Symmetry Analysis in Differential Equations and Application to Cosmology

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**Symmetry Analysis in Differential Equations and Application to Cosmology**" submitted by **Roshni Bhaumik**, who got her name registered on August 8, 2019 (Index No: 7/19/Maths./26) for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon her own work under my supervision and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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DECLARATION BY THE AUTHOR

I hereby declare that the thesis is based on my own work carried out at the Department of Mathematics, Jadavpur University, Kolkata- 700032, India. Also, I declare that, no part of it has not been submitted for any degree/diploma/some other qualification at any other University.

All the figures presented in this thesis have been produced by the author using Maple software. The thesis has been checked several times with extreme care to free it from all discrepancies and typos. Even then the vigilant readers may find some mistakes, and, several portions of this thesis may seem unwarranted or mistaken or incorrect. The author takes the sole responsibility for these unwanted errors which have resulted from her inadequate knowledge in the subject or escaped her notice.

Finally, I state that, to the best of my knowledge, all the assistance taken to prepare this thesis have been properly cited and acknowledged.

Roshni Bhaumik.

Roshni Bhaumik

In memory of my Grandfather

Janaki Nath Bhaumik,

and my Grandmother

Angur Bala Bhaumik

ABSTRACT

The thesis consists of eight chapters. First chapter contains the introduction about symmetry analysis while in next six chapters my research works have been described. In this thesis we have mainly studied about the Noether's theorem and its applications. Here Noether symmetry analysis has been used to determine the classical solutions of various cosmological models. Here we have also discussed the Quantum cosmology for various dark energy models. Solving the Wheeler DeWitt equation we have found the wave function of the Universe for given cosmological models.

- In second chapter, multiscalar field cosmological model has been studied using Noether symmetry approach. In this cosmological model, two minimally coupled scalar fields have been studied.

- Third chapter contains the Chameleon field cosmological model which has been studied using Noether symmetry approach. Also quantum cosmology for this model has been studied.

- In the fourth chapter, $f(T)$-gravity theory has been discussed. Both classical and quantum cosmology have been studied here. Here Big-Bang singularity may be avoided for the present $f(T)$-cosmological model.

- Fifth chapter is actually the extension of the second chapter where quantum cosmology has been studied for the double scalar field cosmological model.

- The sixth chapter contains contains the Einstein-Skyrme model where the Wheeler DeWitt equation is constructed for studying quantum

cosmology of the system and the wave function of the Universe is evaluated using the conserved Noether charge.

- Seventh chapter analyze the Noether symmetry for a cosmological model with variable G and Λ. In this model we have discussed about the solution of the matter-dominated cosmological equations.

Finally, the last chapter contains the brief discussion and future prospects of my work.

PREFACE

The work of this thesis has been carried out at the Department of Mathematics, Jadavpur University, Kolkata- 700032, India. The thesis is based on the following published papers:

- **CHAPTER 2** has been published as "*Multi-scalar field cosmological model and possible solutions using Noether symmetry approach*", Santu Mondal, **Roshni Bhaumik**, Sourav Dutta, Subenoy Chakraborty , **Mod.Phys.Lett.A 36** (2021) 34, 2150246.

- **CHAPTER 3** has been published as "*Noether symmetry analysis in chameleon field cosmology*", **Roshni Bhaumik**, Sourav Dutta, Subenoy Chakraborty, **Int.J.Mod.Phys.A 37** (2022) 05, 2250018.

- **CHAPTER 4** has been published as "*Classical and quantum cosmology in f(T)-gravity theory: A Noether symmetry approach*", **Roshni Bhaumik**, Sourav Dutta, Subenoy Chakraborty, **Int. J. Geom. Meth. Mod. Phys. 19**(2022) 02, 2250027.

- **CHAPTER 5** has been published as "*Quantum Cosmology for double scalar field cosmological model: Symmetry Analysis*", **Roshni Bhaumik**, Sourav Dutta, Subenoy Chakraborty, **Int.J.Mod.Phys.A 37**(2022) 24, 2250154

- **CHAPTER 6** has been communicated as "*Geometric symmetries of the physical space of the Einstein-Skyrme model and quantum cosmology: A Noether symmetry analysis*",

Roshni Bhaumik, Muthusamy Lakshmanan, Subenoy Chakraborty, **Mod.Phys.Lett.A**
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- **CHAPTER 7** has been published as "*Study of Noether Symmetry Analysis for a Cosmological model with Variable G and Λ Gravity Theory*", **Roshni Bhaumik**, Sourav Dutta, Subenoy Chakraborty, **Eur.Phys.J.C 82** (2022) 12, 1124 .

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Roshni Bhaumik.

Roshni Bhaumik

"Don't take rest after your first victory because if you fail in second, more lips are waiting to say that your first victory was just luck ".

Dr. A.P.J. Abdul Kalam

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CHAPTER 1

INTRODUCTION

1.1 Symmetries and Differential Equations

A symmetry can be defined as an exact correspondence in position or shape with respect to a point or line or plane. The word "Symmetry" came from a Latin word "*Symmetria*". Actually symmetry is nothing but an operation which leaves invariant that upon which it acts. The idea of symmetry can be associated with two mathematician Felix Klein and Sophus Lie. They had developed the necessary mathematical tools needed to explain the notion.

To solve some physical problems, differential equations act a major role. But sometimes we encounter with some difficult differential equations which can not be solved easily. Symmetry analysis can be used to solve different types of differential equations. In the late 19th century famous mathematician Sophus Lie had developed Lie symmetry analysis. In 1888, his pioneering work was published with the title "*Theory of transformation groups*" which leads to walk on a new path of Mathematics: the Symmetries. Undoubtedly Lie theory is an important mathematical tool in modern science. In this symmetry analysis one determines a canonical co-ordinate system for a given differential equation and transforms the equation in terms of new co-ordinate system to make the equation simpler to solve.

In General Theory of Relativity (GTR), there are ten non-linear second order partial differential equations (gravitational field equations) which are very difficult to solve exactly. Symmetry analysis can be used to reduce the number of independent variables or the number of independent equations. However, Lie symmetries are restricted to the space of independent variables while there may exist continuous transformation in the space of dependent variables. As a result field equations become invariant.

On the other hand, from the last century symmetry analysis is reigning in the field of global continuous symmetries, internal symmetries of space-time, gauge symmetries and permutation symmetry in Quantum Field theory. In the Noether symmetry analysis the conserved charge can identify the actual one among similar physical processes. Here Noether integral has been chosen as a tool to simplify the given differential equations. Moreover, a new door has been opened in studying Quantum cosmology by Noether symmetry.

In this thesis we have mainly studied about the Noether's theorem and its applications. Here Noether symmetry analysis has been used to determine the classical cosmological solutions of various cosmological models such as multiscalar field cosmological model, Chameleon field cosmological model, Einstein-Skyrme model, cosmological model with variable G and Λ. Here we have also discussed the Quantum cosmology for various dark energy model. Solving the Wheeler DeWitt equation we have found the wave function of the Universe for a given cosmological model.

1.1.1 Symmetry Group

If we make a transformation of a geometric object in such a way that the object is invariant under this transformation then the group of all such transformations form symmetry group [1]. This transformation takes the object to itself. It preserves all the relevant structure of the object. An usual notation for the symmetry group of an object M is $G = \mathrm{Sym}(M)$. In metric space, symmetries of an object form a subgroup of the isometry group of the ambient space (ambient space actually is a space which surrounds an object).

1.1.2 One Parameter Group of Point Transformation

To simplify a given differential equation, it is a common procedure to transform the dependent and independent variables into new variables. Consider 'a' as the independent variable of the given differential equation while 'b' is the dependent variable and we make a point transformation from (a, b) to $(\tilde{a}, \tilde{b})$; where $\tilde{a} = \tilde{a}(a, b)$ and $\tilde{b} = \tilde{b}(a, b)$.

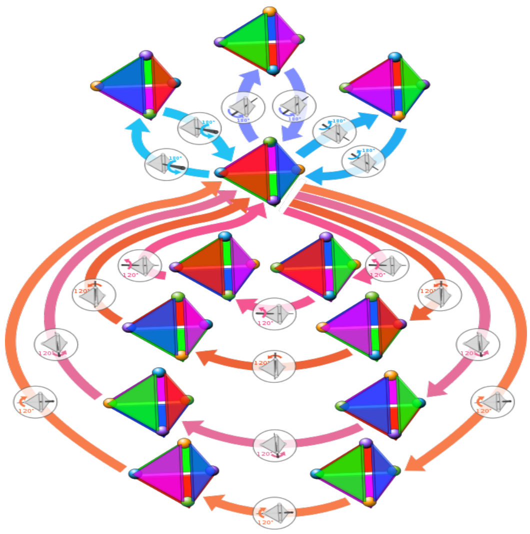

Figure 1.1: The 12 rotations form the symmetry group of the tetrahedron

In the context of symmetry analysis, this point transformation must depend on at least one arbitrary parameter ϵ.

$$\begin{aligned} \tilde{a} &= \tilde{a}(a, b; \epsilon), \\ \tilde{b} &= \tilde{b}(a, b; \epsilon). \end{aligned} \tag{1.1}$$

Moreover this transformation should be invertible and the repeated application give a transformation which belongs to the same family.

$$\tilde{\tilde{a}} = \tilde{\tilde{a}}(\tilde{a}, \tilde{b}; \tilde{\epsilon}) = \tilde{\tilde{a}}(a, b; \tilde{\tilde{\epsilon}}), \tag{1.2}$$

where $\tilde{\tilde{\epsilon}} = \tilde{\tilde{\epsilon}}(\tilde{\epsilon}, \epsilon)$.

We also have $\tilde{a}(a, b; 0) = a$, $\tilde{b}(a, b; 0) = b$. This is the identity transformation.

From the above properties it can be said that the transformation (1.1) is a one parameter group of transformation [2].

Example: The rotation is given by

$$\tilde{a} = a\cos\epsilon - b\sin\epsilon,$$
$$\tilde{b} = a\sin\epsilon + b\cos\epsilon.$$

This is an example of one parameter group.

1.1.3 Invariance

A function $f : A \to \mathbb{R}$ is said to be invariant if it is unaffected by the group transformation:

$$f(g.x) = f(x), \tag{1.3}$$

for all $g \in G$ and $x \in A$. Here G is symmetry group of the system.

Lie symmetry [2] of differential equation is an example of one-parameter point transformation. Under the Lie symmetry approach differential equation remains invariant [2]. Lie symmetry is also useful to study non-linear differential equation. By finding invariant function, Lie symmetry can solve the system of equations and obtain the analytic solutions of the system. This kind of solutions are known as invariant solutions.

Example: Heat conduction equation $\dfrac{\partial b}{\partial t} - \dfrac{\partial^2 b}{\partial a^2} = 0$ is invariant under the following transformations:

$$\begin{aligned}
\tilde{t} &= t + d, \quad \tilde{a} = a, \quad \tilde{b} = b. \\
\tilde{t} &= t, \quad \tilde{a} = a + d, \quad \tilde{b} = b. \\
\tilde{t} &= t, \quad \tilde{a} = a, \quad \tilde{b} = b + df(t, a); \quad \text{where } f(t, a) \text{ satisfies the condition } f_t - f_{xx} = 0.
\end{aligned}$$

1.1.4 Infinitesimal Generator

Let (a, b) be an arbitrary point. We can write

$$\begin{aligned}
\tilde{a}(a, b, \epsilon) &= a + \epsilon\xi(a, b) + ... = a + \epsilon X a + ..., \\
\tilde{b}(a, b, \epsilon) &= b + \epsilon\eta(a, b) + ... = b + \epsilon X b + ...,
\end{aligned} \tag{1.4}$$

where

$$\xi(a,b) = \frac{\partial \tilde{a}}{\partial \epsilon}\Big|_{\epsilon=0},$$
$$\eta(a,b) = \frac{\partial \tilde{b}}{\partial \epsilon}\Big|_{\epsilon=0}, \tag{1.5}$$

and

$$X = \xi(a,b)\frac{\partial}{\partial a} + \eta(a,b)\frac{\partial}{\partial b}. \tag{1.6}$$

The operator X is known as infinitesimal generator of the transformation. And ξ and η are two components of this infinitesimal generator. The term 'generator' implies that if we repeat the transformation we will get the finite transformation i.e, by integrating we will get

$$\frac{\partial \tilde{a}}{\partial \epsilon} = \xi(\tilde{a},\tilde{b}),$$
$$\text{and} \quad \frac{\partial \tilde{b}}{\partial \epsilon} = \eta(\tilde{a},\tilde{b}), \tag{1.7}$$

with the initial values of a, b at $\epsilon = 0$; So we will get the finite transformation.

Now we can rescale ϵ as $\epsilon = g(\hat{\epsilon})$, $g(0) = 0$ and $g'(0) \neq 0$.

Then from equation (1.5) we will get

$$\hat{\xi} = \frac{\partial \tilde{a}}{\partial \hat{\epsilon}}\Big|_{\epsilon=0} = \frac{\partial \tilde{a}}{\partial \epsilon} g'(\hat{\xi})\Big|_{\epsilon=0} = g'(0)\xi.$$

Similarly, $\hat{\eta} = g'(0)\eta$.

1.1.5 Law of transformations

Equation (1.6) represents the infinitesimal generator. Now a natural question arrives, what will be the form of infinitesimal generator when we take $p(a,b)$ and $q(a,b)$ instead of a, b.

We can write the generalized form of infinitesimal generator for more than two variables as follows [2]:

$$X = s^i(a^n)\frac{\partial}{\partial a^i}, \qquad i = 1, 2, ..., N \tag{1.8}$$

(summation over the dummy index i).

Now we make a transformation as

$$a^{i'} = a^{i'}(a^i), \qquad \left|\frac{\partial a^{i'}}{\partial a^i}\right| \neq 0. \tag{1.9}$$

From this transformation, we get,

$$\frac{\partial}{\partial a^i} = \frac{\partial a^{i'}}{\partial a^i}\frac{\partial}{\partial a^{i'}}. \tag{1.10}$$

Then the infinitesimal generator (1.8) can be written as

$$X = s^i(a^n)\frac{\partial a^{i'}}{\partial a^i}\frac{\partial}{\partial a^{i'}} = s^{i'}\frac{\partial}{\partial a^{i'}}, \tag{1.11}$$

where $s^{i'} = \dfrac{\partial a^{i'}}{\partial a^i}s^i$.

Now,

$$Xa^n = s^i\frac{\partial}{\partial a^i}a^n = s^n. \tag{1.12}$$

[using the above property]

So, the infinitesimal generator can be written as

$$X = (Xa^i)\frac{\partial}{\partial a^i} = (Xa^{i'})\frac{\partial}{\partial a^{i'}}. \tag{1.13}$$

From the above discussion it is clear that if one knows the infinitesimal generator in a^i co-ordinate then it will not be difficult to find out the infinitesimal generator in $a^{i'}$ co-ordinate.

So, we can conclude that one can always find a suitable co-ordinates for an arbitrary number N of co-ordinates a^i for which the generator takes the simple form,

$$X = \frac{\partial}{\partial s}. \tag{1.14}$$

This is the normal form of the generator X.

Example: Consider the generator is given by

$$X = a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b}, \tag{1.15}$$

and we want to write this generator in terms of u and v, where $u = \frac{b}{a}$, $v = ab$.

Then from (1.13) we can calculate

$$Xu = \left(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b}\right) u = 0,$$
$$Xv = 2ab = 2v. \tag{1.16}$$

So, the generator takes the form as

$$X = 2v\frac{\partial}{\partial v}, \tag{1.17}$$

for the symmetry vector.

1.1.6 Extensions of transformations

Consider a differential equation ,

$$J(a, b, b', b'', ..., b^{(n)}) = 0, \tag{1.18}$$

$$\text{where } b' = \frac{db}{da} \quad \text{etc.}$$

Now if we want to make a point transformation to the differential equation, atfirst we have to know the form of transformation of $b^{(n)}$. Now the prolongation of the point transformation to the derivatives can be written as [2, 3]

$$\begin{aligned} \tilde{b}' = \frac{d\tilde{b}}{d\tilde{a}} &= \frac{d\tilde{b}(a, b; \epsilon)}{d\tilde{a}(a, b; \epsilon)} \\ &= \frac{b'\left(\frac{\partial \tilde{b}}{\partial b} + \frac{\partial \tilde{b}}{\partial a}\right)}{b'\left(\frac{\partial \tilde{a}}{\partial b} + \frac{\partial \tilde{a}}{\partial a}\right)} \\ &= \tilde{b}'(a, b, b'; \epsilon). \end{aligned} \tag{1.19}$$

Similarly, $\tilde{b}'' = \frac{d\tilde{b}'}{d\tilde{a}} = \tilde{b}''(a, b, b', b''; \epsilon)$ etc.

