we only assume that $|M_T \cap Ext(B_{\mathbb{X}})| \geq 4$.

First let $|V_T \cap Ext(B_{\mathbb{X}})| = 4$. In this case, we show that T is an nu-extreme contraction if and only if (i) holds. Let $V_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2\}$ for some $x_1, x_2 \in S_{\mathbb{X}}$. Clearly, $\{x_1, x_2\}$ is linearly independent. Therefore, from Theorem 6.7, we can conclude that T is nu-extreme contraction, that is, T is nu-smooth of order 4 if and only if x_1 and x_2 are nu-smooth of order

2 with respect to T. Therefore, if $|V_T \cap Ext(B_{\mathbb{X}})| = 4$, then T is an nu-extreme contraction if and only if (i) holds.

Now, let $|V_T \cap Ext(B_{\mathbb{X}})| = 6$. In this case, we show that T is an nu-extreme contraction if and only if either (ii) or (iii) holds. Let $V_T \cap Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3\}$ for some $x_1, x_2, x_3 \in S_{\mathbb{X}}$. Clearly if each x_i is nu-smooth with respect to T, for $1 \leq i \leq 3$, then by Theorem 6.7, T is an nu-smooth of order 3 and hence Tcan not be an nu-extreme contraction. Thus T is an nu-extreme contraction if and only if $|\{x_i : x_i \text{ is not nu-smooth with respect to } T\}| \geq 1$. Now it is easy to observe that T is an nu-extreme contraction if and only either (ii) or (iii) holds.

Next, let $|M_T \cap Ext(B_{\mathbb{X}})| \geq 8$. Then from Theorem 6.12, we can easily conclude that T is an nu-extreme contraction if and only if either (iv) or (v) holds. □

Let us now study some examples to see the relation between the order of smoothness and the order of nu-smoothness of an element of the unit sphere of a finite-dimensional Banach space with respect to some linear operator defined on that Banach space and also see the connection between the extreme points and nu-smoothness of order k for an element with respect to some linear operator defined on that Banach space.

Example 6.16. *Consider the two-dimensional Banach space $\mathbb{X}$ such that $S_{\mathbb{X}}$ is a regular hexagon having vertices $\pm x_1, \pm x_2, \pm x_3$, where $x_1 = (1,0), x_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}), x_3 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let $T \in \mathbb{L}(\mathbb{X})$ be defined by $T(1,0) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $T(0,1) = (0,0)$. Here $\pm J(Tx_1) \cap \pm J(x_1) \neq \emptyset$ and hence $\|T\| = w(T) = 1$. Now, $Tx_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ is an extreme point of $B_{\mathbb{X}}$ but x_1 is not nu-smooth of order 2 with respect to T. In fact x_1 is nu-smooth with respect to T as $J_w(Tx_1) = \{x_1^*\}$, where $x_1^*(x,y) = x + \frac{1}{\sqrt{3}}y$. Clearly x_1 is $2-$smooth.*

Example 6.17. *Consider the two-dimensional Banach space $\mathbb{X}$ such that $S_{\mathbb{X}}$ is a regular hexagon having vertices $\pm x_1, \pm x_2, \pm x_3$, where $x_1 = (1,0), x_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}), x_3 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let $T \in \mathbb{L}(\mathbb{X})$ be defined by $T(1,0) = (0, \frac{\sqrt{3}}{2})$, $T(0,1) = (0,0)$. It is easy to verify that $w(T) = \frac{1}{2}$ and $x_1 \in V_T \cap Ext(B_{\mathbb{X}})$. Now $J_w(Tx_1) = \{x_1^*, -x_3^*\}$, where $x_1^*(x,y) = x + \frac{1}{\sqrt{3}}y$ and $x_3^*(x,y) = -x + \frac{1}{\sqrt{3}}y$. Thus x_1 is not nu-smooth with respect to T, in fact it is nu-smooth of order 2 with respect to T.*

In [34, Th. 2.3] author studies the structure of extreme points in the dual of the space of bounded linear operators defined on a finite-dimensional Banach space. We here obtain the exact structure of the set of extreme points of $B_{\mathbb{L}(\mathbb{X})_w^*}$ using Theorem 6.14 when $\mathbb{X}$ is a two-dimensional polygonal Banach space. For this we need the notion of Birkhoff-James orthogonality. Recall that for $x, y \in \mathbb{X}$, x is said to be Birkhoff-James orthogonal [3, 20] to y, written as $x \perp_B y$ if $\|x + \lambda y\| \geq \|x\|$ for all scalars λ.

Theorem 6.18. *Suppose $\mathbb{X}$ is a two-dimensional polygonal Banach space. Then*

$$Ext(B_{\mathbb{L}(\mathbb{X})^*_w}) = \{x^* \otimes x : x \in Ext(B_{\mathbb{X}}),\ x^* \in Ext(B_{\mathbb{X}^*}),\ |x^*(x)| = 1\}.$$

Proof. By [34, Th. 2.3], we get

$$Ext(B_{\mathbb{L}(\mathbb{X})^*_w}) \subseteq \{x^* \otimes x : x \in Ext(B_{\mathbb{X}}),\ x^* \in Ext(B_{\mathbb{X}^*}),\ |x^*(x)| = 1\}.$$

Here we prove only the reverse inclusion. Let $x \in Ext(B_{\mathbb{X}})$, $x^* \in Ext(B_{\mathbb{X}^*})$ with $|x^*(x)| = 1$. Then x and x^* both are $2-$smooth points as $\dim(\mathbb{X}) = \dim(\mathbb{X}^*) = 2$. Now let us take $Ext\ J(x) = \{x^*, y^*\}$ and $Ext\ J(x^*) = \{x, y\}$. Let $(0 \neq)z \in \ker(x^*)$ and $(0 \neq)z_1 \in \ker(y^*)$. Then from [20, Th. 2.1] it follows that $x \perp_B z$ and $x \perp_B z_1$. Observe that $\{z, z_1\}$ is linearly independent. Now $\mathbb{L}(\mathbb{X})^*_w$ is a 4-dimensional polyhedral Banach space. Thus by Proposition 6.3, $f \in Ext(B_{\mathbb{L}(\mathbb{X})^*_w})$ if and only if f is $4-$smooth. For $1 \leq i \leq 4$, we now define $T_i \in \mathbb{L}(\mathbb{X})$ by:

$$\begin{array}{llll} T_1 x = x & T_2 x = y & T_3 x = x & T_4 x = x \\ T_1 z = 0 & T_2 z = 0 & T_3 z = \dfrac{1}{2} z & T_4 z_1 = 0. \end{array}$$

Using Proposition 6.1, it is easy to observe that $w(T_i) = \|T_i\| = 1$ for all $1 \leq i \leq 4$. Also we have $|(x^* \otimes x)(T_i)| = |x^*(T_i(x)| = 1$, where $T_i \in \mathbb{L}(\mathbb{X})_w = \mathbb{L}(\mathbb{X})^{**}_w$, i.e., $T_i \in \pm J(x^* \otimes x)$. Let $y = ax + bz$, $z = cx + dz_1$. Then $T_4(z) = cT_4(x) = cx$. Finally performing an easy calculation it can be proved that the set $\{T_i : 1 \leq i \leq 4\}$ is linearly independent. Thus dim *span* $J(x^* \otimes x) = 4$ and hence $x^* \otimes x$ is a $4-$smooth point in the polyhedral Banach space $\mathbb{L}(\mathbb{X})^*_w$, where $\dim(\mathbb{L}(\mathbb{X})^*_w) = 4$. Therefore, by Proposition 6.3, we conclude that $x^* \otimes x \in Ext(B_{\mathbb{L}(\mathbb{X})^*_w})$. This completes the proof of the theorem. □

Finally, we end this section with following nice observation which gives a connection between the number of extreme points of $B_{\mathbb{X}}$ and the number of extreme points of $B_{\mathbb{L}(\mathbb{X})^*_w}$, when $\mathbb{X}$ is a two-dimensional polygonal Banach space.

Corollary 6.1. *Suppose $\mathbb{X}$ is a two-dimensional polygonal Banach space such that $|Ext(B_{\mathbb{X}})| = 2n$. Then $|Ext(B_{\mathbb{L}(\mathbb{X})^*_w})| = 4n$.*

Proof. Let $Ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \ldots, \pm x_n\}$. Then $|Ext(B_{\mathbb{X}^*})| = 2n$. By Theorem 6.18 and [34, Th. 2.3], we have

$$Ext(B_{\mathbb{L}(\mathbb{X})^*_w}) = \{x^* \otimes x : x \in Ext(B_{\mathbb{X}}), x^* \in Ext(B_{\mathbb{X}^*}), |x^*(x)| = 1\}.$$

As $|\{x^* \in Ext(B_{\mathbb{X}^*}) : |x^*(x)| = 1\}| = 4$ for each $x \in Ext(B_{\mathbb{X}})$, we have $|Ext(B_{\mathbb{L}(\mathbb{X})^*_w})| = 4n$. This completes the proof. $\square$

CHAPTER 7

BIRKHOFF-JAMES ORTHOGONALITY OF BOUNDED LINEAR OPERATORS

7.1 Introduction

The purpose of the present chapter is to continue the study of orthogonality properties of bounded linear operators between Banach spaces, in light of the seminal result obtained by Bhatia and Šemrl [2] regarding orthogonality of linear operators on Euclidean spaces. Let us first establish the relevant notations and the terminologies in this context.

Letters $\mathbb{X}$ and $\mathbb{Y}$ denote Banach spaces. Throughout the chapter, we work only with real Banach spaces. Let $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$ denote the unit ball and the unit sphere of $\mathbb{X}$ respectively. Let $Ext\ B_{\mathbb{X}}$ denotes the set of all extreme points of $B_{\mathbb{X}}$. For a set $\mathcal{S} \subset \mathbb{X}$, $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$. Let $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ denote the Banach space of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$, endowed with the usual operator norm. We write $\mathbb{L}(\mathbb{X}, \mathbb{Y}) = \mathbb{L}(\mathbb{X})$, if $\mathbb{X} = \mathbb{Y}$. For a bounded linear operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, let M_T denote

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the norm attainment set of T, i.e., $M_T = \{x \in S_{\mathbb{X}} : \|Tx\| = \|T\|\}$. The notion of Birkhoff-James orthogonality in a Banach space is well-known and is used extensively in the study of the geometry of Banach spaces. For $x, y \in \mathbb{X}$, x is said to be orthogonal to y in the sense of Birkhoff-James [20], written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$. Similarly, for $T, A \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, T is said to be Birkhoff-James orthogonal to A, written as $T \perp_B A$, if $\|T + \lambda A\| \geq \|T\|$ for all $\lambda \in \mathbb{R}$. For the n-dimensional Euclidean space $\mathbb{E}^n$, Bhatia and Šemrl [2] proved that for $T, A \in \mathbb{L}(\mathbb{E}^n)$, $T \perp_B A$ if and only if there exists $x \in S_{\mathbb{E}^n}$ such that $\|Tx\| = \|T\|$ and $Tx \perp_B Ax$. For initial ideas readers can look into [21, 22]. We refer the readers to [49, 48] for another approach in this context. The characterization of Birkhoff-james orthogonality of compact operators on complex reflexive banach spaces has been studied in [45]. In recent times, various generalizations of this remarkable theorem has been obtained [61, 63] in the setting of Banach spaces. Our aim is to study the BŠ Property of bounded linear operators between Banach spaces, especially when the domain space and the co-domain spaces are polyhedral. $\mathbb{X}$ is said to be a polyhedral Banach space if $B_{\mathbb{X}}$ has only finitely many extreme points. Equivalently, $\mathbb{X}$ is a polyhedral Banach space if $B_{\mathbb{X}}$ is a polyhedron. Readers can go through [42] for details. In this context, let us mention the following formal definition:

Definition 7.1. *[57] Let $\mathbb{X}$ be a finite-dimensional polyhedral Banach space. Let F be a facet of the unit ball $B_{\mathbb{X}}$ of $\mathbb{X}$. A functional $f \in S_{\mathbb{X}^*}$ is said to be a* supporting functional corresponding to the facet F of the unit ball $B_{\mathbb{X}}$ *if the following two conditions are satisfied:*
(1) *f attains norm at some point v of F,*
(2) $F = (v + \ker f) \cap S_{\mathbb{X}}$.

We also make use of the concept of normal cones in a Banach space in our study.

Definition 7.2. *A subset K of $\mathbb{X}$ is said to be a normal cone in $\mathbb{X}$ if*
(i) $K + K \subset K$, (ii) $\alpha K \subset K$ *for all* $\alpha \geq 0$, *and* (iii) $K \cap (-K) = \{\theta\}$.

Normal cones are important in the study of the geometry of Banach spaces, because there is a natural partial ordering $\geq$ associated with a normal cone K. Namely, for any two elements $x, y \in \mathbb{X}$, $x \geq y$ if $x - y \in K$. It is easy to observe that in a two-dimensional Banach space $\mathbb{X}$, any normal cone K is completely determined by the intersection of K with the unit sphere $S_{\mathbb{X}}$. Keeping this in mind, when we say that K is a normal cone in $\mathbb{X}$, determined by v_1, v_2, what we really mean is that $K \cap S_{\mathbb{X}} = \left\{ \frac{(1-t)v_1 + tv_2}{\|(1-t)v_1 + tv_2\|} : t \in [0,1] \right\}$. Of course, in this case $K = \{\alpha v_1 + \beta v_2 : \alpha, \beta \geq 0\}$.

We are interested in the following: If $\dim \mathbb{X} > 2$ and $M_T \neq D \cup (-D)$, where D is a closed connected subset of $S_{\mathbb{X}}$, then whether $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ satisfies the BŠ Property, where $\mathbb{Y}$ is any Banach space. With this motivation in mind, we introduce the following definition for a Banach space $\mathbb{X}$, which plays a significant role in our study.

Definition 7.3. *Let $\mathbb{X}$ be a Banach space. Given $n \in \mathbb{N}$, we say that $\mathbb{X}$ has Property P_n if for every choice of n vectors $x_1, x_2, \ldots, x_n \in S_{\mathbb{X}}$, $\bigcup\limits_{i=1}^{n} x_i^{\perp} \subsetneq \mathbb{X}$.*

It is clear from the above definition that if $\mathbb{X}$ has Property P_n then $\mathbb{X}$ has Property P_m for all $m \in \mathbb{N}$, with $m \leq n$. We illustrate the connection between Property P_n for a Banach space and bounded linear operators not satisfying the BŠ Property. We further explore Property P_n for different polyhedral Banach spaces.

7.2 Connection between P_n Property and Bhatia-Šemrl Property

We begin this section with the observation that Theorem 2.3 of [62] holds true even if the co-domain space is any Banach space with dimension at least two. Indeed, the said theorem can be stated in the following more general form, by using essentially the same arguments presented in the proof of the original result.