Now, we want to write the extension of the infinitesimal generator X.

$$
\begin{aligned}
\tilde{a} &= a + \epsilon\xi(a,b) + ... = a + \epsilon X a + ..., \\
\tilde{b} &= b + \epsilon\eta(a,b) + ... = b + \epsilon X b + ..., \\
\tilde{b}' &= b' + \epsilon\eta'(a,b,b') + ... = b' + \epsilon X b' + ..., \\
... \quad & ... \quad ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; .... \\
... \quad & ... \quad ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; .... \\
\tilde{b}^{(n)} &= b^{(n)} + \epsilon\eta^{(n)}(a,b,b',...,b^{(n)}) + ... = b^{(n)} + \epsilon X b^{(n)} + ... \quad .
\end{aligned} \tag{1.20}
$$

Here η, η',...,$\eta^{(n)}$ are of the form,

$$
\begin{aligned}
\eta' &= \left.\frac{\partial \tilde{b}'}{\partial \epsilon}\right|_{\epsilon=0} \quad , \\
& .... \\
& .... \\
\eta^{(n)} &= \left.\frac{\partial \tilde{b}^{(n)}}{\partial \epsilon}\right|_{\epsilon=0} \quad .
\end{aligned} \tag{1.21}
$$

Using the equations (1.20) and (1.21) one can get,

$$
\begin{aligned}
\tilde{b}' = b' + \epsilon\eta' + ... = \frac{d\tilde{b}}{d\tilde{a}} &= \frac{db + \epsilon d\eta + ...}{da + \epsilon d\xi + ...} \\
&= \frac{b' + \epsilon\left(\frac{d\eta}{da}\right) + ...}{1 + \epsilon\left(\frac{d\xi}{da}\right) + ...} \\
&= b' + \epsilon\left(\frac{d\eta}{da} - b'\frac{d\xi}{da}\right) + ...,
\end{aligned} \tag{1.22}
$$

$$
\begin{aligned}
\tilde{b}^{(n)} &= b^{(n)} + \epsilon\eta^{(n)} + ... \\
&= \frac{d\tilde{b}^{(n-1)}}{d\tilde{a}} \\
&= b^{(n)} + \epsilon\left(\frac{d\eta^{(n-1)}}{da} - b^{(n)}\frac{d\xi}{da}\right) + ... \quad .
\end{aligned} \tag{1.23}
$$

From which we can get,

$$\begin{aligned}\eta' &= \frac{d\eta}{da} - b'\frac{d\xi}{da} \\ &= \frac{d\eta}{da} + b'\left(\frac{d\eta}{db} - \frac{d\xi}{da}\right) - b'^2\frac{d\xi}{db}, \\ \eta^{(n)} &= \frac{d\eta^{(n-1)}}{da} - b^{(n)}\frac{d\xi}{da}. \end{aligned} \tag{1.24}$$

Using the method of induction, one can show that,

$$\eta^{(n)} = \frac{d^n}{da^n}(\eta - b'\xi) + b^{(n+1)}\xi. \tag{1.25}$$

It is to be noted that $\eta^{(n)}$ is not the nth derivative of η.

Now, if we summarize the result, we can write that if the infinitesimal generator of a point transformation takes the form

$$X = \xi(a,b)\frac{\partial}{\partial a} + \eta(a,b)\frac{\partial}{\partial b}, \tag{1.26}$$

then its extension (prolongation) upto the nth derivative takes the form,

$$X = \xi\frac{\partial}{\partial a} + \eta\frac{\partial}{\partial b} + \eta'\frac{\partial}{\partial b'} + ... + \eta^{(n)}\frac{\partial}{\partial b^{(n)}}, \tag{1.27}$$

where $\eta^{(n)}(a, b, b', ..., b^{(n)})$ are defined by equation (1.24).

1.1.7 Multiple-parameter groups of transformations

Now we will make a transformation which depends on more than one parameter ϵ. We take the transformation as

$$\tilde{a} = \tilde{a}(a, b; \epsilon_N), \qquad \tilde{b} = \tilde{b}(a, b; \epsilon_N), \tag{1.28}$$

$$N = 1, 2, ..., r.$$

We also consider these ϵ_N are independent of each other. This transformation form an $r-$parameter group (G_r) if this transformation contains the identity and includes their repeated application (with possibly different ϵ_N).

So, an infinitesimal generator X_n can be written as

$$X_N = \xi_N\frac{\partial}{\partial a} + \eta_N\frac{\partial}{\partial b}, \tag{1.29}$$

corresponding to each parameter ϵ_N.

where,

$$
\begin{aligned}
\xi_N(a,b) &= \left.\frac{\partial \tilde{a}}{\partial \epsilon_N}\right|_{\epsilon_M=0}, \\
\eta_N(a,b) &= \left.\frac{\partial \tilde{b}}{\partial \epsilon_N}\right|_{\epsilon_M=0}.
\end{aligned}
\tag{1.30}
$$

It is to be noted that a rescaling of ϵ_N rescales the corresponding X_N by a constant factor.

1.1.8 The definition of symmetry

Consider a point transformation $\tilde{a} = \tilde{a}(a,b)$, $\tilde{b} = \tilde{b}(a,b)$ which may or may not depend on some parameters. This transformation is called symmetry transformation of a ordinary differential equation if it transforms one solution into another solution; i.e, image of any solution is again a solution of the ordinary differential equation.

Consider a nth order differential equation

$$J\left(a, b, b', ..., b^{(n)}\right) = 0. \tag{1.31}$$

This ordinary differential equation will not be changed under a symmetry condition. Then we will get

$$J\left(\tilde{a}, \tilde{b}, \tilde{b}', ..., \tilde{b}^{(n)}\right) = 0. \tag{1.32}$$

This definition implies that the existence of a symmetry is independent of the choice of variables in which the differential equation and its solutions are given. So, we can expect simple differential equations to have many symmetries.

The symmetries which do not form a Lie group can be very useful to study differential equations but there is no practicable way to find them. So, from now on we will assume the symmetry transformation contains atleast one parameter ϵ;

$$\begin{aligned} \tilde{a} &= \tilde{a}(a, b; \epsilon), \\ \tilde{b} &= \tilde{b}(a, b; \epsilon), \\ \tilde{b}' &= \tilde{b}'(a, b, b'; \epsilon) \qquad \text{etc.} \end{aligned} \tag{1.33}$$

We call it a Lie point symmetry [2].

1.1.9 Canonical Co-ordinates

Let us assume $\overrightarrow{X}$, a vector field in the augmented space (a, b), given by

$$\overrightarrow{X} = \xi(a, b)\frac{\partial}{\partial a} + \eta(a, b)\frac{\partial}{\partial b}. \tag{1.34}$$

Now we will make a point transformation $(a, b) \rightarrow (p, q)$.

Then the co-ordinate (p, q) will be called canonical co-ordinate of $\overrightarrow{X}$ if

$$\overrightarrow{X}p = 0 \quad \text{and} \quad \overrightarrow{X}q = 1, \tag{1.35}$$

holds.

1.1.10 Noether's Symmetry

Emmy Noether in 1915, first introduced some results which are known as Noether's theorem. According to her results the invariance of the functional of the calculus of variations and action integral in mechanics can be considered to vary under infinitesimal transformation.

- *Noether's first theorem*: The symmetry groups corresponding to a variational problem as an one-one correspondence with the conservation laws in the corresponding Euler-Lagrange equation.

- *Noether's second theorem*: There always exists a non-trivial differential relation between an infinite dimensional variational symmetry group depending on an arbitrary function and the associated non-trivial differential relation among its Euler-Lagrange equations.

Let us assume a point-like Lagrangian for a physical system as

$$L = L\left(q^{\alpha}\left(x^{i}\right), \partial_{j} q^{\alpha}\left(x^{i}\right)\right), \tag{1.36}$$

where $q^{\alpha}\left(x^{i}\right)$ is the generalized co-ordinates.

Imposing symmetry constraints, either one can solve the evolution equations of a physical system or it can be simplified to a great extent.

The Euler-Lagrange equation for the above Lagrangian can be written as

$$\partial_{j}\left(\frac{\partial L}{\partial \partial_{j} q^{\alpha}}\right) = \frac{\partial L}{\partial q^{\alpha}}, \qquad \alpha = 1, 2, ..., N. \tag{1.37}$$

If we cantract the above equation with some unknown function $\lambda^{\alpha}\left(q^{\beta}\right)$,

$$\lambda^{\alpha}\left[\partial_{j}\left(\frac{\partial L}{\partial \partial_{j} q^{\alpha}}\right) - \frac{\partial L}{\partial q^{\alpha}}\right] = 0. \tag{1.38}$$

Thus the Lie derivative of the Lagrangian takes the form as

$$\mathcal{L}_{X} L = \lambda^{\alpha} \frac{\partial L}{\partial q^{\alpha}} + (\partial_{j} \lambda^{\alpha}) \frac{\partial L}{\partial(\partial_{j} q^{\alpha})} = \partial_{j}\left(\lambda^{\alpha} \frac{\partial L}{\partial(\partial_{j} q^{\alpha})}\right). \tag{1.39}$$

Here the vector field X takes the form as [4, 5]

$$X = \lambda^{\alpha} \frac{\partial}{\partial q^{\alpha}} + (\partial_{j} \lambda^{\alpha}) \frac{\partial}{\partial \partial_{j} q^{\alpha}}. \tag{1.40}$$

This vector field is known as the infinitesimal generator of the Noether symmetry. Now, existence of Noether's theorem demands that the Lie derivative of the Lagrangian must vanish i.e, $\mathcal{L}_{X} L = 0$ i.e,

$$\partial_{j}\left(\lambda^{\alpha} \frac{\partial L}{\partial(\partial_{j} q^{\alpha})}\right) = 0. \tag{1.41}$$

There exists a conserved current which is also known as Noether current associated with the system [6, 7, 8].

$$Q^{i} = \lambda^{\alpha} \frac{\partial L}{\partial \partial_{i} q^{\alpha}}, \tag{1.42}$$

satisfying the condition $\partial_i Q^i = 0$.

Also the energy function of the physical system takes the form as

$$E = \frac{\partial L}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L. \tag{1.43}$$

This energy function is also known as Hamiltonian of the system. It is a constant of motion if there is no explicit time dependence in the Lagrangian [6, 7, 8]. Noether symmetry approach is an acceptable model because the conserved quantities have some physical meaning.

1.1.11 Symmetry and Laws of Conservation

- Symmetry under space translations $\Longrightarrow$ Conservation of linear momentum.
- Symmetry under rotations $\Longrightarrow$ Conservation of angular momentum.
- Symmetry under time translations $\Longrightarrow$ Conservation of energy.

1.1.12 Generalized Symmetries

We have dealt with the cases where the infinitesimal transformations ξ and η are functions of a and b only. But Noether's theorem [9] allows for dependency on derivatives of higher order. Let us consider a functional of the form $\mathcal{A} = \mathcal{A}(a, b, b')$. Here ξ and η can include terms in b', b'', b''' etc. Since the Euler-Lagrange equation is only imposed in the determination of the first integral, the derivatives b'', b''',... are not independent. Hence Noether symmetries can be determined without any prior knowledge of the Euler-Lagrange equation. We can apply the same comments for higher order Lagrangian.

For point symmetry we use seperation by powers of b' (for first order Lagrangian). But this is no longer possible because both ξ and η have functional dependency on b'. Here we can seperate terms by the powers of b''.

Let us consider $\xi = \xi(a, b, b')$ and $\eta = \eta(a, b, b')$;

Then the killing-type equation can be written as

$$\frac{\partial f}{\partial a}+b'\frac{\partial f}{\partial b}+b''\frac{\partial f}{\partial b'}=\xi\frac{\partial \mathcal{A}}{\partial a}+\eta\frac{\partial \mathcal{A}}{\partial b}$$
$$+\left(\frac{\partial \eta}{\partial a}+b'\frac{\partial \eta}{\partial b}+b''\frac{\partial \eta}{\partial b'}-b'\frac{\partial \xi}{\partial a}-b'^2\frac{\partial \xi}{\partial b}-b'b''\frac{\partial \xi}{\partial b'}\right)\frac{\partial \mathcal{A}}{\partial b'}$$
$$+\left(\frac{\partial \xi}{\partial a}+b'\frac{\partial \xi}{\partial b}+b''\frac{\partial \xi}{\partial b'}\right)\mathcal{A}. \tag{1.44}$$

Here we will seperate by powers of b''. So, the coefficient of b'' is

$$\frac{\partial f}{\partial b'}=\frac{\partial \eta}{\partial b'}\frac{\partial \mathcal{A}}{\partial b'}-b'\frac{\partial \xi}{\partial b'}\frac{\partial \mathcal{A}}{\partial b'}+\mathcal{A}\frac{\partial \xi}{\partial b'}. \tag{1.45}$$

So the equation (1.44) now simplifies to

$$\frac{\partial f}{\partial a}+b'\frac{\partial f}{\partial b}=\xi\frac{\partial \mathcal{A}}{\partial a}+\eta\frac{\partial \mathcal{A}}{\partial b}+\frac{\partial \mathcal{A}}{\partial b'}\left(\frac{\partial \eta}{\partial a}+b'\frac{\partial \eta}{\partial b}\right)$$
$$-b'\frac{\partial \mathcal{A}}{\partial b'}\left(\frac{\partial \xi}{\partial a}+b'\frac{\partial \xi}{\partial b}\right)+\mathcal{A}\left(\frac{\partial \xi}{\partial a}+b'\frac{\partial \xi}{\partial b}\right). \tag{1.46}$$

We assume that ξ and η are linear in b'.

1.1.13 Higher Dimensional Systems

Here we will discuss the Noether's theorem concerning Lagrangians of systems of more than one degree of freedom. The symmetry

$$G=\xi\frac{\partial}{\partial a}+\eta_i\frac{\partial}{\partial b_i},$$

will be called Noether symmetry if it satisfies the following general formula

$$f'=\xi\frac{\partial \mathcal{A}}{\partial a}+\eta_i\frac{\partial \mathcal{A}}{\partial b_i}+\left(\eta'-b_i'\xi'\right)\frac{\partial \mathcal{A}}{\partial b_i'}.$$

Here ξ and η_i are of the form

$$\xi=\xi\left(a,b_1,...,b_n,b_1',...,b_n'\right),$$
$$\eta_i=\eta_i\left(a,b_1,...,b_n,b_1',...,b_n'\right).$$

So the first integral can be written as

$$I = f - \left[\xi \mathcal{A} + \left(\eta_i - b'_i \xi\right) \frac{\partial \mathcal{A}}{\partial b'_i}\right].$$

Here the repeated index i denotes the summation.

1.1.14 Gauge Function

In the literature, there are some instances in Noether's theorem in which the gauge function is chosen as zero. In this section we have considered a simple Lagrangian where gauge function is taken to be zero.

Consider the Lagrangian as

$$L = \frac{1}{2} b'^2. \tag{1.47}$$

Here the killing-type equation can be written as

$$f' = G^{[1]} L + \xi' L. \tag{1.48}$$

This can be written as

$$0 = \xi \frac{\partial L}{\partial a} + \eta \frac{\partial L}{\partial b} + \left(\eta' - b'\xi'\right) \frac{\partial L}{\partial b'} + \xi' L. \tag{1.49}$$

So we have

$$0 = \left(\frac{\partial \eta}{\partial a} + b' \frac{\partial \eta}{\partial b}\right) b' - b'^2 \left(\frac{\partial \xi}{\partial a} + b' \frac{\partial \xi}{\partial b}\right) + \frac{b'^2}{2} \left(\frac{\partial \xi}{\partial a} + b' \frac{\partial \xi}{\partial b}\right). \tag{1.50}$$

Now equating the coefficients of b'^3 from both side, we get,

$$-\frac{1}{2} \frac{\partial \xi}{\partial b} = 0. \tag{1.51}$$

Similarly, equating the coefficients of b'^2 and b' from both side we get

$$\begin{aligned} \frac{\partial \eta}{\partial b} - \frac{1}{2} \frac{\partial \xi}{\partial a} &= 0, \\ \frac{\partial \eta}{\partial a} &= 0, \end{aligned} \tag{1.52}$$

respectivly. Here we get,

$$\xi = r(a),$$
$$\eta = \frac{1}{2}r'b + s(a),$$
$$0 = \frac{1}{2}r''b + s'. \quad (1.53)$$

From the coefficient of b, we get,

$$\frac{1}{2}r'' = 0 \implies r = R_0 + R_1 a. \quad (1.54)$$

The coefficient of b^0 gives,

$$s' = 0 \implies s = S_0. \quad (1.55)$$

Here R_0, R_1 and S_0 are integration constants.

Thus the Noether point symmetries are

$$G_1 = \frac{\partial}{\partial a},$$
$$G_2 = b\frac{\partial}{\partial b},$$
$$G_3 = a\frac{\partial}{\partial a} + \frac{1}{2}b\frac{\partial}{\partial b}. \quad (1.56)$$

The non-zero commutators are

$$[G_1, G_2] = G_1,$$

$$[G_1, G_3] = 0,$$

$$[G_2, G_3] = 0. \quad (1.57)$$

Instead of five Noether symmetries here we have three Noether symmetries [10], where the gauge function is set to zero to obtain the number of symmetry to be less than the maximal number and hence the gauge function is not set to zero. This causes a great reduction of terms in the determination of Noether integral.

1.1.15 Formulation of Hamiltonian

Under the Legendre transformation,

$$H = p_i \dot{q}_i - L, \tag{1.58}$$

where q_i is the generalised co-ordinate, $L = L(q, \dot{q}, t)$ is the Lagrangian and $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$ is the conjugate momenta.

The second order Euler-Lagrange equation takes the form as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0. \tag{1.59}$$

The first order Hamiltonian's equation of motion can be written as

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}, \\ \dot{p} &= -\frac{\partial H}{\partial q}. \end{aligned} \tag{1.60}$$

Then the Noetherian integral becomes

$$I = f + \tau H - p\eta. \tag{1.61}$$

Here τ and η are the coefficient functions of a generalised symmetry. In the Hamiltonian formulation time t is an independent variable and p and q are dependent variables.

Let us assume an infinitesimal transformation as

$$\begin{aligned} \bar{t} &= t + \epsilon\tau, \\ \bar{q} &= q + \epsilon\eta, \\ \bar{p} &= p + \epsilon\zeta, \end{aligned} \tag{1.62}$$

generated by the differential operator

$$G = \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial q} + \zeta \frac{\partial}{\partial p}. \tag{1.63}$$

The action integral can be written as

$$A = \int_{t_0}^{t_1} \left[p\dot{q} - H(q, p, t) \right] dt, \tag{1.64}$$

where $\dot{q} = \dot{q}(q, p, t)$.

The transformed action (using the infinitesimal transformation) takes the form,

$$\begin{aligned} \bar{A} &= \int_{\bar{t_0}}^{\bar{t_1}} \left\{ \bar{p}\dot{\bar{q}} - H(\bar{q}, \bar{p}, \bar{t}) \right\} d\bar{t} \\ &= \int_{t_0}^{t_1} \left\{ p\dot{q} - H + \epsilon \left[p\psi + \dot{q}\zeta - \eta \frac{\partial H}{\partial q} - \zeta \frac{\partial H}{\partial p} - \tau \frac{\partial H}{\partial t} + \dot{\tau} \left(p\dot{q} - H \right) \right] \right\} dt \\ &\qquad + \epsilon \left\{ \left(p_1 \dot{q}_1 - H_1 \right) \tau_1 - \left(p_0 \dot{q}_0 - H_0 \right) \tau_0 \right\}. \end{aligned} \tag{1.65}$$

Here the subscripts 0 and 1 represents the evaluation at t_0 and t_1 respectively. $\epsilon\psi$ is the infinitesimal change in $\dot{q}$ created by the infinitesimal transformation.

Here we have considered

$$\psi = \tau \frac{\partial \dot{q}}{\partial t} + \eta \frac{\partial \dot{q}}{\partial q} + \zeta \frac{\partial \dot{q}}{\partial p}. \tag{1.66}$$

Here the differential operator G (in equation 1.63) is said to be the Noether symmetry of the action integral if the transformed action integral remains same as the original action integral; i.e, $\bar{A} = A$ which leads us the following result

$$\dot{f} = p\psi + \dot{q}\zeta - \eta \frac{\partial H}{\partial q} - \zeta \frac{\partial H}{\partial p} - \tau \frac{\partial H}{\partial t} + \dot{\tau} \left(p\dot{q} - H \right). \tag{1.67}$$

Here we can write

$$\left(p_1 \dot{q}_1 - H_1 \right) \tau_1 - \left(p_0 \dot{q}_0 - H_0 \right) \tau_0 = - \int_{t_0}^{t_1} \dot{f} dt. \tag{1.68}$$

The transformation in $\dot{q}$ is a differential consequence of the transformation in q and τ. So we have,

$$\psi = \dot{\eta} - \dot{q}\dot{\tau}. \tag{1.69}$$

So the Hamilton's equation of motion for q takes the form

$$\dot{f} = p\dot{\eta} - \eta\frac{\partial H}{\partial q} - \tau\frac{\partial H}{\partial t} - \dot{\tau}H. \tag{1.70}$$

So the first integral will be

$$I = f + \tau H - p\eta. \tag{1.71}$$

The infinitesimal transformation in p is not independent of that in t and $\dot{q}$.

So we have

$$p = \frac{1}{\frac{\partial^2 H}{\partial p^2}}\left\{\dot{\eta} - \dot{\tau}\frac{\partial H}{\partial p} - \tau\frac{\partial^2 H}{\partial p \partial t} - \eta\frac{\partial^2 H}{\partial p \partial q}\right\}. \tag{1.72}$$

This indicates the fact that underlying the Hamiltonian formalism in $2n+1$ variables there is a basic space of $n+1$ dimensions.

1.2 Introduction to Cosmology

The branch of science which deals with the origin and development of Universe is termed as cosmology. More specifically, this branch of astronomy deals with both the nature of the Universe as well as the evolution of the Universe from Big-Bang scenerio to the future states of Universe.

According to NASA, Cosmology is "*the scientific study of the large scale properties of the universe as a whole*".

Since the end of the last century, the observational evidences [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] from cosmic microwave background (CMB), baryon acoustic oscillation (BAO) [24, 25], Supernovae type Ia [11, 13, 26] etc. indicate that our Universe is in accelerating phase not in decelerating phase. To explain the present accelerating phase Cosmologists are divided into two groups. One group introduces some exotic matter within the frame work of Einstein gravity. This exotic matter violets the strong energy condition which is known as Dark Energy (DE). Other group prefers the modification of the gravity theory by introducing extra terms in the Einstein–Hilbert action.

Cosmological constant Λ [27, 28]and Λcold dark matter (ΛCDM) are commonly used as the dark energy (DE) candidate. But due to two severe drawbacks the cosmological constant is not well accepted model of DE rather dynamical DE models [29, 30, 31, 32] are widely used in the literature. These two drawbacks are known as (i) fine-tuning problem [33, 34] and (ii) the coincidence problem [35, 36]. To avoid these problems, cosmologists have found alternative dark energy model to accommodate the recent accelerating phase of the Universe. As a result we get various kinds of dark energy model such as Multiscalar field dark energy model [37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47] , Chameleon scalar field dark energy model [48, 49], Einstein-Skyrme model [50, 51, 52, 53, 54, 55, 56] , Quintessence model [57, 58, 59, 60], K-essence model [61, 62, 63, 64], Tachyon model [65, 66], Phantom model [67], Quintom model [68, 69, 70] etc. In this thesis, we have studied some dark energy models and found the classical solution of those models. We have also discussed the quantum cosmology of those dark energy model by constructing Wheeler DeWitt (WD) equation.

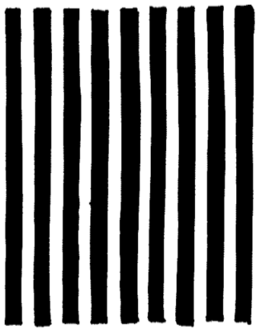

Figure 1.2: Anisotropic but homogeneous on scale larger than the stripe width

Figure 1.3: Isotropic about the origin, but which is inhomogeneous

1.2.1 Homogeneity and Isotropy

In modern physical cosmology, according to the cosmological principle our Universe is homogeneous and isotropic on a large scale. The term 'Homogeneous' indicates that in the Universe, there is no preferred location. Which implies that it looks the same no matter where I can look. The term 'isotropic' represents that there is no preferred direction in the Universe, which means it looks same in every direction in the Universe. Here the term 'large scales' indicates the scales of roughly 100 Mpc or more [71].

On small scales, Universe is neither homogeneous nor isotropic. It is also an important fact that homogeneity does not imply isotropy.

In Figure 1.2, we can see some stripes. This is homogeneous on the scales which is larger than the stripe width but it is not isotropic. On the otherhand, in Figure 1.3 we can see that the bullseye is isotropic around the center but it is not homogeneous.

1.2.2 FLRW Universe

The evolution of the Universe is described by the Einstein field equations. This equation form a bridge between geometry of the space-time and matter distribution of the Universe. The explicit form of Einstein equation takes the form as

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.73}$$

where $\kappa = 8\pi G$, known as gravitational coupling. $T_{\mu\nu}$ denotes the energy momentum tensor. $G_{\mu\nu}$ is known as the Einstein tensor which is defined by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{1.74}$$

Here R is the Ricci scalar defined by

$$R = R_{\mu\nu} g^{\mu\nu}, \tag{1.75}$$

for the FLRW space-time.

1.2.3 Friedmann Equation

The Friedmann equations explain the expansion of the space in homogeneous and isotropic model of the Universe. In 1922, Alexander Friedmann first derived this set of equations from Einstein's field equations. The large scale geometry of the Universe (homogeneous and isotropic) can be described by a metric which is defined by

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - \kappa r^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right]. \tag{1.76}$$

This metric is known as Friedmann-Lemaitre-Robertson-Walker (FLRW) metric [72, 73, 74, 75]. Here $a(t)$ denotes the scale factor of the Universe and κ is the scalar curvature. $\kappa = 1$ deascribes the closed model of the Universe, $\kappa = -1$ describes the open model of the Universe while $\kappa = 0$ denotes the flat model of the Universe.

The Einstein field equations for the non-flat ($\kappa \neq 0$) model can be written as

$$3H^2 + \frac{3\kappa}{a^2} = 8\pi G\rho, \tag{1.77}$$

$$2\dot{H} + 3H^2 + \frac{\kappa}{a^2} = -8\pi Gp. \tag{1.78}$$

Here the overdot indicates the derivative with respect to the cosmic time t. ρ is the energy density and p represents the thermodynamic pressure of the scalar field. H is the usual Hubble parameter defined by $\frac{\dot{a}}{a}$. Hubble parameter measures the rate of expansion of the Universe. The above two equations (1.77) and (1.78) are known as first and second Friedmann equation respectively.

The conserved energy momentum tensor leads the continuity equation

$$\nabla_\mu T^\mu_\nu = 0. \tag{1.79}$$

From the above equation we get,

$$\dot{\rho} + 3H(p + \rho) = 0. \tag{1.80}$$

Eliminating $\frac{\kappa}{a^2}$ from (1.77) and 1.78 we get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3p + \rho). \tag{1.81}$$

$\frac{\ddot{a}}{a}$ measures the acceleration.

If $(3p + \rho) > 0$ then $\frac{\ddot{a}}{a} < 0$ which implies Universe decelerates. If $(3p + \rho) < 0$ then $\frac{\ddot{a}}{a} > 0$ which implies Universe accelerates.

The matter for which $(\rho + 3p) > 0$ i.e, the matter for which strong energy condition is satisfied, is known as normal matter. On the other hand, the matter which violates the strong enegy condition, is known as exotic matter.

One can write the first Friedmann equation as

$$\Omega(t) = 1 + \frac{\kappa}{a^2 H^2}, \tag{1.82}$$

where $\Omega(t) \equiv \frac{\rho}{\rho_c}$ is known as density parameter and $\rho_c = \frac{3H^2}{8\pi G}$ is known as critical density.

from the equation (1.82) we can conclude the following results:

$$\begin{aligned}\Omega(t) &> 1 \implies \rho > \rho_c \implies \kappa = +1, \\ \Omega(t) &< 1 \implies \rho < \rho_c \implies \kappa = -1, \\ \Omega(t) &= 1 \implies \rho = \rho_c \implies \kappa = 0. \end{aligned} \tag{1.83}$$

So, we can classify the geometry of the Universe using matter distribution.

$$q = -\left(1 + \frac{\dot{H}}{H^2}\right). \tag{1.84}$$

This q is a dimensionless quantity which is known as deceleration parameter. If $q < 0$, Universe is accelerating and if $q > 0$, Universe is decelerating.

1.2.4 Dynamics of the Universe Filled with a Perfect Fluid

In this section we have considered the evolution of the Universe filled with a perfect fluid which satisfies the barotropic equation of state as

$$\omega = \frac{p}{\rho}, \tag{1.85}$$

where p is the thermodynamic pressure and ρ is the energy density.

If we put the equation of state from equation (1.85) in the FLRW equation we can find

$$H = \frac{2}{3(1+\omega)(t-t_0)}, \qquad \text{when} \quad \kappa = 0 \tag{1.86}$$

and

$$\rho \propto a^{-3(1+\omega)}. \tag{1.87}$$

Hence the scale factor takes the form as

$$a \propto t^{\frac{2}{3(1+\omega)}}, \tag{1.88}$$

where t is the cosmic time and t_0 is the integration constant. The above solution is not valid for $\omega = -1$.

For radiation dominated Universe i.e, for $\omega = \frac{1}{3}$,

$$\begin{aligned} \rho &\propto a^{-4}, \\ a &\propto (t - t_0)^{\frac{1}{2}}. \end{aligned} \tag{1.89}$$

For dust dominated Universe i.e, for $\omega = 0$,

$$\begin{aligned} \rho &\propto a^{-3}, \\ a &\propto (t - t_0)^{\frac{2}{3}}. \end{aligned} \tag{1.90}$$

For stiff fluid era i.e, for $\omega = 1$,

$$\begin{aligned} \rho &\propto a^{-6}, \\ a &\propto (t - t_0)^{\frac{1}{3}}. \end{aligned} \tag{1.91}$$

1.2.5 Dark Energy

Cosmological Constant: Cosmological constant [27, 28] is a constant coefficient which was introduced by Albert Einstein in 1917 to his field equations of general relativity to achieve a static model of the Universe. But in 1929, after the Hubble's discovery, Einstein abandoned the concept of Cosmological constant. So it was assumed that the it has no effect on the gravitational field. As a result, from the year 1929 to the early 1990's, most cosmologists took the cosmological constant to be zero.

With the cosmological constant the Einstein equation can be written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.92}$$

where Λ is the cosmological constant.

Now the modified Friedmann equations take the form as

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3},$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3p + \rho) + \frac{\Lambda}{3}. \tag{1.93}$$

These equations admit a static solution with positive spatial curvature. This solution is known as "Einstein Static Universe".

In terms of cosmological constant, the energy density and pressure can be written as

$$\rho_\Lambda = \frac{\Lambda}{8\pi G},$$
$$p_\Lambda = -\frac{\Lambda}{8\pi G}. \tag{1.94}$$

So one can get, $p_\Lambda + \rho_\Lambda = 0$.

Hence, the equation of state parameter can be written as

$$\omega_\Lambda = \frac{p_\Lambda}{\rho_\Lambda} = -1. \tag{1.95}$$

Here strong energy condition is violated. So cosmological constant may be a capable candidate for dark energy. There are two main drawbacks of the cosmological constant namely, Coincidence problem and Fine Tuning problem.

Coincidence Problem: The recent observational evidences indicates that energy density of cosmological constant and energy density of matter are of the same order i.e, if the energy density of the cosmological constant is ρ_Λ and energy density of the matter is ρ_m, then $\rho_\Lambda \propto \rho_m$. So we can conclude that this is a special epoch of the evolution of the Universe when the energy densities of both the matter and cosmological constant are roughly equal. This is known as Coincidence problem.

Fine Tuning Problem: According to the recent observations of the cosmologists, cosmological constant is a non-zero quantity which takes the value (predicted value) as

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} = 10^{-47}\ \text{GeV}^4. \tag{1.96}$$

But according to the quantum field theory, the value of cosmological constant should be very high which is approximately 10^{74} GeV^4. So there is a problem in the observed value and the theoretically predicted value. This is one of the major problems related to cosmological constant.

For these two drawbacks of cosmological constant, Cosmologists introduce dynamical dark energy models.

CHAPTER 2

MULTISCALAR FIELD COSMOLOGICAL MODEL AND POSSIBLE SOLUTIONS USING NOETHER SYMMETRY APPROACH

2.1 Prelude

Today it is well accepted that Einstein gravity with normal matter is not sufficient to describe the present evolution of the Universe. To explain the present accelerated expansion as predicted by the series of observational evidences [13, 15, 76] for the last two decades, one needs some exotic matter having large negative pressure (known as Dark energy(DE)). Cosmologists started with cosmological constant as the DE candidate–the simplest and observationally most favourable. However, due to two severe drawbacks of the cosmological constant namely the fine tuning problem and coincidence problem the cosmologists are searching for an alternative choice which is commonly known as dynamical DE model. A quite well-known candidate for dynamical DE is the scalar field model (minimally or non-minimally coupled to gravity)

Usually, in Scalar field cosmology quintessence model, phantom model and even multiscalar field models are commonly used. In particular, multi-scalar filed cosmology have a wide range of applicability such as an alternative models of inflation i.e, hybrid inflation, α–attractors, double inflation and so on [77, 78, 79, 80, 81, 82, 83, 84, 85] . A very common multi-scalar field model is the quintom model having two scalar fields of which one is the quintessence field while the other one is of phantom nature [70] . An existence of the quintom model having a mixed kinetic term has been introduced in ref. [86], considering the space of the kinetic energy to have flat geometry. A commonly used two scalar field model having non vanishing curvature of the 2D space are known as Chiral cosmological models [85, 87, 88] and have analogy with the non-linear sigma cosmological model [89, 90, 91].

In this chapter, a two scalar field model (for multi scalar field models see ref. [37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47]) has been considered having one quintessence field and the other scalar field interact with the quintessence field through the kinetic term. In order to have cosmological solutions of this complicated model Noether symmetry approach has been introduced. The two unknown functions in the model namely the potential function(which drives the dynamics of the quintessence) and the interaction term between the two fields are determined from the consistency conditions of the symmetry analysis, rather than choosing phenomenologically. This type of cosmological model (Chiral cosmological model) is actively studied by several authors. In particular Noether symmetry approach has been proposed for the search of exact solutions in Chiral cosmological models by Paliathanasis et.al., [92]. Also there are other methods for the construction of exact solutions in Chiral cosmology [93, 94, 95]. Moreover, there are several works similar to our analysis for example see ref. [93, 94, 95]. This chapter is organized as follows: Evolution equation of the model are presented in Section-2.2, whereas Section-2.3 deals with the Noether symmetry approach of this model. Section-2.4 describes the analytical solutions and the chapter ends with concluding remarks in Section-2.5.

2.2 Scalar Field Cosmology: Basic Equation

The action of the scalar field $\phi(x^\mu)$ and $\psi(x^\mu)$ coupled to gravity has the form [96]

$$\mathcal{A} = \int d^4x\sqrt{-g}\mathcal{L}\Big(R(x^\mu), \phi(x^\mu), \psi(x^\mu), \bigtriangledown_\nu\phi(x^\mu), \bigtriangledown_\nu\psi(x^\mu)\Big), \tag{2.1}$$

which is characterized by the Lagrangian density given by

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{2}\bigtriangledown_{\mu}\phi\bigtriangledown^{\nu}\phi - \frac{F(\phi)}{2}\bigtriangledown_{\mu}\psi\bigtriangledown^{\mu}\psi - V(\phi), \tag{2.2}$$

where $8\pi G = 1$ is the gravitational coupling, $F(\phi)$ and $V(\phi)$ are the coupling function and the potential function of the scalar field ϕ, R is the scalar curvature.