Theorem 7.1. *Let $\mathbb{X}$ be a two-dimensional Banach space and let $\mathbb{Y}$ be a Banach space of dimension greater than or equal to two. Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ be such that M_T has more than two components. Then T does not satisfy the BŠ Property.*

As a corollary to Theorem 7.1, we can provide an elementary condition on M_T so that T does not satisfy the BŠ Property, when $\mathbb{X}$ is a two-dimensional Banach space.

Corollary 7.1. *Let $\mathbb{X}$ be a two-dimensional Banach space and let $\mathbb{Y}$ be a Banach space of dimension greater than or equal to two. Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ be such that there exist $x, y \in M_T$ with $x \neq \pm y$ and $\frac{x+y}{\|x+y\|}, \frac{x-y}{\|x-y\|} \notin M_T$. Then T does not satisfy the BŠ Property.*

Proof. We claim that M_T has more than two components. Let $u_1 = \frac{x+y}{\|x+y\|}$ and $u_2 = \frac{x-y}{\|x-y\|}$. Consider the following subsets of $S_{\mathbb{X}}$:

$$S_1 = \left\{ \frac{(1-t)u_1 + tu_2}{\|(1-t)u_1 + tu_2\|} : t \in (0,1) \right\},$$

$$S_2 = \left\{ \frac{(1-t)u_1 + t(-u_2)}{\|(1-t)u_1 - tu_2\|} : t \in (0,1) \right\},$$

$$S_3 = -S_1 \text{ and } S_4 = -S_2.$$

Then clearly $S_i,\ i = 1, 2, 3, 4$, are connected subsets of $S_{\mathbb{X}}$ and by the construction of S_i we have, $x \in S_1,\ y \in S_2,\ -x \in S_3$ and $-y \in S_4$. Also, $S_i \cap S_j = \phi$ for all $i, j \in \{1, 2, 3, 4\}$ with

$i \neq j$. As $S_{\mathbb{X}} \setminus \{\pm\frac{x+y}{\|x+y\|}, \pm\frac{x-y}{\|x-y\|}\} = \bigcup_{i=1}^{4} S_i$, $M_T \subseteq \bigcup_{i=1}^{4} S_i$. Also for each disjoint connected set S_i, $S_i \cap M_T \neq \phi$. Therefore, M_T must have more than two components. Hence using Theorem 7.1, we conclude that T does not satisfy the BŠ Property. $\square$

The following example illustrates the applicability of Theorem 7.1 in studying the BŠ Property of a bounded linear operator between Banach spaces.

Example 7.2. *Consider a bounded linear operator $T : \ell_1^2 \to \ell_\infty^3$, defined by $T(x, y) = (x, y, \frac{x+y}{2})$ for all $(x, y) \in \ell_1^2$. Then it is easy to see that $\|T\| = 1$ and $M_T = \{\pm(1, 0), \pm(0, 1)\}$. Therefore by using Theorem 7.1, we conclude that T does not satisfy the BŠ Property.*

If $\mathbb{X}$ is a two-dimensional Banach space, then from Theorem 2.1 of [63] and Theorem 2.3 of [62], it follows that $T \in \mathbb{L}(\mathbb{X})$ satisfies the BŠ Property if and only if $M_T = D \cup (-D)$, where D is a connected subset of $S_{\mathbb{X}}$. If $\dim \mathbb{X} \geq 3$ and M_T has more than two components then it is not known whether T will satisfy the BŠ Property. Our next result gives some insight in that direction, under certain assumptions on the co-domain space $\mathbb{Y}$ and the norm attainment set M_T, for a bounded linear operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$.

Theorem 7.3. *Let $\mathbb{X}$ be an n-dimensional Banach space, where $n \geq 3$ and let $\mathbb{Y}$ be any Banach space. Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, with $|M_T| \geq 4$, be such that the following conditions are satisfied:*
(a) There exists a basis $\{x_1, x_2, x_3, \ldots, x_n\}$ of $\mathbb{X}$ such that $x_1, x_2 \in M_T$.
(b) There exist scalars $\alpha_3, \alpha_4, \ldots, \alpha_n$ and $\beta_3, \beta_4, \ldots, \beta_n$ such that for each $w = c_{1w}x_1 + c_{2w}x_2 + \ldots + c_{nw}x_n \in M_T$, we have,

$$c_{1w} + c_{2w} + \alpha_3 c_{3w} + \ldots + \alpha_n c_{nw} \neq 0 \text{ and } c_{1w} - c_{2w} + \beta_3 c_{3w} + \ldots + \beta_n c_{nw} \neq 0.$$

Then at least one of the following is true:
(i) $\bigcup_{x \in M_T} (Tx)^\perp = \mathbb{Y}$.
(ii) T does not satisfy the BŠ Property.

Proof. Assuming (i) is not true, we show that T does not satisfy the BŠ Property. Under this assumption, $\bigcup_{x \in M_T} (Tx)^\perp \subsetneqq \mathbb{Y}$. Let us take $z \in \mathbb{Y} \setminus \bigcup_{x \in M_T} (Tx)^\perp$. We note that it follows from Proposition 2.1 of [61] that for each $i = 1, 2$, either $z \in (Tx_i)^+$ or $z \in (Tx_i)^-$.

Case I: Let $z \in (Tx_1)^+ \cap (Tx_2)^+$ or $z \in (Tx_1)^- \cap (Tx_2)^-$. Let us define $A : \mathbb{X} \to \mathbb{Y}$ by

$$Ax_1 = z, Ax_2 = -z \text{ and } Ax_i = \beta_i z \text{ for } i = 3, 4, \ldots, n.$$

If $z \in (Tx_1)^+ \cap (Tx_2)^+$, then $Ax_1 \in (Tx_1)^+$ and $Ax_2 \in (Tx_2)^-$. On the other hand, if $z \in (Tx_1)^- \cap (Tx_2)^-$, then $Ax_1 \in (Tx_1)^-$ and $Ax_2 \in (Tx_2)^+$. Therefore, using Theorem 2.2 of [61], we have $T \perp_B A$, in both the cases. We claim that $Tu \not\perp_B Au$ for any $u \in M_T$. Let $u = c_{1u}x_1 + c_{2u}x_2 + \ldots + c_{nu}x_n \in M_T$. Then, $Au = (c_{1u} - c_{2u} + c_{3u}\beta_3 + \ldots + c_{nu}\beta_n)z = \gamma z$, (say). As $\gamma \neq 0$ and $Tu \not\perp_B z$, so we conclude that $Tu \not\perp_B Au$. Thus T does not satisfy the BŠ Property.

Case II: Let $z \in (Tx_1)^+ \cap (Tx_2)^-$ or $z \in (Tx_1)^- \cap (Tx_2)^+$. Let us define $A : \mathbb{X} \to \mathbb{Y}$ by $Ax_1 = z$, $Ax_2 = z$ and $Ax_i = \alpha_i z$ for $i = 3, 4, \ldots, n$. Proceeding in a similar manner, we can conclude that $T \perp_B A$ but there exists no $u \in M_T$ such that $Tu \perp_B Au$. Therefore T does not satisfy the BŠ Property. This completes the proof of the theorem. □

The above theorem indirectly hints at the importance of the notion of Property P_n for a Banach space in the study of the BŠ Property of a bounded linear operator. We state this formally in the following corollary.

Corollary 7.2. *Let $\mathbb{X}$ be an n-dimensional Banach space, where $n \geq 3$ and let $\mathbb{Y}$ be a Banach space such that $\mathbb{Y}$ has Property P_m, for some $m \in \mathbb{N}$. Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ be such that the following conditions are satisfied:*
(a) $|M_T| \geq 4$ and $|T(M_T)| \leq 2m$,
(b) There exists a basis $\{x_1, x_2, x_3, \ldots, x_n\}$ of $\mathbb{X}$ such that $x_1, x_2 \in M_T$,
(c) There exist scalars $\alpha_3, \alpha_4, \ldots, \alpha_n$ and $\beta_3, \beta_4, \ldots, \beta_n$ such that for each $w = c_{1w}x_1 + c_{2w}x_2 + \ldots + c_{nw}x_n \in M_T$, we have that

$$c_{1w} + c_{2w} + \alpha_3 c_{3w} + \ldots + \alpha_n c_{nw} \neq 0 \text{ and } c_{1w} - c_{2w} + \beta_3 c_{3w} + \ldots + \beta_n c_{nw} \neq 0.$$

Then T does not satisfy the BŠ Property.

Proof. As $T(M_T)$ contains at most $2m$ elements which are pairwise scalar multiples of one another and the co-domain space $\mathbb{Y}$ has Property P_m, we must have $\bigcup\limits_{x \in M_T} (Tx)^\perp \subsetneqq \mathbb{Y}$. Then from Theorem 7.3, we conclude that T does not satisfy the BŠ Property. □

We now give an example to illustrate the applicability of the Corollary 7.2 in studying the BŠ Property of a bounded linear operator between Banach spaces. Here we would like to mention that every smooth Banach space of dimension at least 2, has Property P_n, for each $n \in \mathbb{N}$.

Example 7.4. *Consider a bounded linear operator $T : \ell_\infty^3 \to \ell_2^3$, defined by $Tx = \frac{x}{\sqrt{3}}$ for all $x \in \ell_\infty^3$. Then it is easy to see that $\|T\| = 1$, $M_T = \{\pm(1, 1, 1), \pm(-1, 1, 1),$
$\pm(-1, -1, 1), \pm(1, -1, 1)\}$ and $T(M_T) = \{\pm(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), \pm(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}),$*

$\pm(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), \pm(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\}$. Consider $x_1 = (1,1,1), x_2 = (-1,1,1), x_3 = (-1,-1,1)$ and $x_4 = (1,-1,1)$. Clearly $\{x_1, x_2, x_3\}$ forms a basis of ℓ_∞^3. If we choose $\alpha = \beta = 1$, then condition (c) of Corollary 7.2 is satisfied. Also, ℓ_2^3 has Property P_n, for any $n \in \mathbb{N}$, as ℓ_2^3 is a smooth space. Therefore, by using Corollary 7.2, we conclude that T does not satisfy the BŠ Property.

It is worth mentioning that Theorem 2.2 of [62] holds true when the domain space is any finite-dimensional Banach space and the co-domain space is any smooth Banach space of dimension at least two. Indeed, the said theorem can be stated in the following more general form, by using the same arguments presented in the proof of the original result.

Theorem 7.5. *Let $\mathbb{X}$ be a finite-dimensional Banach space and $\mathbb{Y}$ be a smooth Banach space of dimension greater than or equal to two. Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ be such that M_T is a countable set with more than two points. Then T does not satisfy the BŠ Property.*

7.3 Study of P_n Property on finite-dimensional polyhedral Banach spaces

In the remaining part of this Chapter, we focus on Property P_n for polyhedral Banach spaces. Our first observation reveals that given any polyhedral Banach space $\mathbb{X}$, there exists a natural number n_0 such that $\mathbb{X}$ does not have Property P_n, for any $n \geq n_0$.

Theorem 7.6. *Let $\mathbb{X}$ be a finite-dimensional polyhedral Banach space such that $B_{\mathbb{X}}$ has exactly $2n$ extreme points. Then $\mathbb{X}$ does not have Property P_n.*

Proof. Let us denote the extreme points of $B_{\mathbb{X}}$ by $\pm u_1, \pm u_2, \ldots, \pm u_n$. We claim that $\bigcup\limits_{i=1}^{n} u_i^{\perp} = \mathbb{X}$. Let $y \in \mathbb{X}$ be arbitrary. Given $z \in \mathbb{X}$ there exists a scalar $a \in \mathbb{R}$ such that $ay + z \perp_B y$, by Theorem 2.3 of [20]. Take $x = \frac{ay+z}{\|ay+z\|}$, then $x \perp_B y$. If x is an extreme point of $B_{\mathbb{X}}$, then we have nothing more to show. Now, suppose that x is not an extreme point of $B_{\mathbb{X}}$. As $x \perp_B y$, by using Theorem 2.1 of [20], there exists a linear functional $f \in S_{\mathbb{X}^*}$ such that $f(x) = \|x\| = 1$ and $f(y) = 0$. Since f attains norm, it is easy to see that there exists an extreme point u_i of $B_{\mathbb{X}}$ such that $|f(u_i)| = \|f\| = 1$. Therefore, by Theorem 2.1 of [20], we have $u_i \perp_B y$. Thus $\bigcup\limits_{i=1}^{n} u_i^{\perp} = \mathbb{X}$. This completes the proof of the theorem. ∎

Our next theorem shows that we have a definitive answer for two-dimensional polyhedral Banach spaces, regarding Property P_n. To prove the theorem, we need the following lemma:

Lemma 7.1. *Let $\mathbb{X}$ be a two-dimensional polyhedral Banach space. Then for any $x \in Ext\ B_{\mathbb{X}}$, there exists a normal cone K of $\mathbb{X}$ such that $x^{\perp} = K \cup (-K)$. In addition, if the normal cone K is determined by $v_1, v_2 \in S_{\mathbb{X}}$, then $\{(1-t)v_1 + tv_2 : t \in (0,1)\} \cap y^{\perp} = \phi$ for each $y \in Ext\ B_{\mathbb{X}} \setminus \{\pm x\}$.*

Proof. Let $g \in S_{\mathbb{X}^*}$ and $g(x) = \|x\| = 1$, i.e., g is a supporting functional of $B_{\mathbb{X}}$ at x. Let f_1 and f_2 be the two supporting functionals corresponding to the two edges of $S_{\mathbb{X}}$ meeting at x. Now, $x + \ker g$ is a supporting line to $B_{\mathbb{X}}$ at x, that lies entirely within the cone formed by the straight lines $x + \ker f_1$ and $x + \ker f_2$. For $i = 1, 2$, let

$$f_i^+ = \{z \in \mathbb{X} : f_i(z) \geq 0\} \text{ and } f_i^- = \{z \in \mathbb{X} : f_i(z) \leq 0\}.$$

We note that each f_i^+ (f_i^-) is a closed half-space in $\mathbb{X}$. Let $u \in \ker g$ be arbitrary. The sit-

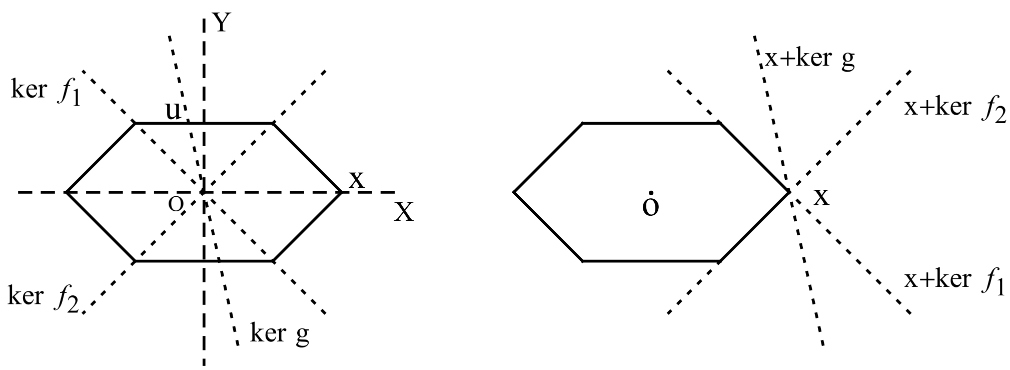

Figure 7.1

uation is illustrated in Figure 1. It follows immediately that either of the following must be true:

(i) $u \in f_1^+$ and $u \in f_2^-$, (ii) $u \in f_1^-$ and $u \in f_2^+$.