Now let us consider the flat FLRW space-time metric given by the line element

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = -dt^2 + a^2(t)(dr^2 + r^2 d\Omega_2^2), \tag{2.3}$$

where $a(t)$ represents the scale factor of the Universe, $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$. So the above point like Lagrangian has the explicit form

$$L = -3a\dot{a}^2 + \frac{1}{2}a^3\dot{\phi}^2 + \frac{a^3}{2}F(\phi)\dot{\psi}^2 - a^3V(\phi). \tag{2.4}$$

Then by varying the action with respect to the scale factor 'a' and the scalar fields 'ϕ' and 'ψ', the Einstein field equations take the form (equation (2.5) is obtained by using the action with respect to the Lapse function [97] chosen as unity in the present context)

$$\frac{1}{2}\Big[F(\phi)\dot{\psi}^2 + \dot{\phi}^2 - 6H^2\Big] + V(\phi) = 0, \tag{2.5}$$

$$\frac{1}{2}\Big[-4\frac{\ddot{a}}{a} + 4H - 2H^2 - F(\phi)\dot{\psi}^2 - \dot{\phi}^2\Big] + V(\phi) = 0, \tag{2.6}$$

$$\frac{1}{2}\Big[2\ddot{\phi} + 6H\dot{\phi} - F'(\phi)\dot{\psi}^2\Big] + V'(\phi) = 0, \tag{2.7}$$

$$\frac{d}{dt}\Big(a^3F(\phi)\dot{\psi}\Big) = 0, \tag{2.8}$$

where an overdot denotes the derivatives with respect to the coamic time 't', 'H' is the usual Hubble parameter defined by $H \equiv \frac{\dot{a}}{a}$ and the conserved quantity of the reduced system is given by (from (2.8))

$$Q := a^3F(\phi)\dot{\psi} = m_1,$$

where m_1 is an integration constant.

2.3 Existence of Noether Symmetry

Noether's first theorem states that every physical system is associated to some conserved quantities provided the Lagrangian of the system is invariant with respect to the Lie derivative [98, 99, 100, 101] along appropriate vector field ($\mathcal{L}_X L = XL$). Further these symmetry constraints help the evolution equations of the physical system to be solvable or to be simplified to a great extend.

For a point like canonical Lagrangian

$$L = L\Big[q^\alpha(x^i)\ \partial_j q^\alpha(x^i)\Big], \tag{2.9}$$

with the generalized coordinates $q^\alpha(x^i)$, the usual Euler-Lagrangian equations i.e.,

$$\partial_j\left(\frac{\partial L}{\partial\ \partial_j q^\alpha}\right) = \frac{\partial L}{\partial q^\alpha}, \quad \alpha = 1, 2....., N \tag{2.10}$$

can be contracted to some unknown function $\lambda^\alpha(q^\beta)$ as

$$\lambda^\alpha\left[\partial_j\left(\frac{\partial L}{\partial\ \partial_j q^\alpha}\right) - \frac{\partial L}{\partial q^\alpha}\right] = 0, \tag{2.11}$$

which can be written as [102, 103]

$$\mathcal{L}_X L = XL = \lambda^\alpha \frac{\partial L}{\partial q^\alpha} + \Big(\partial_j \lambda^\alpha\Big)\frac{\partial L}{\partial\ \partial_j q^\alpha} = \partial_j\left(\lambda^\alpha \frac{\partial L}{\partial\ \partial_j q^\alpha}\right). \tag{2.12}$$

Here

$$X = \lambda^\alpha \frac{\partial}{\partial q^\alpha} + \Big(\partial_j\ \lambda^\alpha\Big)\frac{\partial}{\partial\ \partial_j q^\alpha}, \tag{2.13}$$

is the vector field in the augmented space (the space consists of the dependable variable and their derivatives). Here in the present problem it is a 4D space (t, a, ϕ, ψ). However if the above vector field X is the infinitesimal generator of Noether symmetry, then Noether's theorem tell us that $\mathcal{L}_X L = 0$. As a consequence from equation (2.12) there corresponds a conserved current associated with this symmetry namely [104]

$$Q^i = \lambda^\alpha \frac{\partial L}{\partial\ \partial_i q^\alpha},$$

with $\partial_i Q^i = 0$.

Further, the energy function associated with system is

$$E = \dot{q}^{\alpha}\frac{\partial L}{\partial \dot{q}^{\alpha}} - L.$$

The energy function (also known as Hamiltonian of the system) is a constant of motion provided there is no explicit time dependence in the Lagrangian [6, 7, 8]. Moreover, if the conserved quantity due to the symmetry has some physical analogy [105] then the Noether symmetry approach can be used to identify the reliable model. In the present context, the application of Noether symmetry is two fold–to simplify the evolution equations and to solve the physical problem exactly.

In this section for solving the field equations (2.5–2.7) we use the above symmetry approach. Then according to this symmetry a Lagrangian admits Noether symmetry if there exist a vector valued function $G(t, a, \phi, \psi)$ which satisfies [4, 5]

$$X^{[1]}L + LD_t\xi(t, a, \phi, \psi) = D_tG(t, a, \phi, \psi), \tag{2.14}$$

under the vector field

$$X = \xi(t, a, \phi, \psi)\frac{\partial}{\partial t} + \alpha(t, a, \phi, \psi)\frac{\partial}{\partial a} + \beta(t, a, \phi, \psi)\frac{\partial}{\partial \phi} + \gamma(t, a, \phi, \psi)\frac{\partial}{\partial \psi},$$

where the total derivative operator D_t is given by $D_t \equiv \frac{\partial}{\partial t} + \dot{a}\frac{\partial}{\partial a} + \dot{\phi}\frac{\partial}{\partial \phi} + \dot{\psi}\frac{\partial}{\partial \psi}$, and $X^{[1]}$ is the first prolongation vector given by

$$X^{[1]} = X + (D_t\alpha - \dot{a}D_t\xi)\frac{\partial}{\partial \dot{a}} + (D_t\beta - \dot{\phi}D_t\xi)\frac{\partial}{\partial \dot{\phi}} + (D_t\gamma - \dot{\psi}D_t\xi)\frac{\partial}{\partial \dot{\psi}}, \tag{2.15}$$

and the conserved quantity associated with the vector field X is defined by

$$I = \xi L + (\alpha - \dot{a}\xi)\frac{\partial L}{\partial \dot{a}} + (\beta - \dot{\phi}\xi)\frac{\partial L}{\partial \dot{\phi}} + (\gamma - \dot{\psi}\xi)\frac{\partial L}{\partial \dot{\psi}} - G. \tag{2.16}$$

Now if we assume that, the Lagrangian (2.4) admits the Noether symmetry on the tangent space $(a, \dot{a}, \phi, \dot{\phi}, \psi, \dot{\psi})$ then we get the following set of partial differential equations

$$
\begin{aligned}
\xi_a = \xi_\phi = \xi_\psi &= 0, && (2.17)\\
-3\alpha a^2 V(\phi) - \beta a^3 V'(\phi) - a^3 V(\phi)\xi_t &= G_t, && (2.18)\\
6a\alpha_t = G_a, \ \ -a^3\beta_t = G_\phi, \ \ -a^3 F(\phi)\gamma_t &= G_\psi, && (2.19)\\
-3\alpha - 6a\frac{\partial \alpha}{\partial a} + 3a\frac{\partial \xi}{\partial t} &= 0, && (2.20)\\
\frac{3}{2}\alpha a^2 + a^3\frac{\partial \beta}{\partial \phi} - \frac{1}{2}a^3\frac{\partial \xi}{\partial t} &= 0, && (2.21)\\
\frac{3}{2}\alpha a^2 F(\phi) + \frac{a^3}{2}\beta F'(\phi) + a^3 F(\phi)\frac{\partial \gamma}{\partial \psi} - \frac{1}{2}a^3 F(\phi)\frac{\partial \xi}{\partial t} &= 0, && (2.22)\\
-6a\frac{\partial \alpha}{\partial \phi} + a^3\frac{\partial \beta}{\partial a} &= 0, && (2.23)\\
-6a\frac{\partial \alpha}{\partial \psi} + a^3 F(\phi)\frac{\partial \gamma}{\partial a} &= 0, && (2.24)\\
a^3\frac{\partial \beta}{\partial \psi} + a^3 F(\phi)\frac{\partial \gamma}{\partial \phi} &= 0. && (2.25)
\end{aligned}
$$

Now the existence of the Noether symmetry demands the co-efficients of the infinitesimal generator (i.e., α, β, γ) have to satisfy the overdetermined set of partial differential equations. Thus, by using the method of separation of variables i.e.,

$$
\begin{aligned}
\alpha &= \alpha_1(a)\alpha_2(\phi)\alpha_3(\psi),\\
\beta &= \beta_1(a)\beta_2(\phi)\beta_3(\psi),\\
\gamma &= \gamma_1(a)\gamma_2(\phi)\gamma_3(\psi),
\end{aligned}
$$

we get the explicit form of α, β and γ are given by

$$
\begin{aligned}
\alpha &= \frac{c_\alpha e^{m\phi} + c_\beta e^{-m\phi}}{a^{\frac{1}{2}}}, && (2.26)\\
\beta &= \frac{-4m}{a^{\frac{3}{2}}}\left(c_\alpha e^{m\phi} - c_\beta e^{-m\phi}\right), && (2.27)\\
\gamma &= \gamma_0, \ \xi = m_3 t + m_4, && (2.28)
\end{aligned}
$$

where $c_\alpha, c_\beta, m_3, m_4$ and γ_0 are being arbitrary constants with $m^2 = \frac{3}{8}$.

Also from equation (2.18) and (2.22) we get the unknown potential function and the coupling function which depends on the scalar field (ϕ) having the expression

$$V(\phi) = V_0\Big(c_\alpha e^{m\phi} - c_\beta e^{-m\phi}\Big)^2, \tag{2.29}$$
$$F(\phi) = F_0\Big(c_\alpha e^{m\phi} - c_\beta e^{-m\phi}\Big)^2, \tag{2.30}$$

where V_0, F_0 are being the integration constants. Hence on the tangent space the infinitesimal generator turns out to be

$$\begin{aligned}
X_1 &= t\frac{\partial}{\partial t} + \psi\frac{\partial}{\partial \psi} - \frac{a}{3}\frac{\partial}{\partial a},\\
X_2 &= \frac{\partial}{\partial t},\\
X_3 &= \frac{\partial}{\partial \psi},\\
X_4 &= \frac{2}{a^{\frac{1}{2}}}\cosh(m\phi)\frac{\partial}{\partial a} - \frac{8m}{a^{\frac{3}{2}}}\sinh(m\phi)\frac{\partial}{\partial \phi}.
\end{aligned} \tag{2.31}$$

From the Lagrangian in equation (2.4) one may define the kinetic metric as

$$ds_k^2 = -6a\ da^2 + a^3\ d\phi^2 + 4F_0^2 a^3 \sinh^2\phi\ d\psi^2,$$

and the effective potential $V_{eff}(\phi) = 4V_0 a^3 \sinh^2\phi$. By choosing $u = a^{\frac{3}{2}}$ the above kinetic metric can be written as (with suitable scaling)

$$ds_k^2 = -du^2 + u^2(d\phi^2 + \sinh^2\phi\ d\psi^2).$$

Here the 2D metric h_{AB} with $d\sigma^2 = d\phi^2 + \sinh^2\phi\ d\psi^2$ corresponds to the representation of the $so(3)$ Lie Algebra. Further, it is possible to have a different representation of the $so(3)$ Lie algebra so that the above 2D metric can be written as [92]

$$h_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\phi} \end{pmatrix}.$$

Thus the model can be reduced to describe hyperbolic inflation [106].

2.4 Analytical Solution

In this section our aim is to find the exact cosmological solutions of this multiscalar field model. As the existence of the Noether symmetry implies that there exist a cyclic co-ordiante, so we shall now turn our attention to find a new co-ordiante system in such a way that one of the variables become cyclic. So by point transformation $(t, a, \phi, \psi) \to (s, u, v, w)$ the vector field X transformed into

$$\begin{aligned} \overrightarrow{X}_T &= (i_{\overrightarrow{X}}dt)\frac{\partial}{\partial t} + (i_{\overrightarrow{X}}du)\frac{\partial}{\partial u} + (i_{\overrightarrow{X}}dv)\frac{\partial}{\partial v} + (i_{\overrightarrow{X}}dw)\frac{\partial}{\partial w} \\ &+ \left(\frac{d}{dt}(i_{\overrightarrow{X}}du)\right)\frac{d}{d\dot{u}} + \left(\frac{d}{dt}(i_{\overrightarrow{X}}dv)\right)\frac{d}{d\dot{v}} + \left(\frac{d}{dt}(i_{\overrightarrow{X}}dw)\right)\frac{d}{d\dot{w}}, \end{aligned} \tag{2.32}$$

such that $i_{X_4}ds = 0, i_{X_4}du = 0, i_{X_4}dv = 0$ and $i_{X_4}dw = 1$, i.e.,

$$\begin{aligned} \xi\frac{\partial s(t,a,\phi,\psi)}{\partial t} + \alpha\frac{\partial s(t,a,\phi,\psi)}{\partial a} + \beta\frac{\partial s(t,a,\phi,\psi)}{\partial \phi} + \gamma\frac{\partial s(t,a,\phi,\psi)}{\partial \psi} &= 0, \\ \xi\frac{\partial u(t,a,\phi,\psi)}{\partial t} + \alpha\frac{\partial u(t,a,\phi,\psi)}{\partial a} + \beta\frac{\partial u(t,a,\phi,\psi)}{\partial \phi} + \gamma\frac{\partial u(t,a,\phi,\psi)}{\partial \psi} &= 0, \\ \xi\frac{\partial v(t,a,\phi,\psi)}{\partial t} + \alpha\frac{\partial v(t,a,\phi,\psi)}{\partial a} + \beta\frac{\partial v(t,a,\phi,\psi)}{\partial \phi} + \gamma\frac{\partial v(t,a,\phi,\psi)}{\partial \psi} &= 0, \\ \xi\frac{\partial w(t,a,\phi,\psi)}{\partial t} + \alpha\frac{\partial w(t,a,\phi,\psi)}{\partial a} + \beta\frac{\partial w(t,a,\phi,\psi)}{\partial \phi} + \gamma\frac{\partial w(t,a,\phi,\psi)}{\partial \psi} = 1 &\quad . \end{aligned} \tag{2.33}$$

Now solving equation (2.33) one can get the form of scale factor 'a' and the scale 'ϕ' and 'ψ' in terms of new co-ordiantes (choosing $c_\alpha = c_\beta$)

$$\begin{aligned} u &= a^{\frac{3}{2}}\sinh(m\phi), \\ v &= \psi, \\ w &= \frac{2}{3}a^{\frac{3}{2}}\cosh(m\phi), \\ s &= t. \end{aligned} \tag{2.34}$$

As a consequence the transformed Lagrangian takes the form

$$L = \frac{4}{3}\dot{u}^2 - 3\dot{w}^2 + f_0u^2\dot{v}^2 + 4v_0u^2, \tag{2.35}$$

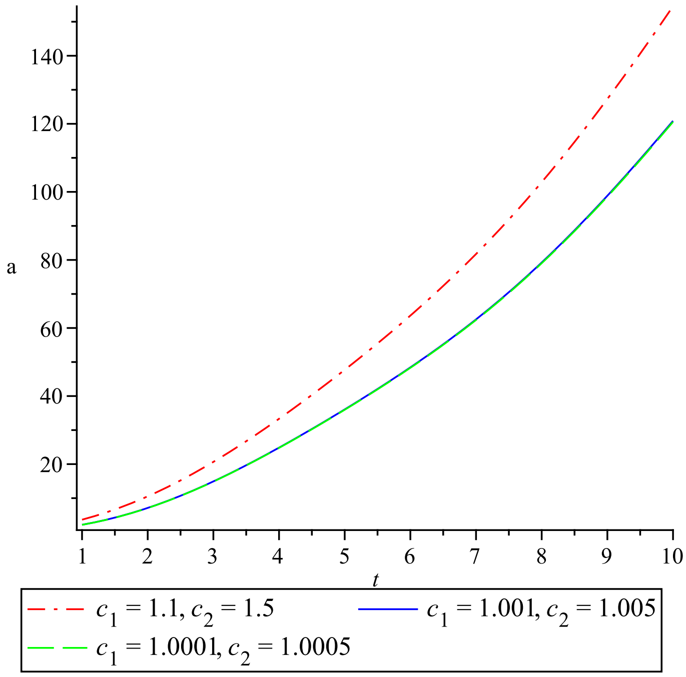

Figure 2.1: Graphical representation of the scale factor with respect to cosmic time t

and the conserved energy in terms of new variable look like

$$E = \frac{4}{3}\dot{u}^2 - 3\dot{w}^2 + f_0 u^2 \dot{v}^2 - 4v_0 u^2. \tag{2.36}$$

Hence the Euler Lagrange equation in the new coordiantes system, using equation (2.35) leads to

$$\begin{aligned} 4\ddot{u} - 3f_0 u\dot{v}^2 - 12uv_0 &= 0, \\ 2f_0 u^2 \dot{v} &= \text{constant}, \\ \ddot{w} &= 0. \end{aligned} \tag{2.37}$$

Thus by solving (2.37) we get the explicit form of the scale factor of the Universe, the scalar fields and the potential and coupling function of the Universe as