Taking $K = f_1^+ \cap f_2^-$, it is easy to see that $-K = f_1^- \cap f_2^+$. Thus for any $u \in \mathbb{X}$, with $x \perp_B u$, we have $u \in K \cup (-K)$. Therefore, $x^{\perp} = \{w \in \mathbb{X} : x \perp_B w\} = K \cup (-K)$. This completes the proof of the first part of the lemma.

Next, suppose K is determined by $v_1, v_2 \in S_{\mathbb{X}}$. From the construction of K it is clear that $v_1 \in \ker f_1 \cap S_{\mathbb{X}}$ and $v_2 \in \ker f_2 \cap S_{\mathbb{X}}$. Let $V = \{(1-t)v_1 + tv_2 : t \in (0,1)\}$. We show that $V \cap y^{\perp} = \phi$, for each $y \in Ext\ B_{\mathbb{X}} \setminus \{\pm x\}$. If possible, suppose that $V \cap y^{\perp} \neq \phi$, for some $y \in Ext\ B_{\mathbb{X}} \setminus \{\pm x\}$. Then there exists $v = (1-t)v_1 + tv_2 \in V$ such that $v \in y^{\perp}$. Since $x \perp_B v$, there exists $f \in S_{\mathbb{X}^*}$ such that $f(x) = \|x\| = 1$ and $f(v) = 0$. On the other hand, since $y \perp_B v$, there exists $h \in S_{\mathbb{X}^*}$ such that $h(y) = \|y\| = 1$ and $h(v) = 0$. Since $\mathbb{X}$ is two-dimensional, it is easy to deduce that $f = \pm h$. Therefore, either x, y are adjacent vertices and $f = h$ is an extreme supporting functional corresponding to the facet $L[x,y] = \{(1-t)x+ty : t \in [0,1]\}$, or, $x, -y$ are adjacent vertices and $f = -h$ is an extreme supporting functional corresponding to the facet $L[x,-y] = \{(1-t)x + t(-y) : t \in [0,1]\}$. As f_1 and f_2 are two supporting functionals corresponding to the two edges of $S_{\mathbb{X}}$ meeting at x, f is equal to either f_1 or f_2. Therefore, v is equal to either v_1 or v_2, which is a contradiction to our assumption that $v \in V$. This completes the proof of the lemma. $\square$

We now prove the desired theorem.

Theorem 7.7. *Let $\mathbb{X}$ be a two-dimensional polyhedral Banach space such that $B_{\mathbb{X}}$ has exactly $2n$ extreme points for some $n \in \mathbb{N}$. Then $\mathbb{X}$ has Property P_{n-1}.*

Proof. If $x \in S_{\mathbb{X}}$ is a non-extreme point of $B_{\mathbb{X}}$, then there exist $x_1, x_2 \in S_{\mathbb{X}}$ such that $x = (1-t)x_1 + tx_2$ for some $t \in (0,1)$ and x_1, x_2 are extreme points of $B_{\mathbb{X}}$. It is easy to observe that $x^\perp \subsetneqq x_1^\perp$ and $x^\perp \subsetneqq x_2^\perp$. Therefore, without loss of generality we may consider any $(n-1)$ extreme points of $B_{\mathbb{X}}$, instead of any $(n-1)$ points in $S_{\mathbb{X}}$, to prove that $\mathbb{X}$ has Property P_{n-1}. Let $Ext\ B_{\mathbb{X}} = \{\pm x_1, \pm x_2, \ldots, \pm x_n\}$. Then from Lemma 7.1, for each $i \in \{1, 2, \ldots, n\}$, there exists normal cone K_i of $\mathbb{X}$ such that $x_i^\perp = K_i \cup (-K_i)$. If the normal cone K_i is determined by $v_{i1}, v_{i2} \in S_{\mathbb{X}}$, then $\{(1-t)v_{i1} + tv_{i2} : t \in (0,1)\} \cap \bigcup\limits_{\substack{j=1 \\ j\neq i}}^{n} x_j^\perp = \phi$. Since x_i is any extreme point of $B_{\mathbb{X}}$, we conclude that the union of the Birkhoff-James orthogonality sets of any $(n-1)$ extreme points of $B_{\mathbb{X}}$ must be a proper subset of $\mathbb{X}$. However, this is clearly equivalent to the fact that $\mathbb{X}$ has Property P_{n-1}. This establishes the theorem. $\square$

As an application of Corollary 7.2, we next give an example of a bounded linear operator T between a three-dimensional polyhedral Banach space $\mathbb{X}$ and a two-dimensional polyhedral Banach space $\mathbb{Y}$ such that T does not satisfy the BŠ Property.

Example 7.8. *Let $\mathbb{X} = \ell_\infty^3$ and let $\mathbb{Y}$ be a two-dimensional real polyhedral Banach space such that $S_{\mathbb{Y}}$ is regular decagon with vertices $(\cos\frac{j\pi}{5}, \sin\frac{j\pi}{5})$, $j \in \{0, 1, 2, \ldots, 9\}$. Consider the linear operator $T : \mathbb{X} \to \mathbb{Y}$, defined by*

$$T(x,y,z) = \Big(\frac{x+y}{2} + \frac{(y-x)\cos\frac{2\pi}{5}}{2}, \frac{(y-x)\sin\frac{2\pi}{5}}{2}\Big).$$

It is easy to check that $\|T\| = 1$, $M_T = \{\pm(1,1,z), \pm(-1,1,z) : z \in [-1,1]\}$ and $T(M_T) = \{\pm(1,0), \pm(\cos\frac{2\pi}{5}, \sin\frac{2\pi}{5})\}$. Consider $x_1 = (1,1,1), x_2 = (-1,1,1)$ and $x_3 = (-1,-1,1)$. Clearly $\{x_1, x_2, x_3\}$ forms a basis of $\mathbb{X}$. If we choose $\alpha = -10$ and $\beta = \frac{-3}{2}$, then condition (c) of Corollary 7.2 is satisfied. From Theorem 7.7, we know that $\mathbb{Y}$ has Property P_4. Therefore, by using Corollary 7.2, we conclude that T does not satisfy the BŠ Property.

For any two Banach spaces $\mathbb{X}, \mathbb{Y}$, it is easy to see that $\mathbb{X} \times \mathbb{Y}$, equipped with the norm $\|(x, y)\| = \max\{\|x\|, \|y\|\}$ for all $(x, y) \in \mathbb{X} \times \mathbb{Y}$, is a Banach space. Let us denote this space by $\mathbb{X} \oplus_\infty \mathbb{Y}$. Similarly, $\mathbb{X} \times \mathbb{Y}$, equipped with the norm $\|(x, y)\| = \|x\| + \|y\|$ for all $(x, y) \in \mathbb{X} \times \mathbb{Y}$, is a Banach space which is denoted by $\mathbb{X} \oplus_1 \mathbb{Y}$. In the following theorems, we study Property P_n for Banach spaces $\mathbb{X} \oplus_\infty \mathbb{Y}$ and $\mathbb{X} \oplus_1 \mathbb{Y}$, where $\mathbb{X}$ is a polyhedral Banach space.

Theorem 7.9. *Let $\mathbb{X}$ be a polyhedral Banach space such that $\mathbb{X}$ does not have Property P_n, for some $n \in \mathbb{N}$. Then $\mathbb{X} \oplus_\infty \mathbb{Y}$ does not have Property P_n, for any Banach space $\mathbb{Y}$. Moreover, if $\mathbb{Y}$ is a polyhedral Banach space such that $\mathbb{Y}$ does not have Property P_m, for some $m \in \mathbb{N}$, then $\mathbb{X} \oplus_\infty \mathbb{Y}$ does not have Property P_r, where $r = \min\{m, n\}$.*

Proof. As $\mathbb{X}$ does not have Property P_n, there exist $x_1, x_2, \ldots, x_n \in S_{\mathbb{X}}$ such that $\bigcup_{i=1}^{n} x_i^{\perp} = \mathbb{X}$. Now we claim that $\bigcup_{i=1}^{n} (x_i, 0)^{\perp} = \mathbb{X} \oplus_\infty \mathbb{Y}$.
Let us first show that $x_i^{\perp} \times \mathbb{Y} \subseteq (x_i, 0)^{\perp}$, for each $i \in \{1, 2, \ldots, n\}$. Let $(x, y) \in x_i^{\perp} \times \mathbb{Y}$. Then for any scalar λ,

$$\begin{aligned} \|(x_i, 0) + \lambda(x, y)\| &= \|(x_i + \lambda x, \lambda y)\| \\ &= \max\{\|x_i + \lambda x\|, \|\lambda y\|\} \\ &\geq \|x_i + \lambda x\| \\ &\geq \|x_i\| = \|(x_i, 0)\|, \end{aligned}$$

as $x \in x_i^{\perp}$. Therefore, $(x, y) \in (x_i, 0)^{\perp}$. Hence $x_i^{\perp} \times \mathbb{Y} \subseteq (x_i, 0)^{\perp}$.
Therefore, $\bigcup_{i=1}^{n} (x_i, 0)^{\perp} \supseteq \bigcup_{i=1}^{n} (x_i^{\perp} \times \mathbb{Y}) = \mathbb{X} \oplus_\infty \mathbb{Y}$. Hence $\bigcup_{i=1}^{n} (x_i, 0)^{\perp} = \mathbb{X} \oplus_\infty \mathbb{Y}$.
Further if $\mathbb{Y}$ does not have Property P_m, then as before we can show that $\mathbb{X} \oplus_\infty \mathbb{Y}$ does not possess Property P_m. Thus $\mathbb{X} \oplus_\infty \mathbb{Y}$ does not have Property P_r, where $r = \min\{m, n\}$. This completes the proof of the theorem. □

Corollary 7.3. *Let $\mathbb{X} = \ell_\infty^n$, for any $n\ (\geq 2) \in \mathbb{N}$. Then $\mathbb{X}$ does not have Property P_2.*

Proof. From Theorem 7.6, it follows that ℓ_∞^2 does not have Property P_2. Also, we know that $\ell_\infty^3 = \ell_\infty^2 \oplus_\infty \mathbb{R}$. Therefore, by using Theorem 7.9, we conclude that ℓ_∞^3 does not have Property P_2. Continuation of this argument proves that ℓ_∞^n, for any $n\ (\geq 2) \in \mathbb{N}$ does not have Property P_2. □

Applying similar arguments, the proofs of the following results are now apparent:

Theorem 7.10. *Let $\mathbb{X}$ be a polyhedral Banach space such that $\mathbb{X}$ does not have Property P_n, for some $n \in \mathbb{N}$. Then $\mathbb{X} \oplus_1 \mathbb{Y}$ does not have Property P_n, for any Banach space $\mathbb{Y}$. Moreover, if*

$\mathbb{Y}$ is a polyhedral Banach space such that $\mathbb{Y}$ does not have Property P_m, for some $m \in \mathbb{N}$, then $\mathbb{X} \oplus_1 \mathbb{Y}$ does not have Property P_r, where $r = \min\{m, n\}$.

Corollary 7.4. *Let $\mathbb{X} = \ell_1^n$, where $n(\geq 2) \in \mathbb{N}$. Then $\mathbb{X}$ does not have Property P_2.*

Let $\mathbb{X}$ be a three-dimensional polyhedral Banach space such that $B_{\mathbb{X}}$ is a prism with vertices $(\cos\frac{j\pi}{n}, \sin\frac{j\pi}{n}, \pm 1)$, $j \in \{0, 1, 2, \ldots, 2n-1\}$, $n \geq 2$. Then it is trivial to see that $\mathbb{X} = \mathbb{Y} \oplus_\infty \mathbb{R}$, where $\mathbb{Y}$ is a two-dimensional polyhedral Banach space so that the extreme points of $B_{\mathbb{Y}}$ are given by $(\cos\frac{j\pi}{n}, \sin\frac{j\pi}{n})$, $j \in \{0, 1, 2, \ldots, 2n-1\}$, $n \geq 2$. Therefore, by using Theorem 7.6 and Theorem 7.9, we can conclude that $\mathbb{X}$ does not have Property P_n. However, in the next theorem we show that $\mathbb{X}$ does not have Property P_2, for any $n \geq 2$.

Theorem 7.11. *Let $\mathbb{X}$ be a three-dimensional polyhedral Banach space such that $B_{\mathbb{X}}$ is a prism with vertices $(\cos\frac{j\pi}{n}, \sin\frac{j\pi}{n}, \pm 1)$, $j \in \{0, 1, 2, \ldots, 2n-1\}$, $n \geq 2$. Then $\mathbb{X}$ does not have Property P_2.*

Proof. Let the vertices of $B_{\mathbb{X}}$ be $v_{\pm(j+1)}$, $j \in \{0, 1, 2, \ldots, 2n-1\}$, where $v_{\pm(j+1)} = (\cos\frac{j\pi}{n}, \sin\frac{j\pi}{n}, \pm 1)$. The unit sphere $S_{\mathbb{X}}$ is shown in Figure 2.