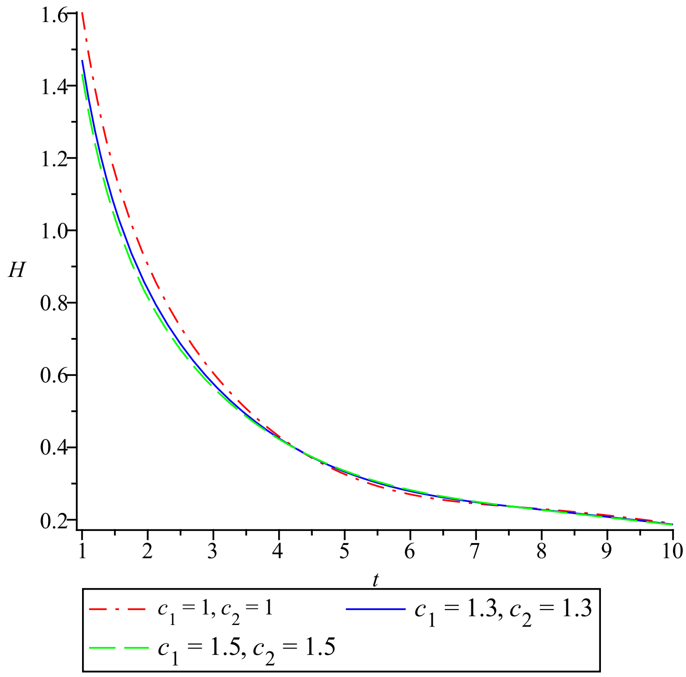

Figure 2.2: Represents the Hubble parameter with respect to cosmic time t

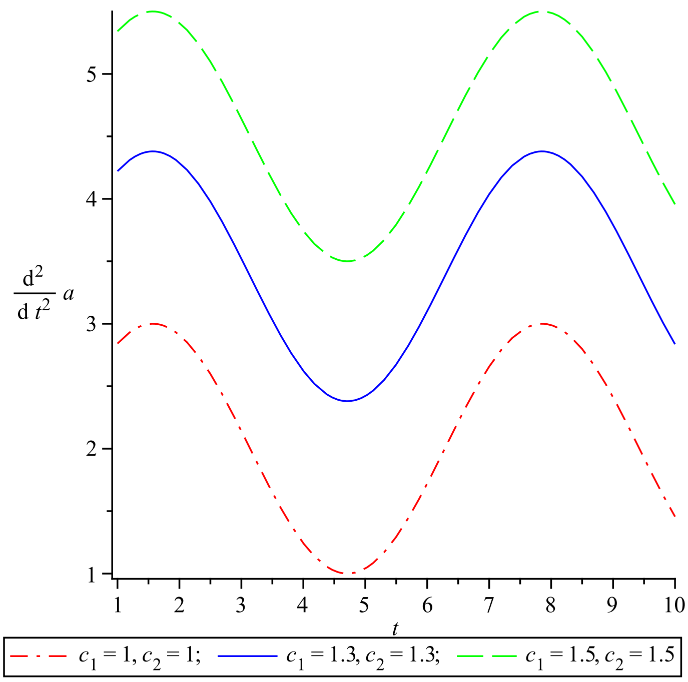

Figure 2.3: Presents the acceleration parameter with respect to cosmic time t

$$
\begin{aligned}
a &= \left[\left(c_1 t + c_2\right)^2 - \sin t - c_2^2\right]^{\frac{1}{3}}, \\
\phi &= \tanh^{-1}\left\{\frac{\sqrt{\sin t + c_2^2}}{(c_1 t + c_2)}\right\}, \\
\psi &= \tan^{-1}\left\{\frac{s\tan(\frac{t}{2}) + 2}{\sqrt{s^2 - 1}}\right\}, \\
V(\phi) &= V_0 \sinh^2\left[\tanh^{-1}\left\{\frac{\sqrt{\sin t + c_2^2}}{(c_1 t + c_2)}\right\}\right], \\
F(\phi) &= F_0 \sinh^2\left[\tanh^{-1}\left\{\frac{\sqrt{\sin t + c_2^2}}{(c_1 t + c_2)}\right\}\right], \qquad (2.38)
\end{aligned}
$$

with c_1, c_2, s, V_0, F_0 as integration constants.

2.5 Concluding remarks

In this chapter, a complicated coupled cosmological model has been solved analytically using symmetry analysis. The application of Noether symmetry to any physical system has two fold advantages namely (i) the symmetry conditions determine the symmetry vector field as well as any unknown parameter or function in the system (instead of choosing phenomenologically), (ii) using suitable transformation in the augmented space (choosing a cyclic coordinate) the Lagrangian as well as the evolution equations are simplified to a great extend so that analytic solutions can be obtained.

In the previous section coupled two scalar field cosmological model has been solved completely using Noether symmetry analysis and the cosmological parameters namely the scale factor, Hubble parameter and the acceleration parameter have been shown graphically in Figures (2.1–2.3). The figure shows that the present cosmological model is an expanding model of the Universe in an accelerating phase. Also the Hubble parameter gradually decreases with time in accordance with observational evidences and it does not vanish in finite time. The model is useful to study hyperbolic inflation. Further from the point of view of stability criterion we have analyzed the solution for different choices of the parameter involved with differ infinitesimally (see Figure.(2.1)). It is found that the solution also differ infinitesimally. So the solution can be consider as stable. Finally one may conclude that symmetry analysis is a powerful mathematical tool which is useful to analyze complicated phenomenological cosmological model completely.

CHAPTER 3

NOETHER SYMMETRY ANALYSIS IN CHAMELEON FIELD COSMOLOGY

3.1 Prelude

Standard cosmology has been facing a great challenge since the end of the last century. The observational evidences since 1998 [11, 13, 18, 26, 107] are not in favour of decelerated expansion (prediction of standard cosmology) rather they are in favour of accelerated expansion. So far there are two options proposed by the cosmologists to accommodate these observational evidences. One of the possibilities is to introduce some exotic matter (known as dark energy) within the frame work of Einstein gravity. This mysterious matter component is totally of unknown nature except its large negative pressure. At first cosmologists took cosmological constant as the dark energy (DE) candidate. But due to two severe drawbacks (namely discrepency in its predicted and observed value and coincidence problem [34]) the cosmological constant is not well accepted model DE rather dynamical DE models [29, 30, 31, 32] are widely used in the literature. The chapter is an example of using dynamical DE model. Usually, a scalar field having potential $V(\phi)$ is chosen as DE candidate so that the pressure component $p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$ can evolve to have the required negative value for observed accelerated expansion. Here the scalar field (chosen as dynamical DE) is non-minimal coupled to dark matter (DM) through an interference term in the action [108]. As a result, there is a new source term in the matter conservation equation. This kind of DE model is termed as chameleon field. This model is quite useful to obtain accelerated expansion of the Universe and other interesting cosmological consequences [109] (for details see the review [110]).

On the other hand, since the last century, symmetry analysis has a significant role in studying global continuous symmetries (i.e, translation, rotation etc.) as well as in local gauge symmetries, internal symmetries of the space-time (in cosmology) and permutation symmetry in quantum field theory [111, 112]. In particular, geometrical symmetries of the space-time and symmetries of the physical system have great role in analyzing any physical motion. From the perspective of Noether symmetry, the conserved charge has been used to identify the actual one among similar physical processes. Further, in Noether symmetry approach, the Noether integral (i.e, the first integral) has been chosen as a tool for simplification of a system of differential equations or for the integrability of the system [6, 7, 113, 114, 115, 116].

In addition, an advantage of using Noether symmetry to any physical system involving arbitrary physical parameters or some arbitrary functions of the field variables is that symmetry analysis uniquely determine these physical parameters or arbitrary functions involved (for details see ref. [117]). Also since recent past symmetry analysis has been used for physical systems in Riemannian spaces [2, 118, 119, 120, 121, 122, 123, 124, 125].

Moreover, Noether symmetry analysis has opened new windows in studying quantum cosmology with suitable operator ordering, the Wheeler DeWitt (WD) equation so constructed on the minisuperspace is associated with Lie point symmetries. It is possible to have a subset of the general solutions of the WD equation having oscillatory behaviours [103, 126] by imposing Noether symmetries. The Noether symmetries with Hartle criterion can identify those classical trajectories in minisuperspace [127, 128] which are solutions of the cosmological evolution equations i.e, one may consider Noether symmetries as a bridge between quantum cosmology and classical observable Universe.

In this chapter, Noether symmetry analysis is used to both classical and quantum cosmology for chameleon field DE model. By imposing Noether symmetry to the Lagrangian and making canonical transformation of the dynamical variables, it is possible to have classical solutions of the coupled non-linear Einstein field equations. WD equation is constructed for the present chameleon DE cosmological model in the background of FLRW space-time and Noether symmetry is used as a tool to solve the WD equation. This chapter is organised as follows: a brief overview of Noether symmetry are described in Section–3.2 whereas Section–3.3 presents the Noether symmetry and cosmological solutions to Chameleon field dark energy model and Section-3.4 deals with quantum cosmology in the minisuperspace approach: a general prescription and the formation of WD equation in the present cosmological model and possible solution with Noether symmetry are presented in Section-3.5, finally the chapter ends with a conclusion in Section-3.6.

3.2 A Brief Overview of Noether Symmetry approach

Noether's first theorem states that every differentiable symmetry of the action of a physical system with conservative forces has a corresponding conservation law i.e, the lie derivative [129, 130] of the Lagrangian [102, 103] of any physical system, associated to some conserved quantities, will be invariant along an appropriate vector field ($\mathcal{L}_{\overrightarrow{X}} f = \overrightarrow{X}(f)$).

The Noether symmetry approach has already been discussed in Section 2.3.

In the context of quantum cosmology, Hamiltonian formulation is very useful and Noether symmetry condition is rewritten as [131]

$$\mathcal{L}_{\overrightarrow{X}_H} H = 0, \tag{3.1}$$

with

$$\overrightarrow{X}_H = \dot{q}\frac{\partial}{\partial q} + \ddot{q}\frac{\partial}{\partial \dot{q}}.$$

In minisuperspace models of quantum cosmology, symmetry analysis determines appropriate interpretation of the wave function. The conserved canonically conjugate momenta due to Noether symmetry can be written as

$$\Pi_l = \frac{\partial L}{\partial q^l} = \Sigma_l, \tag{3.2}$$

$l = 1, 2, ...m$,

with 'm' denoting the no of symmetries. Also the operator version (i.e, quantization) of equation (3.2) i.e,

$$-i\partial_{q^l} |\psi\rangle = \Sigma_l |\psi\rangle, \tag{3.3}$$

identifies a translation along q^l-axis through symmetry analysis. Also equation (3.3) has oscillatory solution for real conserved quantity Σ_l i.e,

$$|\psi\rangle = \sum_{l=1}^{m} e^{i\Sigma_l q^l} \left|\phi(q^k)\right\rangle, k < n. \tag{3.4}$$

Here the index 'k' stands for directions along which there is no symmetry with n, the dimension of the minisuperspace. In the above expression for the wave function $|\phi(q^k)\rangle$ denotes the non-oscillatory part of the wave function with $m + k = n$, the dimension of the minisuperspace. Thus oscillatory part of the wave function implies existence of Noether symmetry and the conjugate momenta along the symmetry directions should be conserved and vice-versa (Hartle [132]). Due to symmetries the first integrals of motion identify the classical trajectories. In fact, for 2D minisuperspace, it is possible to have complete solution of the system by Noether symmetry.

Now, the Wheeler-DeWitt equation, the fundamental equation in quantum cosmology can be written as $\hat{H}\psi = 0$, where $\hat{H}$ is the operator version of the Hamiltonian of the system and ψ is the wave function of the Universe. Also one has to take into account the operator ordering problem in course of quantization process.

3.3 Noether symmetry and cosmological solutions to chameleon field Dark Energy Model

This section is devoted to study chameleon field dark energy cosmological model. This model consists of a canonical scalar field (having self interaction potential) non-minimally coupled to dark matter (DM). So the potential function and the coupling function are the two unknown functions of the scalar field. The action integral of the model has the explicit form [48, 49]

$$I = \int \left[\frac{R}{16\pi G} + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi) + f(\phi)L_m\right]\sqrt{-g}d^4x, \tag{3.5}$$

where as usual R is the Ricci scalar, G is the Newtonian gravitational constant and ϕ is the chameleon scalar field having potential $V(\phi)$. Here L_m is the Lagrangian for dark matter which is non-minimally coupled to the chameleon scalar field with $f(\phi)$ (an analytic function), the coupling function. By choosing the dark matter to be an ideal gas the matter

Lagrangian can be chosen as $L_m \simeq \rho_m$ [133].

In the background of flat FLRW space-time the point-like Lagrangian for the above cosmological model takes the form

$$L(a,\dot{a},\phi,\dot{\phi}) = 3a\dot{a}^2 - a^3\left(\frac{\dot{\phi}^2}{2} - V(\phi)\right) - \rho_0\omega f(\phi)a^{-3\omega}, \tag{3.6}$$

where $\omega = \frac{p}{\rho}$ is the equation of state parameter for hot dark matter chosen as perfect fluid and ρ_0 is the integration constant. Now the Euler-Lagrange equations (i.e, the Einstein field equations) for the Lagrangian (3.6) are given by

$$3\frac{\dot{a}^2}{a^2} = \rho_0 f a^{-3(\omega+1)} + \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{3.7}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_0\omega f a^{-3(\omega+1)}, \tag{3.8}$$

where an over dot indicates differentiation with respect to the cosmic time 't'. Furthermore, the equation of motion $T^{\mu\nu}_{;\nu} = 0$ for the cosmological fluid with energy momentum tensor $T_{\mu\nu} = T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu}$ is given by

$$\dot{\phi}\ddot{\phi} + 3H\dot{\phi}^2 + v(\phi)\dot{\phi} + \rho_m f'(\phi)\dot{\phi} + \dot{\rho_m} f(\phi) + 3H(1+\omega)\rho_m f(\phi) = 0. \tag{3.9}$$

One may note that among these three evolution equations (3.7)-(3.9), only two are independent while (3.7) is termed as constraint equation.

As in the present cosmological model there is the interaction term $f(\phi)$ so one has $\left(T^{(\phi)\mu\nu}\right)_{;\nu} = -Q$, $\left(T^{(m)\mu\nu}\right)_{;\nu} = Q$ or eqivalently

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) + f'(\phi)\rho_m = -\frac{Q}{\dot{\phi}}, \tag{3.10}$$

and

$$\dot{\rho_m} f(\phi) + 3H(1+\omega)\rho_m f(\phi) = Q. \tag{3.11}$$

As the present model reduces to that of Weyl integrable gravity with $f(\phi) = f_0 e^{\lambda\phi}$ [134] and setting $Q = \alpha\rho_m f'(\phi)\dot{\phi}$ (where α is a non zero constant), equation (3.10) and (3.11) become

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) + (1+\alpha)f'(\phi)\rho_m = 0, \tag{3.12}$$

and

$$\dot{\rho_m} f(\phi) + 3H(1+\omega)\rho_m f(\phi) - \alpha f'(\phi)\dot{\phi}\rho_m = 0. \tag{3.13}$$

The matter conservation equation (3.13) can be integrated to have

$$\rho_m(t) = \rho_0 a^{-3(1+\omega)}\{f(\phi)\}^{\alpha}. \tag{3.14}$$

Thus the scalar field evolution equation (i.e, the modified Klein-Gordon equation) (3.12) becomes

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = -\rho_0 a^{-3(1+\omega)}\left[\{f(\phi)\}^{\alpha+1}\right]. \tag{3.15}$$

The configuration space for the present model is a 2D space (a, ϕ) and the infinitesimal generator for the Noether symmetry takes the form

$$\overrightarrow{X} = p\frac{\partial}{\partial a} + q\frac{\partial}{\partial \phi} + \dot{p}\frac{\partial}{\partial \dot{a}} + \dot{q}\frac{\partial}{\partial \dot{\phi}}, \tag{3.16}$$

where $p = p(a, \phi)$ and $q = q(a, \phi)$ are the unknown coefficients with $\dot{p} = \frac{\partial p}{\partial a}\dot{a} + \frac{\partial p}{\partial \phi}\dot{\phi}$ and similarly for $\dot{q}$.

These coefficients of the Noether symmetry vector are determined from an overdetermined system of partial differential equation, obtained by imposing Noether symmetry to the Lagrangian i.e,

$$\mathcal{L}_{\overrightarrow{X}} L = 0,$$

i.e,

$$\begin{aligned} p + 2a\frac{\partial p}{\partial a} &= 0, \\ 3p + 2a\frac{\partial q}{\partial \phi} &= 0, \\ 6\frac{\partial p}{\partial \phi} - a^2\frac{\partial q}{\partial a} &= 0, \end{aligned} \tag{3.17}$$

with a differential equation for the potential and coupling function as

$$3\rho_0\omega p a^{-3\omega-1}F(\phi) + 3pa^2V(\phi) + qa^3V'(\phi) - \rho_0 a^{-3\omega}qF'(\phi) = 0, \tag{3.18}$$

where $F(\phi) = \{f(\phi)\}^{\alpha+1}$

The above set of partial differential equations (3.17) are solvable using the method of separation of variables i.e, $p(a,\phi) = p_1(a)p_2(\phi)$, $q(a,\phi) = q_1(a)q_2(\phi)$ as

$$p = a^{-\frac{1}{2}}\left(c_p e^{m\phi} + c_q e^{-m\phi}\right),$$

$$q = -4ma^{-\frac{3}{2}}\left(c_p e^{m\phi} - c_q e^{-m\phi}\right), \tag{3.19}$$

where $m^2 = \dfrac{3}{8}$; c_p, c_q and q_0 are arbitrary constants. Using the above solutions (3.19) into (3.18), the solutions for $V(\phi)$ and $f(\phi)$ can take the form (with $\omega = -1$)

$$V(\phi) - \rho_0 F(\phi) = k\left(c_p e^{m\phi} - c_q e^{-m\phi}\right)^2, \tag{3.20}$$

where k is a positive integration constant.

Thus, the infinitesimal generator of the Noether symmetry is determined (except for arbitrary integration constants) by imposing symmetry condition which in turn determines the potential function and the coupling function.

Another important issue related to Noether symmetry is the conserved quantities associated with it. In general for a field theory in curved space there is no well-defined notion of energy. However, the conserved quantity derived from Noether's theorem is the energy-momentum tensor. In particular, when the system has time-like killing vector then there is an associated conserved energy. Though FLRW space-time has no time-like killing vector field, but the Lagrangian density is explicit time independent. Hence in analogy with point-like Lagrangian, it is possible to define an energy which will be conserved in nature. Thus in the context of Noether symmetry to the present cosmological model one can have two conserved quantities namely conserved charge and conserved energy having explicit form

$$Q = 6\dot{a}a^{\frac{1}{2}}\left(c_p e^{m\phi} + c_q e^{-m\phi}\right) + a^3\dot{\phi}\left\{4ma^{-\frac{3}{2}}\left(c_p e^{m\phi} - c_q e^{-m\phi}\right)\right\},$$

$$E = 3a\dot{a}^2 - \frac{1}{2}a^3\dot{\phi}^2 - a^3 V(\phi) + \rho_0 F(\phi)a^{-3\omega}. \tag{3.21}$$

Usually, associated with Noether symmetry there is a conserved current, whose time component integrating over spatial volume gives a conserved charge. But in the present context as all the variables are time dependent only so Q defined in (3.21) is the Noether charge. Moreover, the above conserved charge can be expressed geometrically as the inner product of the infinitesimal generator with cartan one form [135] as

$$Q = i_{\overrightarrow{X}} \theta_L, \tag{3.22}$$

where $i_{\overrightarrow{X}}$ denotes the inner product with the vector field $\overrightarrow{X}$ and

$$\theta_L = \frac{\partial L}{\partial a} da + \frac{\partial L}{\partial \phi} d\phi, \tag{3.23}$$

is termed as cartan one form.