A simple computation reveals the explicit expression for the norm function on $\mathbb{X}$. Given any

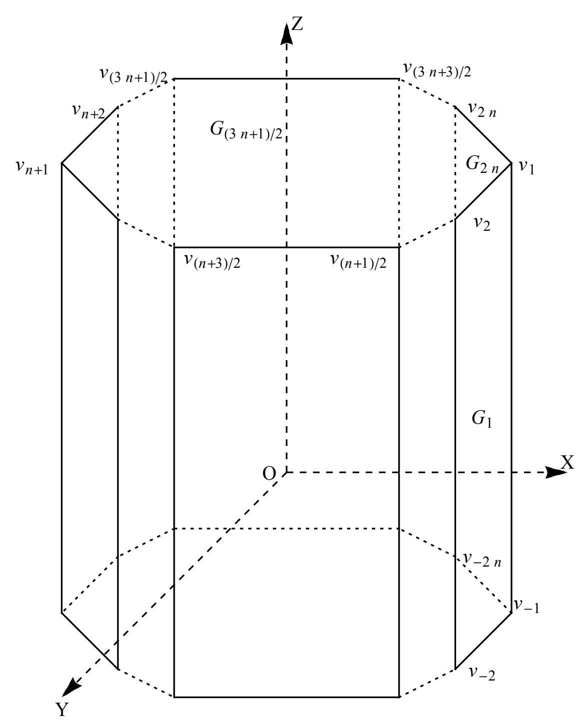

Figure 7.2

$(x, y, z) \in \mathbb{X}$, we have,

$$\|(x,y,z)\| = \max_{0\leq j\leq 2n-1}\left\{\frac{|\cos\frac{(2j+1)\pi}{2n}||x|}{\cos\frac{\pi}{2n}} + \frac{|\sin\frac{(2j+1)\pi}{2n}||y|}{\cos\frac{\pi}{2n}}, |z|\right\}.$$

We claim that $v_1^{\perp} \cup v_{n+1}^{\perp} = \mathbb{X}$. Let $(x, y, z) \in \mathbb{X}$ be such that $x \geq 0, z \leq 0$ and y is arbitrary. Now, for any scalar $\lambda \geq 0$,

$$\begin{aligned}
\|(1,0,1)+\lambda(x,y,z)\| &= \|(1+\lambda x, \lambda y, 1+\lambda z)\| \\
&= \max_{0\leq j\leq 2n-1}\Bigg\{\frac{|\cos\frac{(2j+1)\pi}{2n}||1+\lambda x|}{\cos\frac{\pi}{2n}} + \frac{|\sin\frac{(2j+1)\pi}{2n}||\lambda y|}{\cos\frac{\pi}{2n}}, \\
&\qquad |1+\lambda z|\Bigg\} \\
&\geq |1+\lambda x| + \tan\frac{\pi}{2n}|\lambda y| \\
&\geq 1 = \|(1,0,1)\|.
\end{aligned}$$

Also, for any scalar $\lambda \leq 0$,

$$\begin{aligned}
\|(1,0,1)+\lambda(x,y,z)\| &= \|(1+\lambda x, \lambda y, 1+\lambda z)\| \\
&= \max_{0\leq j\leq 2n-1}\Bigg\{\frac{|\cos\frac{(2j+1)\pi}{2n}||1+\lambda x|}{\cos\frac{\pi}{2n}} + \frac{|\sin\frac{(2j+1)\pi}{2n}||\lambda y|}{\cos\frac{\pi}{2n}}, \\
&\qquad |1+\lambda z|\Bigg\} \\
&\geq |1+\lambda z| \\
&\geq 1 = \|(1,0,1)\|.
\end{aligned}$$

Therefore, $(1,0,1)\perp_B(x,y,z)$, for all $x \geq 0, z \leq 0$ and for any y. From the homogeneity property of Birkhoff-James orthogonality, it follows that $(1,0,1)\perp_B(x,y,z)$, for all $x \leq 0, z \geq 0$ and for any y.

Let $(x, y, z) \in \mathbb{X}$ be such that $x \geq 0, z \geq 0$. Now for any $\lambda \geq 0$,

$$\begin{aligned}
\|(-1,0,1)+\lambda(x,y,z)\| &= \|(-1+\lambda x, \lambda y, 1+\lambda z)\| \\
&= \max_{0\leq j\leq 2n-1}\Bigg\{\frac{|\cos\frac{(2j+1)\pi}{2n}||-1+\lambda x|}{\cos\frac{\pi}{2n}} + \frac{|\sin\frac{(2j+1)\pi}{2n}||\lambda y|}{\cos\frac{\pi}{2n}}, \\
&\qquad |1+\lambda z|\Bigg\} \\
&\geq |1+\lambda z| \\
&\geq 1 = \|(-1,0,1)\|.
\end{aligned}$$

Also, for any $\lambda \leq 0$,

$$\begin{aligned}
\|(-1,0,1)+\lambda(x,y,z)\| &= \|(-1+\lambda x, \lambda y, 1+\lambda z)\| \\
&= \max_{0\leq j\leq 2n-1}\Big\{\frac{|\cos\frac{(2j+1)\pi}{2n}||-1+\lambda x|}{\cos\frac{\pi}{2n}}+\frac{|\sin\frac{(2j+1)\pi}{2n}||\lambda y|}{\cos\frac{\pi}{2n}}, \\
&\qquad |1+\lambda z|\Big\} \\
&\geq |-1+\lambda x|+\tan\frac{\pi}{2n}|\lambda y| \\
&\geq 1 = \|(-1,0,1)\|.
\end{aligned}$$

Therefore, $(-1,0,1)\perp_B(x,y,z)$, for all $x\geq 0, z\geq 0$ and for any y. From the homogeneity property of Birkhoff-James orthogonality, it follows that $(-1,0,1)\perp_B(x,y,z)$, for all $x\leq 0, z\leq 0$ and for any y.
Hence for any $(x,y,z)\in\mathbb{X}$, either $(x,y,z)\in v_1^{\perp}$ or $(x,y,z)\in v_{n+1}^{\perp}$, i.e., $v_1^{\perp}\cup v_{n+1}^{\perp}=\mathbb{X}$. Therefore, $\mathbb{X}$ does not have Property P_2. This completes the proof of the theorem. $\square$

Remark 7.12. *Theorem 7.11 shows that the space $\mathbb{X}\oplus_\infty\mathbb{Y}$ may not have Property P_n even if one of space has Property P_n.*

Our Previous results point to the fact that there is a large family of three-dimensional polyhedral Banach spaces not having Property P_2. The concerned spaces have been constructed by taking ℓ_∞ sum of two-dimensional polyhedral Banach spaces with $\mathbb{R}$. In the next theorem, we give another such example of a three-dimensional Polyhedral Banach space, not having Property P_2, which cannot be constructed by taking ℓ_∞ sum of lower dimensional Banach space.

Theorem 7.13. *Let $\mathbb{X}$ be a three-dimensional polyhedral Banach space such that $B_{\mathbb{X}}$ is a polyhedron obtained by gluing two pyramids at the opposite base faces of a right prism having square base, with vertices $\pm(1,1,1),\pm(-1,1,1),\pm(-1,-1,1), \pm(1,-1,1),\pm(0,0,2)$. Then $\mathbb{X}$ does not have Property P_2.*

Proof. Let the vertices of $B_{\mathbb{X}}$ be $v_{\pm j}, j\in\{1,2,3,4\}$ and $w_{\pm 1}$, where $v_{\pm 1}=(1,1,\pm 1)$, $v_{\pm 2}=(-1,1,\pm 1)$, $v_{\pm 3}=(-1,-1,\pm 1)$, $v_{\pm 4}=(1,-1,\pm 1)$ and $w_{\pm 1}=(0,0,\pm 2)$. The unit sphere $S_{\mathbb{X}}$ is shown in Figure 3.

Given any $(x,y,z)\in\mathbb{X}$, the expression for the norm function on $\mathbb{X}$ turns out to be the following:

$$\|(x,y,z)\| = \max\left\{|x|,|y|,\frac{|x|}{2}+\frac{|z|}{2},\frac{|y|}{2}+\frac{|z|}{2}\right\}.$$

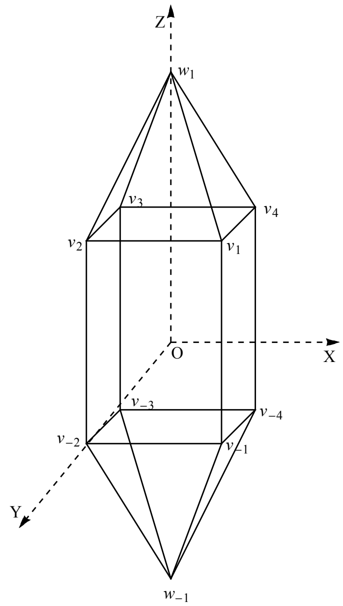

Figure 7.3

We claim that, $v_1^{\perp} \cup v_4^{\perp} = \mathbb{X}$. Let $(x, y, z) \in \mathbb{X}$ be such that $x \geq 0, y \geq 0$. Now, for any $\lambda \geq 0$,

$$\begin{aligned}
\|(1,-1,1)+\lambda(x,y,z)\| &= \|(1+\lambda x, -1+\lambda y, 1+\lambda z)\| \\
&= \max\Big\{|1+\lambda x|, |-1+\lambda y|, \frac{|1+\lambda x|}{2}+\frac{|1+\lambda z|}{2}, \\
&\qquad \frac{|-1+\lambda y|}{2}+\frac{|1+\lambda z|}{2}\Big\} \\
&\geq |1+\lambda x| \\
&\geq 1 = \|(1,-1,1)\|.
\end{aligned}$$

Also, for any $\lambda \leq 0$,

$$\begin{aligned}
\|(1,-1,1)+\lambda(x,y,z)\| &= \|(1+\lambda x, -1+\lambda y, 1+\lambda z)\| \\
&= \max\Big\{|1+\lambda x|, |-1+\lambda y|, \frac{|1+\lambda x|}{2}+\frac{|1+\lambda z|}{2}, \\
&\qquad \frac{|-1+\lambda y|}{2}+\frac{|1+\lambda z|}{2}\Big\} \\
&\geq |-1+\lambda y| \\
&\geq 1 = \|(1,-1,1)\|.
\end{aligned}$$

Therefore, $(1,-1,1)\perp_B(x,y,z)$, for all $x \geq 0, y \geq 0$ and for any z. From the homogeneity property of Birkhoff-James orthogonality, it follows that $(1,-1,1)\perp_B(x,y,z)$, for all $x \leq 0, y \leq 0$ and for any z.

Let $(x,y,z) \in \mathbb{X}$ be such that $x \geq 0, y \leq 0$. For any $\lambda \geq 0$,

$$\begin{aligned}
\|(1,1,1)+\lambda(x,y,z)\| &= \|(1+\lambda x, 1+\lambda y, 1+\lambda z)\| \\
&= \max\Big\{|1+\lambda x|, |1+\lambda y|, \frac{|1+\lambda x|}{2}+\frac{|1+\lambda z|}{2}, \\
&\qquad \frac{|1+\lambda y|}{2}+\frac{|1+\lambda z|}{2}\Big\} \\
&\geq |1+\lambda x| \\
&\geq 1 = \|(1,1,1)\|.
\end{aligned}$$

Also, for any $\lambda \leq 0$,

$$\begin{aligned}
\|(1,1,1)+\lambda(x,y,z)\| &= \|(1+\lambda x, 1+\lambda y, 1+\lambda z)\| \\
&= \max\Big\{|1+\lambda x|, |1+\lambda y|, \frac{|1+\lambda x|}{2}+\frac{|1+\lambda z|}{2}, \\
&\qquad \frac{|1+\lambda y|}{2}+\frac{|1+\lambda z|}{2}\Big\} \\
&\geq |1+\lambda y| \\
&\geq 1 = \|(1,1,1)\|.
\end{aligned}$$

Therefore, $(1,1,1)\perp_B(x,y,z)$, for all $x \geq 0, y \leq 0$ and for any z. From the homogeneity property of Birkhoff-James orthogonality, it follows that $(1,1,1)\perp_B(x,y,z)$, for all $x \leq 0, y \geq 0$ and for any z.
Hence for any $(x,y,z) \in \mathbb{X}$, either $(x,y,z) \in v_1^{\perp}$ or $(x,y,z) \in v_4^{\perp}$, i.e., $v_1^{\perp} \cup v_4^{\perp} = \mathbb{X}$. Therefore, $\mathbb{X}$ does not have Property P_2. This completes the proof of the theorem. $\square$

We next give an example of a three-dimensional polyhedral Banach space which has Property P_2 but does not have Property P_3.

Theorem 7.14. *Let $\mathbb{X}$ be a three-dimensional polyhedral Banach space such that $B_{\mathbb{X}}$ is a polyhedron with vertices $(\cos\frac{j\pi}{n}, \sin\frac{j\pi}{n}, \pm 1), (0,0,\pm 2)$, $j \in \{0,1,2,\ldots,2n-1\}$, $n \geq 3$. Then $\mathbb{X}$ has Property P_2 but $\mathbb{X}$ does not have Property P_3.*

Proof. We prove the theorem by assuming that n is an odd integer. Similar calculations hold true, when n is an even integer. Let the vertices of $B_{\mathbb{X}}$ be $v_{\pm(j+1)}$, $j \in \{0,1,2,\ldots,2n-1\}$ and $w_{\pm 1}$, where $v_{\pm(j+1)} = (\cos\frac{j\pi}{n}, \sin\frac{j\pi}{n}, \pm 1)$ and $w_{\pm 1} = (0,0,\pm 2)$. The unit sphere $S_{\mathbb{X}}$ is shown in Figure 4. Let G_{j+1} denote the facet of $B_{\mathbb{X}}$ containing $v_{j+1}, v_{-j-1}, v_{j+2}, v_{-j-2}$, where $v_{2n+1} = v_1$ and $v_{-2n-1} = v_{-1}$. Let $F_{\pm(j+1)}$ denote the facet of $B_{\mathbb{X}}$ containing $v_{\pm(j+1)}, v_{\pm(j+2)}, w_{\pm 1}$. For each $j \in \{0,1,2,\ldots,2n-1\}$, let $g_{j+1}, f_{\pm(j+1)}$ be the supporting functionals corresponding to the facets $G_{j+1}, F_{\pm(j+1)}$ respectively, i.e., $(v_{j+1}+\ker g_{j+1}) \cap S_{\mathbb{X}} = G_{j+1}$ and $(v_{\pm(j+1)}+\ker f_{\pm(j+1)}) \cap$

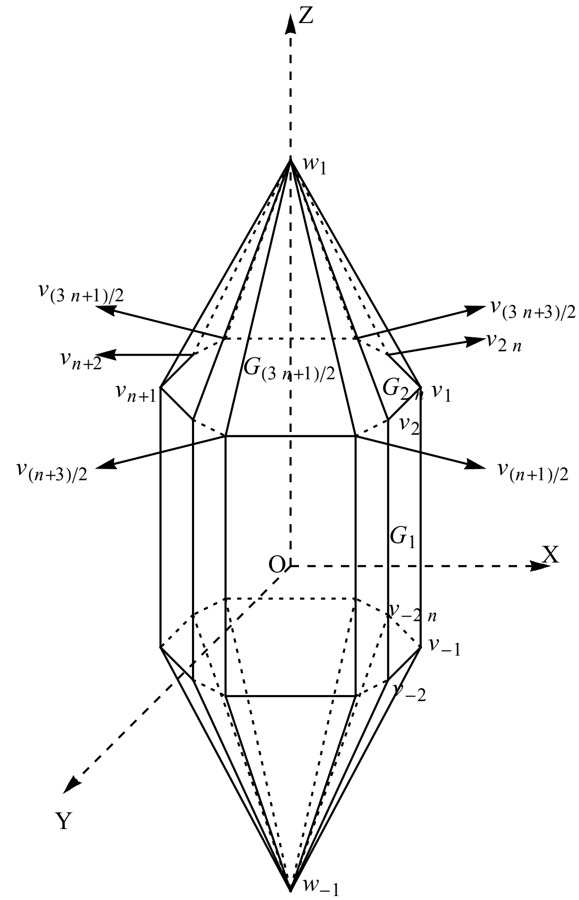

Figure 7.4

$S_{\mathbb{X}} = F_{\pm(j+1)}$.
For every $v_{j+1} \in S_{\mathbb{X}},\ j = 0, 1, 2, \ldots, 2n-1$, there are four adjacent facets $G_j, G_{j+1},\ F_j, F_{j+1}$. Here we assume that $G_0 = G_{2n}$ and $F_0 = F_{2n}$. Therefore by Lemma 2.1 of [57], extreme supporting functionals corresponding to the vertices v_{j+1},
$j = 0, 1, 2, \ldots, 2n-1$, are $g_j, g_{j+1}, f_j, f_{j+1}$. Here also we assume that $g_0 = g_{2n}$ and $f_0 = f_{2n}$. Now consider the following subsets of $S_{\mathbb{X}^*}$:
$H_{j+1} = \Big\{h \in S_{\mathbb{X}^*} : h = \lambda_1 g_j + \lambda_2 g_{j+1} + \lambda_3 f_j + \lambda_4 f_{j+1}, \lambda_k \geq 0\ \forall\ k \in \{1, 2, 3, 4\}$
and $\sum\limits_{k=1}^{4} \lambda_k = 1\Big\}$, for each $j = 0, 1, 2, \ldots, 2n-1$. Then by using Theorem 2.1 of [20] and Lemma 2.1 of [57], we conclude that $v_{j+1}^{\perp} = \bigcup\limits_{h \in H_{j+1}} \ker h$, for each $j = 0, 1, 2, \ldots, 2n-1$. Again, for w_1, there are $2n$ adjacent facets $F_{j+1},\ j = 0, 1, 2, \ldots, 2n-1$. Therefore, as before, extreme supporting functionals corresponding to the vertex w_1 are $f_{j+1},\ j = 0, 1, 2, \ldots, 2n-1$. Now, consider the following subset of $S_{\mathbb{X}^*}$.