On the otherhand, this geometric inner product representation is useful to find out cyclic variables in the Lagrangian. In context of solving coupled non-linear evolution equations, determination of cyclic variables will be very useful as not only the Lagrangian but also the evolution equations will be simplified to a great extend.

In the present context the transformation of the 2D augmented space: $(a, \phi) \to (u, v)$ transform the symmetry vector as

$$\overrightarrow{X_T} = \left(i_{\overrightarrow{X}} du\right) \frac{\partial}{\partial u} + \left(i_{\overrightarrow{X}} dv\right) \frac{\partial}{\partial v} + \left\{\frac{d}{dt}\left(i_{\overrightarrow{X}} du\right)\right\} \frac{d}{d\dot{u}} + \left\{\frac{d}{dt}\left(i_{\overrightarrow{X}} dv\right)\right\} \frac{d}{d\dot{v}}. \tag{3.24}$$

Geometrically, $\overrightarrow{X_T}$ may be interpreted as the lift of a vector field on the augmented space. Now, without any loss of generality we restrict the above point transformation to [135]

$$i_{\overrightarrow{X}} du = 1 \quad \text{and} \quad i_{\overrightarrow{X}} dv = 0, \tag{3.25}$$

so that

$$\overrightarrow{X_T} = \frac{\partial}{\partial u} \quad \text{and} \quad \frac{\partial L_T}{\partial u} = 0, \tag{3.26}$$

i.e, u is the cyclic variable. The above geometric process of identification of cyclic variables can be interpreted as to choose the transformed infinitesimal generator along any co-ordinate line (identified as the cyclic variable) [114].

Now the explicit form of the above point transformation (3.25) are the first order linear partial differential equations having solution

Case I: $c_p = c_q$

$$
\begin{aligned}
u &= \frac{2}{3} a^{\frac{3}{2}} \cosh m\phi, \\
v &= a^{\frac{3}{2}} \sinh m\phi.
\end{aligned} \tag{3.27}
$$

<u>Case II:</u> $c_p \neq c_q$

$$
\begin{aligned}
u &= \frac{1}{6c_pc_q} a^{\frac{3}{2}} \left(c_p e^{m\phi} + c_q e^{-m\phi}\right), \\
v &= a^{\frac{3}{2}} \left(c_p e^{m\phi} - c_q e^{-m\phi}\right).
\end{aligned} \tag{3.28}
$$

The simplified Lagrangian in the new variables has the form:

$$
L = 3\dot{u}^2 - \frac{4}{3}\dot{v}^2 + 4kc_p^2 v^2, \qquad \text{(Case I)} \tag{3.29}
$$

$$
= 12c_pc_q\dot{u}^2 - \frac{1}{3c_pc_q}\dot{v}^2 + kv^2. \qquad \text{(Case II)} \tag{3.30}
$$

The conserved quantities in the new variables can be expressed as

$$
\begin{aligned}
Q &= 6\dot{u}, \\
E &= 3\dot{u}^2 - \frac{4}{3}\dot{v}^2 - 4kc_p^2 v^2. \qquad \text{(Case I)}
\end{aligned}
$$

and

$$
\begin{aligned}
Q &= 24c_pc_q\dot{u}, \\
E &= 12c_pc_q\dot{u}^2 - \frac{1}{3c_pc_q}\dot{v}^2 - kv^2. \qquad \text{(Case II)}
\end{aligned}
$$

Now solving the Euler-Lagrange equations for the new Lagrangian the new augmented variables has the form:

$$
\begin{aligned}
u &= At + B, \\
v &= k_1 \cos\sqrt{3k}c_pt + k_2 \sin\sqrt{3k}c_pt, \qquad \text{(Case I)}
\end{aligned}
$$

and

$$\begin{aligned} u &= rt + s, \\ v &= k_1' \cos\sqrt{3c_pc_qk}t + k_2' \sin\sqrt{3c_pc_qk}. \qquad \text{(Case II)} \end{aligned}$$

Hence the cosmic scale factor and the chameleon scalar field has the explicit expression:

$$\begin{aligned} a(t) &= \left[\frac{9}{4}(At+B)^2 - \left(k_1\cos\sqrt{3k}c_pt + k_2\sin\sqrt{3k}c_pt\right)^2\right]^{\frac{1}{3}}, \\ \phi(t) &= \frac{2\sqrt{2}}{3}\tanh^{-1}\left[\frac{2\left(k_1\cos\sqrt{3k}c_pt + k_2\sin\sqrt{3k}c_pt\right)}{3(At+B)}\right], \qquad \text{(Case I)} \end{aligned}$$

and

$$\begin{aligned} a(t) &= \left[9c_pc_q(rt+s)^2 - \frac{1}{4c_pc_q}\left(k_1'\cos\sqrt{3c_pc_qk}t + k_2'\sin\sqrt{3c_pc_qk}t\right)^2\right]^{\frac{1}{3}}, \\ \phi(t) &= \frac{2\sqrt{2}}{\sqrt{3}}\ln\frac{6c_pc_q(rt+s) + \left(k_1'\cos\sqrt{3c_pc_qk}t + k_2'\sin\sqrt{3c_pc_qk}t\right)}{2c_p\left[9c_pc_q(rt+s)^2 - \frac{1}{4c_pc_q}\left(k_1'\cos\sqrt{3c_pc_qk}t + k_2'\sin\sqrt{3c_pc_qk}t\right)^2\right]^{\frac{1}{2}}}. \qquad \text{(Case II)} \end{aligned}$$

In the above solutions (A, B, k_1, k_2) and (r, s, k_1', k_2') are arbitrary integration constants.

3.4 Quantum Cosmology in the Minisuperspace Approach: A general Prescription

Minisuperspaces are considered as restrictions of geometrodynamics of the superspace and physically important and interesting models are defined on minisuperspaces. In cosmology, the simplest and widely used minisuperspace models are homogeneous and isotropic merics and matter fields and consequently the lapse function is homogeneous (i.e, $N = N(t)$) while shift function vanishes identically. So in 4D manifold, using $(3+1)$-decomposition the metric can be written as

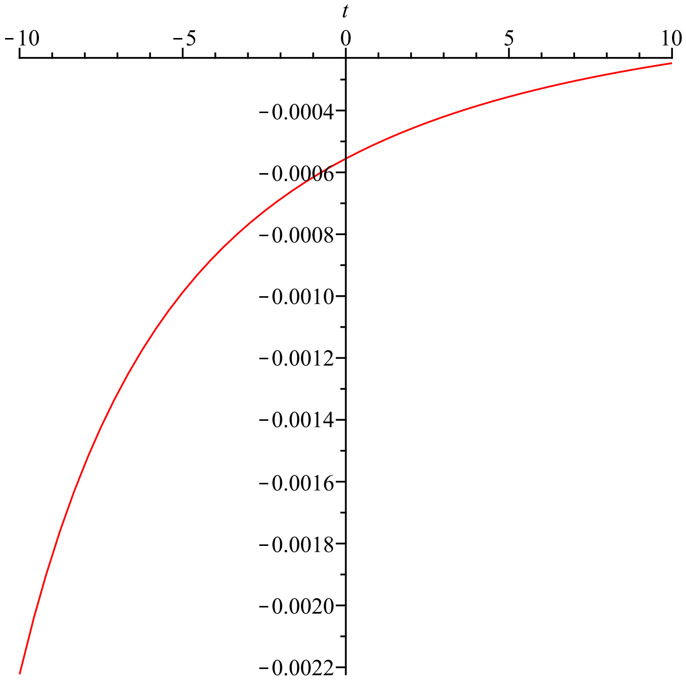

Figure 3.1: Graphical representation of $\frac{\ddot{a}}{a}$ with respect to cosmic time t when $c_p = c_q$

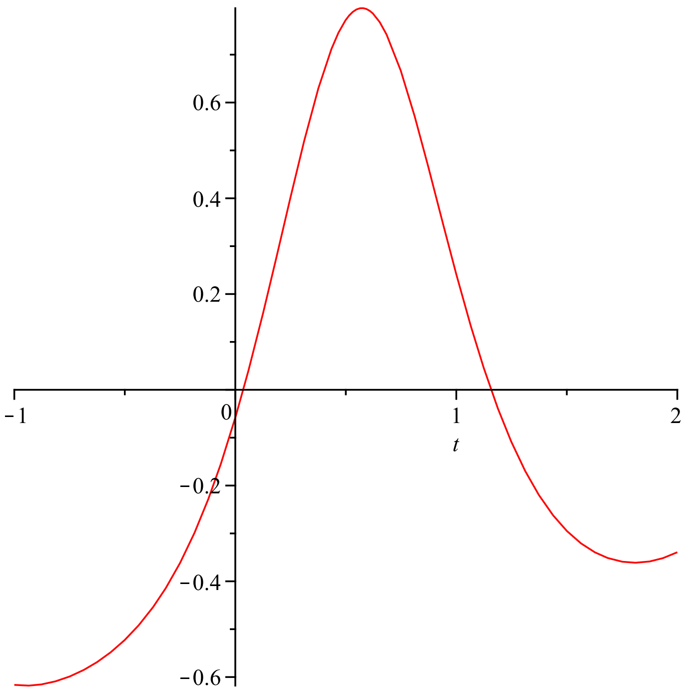

Figure 3.2: Represents $\frac{\ddot{a}}{a}$ with respect to cosmic time t when $c_p \neq c_q$

$$ds^2 = -N^2(t)dt^2 + h_{ab}(x,t)dx^a dx^b, \tag{3.31}$$

and the Einstein-Hilbert action can be written as

$$I(h_{ab}, N) = \frac{m^2{}_p}{16\pi} \int dt\ d^3x N\sqrt{h}\left[k_{ab}k^{ab} - k^2 + (3)_R - 2\Lambda\right], \tag{3.32}$$

where k_{ab} is the extrinsic curvature of the 3 space, $k = k_{ab}h^{ab}$ is the trace of the extrinsic curvature, $(3)_R$ is the curvature scalar of the three space and Λ is the cosmological constant.

Now due to homogeneity of the three space, the metric h_{ab} is characterized by a finite number of time functions $q^\alpha(t)$, $\alpha = 0, 1, 2, ..., n-1$ and the above action can be written in the form of a relativistic point particle with self interacting potential in a nD curved space-time as [48, 49]

$$I\left(q^\alpha(t), N(t)\right) = \int_0^1 dtN\left[\frac{1}{2N^2}f_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta - V(q)\right]. \tag{3.33}$$

So the equation of motion of the (equivalent) relativistic particle can be written as (considering variation of the action with respect to the field variables $q^\alpha(t)$)

$$\frac{1}{N}\frac{d}{dt}\left(\frac{\dot{q}^\alpha}{N}\right) + \frac{1}{N^2}\Gamma^\alpha_{\mu\nu}\dot{q}^\mu\dot{q}^\nu + f^{\alpha\beta}\frac{\partial\nu}{\partial q^\beta} = 0, \tag{3.34}$$

with $\Gamma^\alpha_{\beta\gamma}$, the christoffel symbols in the minisuperspace. Also there is a constraint equation obtained by variation with respect to the lapse function as

$$\frac{1}{2N^2}f_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + V(q) = 0. \tag{3.35}$$

For canonical quantization scheme one has to switch over to Hamiltonian formulation. The momenta canonical to q^α is given by

$$p_\alpha = \frac{\partial L}{\partial q^\alpha} = f_{\alpha\beta}\frac{\dot{q}^\beta}{N}. \tag{3.36}$$

So the Hamiltonian is defined as

$$H = p_\alpha\dot{q}^\alpha - L = N\left[\frac{1}{2}f^{\alpha\beta}p_\alpha p_\beta + V(q)\right] = N\mathcal{H}, \tag{3.37}$$

with $f^{\alpha\beta}$ the inverse metric. Using the definition of p_α (i.e, equation (3.36)) into the constraint equation (3.35) one obtains

$$\mathcal{H}(q^\alpha, p_\alpha) \equiv \frac{1}{2} f^{\alpha\beta} p_\alpha p_\beta + V(q) = 0. \tag{3.38}$$

Now, writing p_α as $-i\hbar\frac{\partial}{\partial q_\alpha}$ in quantization scheme, the operator version of the above constraint equation (3.38) on a time independent function (the wave function of the Universe), one gets the Wheeler-DeWitt (WD) equation in quantum cosmology as

$$\mathcal{H}\left(q^\alpha, -i\hbar\frac{\partial}{\partial q^\alpha}\right)\psi(q^\alpha) = 0. \tag{3.39}$$

In general, the minisuperspace metric depends on q^α, so the above WD equation has operator ordering problem. However by imposing the quantization in minisuperspace to be covariant in nature, one may resolve the above operator ordering problem. Further, in the context of quantam cosmology for probability measure, $\exists$ a conserved current for hyperbolic type of partial differential equation as

$$\overrightarrow{J} = \frac{i}{2}(\psi^*\nabla\psi - \psi\nabla\psi^*), \tag{3.40}$$

with $\overrightarrow{\nabla}.\overrightarrow{J} = 0$. Here ψ is the solution of the hyperbolic type WD differential equation. Thus it is possible to define the probability measure on the minisuperspace as

$$dp = |\psi(q^\alpha)|^2 dV, \tag{3.41}$$

where dV is a volume element on minisuperspace.

3.5 Formation of WD Equation in the Present Cosmological Model and possible solution with Noether Symmetry

In the present cosmological model, the 2D configuration space $\{a, \phi\}$ is associated with conjugate momenta, given by

$$\begin{aligned} p_a &= \frac{\partial L}{\partial \dot{a}} = 6a\dot{a}, \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = -a^3\dot{\phi}. \end{aligned} \tag{3.42}$$

So the Hamiltonian of the system (also known as Hamiltonian constraint) can be expressed as

$$\mathcal{H} = \frac{1}{12a}p_a^2 - \frac{1}{2a^3}p_\phi^2 - a^3V(\phi) + \rho_0\omega f(\phi)a^{-3\omega}, \tag{3.43}$$

with equivalent Hamilton's equations of motion

$$\begin{aligned}
\dot{a} &= \frac{1}{6a}p_a, \\
\dot{\phi} &= -\frac{1}{a^3}p_\phi, \\
\dot{p_a} &= \frac{1}{12a^2}p_a^2 - \frac{3}{2a^4}p_\phi^2 + 3a^2V(\phi) + 3\rho_0\omega^2 f(\phi)a^{-3\omega-1}, \\
\dot{p_\phi} &= -a^3V'(\phi) + \rho_0\omega f'(\phi)a^{-3\omega}.
\end{aligned} \tag{3.44}$$

Further, the Lagrangian (i.e, equation 3.6) of the system can be interpreted geometrically, dividing it into two parts. The first two terms are known as kinetic part and the remaining two terms constitute the dynamic part. Also the kinetic part may be viewed as a 2D pseudo-Riemannian space with line element

$$ds^2 = -6ada^2 + a^2d\phi^2. \tag{3.45}$$

This 2D Lorentzian manifold (a, ϕ) is known as minisuperspace (in quantum cosmology). The wave function of the Universe in quantum cosmology is a solution of the WD equation, a 2nd order hyperbolic partial differential equation defined over minisuperspace and it is the operator version of the Hamiltonian constraint.

Further, in the context of WKB approximation one can write the wave function as $\psi(x^k) \sim e^{i\delta(x^k)}$ and hence the WD equation (3.39) becomes first order non-linear partial differential equation which is nothing but (null) Hamilton-Jacobi (H-J) equation in the same geometry.

In quantization of the model one has to construct the WD equation $\hat{\mathcal{H}}\psi(u, v) = 0$, with $\hat{\mathcal{H}}$ the operator version of the Hamiltonian (3.43) and $\psi(u, v)$, the wave function of the Universe. In course of conversion to the operator version there is a problem related to the ordering of a variable and its conjugate momentum [136]. In the first term of the Hamiltonian (3.43) there is a product of 'a' and 'p_a', so one has to consider the ordering consideration: $p_a \rightarrow -i\partial_a, \;\; p_\phi \rightarrow -i\partial_\phi$. As a result there is a two parameter family of Wheeler DeWitt

(WD) equation

$$\left[-\frac{1}{12}\frac{1}{a^l}\frac{\partial}{\partial a}\frac{1}{a^m}\frac{\partial}{\partial a}\frac{1}{a^n}+\frac{1}{a^3}\frac{\partial^2}{\partial\phi^2}-a^3V(\phi)+\rho_0 wf(\phi)a^{-3w}\right]\psi(a,\phi)=0, \tag{3.46}$$

with the triplet of real numbers (l, m, n) satisfying $l+m+n=1$. Due to infinite possible choices for (l, m, n) one may have infinite number of possible ordering. Also the semi classical limit namely the Hamilton Jacobi equation (obtained by substituting $\psi = \exp(is)$) does not regard to the above triplet. In fact, the following choices are commonly used

i) $l = 2, m = -1, n = 0$: D'Alembert operator ordering.
ii) $l = 0 = n, m = 1$: Vilenkin operator ordering.
iii) $l = 1, m = 0 = n$: no ordering.

Thus factor ordering affects the behaviour of the wave function while semi classical results will not be influenced by the above ordering problem. Now choosing the third option (i.e., no ordering) the WD equation for the present model has the explicit form

$$\left[-\frac{1}{12a}\frac{\partial^2}{\partial a^2}+\frac{1}{2a^3}\frac{\partial^2}{\partial\phi^2}-a^3V(\phi)+\rho_0 wf(\phi)a^{-3w}\right]\psi(a,\phi)=0. \tag{3.47}$$

The general solution of the above second order hyperbolic partial differential equation is known as the wave function of the Universe. This solution can be constructed from the separation of the Eigen functions of the above WD operator as

$$\psi(a,\phi)=\int W(Q)\psi(a,\phi,Q)\ dQ, \tag{3.48}$$

with ψ an eigen function of the WD operator, $W(Q)$ a weight function and Q, the conserved charge. Now it is desirable to have wave function in quantum cosmology that is consistent with classical theory. In otherwords, one has to construct a coherent wave packet having good asymptotic behaviour in the minisuperspace and maximize around the classical trajectory. As the minisuperspace variables $\{a, \phi\}$ are highly coupled in the WD operator so it is not possible to have any explicit solution of the WD equation even with separation of variable method. Thus one may analyze the present model in the context of quantum cosmology using the new variables (u, v) (obtained by point transformation) in the augmented space

Case-I: $c_p = c_q$

In this case the Lagrangian is given by equation (3.29) for which u is the cyclic variable. So one has

$$\begin{aligned} p_1 &= \frac{\partial L}{\partial \dot{u}} = 6\dot{u} = \text{Conserved}, \\ p_2 &= \frac{\partial L}{\partial \dot{v}} = -\frac{8}{3}\dot{v}. \end{aligned} \tag{3.49}$$

Hence the Hamiltonian of the system takes the form

$$\mathcal{H} = \frac{1}{12}p_u^2 - \frac{3}{16}p_v^2 - 4kc_p^2 v^2. \tag{3.50}$$

Thus the WD equation takes the form

$$\left[-\frac{1}{12}\frac{\partial^2}{\partial u^2} + \frac{3}{16}\frac{\partial^2}{\partial v^2} - 4kc_p^2 v^2 \right] \chi(u, v) = 0. \tag{3.51}$$

The operator version of the conserved momentum in equation (3.49) can be written as

$$i\frac{\partial \chi(u, v)}{\partial u} = \Sigma_0 \; \chi(u, v). \tag{3.52}$$

Now writing $\chi(u, v) = A(u)B(v)$, one has

$$\begin{aligned} i\frac{dA}{du} &= \Sigma_0 \; A, \\ \text{i.e., } A(u) &= A_0 \exp(-i\Sigma_0 \; u), \end{aligned} \tag{3.53}$$

with A_0, the constant of integration. Using equation (3.53), the WD equation (3.51) becomes a second order ordinary differential equation in B as

$$\begin{aligned} \frac{3}{16}\frac{d^2 B}{dv^2} - 4kc_p^2 v^2 B + \frac{\Sigma_0^2}{12}B &= 0, \\ \text{i.e., } \frac{d^2 B}{dv^2} - (\lambda v^2 - \mu)B &= 0, \end{aligned} \tag{3.54}$$

with $\lambda = \frac{64}{3}kc_p^2, \; \mu = \frac{4}{9}\Sigma_0^2$.

Case-II: $c_p \neq c_q$

The Lagrangian of the system (given by equation (3.30)) shows the variable 'u' to be cyclic and the conserved momentum has the expression:

$$p_u = \frac{\partial L}{\partial \dot{u}} = 24 c_p c_q \dot{u} = \Lambda_0, \text{ a constant,} \tag{3.55}$$

while the momentum conjugate to the variable 'v' is given by

$$p_2 = \frac{\partial L}{\partial \dot{v}} = -\frac{2}{3 c_p c_q} \dot{v}. \tag{3.56}$$

Hence the Hamiltonian of the system is expressed as

$$\mathcal{H} = \frac{1}{48 c_p c_q} p_u^2 - \frac{3 c_p c_q}{4} p_v^2 - k v^2, \tag{3.57}$$

and consequently the WD equation takes the form

$$\left[\frac{1}{48 c_p c_q} \frac{\partial^2}{\partial u^2} + \frac{3 c_p c_q}{4} \frac{\partial^2}{\partial v^2} - k v^2\right] \xi(u, v) = 0. \tag{3.58}$$

The operator version of the conserved momentum as before shows

$$\begin{aligned} \xi(u, v) &= C(u) D(v), \\ \text{with } C(u) &= C_0 \exp(-i \Lambda_0 u). \end{aligned} \tag{3.59}$$

Thus from the above WD equation (3.58) and using the separation of variables, the differential equation for D reduces to

$$\frac{d^2 D}{dv^2} - (l v^2 - s) D = 0, \tag{3.60}$$

with $l = \frac{4k}{3 c_p c_q}$, $s = \frac{\Lambda_0^2}{36 c_p^2 c_q^2}$.

The solution of this second order differential equation takes the form

$$\begin{aligned} D(v) &= C_1 \sqrt{v}\ J_{\frac{1}{4}}\left(\frac{1}{2}\sqrt{-l} v^2\right) + C_2 \sqrt{v}\ Y_{\frac{1}{4}}\left(\frac{1}{2}\sqrt{-l} v^2\right) \quad \text{when}(s = 0), \\ &= \frac{C_1\ M_{\frac{1}{4}\frac{s}{\sqrt{l}}, \frac{1}{4}}\left(\sqrt{l} v^2\right)}{\sqrt{v}} + \frac{C_2\ W_{\frac{1}{4}\frac{s}{\sqrt{l}}, \frac{1}{4}}\left(\sqrt{l} v^2\right)}{\sqrt{v}} \text{when}(s \neq 0). \end{aligned} \tag{3.61}$$

where J and Y are usual Bessel functions and M and W are known as Whittaker functions. We have represented the wave function graphically both for zero and non-zero s in Figure

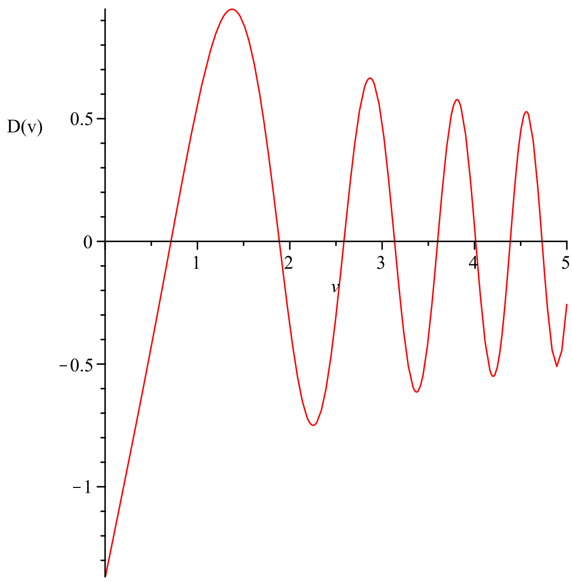

Figure 3.3: Represents the wave function when $s = 0$

3.3 and Figure 3.4. From both the figure (Figure 3.3) and (Figure 3.4), we see that at $u = 0$, $v = 0$ (i.e, $l = 0$), wave function has finite non-zero value. In the present model it is possible to avoid the Big Bang singularity using quantum cosmology near the initial singularity.

3.6 Conclusion

This chapter is an example where symmetry analysis particularly Noether symmetry has been extensively used both in classical and quantum cosmology. Here Chameleon field dark energy model has been considered in the background of homogeneous and isotropic flat FLRW space–time. Although the full quantum theory is described on the infinite dimensional superspace, but here we shall confine to minisuperspace which is a 2D Lorentzian manifold.

Although the Einstein field equations are non–linear coupled differential equation, but using a transformation in the augmented space and introducing geometric inner product it is possible to identify the cyclic variable(s) so that the field equations simplified to a great extend and consequently classical cosmological solutions are evaluated. There are two sets of solutions for two different choices of the arbitrary constants involved. Both the solutions show an expanding model of the Universe with accelerating and decelerating phases (depending on the choices of the arbitrary constants involved). In particular the present model describes the decelerating phase only for the choice $c_p = c_q$ (Figure 3.1), while the model makes a transition from decelerating phase to accelerating phase and then again it goes to decelerating phase

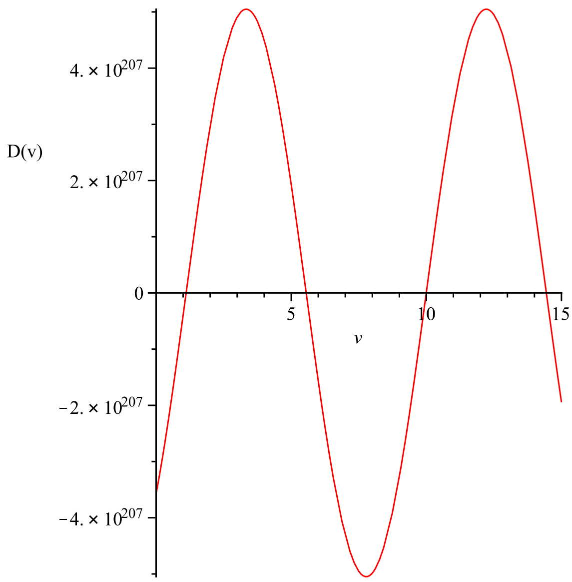

Figure 3.4: Graphical representation of the wave function when $s \neq 0$

for the choice $c_p \neq c_q$ (Figure 3.2).

On the otherhand the application of Noether symmetry to the minisuperspace shows the path for solving WD equation. The conserved momentum due to Noether symmetry, after converting to quantum version shows an oscillatory solution to the WD equation and consequently it gives the semi classical limit of quantum cosmology. Further, the non–oscillatory part of the WD equation is an ordinary differential equation having solution in the form of Bessel function or Whittaker function. The graphical presentation of this part of the solution has been shown in figures (3.3 and 3.4), which clearly show that the present quantum cosmological model can overcome the Big Bang singularity i.e., the present model may describe the early era of evolution without any singularity. Finally, one may conclude that Noether symmetry analysis is very useful in describing quantum cosmology in minisuperspace model and also leads to possible solution of the WD equation.

CHAPTER 4

CLASSICAL AND QUANTUM COSMOLOGY IN $F(T)$-GRAVITY THEORY: A NOETHER SYMMETRY APPROACH

4.1 Prelude

In the context of recent series of observational evidences [11, 13, 15, 137, 138] which predict that our Universe is going through an era of accelerated expansion, a group of cosmologists are in favour of modifying Einstein gravity to accommodate these predictions. There are several modified gravity theories in the literature among which the popular one is the $f(R)$-gravity theory [139, 140]. In this gravity theory, the scalar curvature R in the Einstein-Hilbert action is replaced by an arbitrary function $f(R)$. In recent years, another gravity theory gets much attention and is known as teleparallel gravity. Here the gavitational interactions [141, 142, 143] are described by torsion (instead of curvature). Such a gravity model was first proposed by Einstein with a view to unify electromagnetism and gravity over Weitzenböck non-Riemannian manifold. So the Levi-civita connection is replaced by Weitzenb$\ddot{o}$ck connection in the underlying Riemann-cartan space-time. As a result, pure geometric nature of the gravitational interaction is violated and torsion behaves as force. Hence gravity may be considered as a gauge theory of the translation group [144]. Although there are conceptual differences between GR and teleparallel gravity theory, still at classical level both of them have equivalent dynamics.

In analogy to $f(R)$-gravity theory, a generalization to teleparallel gravity has been formulated [5, 145, 146, 147, 148, 149, 150, 151, 152, 153] by replacing the torsion scalar T by a generic function $f(T)$. Linder termed this modified gravity as $f(T)$-gravity theory. For a comparative study with $f(R)$-gravity theory, there are two important differences namely (a) the field equations in $f(T)$-gravity theory are second order while one has fourth order equations in $f(R)$-gravity [154] (b) Although $f(R)$-gravity theory obey local lorentz invariance but not by $f(T)$-gravity theory. As a result, in $f(T)$-gravity theory all 16 components of the vierbien are independent and a gauge choice [155] can not fix six of them. Further, the four linearly independent vierbeins (i.e, tetrad) fields are the dynamical object in $f(T)$-gravity theory. Also these tetrad fields form the orthogonal bases for the tangent space at each point of space-time. The name "teleparallel" is justified as the vierbeins are parallel vector fields (for a review see ref. [156, 157, 158, 159, 160, 161, 162]).

The geometrical symmetries namely Lie point and Noether symmetries related to space-time are usually very useful to solve/study physical systems. The conserved charges in Noether symmetry are considered as a selection criterion to discriminate similar physical processes [101, 105, 135, 163, 164, 165, 166, 167]. Mathematically a given system of differential equation can either be simplified or to have a first integral (Norther integral) by imposing Noether symmetry to the system. Further, it is possible to constrain or determine physical parameters involved in a physical system by imposing Noether symmetry to it [6, 117]. In recent years, symmetry analysis has been widely used to the physical systems in Riemannian spaces [118, 119, 120, 122, 123, 124, 125] and specially in the context of cosmology [7, 113, 114, 168, 169, 170]. This chapter is an example of it. Usually, evolution equations are simplified to a great extend by determining a cyclic variable in the augmented space. As a consequence, analytic solutions are possible with new variables (in the augmented space) and are analyzed from cosmological context.

On the other hand, Noether symmetry can also be used in quantum cosmology to identify a typical subset of the general solution of the Wheeler-DeWitt (WD) equation having oscillatory behaviour [8, 131, 103, 126]. Also in Minisuperspace geometry, symmetry analysis identifies equations of classical trajectories [127, 128]. Hence classical observable universe can be related to the quantum cosmology through the application of Noether symmetry analysis.

In this chapter, both classical and quantum cosmology have been studied for $f(T)$-gravity theory using the Noether symmetry analysis in the background of homogeneous and isotropic flat FLRW space-time model. The chapter is organized as follows: Section 4.2 describes the brief review of conformal symmetry and classical cosmology in f(T) gravity and Noether symmetry is presented in Section 4.3, where as Section 4.4 presents a general description of quantum cosmology: minisuperspace approach and chapter ends with a summary in Section 4.5.

4.2 Conformal Symmetry: A brief review

In differential geometry, conformal invariance gives rich geometrical structures. A vector field ξ^α is a Conformal Killing Vector (CKV) of the metric g_{ij} if

$$\mathcal{L}_{\overrightarrow{\xi}} g_{ij} = \mu(x^k) g_{ij}, \tag{4.1}$$

where μ is an arbitrary function of the space and notationally $\mathcal{L}_{\overrightarrow{\xi}}$ indicates Lie derivative with respect to the vector field $\overrightarrow{\xi}$. In particular if

(i) $\mu(x^k) = \mu_0 (\neq 0)$, a constant: ξ^α - homothetic vector field.
(ii) $\mu(x^k) = 0$: ξ^α - killing vector field.

The above three class of vector fields individually form an algebra as follows:

(a) The class of conformal killing vectors form an algebra, known as conformal algebra (CA) of the metric [171].

(b) The class of homothetic vector fields form an algebra, known as homothetic algebra (HA).

(c) The class of killing vector fields form an algebra, known as killing algebra (KA).

These three algebras are related as

$$KA \subseteq HA \subseteq CA. \tag{4.2}$$

Further, for a $n(>2)$ dimensional manifold of constant curvature, the dimension of these three algebras are $\frac{(n+1)(n+2)}{2}$, $\frac{n(n+1)}{2}+1$ and $\frac{n(n+1)}{2}$ respectively.

In a given space, two metrics g and g' are said to be conformally related if $\exists$ a function $\Pi(x^k)$ so that

$$g'_{ij} = \Pi(x^k) g_{ij}. \tag{4.3}$$

It is to be noted that two conformally related metrics have the same conformal algebra but subalgebras are not necessarily same. In fact if $\overrightarrow{\xi_0}$ is a conformal killing vector for the conformally related metrics g and g' then the corresponding conformal functions $\mu(x^k)$ and $\mu'(x^k)$ are related by the relation

$$\mu'(x^k) = \mu(x^k) + \mathcal{L}_{\overrightarrow{\xi_0}}(\ln \Pi). \tag{4.4}$$

As Noether symmetries follow the homothetic algebra of the metric so two conformally related physical systems are not identical.

In the context of conformal Lagrangian, Tsamparlis et al. [171] have shown that the equations of motion (i.e, Euler-Lagrange equations) corresponding to two conformal Lagrangians transform covariantly under the conformal transformation provided Hamiltonian (i.e, the total energy) is zero. So systems with vanishing energy are conformally related and corresponding equations of motion are conformally invariant. Further, in quantum cosmology, due to the Hamiltonian constraint the total energy of the system has been zero, and consequently, one has conformally invariant systems with respect to the equations of motion.