$H = \Big\{h \in S_{\mathbb{X}^*} : h = \sum\limits_{k=0}^{2n-1} \lambda_{k+1} f_{k+1}, \lambda_{k+1} \geq 0 \ \forall \ k \in \{0, 1, 2, \ldots, 2n-1\}$ and $\sum\limits_{k=0}^{2n-1} \lambda_{k+1} = 1\Big\}$. From this, we conclude that $w_1^\perp = \bigcup\limits_{h \in H} \ker h$. Now, for any $(x, y, z) \in \mathbb{X}$, we have,

$\|(x, y, z)\| = \max\limits_{0 \leq j \leq \frac{n-1}{2}} \Big\{ \frac{\cos\frac{(2j+1)\pi}{2n}|x|}{\cos\frac{\pi}{2n}} + \frac{\sin\frac{(2j+1)\pi}{2n}|y|}{\cos\frac{\pi}{2n}}, \frac{\cos\frac{(2j+1)\pi}{2n}|x|}{2\cos\frac{\pi}{2n}} + \frac{\sin\frac{(2j+1)\pi}{2n}|y|}{2\cos\frac{\pi}{2n}} + \frac{|z|}{2} \Big\}$. Using the above expression of the norm function, we can compute the Birkhoff-James orthogonality set of the extreme points of $B_{\mathbb{X}}$. For the extreme points of $B_{\mathbb{X}}$, lying above the plane $z = 0$, we have,

$v_1^\perp = (1, 0, 1)^\perp = \pm\Big[\{(x, y, z) \in \mathbb{X} : x \geq 0, y \geq 0, z \geq 0, x - \tan(\frac{\pi}{2n})y \leq 0\}$
$\cup \{(x, y, z) \in \mathbb{X} : x \leq 0, y \geq 0, z \geq 0, \frac{x}{2} + \frac{\tan(\frac{\pi}{2n})y}{2} + \frac{z}{2} \geq 0\}$
$\cup \{(x, y, z) \in \mathbb{X} : x \leq 0, y \leq 0, z \geq 0, \frac{x}{2} - \frac{\tan(\frac{\pi}{2n})y}{2} + \frac{z}{2} \geq 0\}$
$\cup \{(x, y, z) \in \mathbb{X} : x \geq 0, y \leq 0, z \geq 0, x + \tan(\frac{\pi}{2n})y \leq 0\}\Big]$.

Now, for each $j \in \{1, 2, \ldots, \frac{n-1}{2}\}$, we have

$v_{(j+1)}^\perp = (\cos\frac{j\pi}{n}, \sin\frac{j\pi}{n}, 1)^\perp = \pm\Big[\{(x, y, z) \in \mathbb{X} : x \leq 0, y \geq 0, z \geq 0,$
$\frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \frac{z}{2} \geq 0, \frac{\cos\frac{(2j-1)\pi}{2n}}{\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j-1)\pi}{2n}}{\cos\frac{\pi}{2n}}y \leq 0\}$
$\cup \{(x, y, z) \in \mathbb{X} : x \leq 0, y \leq 0, z \geq 0, \frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \frac{z}{2} \geq 0\}$
$\cup \{(x, y, z) \in \mathbb{X} : x \leq 0, y \leq 0, z \geq 0, \frac{\cos\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \frac{z}{2} \geq 0\}$
$\cup \{(x, y, z) \in \mathbb{X} : x \geq 0, y \leq 0, z \geq 0, \frac{\cos\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \frac{z}{2} \geq 0,$
$\frac{\cos\frac{(2j+1)\pi}{2n}}{\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j+1)\pi}{2n}}{\cos\frac{\pi}{2n}}y \leq 0\}\Big]$.

Again, for each $j \in \{1, 2, \ldots, \frac{n-1}{2}\}$, we have

$v_{(j+\frac{n+1}{2})}^\perp = (-\cos\frac{j\pi}{n}, \sin\frac{j\pi}{n}, 1)^\perp = \pm\Big[\{(x, y, z) \in \mathbb{X} : x \geq 0, y \geq 0, z \geq 0,$
$-\frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \frac{z}{2} \geq 0, -\frac{\cos\frac{(2j-1)\pi}{2n}}{\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j-1)\pi}{2n}}{\cos\frac{\pi}{2n}}y \leq 0\}$
$\cup \{(x, y, z) \in \mathbb{X} : x \leq 0, y \leq 0, z \geq 0, -\frac{\cos\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \frac{z}{2} \geq 0,$
$-\frac{\cos\frac{(2j+1)\pi}{2n}}{\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j+1)\pi}{2n}}{\cos\frac{\pi}{2n}}y \leq 0\}$
$\cup \{(x, y, z) \in \mathbb{X} : x \geq 0, y \leq 0, z \geq 0, -\frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \frac{z}{2} \geq 0\}$
$\cup \{(x, y, z) \in \mathbb{X} : x \geq 0, y \leq 0, z \geq 0, -\frac{\cos\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \frac{z}{2} \geq 0\}\Big]$.

Now, $v_{(n+1)}^\perp = (-1, 0, 1)^\perp = \pm\Big[\{(x, y, z) \in \mathbb{X} : x \geq 0, y \geq 0, z \geq 0,$
$-\frac{x}{2} + \frac{\tan(\frac{\pi}{2n})y}{2} + \frac{z}{2} \geq 0\}$
$\cup \{(x, y, z) \in \mathbb{X} : x \leq 0, y \geq 0, z \geq 0, -x - \tan(\frac{\pi}{2n})y \leq 0\}$
$\cup \{(x, y, z) \in \mathbb{X} : x \leq 0, y \leq 0, z \geq 0, -x + \tan(\frac{\pi}{2n})y \leq 0\}$
$\cup \{(x, y, z) \in \mathbb{X} : x \geq 0, y \leq 0, z \geq 0, -\frac{x}{2} - \frac{\tan(\frac{\pi}{2n})y}{2} + \frac{z}{2} \geq 0\}\Big]$.

Again, for each $j \in \{1, 2, \ldots, \frac{n-1}{2}\}$, we have

$v_{(j+n+1)}^\perp = (-\cos\frac{j\pi}{n}, -\sin\frac{j\pi}{n}, 1)^\perp = \pm\Big[\{(x, y, z) \in \mathbb{X} : x \geq 0, y \geq 0, z \geq 0,$

$-\frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x-\frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y+\frac{z}{2}\geq 0\}$
$\cup\{(x,y,z)\in\mathbb{X}: x\geq 0, y\geq 0, z\geq 0, -\frac{\cos\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x-\frac{\sin\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y+\frac{z}{2}\geq 0\}$
$\cup\{(x,y,z)\in\mathbb{X}: x\leq 0, y\geq 0, z\geq 0, -\frac{\cos\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x-\frac{\sin\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y+\frac{z}{2}\geq 0,$
$-\frac{\cos\frac{(2j+1)\pi}{2n}}{\cos\frac{\pi}{2n}}x-\frac{\sin\frac{(2j+1)\pi}{2n}}{\cos\frac{\pi}{2n}}y\leq 0\}$
$\cup\{(x,y,z)\in\mathbb{X}: x\geq 0, y\leq 0, z\geq 0, -\frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x-\frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y+\frac{z}{2}\geq 0,$
$-\frac{\cos\frac{(2j-1)\pi}{2n}}{\cos\frac{\pi}{2n}}x-\frac{\sin\frac{(2j-1)\pi}{2n}}{\cos\frac{\pi}{2n}}y\leq 0\}\Big].$
Also, for each $j\in\{1,2,\ldots,\frac{n-1}{2}\}$, we have
$v^{\perp}_{(j+\frac{3n+1}{2})}=(\cos\frac{j\pi}{n},-\sin\frac{j\pi}{n},1)^{\perp}=\pm\Big[\{(x,y,z)\in\mathbb{X}: x\geq 0, y\geq 0, z\geq 0,$
$\frac{\cos\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x-\frac{\sin\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y+\frac{z}{2}\geq 0, \frac{\cos\frac{(2j+1)\pi}{2n}}{\cos\frac{\pi}{2n}}x-\frac{\sin\frac{(2j+1)\pi}{2n}}{\cos\frac{\pi}{2n}}y\leq 0\}$
$\cup\{(x,y,z)\in\mathbb{X}: x\leq 0, y\geq 0, z\geq 0, \frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x-\frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y+\frac{z}{2}\geq 0\}$
$\cup\{(x,y,z)\in\mathbb{X}: x\leq 0, y\geq 0, z\geq 0, \frac{\cos\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x-\frac{\sin\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y+\frac{z}{2}\geq 0\}$
$\cup\{(x,y,z)\in\mathbb{X}: x\leq 0, y\leq 0, z\geq 0, \frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x-\frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y+\frac{z}{2}\geq 0,$
$\frac{\cos\frac{(2j-1)\pi}{2n}}{\cos\frac{\pi}{2n}}x-\frac{\sin\frac{(2j-1)\pi}{2n}}{\cos\frac{\pi}{2n}}y\leq 0\}\Big].$

Now, $w_1^{\perp}=(0,0,2)^{\perp}=\pm\Big[A\cup B\cup C\cup D\Big]$, where $A=\bigcup_{j=0}^{\frac{n-1}{2}}A_j$, $A_j=\{(x,y,z)\in\mathbb{X}: x\geq 0, y\geq 0, z\geq 0, -\frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x-\frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y+\frac{z}{2}\leq 0\}$, $B=\bigcup_{j=0}^{\frac{n-1}{2}}B_j$, $B_j=\{(x,y,z)\in\mathbb{X}: x\leq 0, y\geq 0, z\geq 0, \frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x-\frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y+\frac{z}{2}\leq 0\}$, $C=\bigcup_{j=0}^{\frac{n-1}{2}}C_j$, $C_j=\{(x,y,z)\in\mathbb{X}: x\leq 0, y\leq 0, z\geq 0, \frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x+\frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y+\frac{z}{2}\leq 0\}$, $D=\bigcup_{j=0}^{\frac{n-1}{2}}D_j$, $D_j=\{(x,y,z)\in\mathbb{X}: x\geq 0, y\leq 0, z\geq 0, -\frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x+\frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y+\frac{z}{2}\leq 0\}$. From the above expressions of the Birkhoff-James orthogonality sets of extreme points of $B_{\mathbb{X}}$, it follows that, for any two extreme points $u,v\in S_{\mathbb{X}}$, $u^{\perp}\cup v^{\perp}\subsetneqq\mathbb{X}$. Therefore $\mathbb{X}$ has Property P_2.
Again, by using the same expressions, we can show that $(1,0,1)^{\perp}\cup(-1,0,1)^{\perp}\cup(0,0,2)^{\perp}=\mathbb{X}$. Therefore, $\mathbb{X}$ does not have Property P_3. This completes the proof of the theorem. □

Finally, we give an example of a linear operator T between a three-dimensional polyhedral Banach space and a three-dimensional polyhedral Banach space having Property P_2, such that T does not satisfy the BŠ Property.

Example 7.15. *Let $\mathbb{X}=\ell_\infty^3$ and let $\mathbb{Y}$ be a three-dimensional polyhedral Banach space such that $B_{\mathbb{Y}}$ is a polyhedron with vertices* $(1,0,\pm1),(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\pm1)$,
$(0,1,\pm1),(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}},\pm1),(-1,0,\pm1),(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}},\pm1),(0,-1,\pm1),(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}},\pm1)$,

$(0,0,\pm 2)$. *Consider a bounded linear operator* $T : \mathbb{X} \to \mathbb{Y}$, *defined by*

$$T(x,y,z) = \Big(\frac{x+y}{2}, \frac{y-x}{2}, y\Big).$$

Then it is easy to check that $\|T\| = 1$, $M_T = \{\pm(1,1,z), \pm(-1,1,z) : z \in [-1,1]\}$ *and* $T(M_T) = \{\pm(1,0,1), \pm(0,1,1)\}$. *Consider* $x_1 = (1,1,1), x_2 = (-1,1,1)$ *and* $x_3 = (-1,-1,1)$. *Clearly* $\{x_1, x_2, x_3\}$ *forms a basis of* $\mathbb{X}$. *If we choose* $\alpha = -10$ *and* $\beta = \frac{-3}{2}$, *then condition (c) of Corollary 7.2 is satisfied. From Theorem 7.14, we know that* $\mathbb{Y}$ *has Property* P_2. *Therefore, by using Corollary 7.2, we conclude that* T *does not satisfy the BŠ Property.*

In view of the methods employed to study the BŠ Property of linear operators and the results obtained in the present chapter, it is perhaps appropriate to end it with the following remark:

Remark 7.16. *We have illustrated the important role played by Property* P_n *in determining the BŠ Property of linear operators. Indeed, using this concept, we have extended the previously obtained results in [62]. It is worth mentioning in this connection that Example 7.4, Example 7.8 and Example 7.15 provided in this article are beyond the scope of the Proposition* 2.1 *of [62]. We note that Property* P_n *is essentially a structural concept, associated especially with polyhedral Banach spaces. Therefore, it might be interesting to further study various polyhedral Banach spaces in light of the newly introduced concept of Property* P_n.

CHAPTER 8

BIRKHOFF-JAMES ORTHOGONALITY OF BOUNDED LINEAR OPERATORS-II

8.1 Introduction

Birkhoff-James orthogonality plays a central role in determining the geometry of normed linear spaces in general, and spaces of operators, in particular. One of the most interesting aspects of Birkhoff-James orthogonality is the relation between orthogonality of operators and that of norming elements in the ground space. The purpose of this chapter is to continue the investigation of a certain property from [50]. Before proceeding further, let us fix the notations and the terminologies.

Letters $\mathbb{X}$ and $\mathbb{Y}$ denote normed linear spaces. Throughout the present chapter, we will assume the underlying scalar field to be $\mathbb{R}$. Let $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$ denote the unit ball and the unit sphere of $\mathbb{X}$, respectively. Let $B[x, r] = \{z \in \mathbb{X} : \|x - z\| \leq r\}$ and $B(x, r) = \{z \in \mathbb{X} : \|x - z\| < r\}$ denote the closed ball and the open ball centered at x and

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radius $r > 0$, respectively. For a subset A of $\mathbb{X}$, let $|A|$ denote the cardinality of A. Let $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ be the normed space of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$, endowed with the usual operator norm. We write $\mathbb{L}(\mathbb{X}, \mathbb{Y}) = \mathbb{L}(\mathbb{X})$, if $\mathbb{X} = \mathbb{Y}$. An element $x(\neq 0)$ is said to be smooth point of $\mathbb{X}$ if there is unique $f \in S_{\mathbb{X}^*}$ such that $f(x) = ||x||$. A normed linear space $\mathbb{X}$ is said to be smooth if every non-zero element of $\mathbb{X}$ is a smooth point. Let $Ext\ B_{\mathbb{X}}$ denote the collection of all extreme points of the unit ball $B_{\mathbb{X}}$. For a bounded linear operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, let M_T denote the norm attainment set of T, i.e. $M_T = \{x \in S_{\mathbb{X}} : \|Tx\| = \|T\|\}$. For any two points $x_1, x_2 \in \mathbb{X}$, $L[x_1, x_2]$ denotes the closed line segment joining x_1 and x_2, i.e. $L[x_1, x_2] = \{(1-t)x_1 + tx_2 : t \in [0, 1]\}$. For $x, y \in \mathbb{X}$, x is said to be orthogonal to y in the sense of Birkhoff-James [20], written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$. Similarly, for $T, A \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, T is said to be Birkhoff-James orthogonal to A, written as $T \perp_B A$, if $\|T + \lambda A\| \geq \|T\|$ for all $\lambda \in \mathbb{R}$. For an element $x \in \mathbb{X}$, by $x^{\perp}$ we mean the collection of all elements $y \in \mathbb{X}$ such that $x \perp_B y$, i.e., $x^{\perp} = \{y \in \mathbb{X} : x \perp_B y\}$. For studying orthogonality of operators between Banach spaces, the following definition from [61] is very helpful. Given $x, y \in \mathbb{X}$, we say that $y \in x^+$ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \geq 0$. Similarly, we say that $y \in x^-$ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \leq 0$. For an immediate application of these notions towards studying bounded linear operators, which is also relevant to the present work, we refer the readers to [60]. In connection to the conjecture proposed by Bhatia and Šemrl, the term "Bhatia-Šemrl (BŠ) Property" was first coined in [62] and then extended in [50]. We mention the same, for the convenience of the readers.

Our main objective is the continuation of the study in [50]. Indeed, we focus on the following problem: If $\mathbb{X}$ is a finite-dimensional Banach space with dim $\mathbb{X} > 2$ and if $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is such that $M_T \neq D \cup (-D)$, where D is a connected subset of $S_{\mathbb{X}}$, then whether T satisfies the BŠ Property or not, for any normed linear space $\mathbb{Y}$. In this connection, Property P_n was introduced in [50].

Definition 8.1. *[50, Defn. 1.6] Let $\mathbb{X}$ be a Banach space. Given $n \in \mathbb{N}$, we say that $\mathbb{X}$ has Property P_n if for every choice of n vectors $x_1, x_2, \ldots, x_n \in S_{\mathbb{X}}$, $\bigcup_{i=1}^{n} x_i^{\perp} \subsetneqq \mathbb{X}$.*

Trivially, if $\mathbb{X}$ has Property P_n then $\mathbb{X}$ has Property P_m for all $m \in \mathbb{N}$, with $m \leq n$. Let us now introduce the definition of BŠ pair which plays a crucial role in the whole scheme of things.

Definition 8.2. *Let $\mathbb{X}, \mathbb{Y}$ be normed linear spaces. We say that the pair $(\mathbb{X}, \mathbb{Y})$ is a BŠ pair if for every $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, T satisfies the BŠ Property if and only if $M_T = D \cup (-D)$, where D is a non-empty connected subset of $S_{\mathbb{X}}$.*

Observe that the existence of BŠ pairs substantiates the Conjecture 1.1 to be true. In this chapter, we investigate operators T which satisfy the BŠ Property. We also exhibit BŠ pairs of spaces $(\mathbb{X}, \mathbb{Y})$. Indeed, we show that $(\ell_1^n, \mathbb{Y})$ is a BŠ pair for any normed linear space $\mathbb{Y}$. This proves the validity of Conjecture 1.1, whenever the domain space is ℓ_1^n. Further, we study the BŠ Property of operators on polyhedral Banach spaces. We also characterize the space ℓ_∞^3 among all 3-dimensional polyhedral Banach spaces having exactly eight extreme points in the unit ball. Recall that a finite-dimensional Banach space $\mathbb{X}$ is said to be polyhedral if $B_{\mathbb{X}}$ has only finitely many extreme points.

8.2 Bhatia-Šemrl Property with norm attainment set of a bounded linear operator

We begin with the following theorem which gives a nice connection between Property P_n and the BŠ Property.

Theorem 8.1. *Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, where dim $\mathbb{X} = n \geq 2$ and $\mathbb{Y}$ has Property P_m, for some $m \geq 2$. If $4 \leq |M_T| \leq 2m$, then T does not satisfy the BŠ Property.*

Proof. Let $M_T = \{\pm x_1, \pm x_2, \ldots, \pm x_k\}$, where $2 \leq k \leq m$. Clearly, as any two elements $x_p, x_q,\ p \neq q,\ p, q \in \{1, 2, \ldots, k\}$, are linearly independent, we can extend $\{x_1, x_2\}$ to a basis $\{x_1, x_2, y_3, \ldots, y_n\}$ of $\mathbb{X}$. Then for each $x_i \in M_T$, $i = 1, 2, \ldots, k$, we can write $x_i = c_1^i x_1 + c_2^i x_2 + c_3^i y_3 + \ldots + c_n^i y_n$, where c_j^i's are real scalars. Claim that we can find n scalars $\alpha_j > 0$, $j = 1, 2, \ldots, n$, such that $c_1^i\alpha_1 + c_2^i\alpha_2 + \ldots + c_n^i\alpha_n \neq 0$ and $c_1^i\alpha_1 - c_2^i\alpha_2 + \ldots + c_n^i\alpha_n \neq 0$, for all $i = 1, 2, \ldots, k$. Otherwise, if $c_1^i\alpha_1 + c_2^i\alpha_2 + \ldots + c_n^i\alpha_n = 0$, then $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ belongs to the hyperspace $H_1^i = \{(z_1, z_2, \ldots, z_n) : c_1^i z_1 + c_2^i z_2 + \ldots + c_n^i z_n = 0\}$ and if $c_1^i\alpha_1 - c_2^i\alpha_2 + \ldots + c_n^i\alpha_n = 0$, then $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ belongs to the hyperspace $H_2^i = \{(z_1, z_2, \ldots, z_n) : c_1^i z_1 - c_2^i z_2 + \ldots + c_n^i z_n = 0\}$. These collections of hyperspaces are finite and so $\mathbb{X} \neq \bigcup_{i=1}^k (H_1^i \cup H_2^i)$. Therefore, our claim is established.

Now, $\bigcup_{x \in M_T} (Tx)^\perp = \bigcup_{i=1}^{k} (Tx_i)^\perp \subsetneqq \mathbb{Y}$, as $\mathbb{Y}$ has property P_m and $k \leq m$. Let us take $z \in \mathbb{Y} \setminus \bigcup_{i=1}^{k} (Tx_i)^\perp$. From [61, Prop. 2.1], it follows that $z \in (Tx_1)^+$ or $z \in (Tx_1)^-$ and $z \in (Tx_2)^+$ or $z \in (Tx_2)^-$. Accordingly we consider the following cases.

Case I: Let $z \in (Tx_1)^+ \cap (Tx_2)^+$ or $z \in (Tx_1)^- \cap (Tx_2)^-$. Let us define a linear operator $A : \mathbb{X} \to \mathbb{Y}$ by

$$Ax_1 = \alpha_1 z, Ax_2 = -\alpha_2 z \text{ and } Ay_j = \alpha_j z \text{ for } j = 3, 4, \ldots, n.$$

If $z \in (Tx_1)^+ \cap (Tx_2)^+$, then $Ax_1 \in (Tx_1)^+$ and $Ax_2 \in (Tx_2)^-$. Also, if $z \in (Tx_1)^- \cap (Tx_2)^-$, then $Ax_1 \in (Tx_1)^-$ and $Ax_2 \in (Tx_2)^+$. In both the cases, it follows from [61, Th. 2.2] that $T \perp_B A$. Clearly $Ax_i = (c_1^i\alpha_1 - c_2^i\alpha_2 + \ldots + c_n^i\alpha_n)z$, for all $i = 1, 2, \ldots, k$. As $c_1^i\alpha_1 - c_2^i\alpha_2 + \ldots + c_n^i\alpha_n \neq 0$ and $Tx_i \not\perp_B z$, for all $i = 1, 2, \ldots, k$, we conclude that $Tx_i \not\perp_B Ax_i$, for all $i = 1, 2, \ldots, k$. Thus T does not satisfy the BŠ Property.

Case II: Let $z \in (Tx_1)^+ \cap (Tx_2)^-$ or $z \in (Tx_1)^- \cap (Tx_2)^+$. Let us define a linear operator $A : \mathbb{X} \to \mathbb{Y}$ by $Ax_1 = \alpha_1 z$, $Ax_2 = \alpha_2 z$ and $Ay_j = \alpha_j z$ for $j = 3, 4, \ldots, n$. Proceeding similarly as in Case I, we can conclude that $T \perp_B A$ but there exists no $x \in M_T$ such that $Tx \perp_B Ax$. Therefore, T does not satisfy the BŠ Property. □

Remark 8.2. *We note that Theorem 8.1 improves [50, Cor. 2.5].*

Our next example illustrates the applicability of Theorem 8.1 in studying the BŠ Property of bounded linear operators.

Example 8.3. *Let $\mathbb{X} = \ell_\infty^n$ and let $\mathbb{Y} = \ell_2^n$. Consider a bounded linear operator $T : \mathbb{X} \to \mathbb{Y}$, defined by*

$$Tx = \frac{x}{\sqrt{n}}, x \in \mathbb{X}.$$

Then it is easy to check that $\|T\| = 1$ and $M_T = Ext\ B_{\mathbb{X}}$. Clearly $\mathbb{Y}$ has Property P_m for any $m \in \mathbb{N}$. Therefore, by using Theorem 8.1, we conclude that T does not satisfy the BŠ Property.

We next present a generalized version of [62, Lemma 2.1], which will be essential for our purpose of studying orthogonality of bounded linear operators.

Lemma 8.1. *Let M be a countable subset of a Banach space $\mathbb{X}$ of dimension $n \geq 2$. Then for any given $m \in \{1, 2, \ldots, n\}$, there exist $(n - m)$ linearly independent vectors $y_{m+1}, y_{m+2}, \ldots, y_n$ such that $\{x_1, x_2, \ldots, x_m, y_{m+1}, y_{m+2}, \ldots, y_n\}$ is a basis of $\mathbb{X}$, whenever $\{x_1, \ldots, x_m\}$ is any linearly independent set in M.*

Proof. For $m = 2$, the proof of the lemma directly follows from the proof of Lemma 2.1 of [62]. All the other cases can be proved similarly. □

We next obtain another class of operators not satisfying the BŠ Property.

Theorem 8.4. *Let $\mathbb{X}$ be an n-dimensional Banach space and let $\mathbb{Y}$ be any smooth Banach space. Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ be such that M_T satisfies the following conditions:*

(1) M_T has more than two and countably many components.

(2) M_T has at most two non-singleton components $\pm D_i$, for some $i \in \mathbb{N}$. If $D_i = -D_i$, then M_T

has exactly one non-singleton component. All other components of M_T are singleton.
(3) *M_T contains at least one pair of singleton components $\pm D_j$, $j(\neq i) \in \mathbb{N}$ such that $\pm D_j \cap span\{D_i\} = \phi$.*
Then T does not satisfy the BŠ Property.

Proof. If M_T does not contain any non-singleton component, then the desired result follows from [50, Th. 2.7]. Without loss of generality, we assume that $M_T = \bigcup\limits_{\substack{j=1\\ j\neq i}}^{\infty}(\pm D_j)\bigcup(\pm D_i)$, where D_i is the non-singleton component and D_j, $j \neq i$, are the singleton components of M_T. Without loss of generality we assume that $\pm D_1$ are the non-singleton components and $\pm D_2$ are the singleton components such that $\pm D_2 \cap span\{D_1\} = \phi$. Let us assume that dim $(span\ D_1) = l$, where $l < n$. Let $x_1, x_2, \ldots, x_l \in D_1$ be linearly independent and let $\pm D_2 = \{\pm x_{l+1}\} \subseteq S_{\mathbb{X}}$. We would like to apply Lemma 8.1 in our present setting. Let $M = \{x_1, x_2, \ldots, x_l\} \cup \{z \in M_T : z \notin span\{x_1, x_2, \ldots, x_l\}\} = P \cup Q$, where $P = \{x_1, x_2, \ldots$
$\ldots, x_l\}$ and $Q = \{z \in M_T : z \notin span\{x_1, x_2, \ldots, x_l\}\}$. Clearly M is a countable set, as $\pm D_1 \subseteq span\{x_1, x_2, \ldots, x_l\}$ and $M_T \setminus \{\pm D_1\}$ is a countable set. Let $m = l+1$. Therefore, by Lemma 8.1, we can fix $n-(l+1)$ elements $z_{l+2}, z_{l+3}, \ldots, z_n \in \mathbb{X}$ such that $\{x_1, x_2, \ldots, x_l, z, z_{l+2}, z_{l+3}, \ldots, z_n\}$ is a basis of $\mathbb{X}$ for all $z \in Q$. Then $\{x_1, x_2, \ldots, x_l,$
$x_{l+1}, z_{l+2}, z_{l+3}, \ldots, z_n\}$ is a basis of $\mathbb{X}$ and so we can fix scalars $c_{x,k}(x \in \mathbb{X}, 1 \leq k \leq n)$ such that for each $x \in \mathbb{X}$,

$$x = \sum_{i=1}^{l+1} c_{x,i} x_i + \sum_{j=l+2}^{n} c_{x,j} z_j.$$

Also it follows that if $z \in Q$, then $c_{z,l+1} \neq 0$. For each $v \in \mathbb{Y}$, let A_v be the linear operator defined by

$$\begin{aligned} A_v x_k &= T x_k,\ k = 1, 2, \ldots, l, \\ A_v x_{l+1} &= v, \\ \text{and } A_v z_k &= T z_k,\ k = l+2, l+3, \ldots, n. \end{aligned}$$

We will show that there is a non-zero $v \in \mathbb{Y}$ such that $T \perp_B A_v$ but $Tx \not\perp_B A_v x$ for each $x \in M_T$. For any $\lambda \geq 0$ and $v \in B(-Tx_{l+1}, \|T\|)$, we have

$$\|T + \lambda A_v\| \geq \|(T + \lambda A_v)x_1\| = \|(1+\lambda)Tx_1\| = (1+\lambda)\|T\| \geq \|T\|$$

and

$$\begin{aligned}
\|T-\lambda A_v\| \geq \|(T-\lambda A_v)x_{l+1}\| &= \|Tx_{l+1}-\lambda v\| \\
&\geq \|(1+\lambda)Tx_{l+1}-\lambda(Tx_{l+1}+v)\| \\
&\geq (1+\lambda)\|Tx_{l+1}\|-\lambda\|(Tx_{l+1}+v)\| \\
&\geq (1+\lambda)\|T\|-\lambda\|T\| = \|T\|.
\end{aligned}$$

Therefore, $T\perp_B A_v$ for each $v \in B(-Tx_{l+1}, \|T\|)$. We next show that there is at least one $v \in B(-Tx_{l+1}, \|T\|)$ such that $Tx \not\perp_B A_v x$ for each $x \in M_T$. Firstly, $Tx \not\perp_B A_v x$ for each $x \in M_T \cap span\{x_1, x_2, \ldots, x_l\}$, since $A_v x = Tx \neq 0$ for all $x \in M_T \cap span\{x_1, x_2, \ldots, x_l\}$. Let $x \in M_T \setminus span\{x_1, x_2, \ldots, x_l\}$, i.e., $x \in Q$. Define $H_x = \{y \in \mathbb{Y} : Tx\perp_B y\}$. By smoothness of $\mathbb{Y}$, H_x is an unique closed hyperspace of $\mathbb{Y}$. Hence H_x is nowhere dense set of $\mathbb{Y}$. Put $P_x = \{v \in \mathbb{Y} : A_v x \in H_x\}$. Since $A_v x = c_{x,1}Tx_1 + c_{x,2}Tx_2 + \ldots + c_{x,l+1}v + c_{x,l+2}Tz_{l+2} + \ldots + c_{x,n}Tz_n$ and $c_{x,l+1} \neq 0$, we have

$$P_x = \frac{1}{c_{x,l+1}}\left(H_x - (c_{x,1}Tx_1 + c_{x,2}Tx_2 + \ldots + c_{x,l+2}Tz_{l+2} + \ldots + c_{x,n}Tz_n)\right),$$

for each $x \in Q$. The set $P_x = \{v \in \mathbb{Y} : Tx\perp_B A_v x\}$, being homeomorphic to H_x, is also nowhere dense. As Q is countable, it follows from Baire category theorem that the non-empty open set $B(-Tx_{l+1}, \|T\|)$ contains an element v such that $v \notin P_x$ for each $x \in Q$. Thus we found $v \in B(-Tx_{l+1}, \|T\|)$ such that $T\perp_B A_v$ but $Tx \not\perp_B A_v x$ for all $x \in M_T$. Hence T does not satisfy the BŠ Property. □

Remark 8.5. *We note that Theorem 8.4 improves on [50, Th. 2.7].*

For linear operators between a polyhedral Banach space and a smooth Banach space, we have the following corollary.

Corollary 8.1. *Let $\mathbb{X}$ be an n-dimensional polyhedral Banach space and let $\mathbb{Y}$ be any smooth Banach space. Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ be such that M_T satisfies the following conditions:*
(1) *M_T has more than two components.*
(2) *M_T has at most two non-singleton components $\pm D_i$, for some $i \in \mathbb{N}$. If $D_i = -D_i$, then M_T has exactly one non-singleton component. All the other components of M_T are singleton.*
(3) *M_T contains at least one pair of singleton components $\pm D_j$, $j(\neq i) \in \mathbb{N}$ such that $\pm D_j \cap span\{D_i\} = \phi$.*
Then T does not satisfy the BŠ Property.

Proof. We note that in a finite-dimensional polyhedral Banach space $\mathbb{X}$, $B_{\mathbb{X}}$ contains finitely many extreme points and each component of M_T contains extreme points of $B_{\mathbb{X}}$. Therefore, M_T

has finitely many components. Thus M_T satisfies all the conditions of Theorem 8.4 and so, T does not satisfy the BŠ Property. $\square$

8.3 Classification of spaces which are BŠ pairs

In the remaining section of this chapter we obtain some of the pairs of spaces which satisfied the condition to be BŠ pairs. First we present one of the main results of the chapter that shows that $\mathbb{X} = \ell_1^n$ acts as universal domain space for the pair $(\mathbb{X}, \mathbb{Y})$ to be a BŠ pair.

Theorem 8.6. *Given any normed linear space $\mathbb{Y}$, the pair $(\ell_1^n, \mathbb{Y})$ is a BŠ pair.*

Proof. If $\dim \mathbb{Y} = 1$ then given any $T \in \mathbb{L}(\ell_1^n, \mathbb{Y})$, it is easy to check that $M_T = D \cup (-D)$, where D is a connected subset of $S_{\ell_1^n}$. From this it follows that the pair $(\ell_1^n, \mathbb{Y})$ is a BŠ pair. Let us assume that $\dim \mathbb{Y} > 1$. Let $T \in \mathbb{L}(\ell_1^n, \mathbb{Y})$ be such that $M_T \neq D \cup (-D)$, where D is a connected subset of $S_{\ell_1^n}$. We prove that T does not satisfy the BŠ Property. Since M_T is not of the form $D \cup (-D)$, it must be of the form $M_T = \left(\bigcup\limits_{i=1}^{k} D_i' \right) \cup \left(\bigcup\limits_{j=1}^{l} E_j \right)$, where D_i and E_j are components of M_T, $D_i' = D_i \cup (-D_i)$, $E_j = -E_j$ and $D_i \neq -D_i$. Let us now complete the proof of the theorem by considering the following three exhaustive cases.

Case I: $l = 0$. If $k = 1$, then $M_T = D_1 \cup (-D_1)$, which contradicts our hypothesis. So we assume $k \geq 2$. Then $M_T = \bigcup\limits_{i=1}^{k} D_i'$. Since D_i is a connected subset of $S_{\mathbb{X}}$, where $\mathbb{X} = \ell_1^n$, an easy application of the Krein-Milman theorem shows that each D_i must contain at least one extreme point of $B_{\mathbb{X}}$. Let $D_i \cap \mathrm{E}_{\mathbb{X}} = \{e_{i1}, e_{i2}, \ldots e_{im_i}\}$, $1 \leq i \leq k$, i.e., $\left| D_i \cap \mathrm{E}_{\mathbb{X}} \right| = m_i$. Clearly, $\sum\limits_{i=1}^{k} m_k \leq n$. Let us write $X_1 = span\{e_{11}, e_{12}, ..., e_{1m_1}\}$ and $X_2 = span\{e_{21}, e_{22}, ..., e_{2m_2}, ..., e_{k1}, e_{k2}, .., e_{km_k}\}$. It is easy to see that $D_1 \cup (-D_1) = D_1' \subseteq X_1$ and $\bigcup\limits_{i=2}^{k} D_i' \subseteq X_2$. Thus we have $M_T \subseteq X_1 \cup X_2$ and moreover it is immediate that $X_1 \cap X_2 = \{\theta\}$, as $\mathbb{X} = \ell_1^n$. Therefore, M_T is partitioned into two non-empty subsets $Y_1 = M_T \cap X_1$ and $Y_2 = M_T \cap X_2$ of $\mathbb{X}$, which are contained in complementary subspaces of $\mathbb{X}$. Then by [62, Prop. 2.1], we conclude that T does not satisfy the BŠ Property.

Case II: $k = 0$. If $l = 1$, then $M_T = E_1$, a contradiction to our hypothesis. So we assume $l \geq 2$. In this case $M_T = \bigcup_{i=1}^{l} E_j$. Proceeding similarly as in Case I, we can show that that T does not satisfy BŠ property.

Case III: $l \geq 1, k \geq 1$. In this case $M_T = \left(\bigcup_{i=1}^{k} D'_i\right) \cup \left(\bigcup_{j=1}^{l} E_j\right)$ and once again proceeding as above, we conclude that T does not satisfy the BŠ property.

Thus, in all the possible cases, we can conclude that $T \in \mathbb{L}(\ell_1^n, \mathbb{Y})$ satisfies the BŠ Property if and only if $M_T = D \cup (-D)$, where D is a connected subset of $S_{\ell_1^n}$. $\square$

Remark 8.7. *Observe that ℓ_1^n is the unique (upto isometric isomorphisms) n-dimensional Banach space having the minimum possible number of extreme points of its unit ball. This is the fundamental reason that the above theorem works exclusively for ℓ_1^n spaces.*

In the next theorem, we obtain another class of BŠ pairs of Banach spaces, when the domain space is ℓ_∞^3. For the sake of convenience of the reader, we give the definition of adjacent edges of the unit sphere of a finite-dimensional polyhedral Banach space.

Definition 8.3. *Let $\mathbb{X}$ be finite-dimensional polyhedral Banach space. Two edges E_1, E_2 of $S_{\mathbb{X}}$ are said to be adjacent if $E_1 \cap E_2 = \{v\}$, where v is an extreme point of $B_{\mathbb{X}}$. Similarly, the edges $E_1, E_2, \ldots E_n$ are said to be adjacent if $E_1 \cap E_2 \cdots \cap E_n = \{v\}$. An extreme point of $B_{\mathbb{X}}$ is also called a vertex of $S_{\mathbb{X}}$. If v is a vertex and $v \in E_1 \cap E_2 \cdots \cap E_n$, then we also say that the edges $E_1, E_2, \ldots E_n$ are adjacent to the vertex v.*

Theorem 8.8. *Given any strictly convex and smooth Banach space $\mathbb{Y}$, the pair $(\ell_\infty^3, \mathbb{Y})$ is a BŠ pair.*

Proof. Observe that $B_{\ell_\infty^3}$ has eight vertices $\pm v_1 = \pm(1,1,1), \pm v_2 = \pm(-1,1,1)$, $\pm\, v_3 = (-1,-1,1)$ and $\pm v_4 = \pm(1,-1,1)$ and twelve edges $\pm E_{12} = \pm L[v_1, v_2]$, $\pm\, E_{23} = \pm L[v_2, v_3], \pm E_{34} = \pm L[v_3, v_4], \pm E_{41} = \pm L[v_4, v_1], \pm E_{1(-3)} = \pm L[v_1, -v_3]$, $\pm\, E_{2(-4)} = \pm L[v_2, -v_4]$. We prove that if M_T is not of the form $D \cup (-D)$, where D is a connected subset of $S_{\ell_\infty^3}$, then T does not satisfy the BŠ Property. Given any $T \in \mathbb{L}(\ell_\infty^3, \mathbb{Y})$, if M_T is not of the form $D \cup (-D)$, then it is easy to see that M_T must be one of the following forms:

(i) M_T contains exactly two pairs of vertices of $B_{\ell_\infty^3}$ and no other points of $B_{\ell_\infty^3}$.
(ii) M_T contains exactly three pairs of vertices of $B_{\ell_\infty^3}$ and no other points of $B_{\ell_\infty^3}$.
(iii) M_T contains exactly four pairs of vertices of $B_{\ell_\infty^3}$ and no other points of $B_{\ell_\infty^3}$.
(iv) M_T contains exactly one pair of vertices and exactly one pair of edges of $B_{\ell_\infty^3}$ such that the vertices do not belong to the concerned edges. M_T contains no other points of $B_{\ell_\infty^3}$.
(v) M_T contains exactly two pairs of vertices and exactly one pair of edges of $B_{\ell_\infty^3}$ such that the vertices do not belong to any of the concerned edges. M_T contains no other points of $B_{\ell_\infty^3}$.
(vi) M_T contains exactly two pairs of non-adjacent edges of $B_{\ell_\infty^3}$ and no other points of $B_{\ell_\infty^3}$.

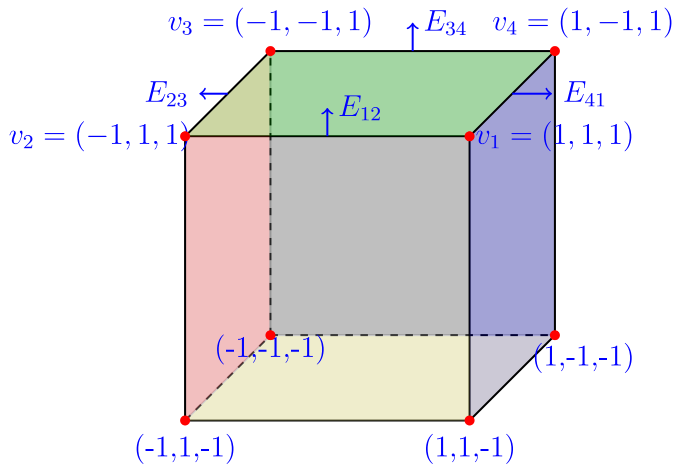

Figure 8.1: Unit sphere of ℓ_∞^3

If M_T is of the form described in either of the Cases $(i),(ii),(iii)$, then T does not satisfy the BŠ Property and the proof of it follows directly from [50, Th. 2.7]. Also, if M_T is of the form described in either of the Cases $(iv),(v)$, then T does not satisfy the BŠ Property and the proof of it follows directly form Corollary 8.1. Here we only consider the Case (vi). Without loss of generality, we may assume that $M_T = \pm E_{12} \cup \pm E_{34}$, for some $T \in \mathbb{L}(\ell_\infty^3, \mathbb{Y})$. As $\mathbb{Y}$ is strictly convex, we must have $T(\pm E_{12}) = \pm u_1$, $T(\pm E_{34}) = \pm u_2$, where $u_1, u_2 \in \mathbb{Y}$ are linearly independent and $\|u_1\| = \|u_2\|$. Now we define a linear operator $A : \ell_\infty^3 \to \mathbb{Y}$ by $A(\pm E_{12}) = T(\pm E_{12}) = \pm u_1$, $A(\pm v_3) = -T(\pm v_3) = \mp u_2$. Then $A(\pm E_{34}) = -T(\pm E_{34}) = \mp u_2$, since $v_4 = v_1 - v_2 + v_3$. Here $Av_1 \in (Tv_1)^+$ and $Av_3 \in (Tv_3)^-$. Therefore, using [61, Th. 2.2] we get $T \perp_B A$. From the construction of the operator A, it is clear that $Tx \not\perp_B Ax$, for all $x \in M_T$. This completes the proof of the theorem. $\square$

In order to obtain further examples of BŠ pairs of polyhedral Banach spaces $(\mathbb{X}, \mathbb{Y})$, we require the following lemma.

Lemma 8.2. *Let $\mathbb{X}$ be any finite-dimensional Banach space and let $\mathbb{Y}$ be any polyhedral Banach space such that $B_{\mathbb{Y}}$ has exactly $2m$ facets. Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ be such that M_T is not of the form $D \cup (-D)$, where D is a connected subset of $S_{\mathbb{X}}$. Then M_T can have at most $2m$ components.*

Proof. Suppose on the contrary that M_T has $(2m+1)$ components, say $D_1, D_2, \ldots, D_{2m+1}$. Let us consider a subset $\{x_1, x_2, \ldots, x_{2m+1}\}$ of M_T, where $x_i \in D_i$ for $i = 1, 2, \ldots, 2m+1$. Let $Tx_1 \in F$, where F is a facet of $B_{\mathbb{Y}}$. Then we must have $Tx_i \notin F$ for $i \in \{2, 3, \ldots, 2m+1\}$. If not then, $(1-\lambda)x_1 + \lambda x_i \in M_T$, for all $\lambda \in [0,1]$, contradicting the fact that D_1 and $D_i (i > 1)$

are distinct components of M_T. Thus Tx_i and Tx_j $(i \neq j)$ can not belong to the same facet of $B_{\mathbb{Y}}$. Let us denote the facets of $B_{\mathbb{Y}}$ as $F_1, F_2, \ldots, F_{2m}$ such that $Tx_i \in F_i$ for $i = 1, 2, \ldots, 2m$. Thus Tx_{2m+1} can not belong to any facet of $B_{\mathbb{Y}}$, which is a contradiction to the fact that $x_{2m+1} \in M_T$. This completes the proof of the lemma. $\square$

Using the above lemma, we obtain the following theorem:

Theorem 8.9. *$(\ell_\infty^3, \ell_\infty^2)$ is a BŠ pair.*

Proof. $B_{\ell_\infty^3}$ has eight vertices $\pm v_1 = \pm(1,1,1), \pm v_2 = \pm(-1,1,1), \pm v_3 = (-1,-1,1)$ and $\pm v_4 = \pm(1,-1,1)$ and twelve edges $\pm E_{12} = \pm L[v_1, v_2], \pm E_{23} = \pm L[v_2, v_3], \pm E_{34}$ $= \pm L[v_3, v_4], \pm E_{41} = \pm L[v_4, v_1], \pm E_{1(-3)} = \pm L[v_1, -v_3], \pm E_{2(-4)} = \pm L[v_2, -v_4]$. Let $T \in \mathbb{L}(\ell_\infty^3, \ell_\infty^2)$ be such that M_T is not of the form $D \cup (-D)$, where D is a connected subset of $S_{\mathbb{X}}$. Then by Lemma 8.2, M_T must be one of the following forms:

(i) M_T contains exactly two pairs of vertices of $B_{\ell_\infty^3}$ and no other points of $B_{\ell_\infty^3}$.
(ii) M_T contains exactly one pair of edges and exactly one pair of vertices of $B_{\ell_\infty^3}$ such that the concerned vertices do not belong to the concerned edges. M_T contains no other points of $B_{\ell_\infty^3}$.
(iii) M_T contains exactly two pairs of edges of $B_{\ell_\infty^3}$ and no other points of $B_{\ell_\infty^3}$.

If M_T is of the form described in either of the Cases (i) and (ii), then T does not satisfy the BŠ Property and the proof of it follows directly from [62, Prop. 2.1]. We only consider the case (iii) in which M_T contains exactly two pairs of edges of $B_{\ell_\infty^3}$. Without loss of generality, we may and do assume that $M_T = \pm E_{12} \cup \pm E_{34}$. Clearly, we have $T(M_T) \cap E_{\ell_\infty^2} = \phi$ and hence $Tx \in Sm(S_{\ell_\infty^2})$ for any $x \in M_T$, where $Sm(S_{\ell_\infty^2})$ denotes the collection of all smooth points of $S_{\ell_\infty^2}$. As $E_{12} \subseteq M_T$ and $E_{34} \subseteq M_T$, we have Tv_1 and Tv_2 belong to the same edge of $B_{\ell_\infty^2}$ and also Tv_3 and Tv_4 belong to the same edge of $B_{\ell_\infty^2}$. Let us define an operator $A \in \mathbb{L}(\ell_\infty^3, \ell_\infty^2)$ as follows:

$$Av_1 = Tv_1, Av_2 = Tv_1, Av_3 = -Tv_3.$$

Clearly, $Av_1 \in (Tv_1)^+$ and $Av_3 \in (Tv_3)^-$. Therefore, using [61, Th. 2.2], we get that $T \perp_B A$. Now, we have $A(E_{12}) = Tv_1$ and $A(E_{34}) = -Tv_3$, as $v_4 = v_1 - v_2 + v_3$. Then it is easy to check that $Tu \not\perp_B Au$ for any $u \in M_T$. Hence T does not satisfy the BŠ Property.
Therefore, the pair $(\ell_\infty^3, \ell_\infty^2)$ is a BŠ pair. $\square$

Using similar arguments, we can also prove the next result, the proof of which is omitted as it follows in similar spirit to the above theorem.

Theorem 8.10. *$(\ell_\infty^3, \ell_\infty^3)$ is a BŠ pair.*

Remark 8.11. *Given $m, n \in \mathbb{N}$ such that $m, n > 3$, it is still unknown whether the pair $(\ell_\infty^m, \ell_\infty^n)$ is a BŠ pair.*

We would like to end the chapter with a related result that characterizes the unit cube among all 3-dimensional convex polyhedrons having eight vertices. This is of independent interest and illustrates the connection between operator norm attainment and geometries of the domain space and the co-domain space.

Theorem 8.12. *Let $\mathbb{X}$ be a three-dimensional polyhedral Banach space such that $B_{\mathbb{X}}$ has exactly eight vertices. Then $\mathbb{X}$ is isometrically isomorphic with ℓ_∞^3 if and only if given any strictly convex Banach space $\mathbb{Y}$ and given any two pairs of non-adjacent edges $\pm E_1$ and $\pm E_2$ of $S_{\mathbb{X}}$, there is a rank two linear operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ such that $M_T = \pm E_1 \cup \pm E_2$.*

Proof. Let us first prove the necessary part of the theorem. We use the notations for the vertices and the edges of $B_{\ell_\infty^3}$ as used in Theorem 2.4 (see Figure 1). Now for any two pairs of non-adjacent edges of $B_{\ell_\infty^3}$, the following three cases may arise:

(i) $\pm E_{12} = \pm L[v_1, v_2]$ and $\pm E_{34} = \pm L[v_3, v_4]$.
(ii) $\pm E_{14} = \pm L[v_1, v_4]$ and $\pm E_{23} = \pm L[v_2, v_3]$.
(iii) $\pm E_{1(-3)} = \pm L[v_1, -v_3]$ and $\pm E_{2(-4)} = \pm L[v_2, -v_4]$.

We only consider the Case (i), as the other two cases will follow similarly. Define a linear operator $T : \ell_\infty^3 \to \mathbb{Y}$ by $T(v_1) = T(v_2) = u_1$, $T(v_3) = u_2$, where $u_1, u_2 \in S_{\mathbb{Y}}$ are linearly independent. So, we have $T(\pm E_{12}) = \pm u_1$ and $T(\pm E_{34}) = \pm u_2$. Hence it can be easily shown that $\|T\| = 1, M_T = \pm E_{12} \cup \pm E_{34}$ and T is a rank two linear operator.

Next we prove the sufficient part of the theorem. Let $\pm v_1, \pm v_2, \pm v_3, \pm v_4$ be the eight vertices of $B_{\mathbb{X}}$. Using the Krein-Milman theorem, we conclude that the set $\{v_1, v_2, v_3, v_4\}$ contains a basis of $\mathbb{X}$. Therefore, the following two cases arise in this context:

$Case(i)$. Any three elements of the set $\{v_1, v_2, v_3, v_4\}$ are linearly independent.
$Case(ii)$. Case (i) is not satisfied.

Case (i) : Let $\{v_1, v_2, v_3\}$ be a basis of $\mathbb{X}$ and let $v_4 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, where $\alpha_i \in \mathbb{R}, i = 1, 2, 3$. Then each α_i is non-zero, as any three elements of the set $\{v_1, v_2, v_3, v_4\}$ are linearly independent. In a three-dimensional polyhedral Banach space every vertex has at least two adjacent edges. Let $\pm E_1 = \pm L[v_1, v_2]$ and $\pm E_2 = \pm L[v_3, v_4]$. Let T_1 be a rank two linear operator such that $M_{T_1} = \pm E_1 \cup \pm E_2$. As $\mathbb{Y}$ is a strictly convex Banach space and $\pm E_1 \subseteq M_{T_1}$, $T_1(\pm E_1) = \pm u_1$ for some non-zero $u_1 \in \mathbb{Y}$. Also $T_1(\pm E_2) = \pm u_2$, for some non-zero $u_2 \in \mathbb{Y}$. As T_1 is of rank two, u_1, u_2 are linearly independent. As $v_4 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, we have $u_2 = T_1(v_4) = (\alpha_1 + \alpha_2)u_1 + \alpha_3 u_2$. Therefore, $(1-\alpha_3)u_2 = (\alpha_1+\alpha_2)u_1$. So we must have $\alpha_1 = -\alpha_2$ and $\alpha_3 = 1$, as u_1, u_2 are linearly independent. Hence $v_4 = \alpha_1(v_1 - v_2) + v_3$. Without loss of generality, we assume that $L[v_1, v_3], L[v_2, v_4]$ are two edges of $B_{\mathbb{X}}$. Now for this two pairs of non-adjacent edges $\pm E_1^{'} = \pm L[v_1, v_3]$ and $\pm E_2^{'} = \pm L[v_2, v_4]$, there is a rank two linear operator $T_2 \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ such that $M_{T_2} = \pm E_1^{'} \cup \pm E_2^{'}$. In a similar argument like above, we have $T_2(\pm E_1^{'}) = u_1^{'}$ for some non-zero $u_1^{'} \in \mathbb{Y}$ and $T_2(\pm E_2^{'}) = u_2^{'}$ for some non-zero $u_2^{'} \in \mathbb{Y}$. Therefore, we have $u_2^{'} = T_2(v_4) = \alpha_1 u_1^{'} - \alpha_1 u_2^{'} + u_1^{'}$. So $\alpha_1 = -1$, as $u_1^{'}, u_2^{'}$ are linearly independent. Hence $v_4 = -v_1 + v_2 + v_3$. Now we define a linear map $S : \mathbb{X} \to \ell_\infty^3$ by

$$S(v_1) = (1,1,1), S(v_2) = (-1,1,1), S(v_3) = (1,1,-1).$$

Then $S(v_4) = (-1,1,-1)$. Clearly, S is an isomorphism, as it maps a basis to a basis and $\|S\| = 1$, as $S(Ext\ B_{\mathbb{X}}) = Ext\ B_{\ell_\infty^3}$. Also $\|S^{-1}\| = 1$, as $S^{-1}(Ext\ B_{\ell_\infty^3}) = Ext\ B_{\mathbb{X}}$. Thus we have

$$\|x\| = \|S^{-1}S(x)\| \leq \|S(x)\| \leq \|x\|.$$

Therefore, S is an isometric isomorphism.

Case (ii) : Without loss of generality, we assume that v_1, v_2, v_3 are linearly dependent. Consider the subspace spanned by v_1, v_2. Let $\mathbb{Z} = span\{v_1, v_2\}$. Then $\dim \mathbb{Z} = 2$ and $\pm v_4 \notin \mathbb{Z}$, otherwise $\dim \mathbb{X} = 2$, which is a contradiction. Now, $\pm v_1, \pm v_2, \pm v_3 \in S_{\mathbb{X}} \cap \mathbb{Z}$. Therefore, $B_{\mathbb{X}}$ is of the form of hexagonal pyramid, where $\pm v_1, \pm v_2, \pm v_3$ are the vertices of the hexagonal base. Now for any two edges E_1 and E_2 on the hexagonal base, either E_1 and E_2 are adjacent or E_1 and $-E_2$ are adjacent. Therefore, two pairs of non-adjacent edges $\pm E_1$ and $\pm E_2$ are not possible from hexagonal base of $B_{\mathbb{X}}$. Also any two edges, which are not in the subspace $\mathbb{Z}$, are adjacent as they have a common vertex, either v_4 or $-v_4$. Hence there is only one possibility for two pairs of non-adjacent edges, one edge is in the two-dimensional subspace $\mathbb{Z}$ and one edge is not in the two-dimensional subspace $\mathbb{Z}$. Without loss of generality, we assume $\pm E_1 = \pm L[v_1, v_2]$ and $\pm E_2 = \pm L[v_3, v_4]$. Now we claim that for any operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, with $\pm E_1 \cup \pm E_2 \subseteq M_T$, T is a rank one linear operator. As $\mathbb{Y}$ is strictly convex and $\pm E_1 \subseteq M_T$, we must have $T(\pm E_1) = \pm u_1$ for some non-zero $u_1 \in \mathbb{Y}$. Also for similar reason $T(\pm E_2) = \pm u_2$ for some non-zero $u_2 \in \mathbb{Y}$. As v_1, v_2, v_3 are linearly dependent, we have $v_3 = \alpha_1 v_1 + \alpha_2 v_2$, for some non-zero $\alpha_1, \alpha_2 \in \mathbb{R}$. Then $T(v_3) = (\alpha_1 + \alpha_2)u_1$. As $\|Tv_3\| = \|T\| = \|u_1\|$, we must have

$|\alpha_1 + \alpha_2| = 1$. Hence $u_2 = \pm u_1$. Therefore, T is a rank one linear operator. So there dost not exist any rank two linear operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ such that $M_T = \pm E_1 \cup \pm E_2$, where $\pm E_1$ and $\pm E_2$ are any two pairs of non-adjacent edges and hence Case-(ii) is not possible.

This establishes the theorem. $\square$

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