4.3 Classical Cosmology in $f(T)$-gravity and Noether Symmetry

In the background of flat FLRW space-time model, the point like Lagrangian in $f(T)$ gravity theory takes the form

$$L = a^3 f(T) - a^3 T f_T(T) - 6a\dot{a}^2 f_T(T) - Da^{-3\omega}, \tag{4.5}$$

where $f(T)$ is a regular function of the torsion scalar T, a is the scale factor and D is a constant of integration. Here matter field is chosen as perfect fluid with $\omega = \frac{p}{\rho}$, the constant equation of state parameter. The modified Friedmann equations are [172]

$$H^2 = \frac{1}{(2f_T + 1)}\left[\frac{\rho}{3} - \frac{f}{6}\right], \tag{4.6}$$

and

$$2\dot{H} = \frac{(\rho + p)}{1 + f_T + 2Tf_{TT}}. \tag{4.7}$$

In the present Lagrangian system we have 2D configuration space $\{a, T\}$ and the momenta conjugate to configuration variables are

$$p_a = \frac{\partial L}{\partial \dot{a}} = -12a\dot{a}f_T(T), \tag{4.8}$$

$$p_T = \frac{\partial L}{\partial \dot{T}} = 0. \tag{4.9}$$

Using Legendre transformation, the Hamiltonian of the system is

$$H = -\frac{1}{24}\frac{{p_a}^2}{af_T(T)} - a^3 f(T) + a^3 T f_T(T) + Da^{-3\omega}. \tag{4.10}$$

So the Hamilton's equations of motion are

$$\begin{aligned}
\dot{a} &= \{a, H\} = -\frac{1}{12}\frac{p_a}{af_T(T)}, \\
\dot{T} &= \{T, H\} = 0, \\
\dot{p_a} &= \{p_a, H\} = -\frac{1}{24}\frac{{p_a}^2}{a^2 f_T(T)} + 3a^2 f(T) - 3a^2 T f_T(T) + 3\omega D a^{-3\omega-1}, \\
\dot{p_T} &= \{p_T, H\} = 6a\dot{a}^2 f_{TT}(T) - a^3 T f_{TT}(T).
\end{aligned} \tag{4.11}$$

We shall now impose Noether symmetry to the above physical system. According to Noether's theorem $\exists$ a vector field $\overrightarrow{X}$ about which the Lie derivative of the Lagrangian should be zero i.e,

$$\mathcal{L}_{\overrightarrow{X}} L = 0, \tag{4.12}$$

where the infinitesimal generator $\overrightarrow{X}$ has the form

$$\overrightarrow{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial T} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{T}}. \tag{4.13}$$

Here $\alpha = \alpha(a, T)$ and $\beta = \beta(a, T)$ are the functions in the configuration space with

$\dot{\alpha} = \frac{\partial \alpha}{\partial a}\dot{a} + \frac{\partial \alpha}{\partial T}\dot{T}$ and so on.

Now from the Noether symmetry condition (4.12) one obtains the following partial differential equations:

$$-6\alpha f'(T) - 6a\beta f''(T) - 12af'(T)\frac{\partial \alpha}{\partial a} = 0, \tag{4.14}$$

$$-12af'(T)\frac{\partial \alpha}{\partial T} = 0, \tag{4.15}$$

and

$$3\alpha a^2 f(T) - 3\alpha a^2 T f'(T) + 3\omega D\alpha a^{-3\omega-1} - \beta a^3 f''(T)T = 0. \tag{4.16}$$

Using seperation of variables for the coefficients α, β of the symmetry vector, the above set of partial differential equations are solvable to give

$$\begin{aligned} \alpha(a,T) &= ca^{1-\frac{3k}{2}}, \\ \beta(a,T) &= -3kca^{-\frac{3k}{2}}T, \end{aligned} \tag{4.17}$$

and also $f(T)$ has the solution

$$f(T) = f_0 T^{\frac{1}{k}}, \tag{4.18}$$

with the equation of state parameter $\omega = 0$.

In the solution, c, k and f_0 are arbitrary integration constants.

In order to solve the modified Friedmann equations we make a transformation in the configuration space $(a,T) \rightarrow (u,v)$ so that one of the transformed variable (say u) becomes cyclic and consequently the transformed Lagrangian becomes much simpler in form. Hence the evolution equations become very simple to have analytic solutions. So the infinitesimal generator (i.e, the vector field $\overrightarrow{X}$) due to this point transformation becomes

$$\overrightarrow{X}_T = \left(i_{\overrightarrow{X}}du\right)\frac{\partial}{\partial u} + \left(i_{\overrightarrow{X}}dv\right)\frac{\partial}{\partial v} + \left\{\frac{d}{dt}\left(i_{\overrightarrow{X}}du\right)\right\}\frac{\partial}{\partial \dot{u}} + \left\{\frac{d}{dt}\left(i_{\overrightarrow{X}}dv\right)\right\}\frac{\partial}{\partial \dot{v}}. \tag{4.19}$$

Thus $\overrightarrow{X}_T$ may be considered as the lift of a vector field defined on the augmented space.

Now, without any loss of generality one may restrict the above point transformation to be

$$i_{\overrightarrow{X}} du = 1,$$

and

$$i_{\overrightarrow{X}} dv = 0, \tag{4.20}$$

so that

$$\overrightarrow{X}_T = \frac{\partial}{\partial u},$$

with

$$\frac{\partial L_T}{\partial u} = 0. \tag{4.21}$$

Here $i_{\overrightarrow{X}}$ stands for the inner product with the vector field $\overrightarrow{X}$. Usually with Noether symmetry there is an associated conserved current. The time component of it when integrated over spatial volume gives a conserved charge, which in geometric notion can be defined as

$$q = i_{\overrightarrow{X}} \theta_L,$$

where the cartan one form θ_L is defined as [169]

$$\theta_L = \frac{\partial L}{\partial a} da + \frac{\partial L}{\partial T} dT. \tag{4.22}$$

Now the first order partial differential equations corresponding to equation (4.20) are

$$\alpha \frac{\partial u}{\partial a} + \beta \frac{\partial u}{\partial T} = 1,$$

and

$$\alpha \frac{\partial v}{\partial a} + \beta \frac{\partial v}{\partial T} = 0. \tag{4.23}$$

Using the solutions for α and β from equation (4.17) the solution for u and v gives

$$u = \frac{2}{3kc} a^{\frac{3k}{2}},$$

$$v = \ln\left(aT^{\frac{1}{3k}}\right), \tag{4.24}$$

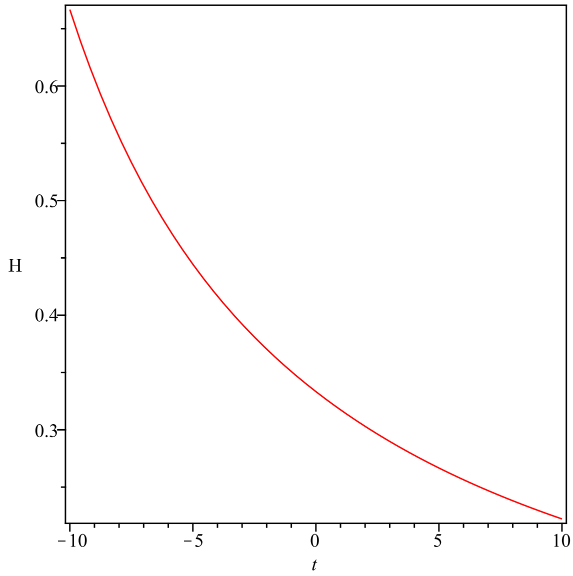

Figure 4.1: H vs t

and the transformed Lagrangian takes the form

$$L = f_0 \left(1 - \frac{1}{k}\right) e^{3v} - \frac{6f_0}{k} c^2 \dot{u}^2 e^{(3-3k)v} - D. \tag{4.25}$$

The solution of the corresponding Euler-Lagrange equations take the form

$$\begin{aligned} v &= B, \\ u &= Ft + G, \end{aligned} \tag{4.26}$$

with B, F and G are arbitrary constants.
So the classical cosmological solution in original variables can be written as

$$\begin{aligned} a &= \left\{\frac{3kc}{2}(Ft+G)\right\}^{\frac{2}{3k}}, \\ T &= \frac{4B_0}{9k^2c^2(Ft+G)^2}, \\ f(T) &= f_0 \left\{\frac{4B_0}{9k^2c^2(Ft+G)^2}\right\}^{\frac{1}{k}}, \end{aligned} \tag{4.27}$$

where B_0 is an arbitrary constant.

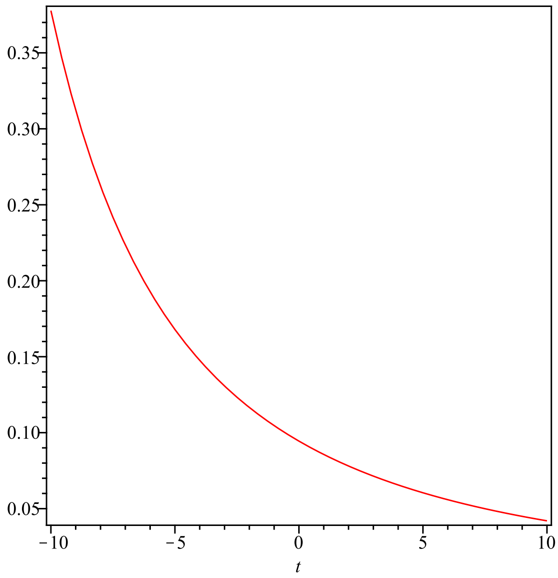

Figure 4.2: $\frac{\ddot{a}}{a}$ vs t

The above cosmological solution indicates power-law expansion of the Universe with Hubble parameter decreases with respect to cosmic time as $\dfrac{1}{t}$ (see Figure 4.1) and the Universe is in an accelerating phase (see Figure 4.2) with rate of acceleration decreases with the evolution.

4.4 A general description of Quantum Cosmology: Minisuperspace Approach

The general description of quantum cosmology has been described in section 3.4.

In this problem the Lagrangian of the system in the transformed variables is given by (4.25). So the canonically conjugate momenta are

$$\begin{aligned} p_u &= \frac{\partial L}{\partial \dot{u}} = -\frac{12f_0}{k}c^2\dot{u}e^{(3-3k)v}, \\ p_v &= \frac{\partial L}{\partial \dot{v}} = 0. \end{aligned} \tag{4.28}$$

Hence the Hamiltonian of the system is

$$H = p_u\dot{u} + p_v\dot{v} - L = -\frac{k}{24f_0c^2}e^{(3k-3)v}p_u^2 - f_0\left(1-\frac{1}{k}\right)e^{3v} + D. \tag{4.29}$$

The above Hamiltonian is a very special type of Hamiltonian having only one dynamical variable u which is also cyclic in nature (v can not be considered as dynamical variable as $\dot{v}$ i.e, p_v does not appear in the Hamiltonian).

Hence the WD equation which is the operator version of the above Hamiltonian takes the form

$$\frac{d^2\phi}{du^2} + l\phi = 0, \tag{4.30}$$

with $l = \frac{f_0\left(1-\frac{1}{k}\right)e^{3v} - D}{\frac{k}{24f_0c^2}e^{(3k-3)v}}$, a constant.

So the solution of WD equation can be written as

$$\begin{aligned} \phi &= A_1e^{\sqrt{l}u} + A_2e^{-\sqrt{l}u}, \quad \text{when } l > 0, \\ &= B_1\cos\sqrt{-l}u + B_2\sin\sqrt{-l}u, \quad \text{when } l < 0, \\ &= C_1u + C_2, \quad \text{when } l = 0, \end{aligned} \tag{4.31}$$

where A_i's, B_i's and C_i's ($i = 1, 2$) are the constants of integration.

4.5 Summary

This chapter deals with $f(T)$ cosmology from the point of view of symmetry analysis. In particular Noether symmetry has been used both in classical and quantum cosmology with $f(T)$ gravity theory. Using Noether symmetry condition to the Lagrangian of the present model along the symmetry vector, the coefficients of the symmetry vector are not only determined, it is possible to determine the explicit form of the $f(T)$ function. Using a transformation of variables in the configuration (satisfying condition (4.20)) the Lagrangian simplifies to a great extend and it is possible to have cosmological solution having power-law nature. In quantum cosmology, the WD equation simplifies to a great extend due to only one dynamical variable having cyclic nature. From the nature of the wave function one can infer that Big-Bang singularity may be avoided quantum mechanically for the present $f(T)$-cosmological model. So one may conclude that $f(T)$ cosmology can avoid the Big-Bang singularity at the very early era of evolution of the Universe.

CHAPTER 5

QUANTUM COSMOLOGY FOR DOUBLE SCALAR FIELD COSMOLOGICAL MODEL: SYMMETRY ANALYSIS

5.1 Prelude

In cosmology, scalar field models are commonly used to describe quintessence model, phantom model. Further, multi-scalar field cosmology has variety of applications namely an alternative inflationary model (known as hybrid inflation), α-attractors and double inflation [77, 78, 79, 80, 81, 82, 83, 84, 85]. Quintom model is an example of multi-scalar field cosmology where there are two scalar fields of which one is a quintessence field while the other is a phantom field. A further extension of Quintom model having a mixed kinetic term has been considered in ref [86] with flat geometry for the space of kinetic energy. Chiral cosmological model [85, 87, 88] is also a two scalar field model with non-vanishing curvature of the 2D space and nonlinear sigma model has analogy with it. In recent past a typical two scalar field model in which one is the usual quintessence field while the other one interacts through the kinetic term. Noether symmetry approach has been used not only to determine the unknown coupling function and the potential function of the scalar field model but also to have the classical cosmological solution.

On the other-hand symmetry analysis of a physical system is associated to problems of differential geometry. The Noether point symmetries of the Lagrangian of a physical system are associated with the homothetic vectors of the metric of the space generated by the dynamic fields. Further, symmetry analysis has been rigorously used in formulation of quantum cosmology. The conformal killing vector field or homothetic vector field of the minisuperspace are associated with the symmetry analysis of the Wheeler DeWitt (WD) equation. By choosing appropriate operator ordering the WD equation can be constructed on the minisuperspace and the existence of Lie point symmetries are associated with the construction of the WD equation. On the other-hand, Noether point symmetries are associated with the oscillatory part of the general solution of the WD equation [103, 126]. According to Hartle [132], the Noether symmetries in minisuperspaces identify those classical trajectories which are solutions of the cosmological evolution equations [127, 128]. Thus a classical-quantum bridge has been provided by the Noether symmetries to associate quantum cosmology with classically observable Universe. In particular the conserved charge associated with Noether symmetry of the WD equation helps to solve the WD equation by identifying the oscillatory part of the solution.

This chapter deals with two scalar field cosmological model (in which the fields are coupled in the kinetic term). From differential geometric point of view, the homothetic vector fields as well as the killing vector fields (translational and rotational) in the 3D augmented space have been evaluated. Finally quantum cosmology has been extensively discussed. The plan of the chapter is as follows: Section 5.2 presents a review of the Noether symmetry approach to the present physical system and also cosmological solutions has been discussed. In section 5.3, both homothetic and killing vectors in context of the present physical model has been evaluated. A detailed study of quantum cosmology has been done in section 5.4. Finally, in section 5.5, a brief conclusion has been presented.

5.2 Noether Symmetry approach and Cosmological Solution: A review

The present two scalar field model coupled to gravity has the Lagrangian density

$$\mathcal{L} = \frac{R}{2} - \frac{1}{2}(\nabla_\mu \phi)(\nabla^\mu \phi) - \frac{F(\phi)}{2}(\nabla_\mu \psi)(\nabla^\mu \psi) - V(\phi), \tag{5.1}$$

by choosing the gravitational coupling $8\pi G = 1$. Here $F(\phi)$ and $V(\phi)$ are the coupling function and the potential function of the scalar field ϕ.

In the background of homogeneous and isotropic flat FLRW model the above Lagrangian takes the form

$$\mathcal{L} = -3a\dot{a}^2 + \frac{1}{2}a^3\dot{\phi}^2 + \frac{a^3}{2}F(\phi)\dot{\psi}^2 - a^3V(\phi), \tag{5.2}$$

and the corresponding evolution equations (i.e, Euler-Lagrangian) read as

$$\begin{aligned} \frac{1}{2}\left[F(\phi)\dot{\psi}^2 + \dot{\phi}^2 - 6H^2\right] + V(\phi) &= 0, \\ -2\frac{\ddot{a}}{a} + 2H - H^2 - \frac{1}{2}F(\phi)\dot{\psi}^2 - \frac{1}{2}\dot{\phi}^2 + V(\phi) &= 0, \\ \ddot{\phi} + 3H\dot{\phi} - \frac{1}{2}F'(\phi)\dot{\psi}^2 + V'(\phi) &= 0, \\ \frac{d}{dt}\left(a^3F(\phi)\dot{\psi}\right) &= 0. \end{aligned} \tag{5.3}$$

Here an over-dot denotes the derivatives with respect to cosmic time t and $H\left(=\frac{\dot{a}}{a}\right)$ is the usual Hubble parameter. From the last equation of the set of equations (5.3) one has the conserved quantity

$$Q = a^3F(\phi)\dot{\psi} = C, \quad \text{an integration constant.} \tag{5.4}$$

Now without choosing phenomenologically, the unknown functions namely $F(\phi)$ and $V(\phi)$ can be determined by imposing Noether symmetry to the Lagrangian. So according to the symmetry condition the constituent equations give the unknown functions (for details see ref. [173]) as

$$V(\phi) = V_0 \sinh^2\left(\frac{\sqrt{3}\phi}{2\sqrt{2}}\right),$$
$$\text{and} \quad F(\phi) = F_0 \sinh^2\left(\frac{\sqrt{3}\phi}{2\sqrt{2}}\right), \tag{5.5}$$

with V_0, F_0 as constants of integration.

Now to solve the above coupled non-linear evolution equations one has to make a transformation in the augmented space (t, a, ϕ, ψ). Due to the existence of Noether symmetry it is possible to make such transformation with one of the variables becomes cyclic, by imposing the inner product condition:

$$i_{\overrightarrow{X}} ds = 0, \quad i_{\overrightarrow{X}} du = 0, \quad i_{\overrightarrow{X}} dv = 0 \quad \text{and} \quad i_{\overrightarrow{X}} dw = 1.$$

Here (s, u, v, w) are the transformed variables and $\overrightarrow{X}$ is the symmetry vector corresponding to the Noether symmetry. As a result, the transformed variables take the form

$$u = a^{\frac{3}{2}} \sinh\left(\frac{\sqrt{3}\phi}{2\sqrt{2}}\right), \quad v = \psi, \quad w = \frac{2}{3} a^{\frac{3}{2}} \cosh\left(\frac{\sqrt{3}\phi}{2\sqrt{2}}\right), \quad s = t, \tag{5.6}$$

and the transformed Lagrangian can be written as

$$L = \frac{4}{3}\dot{u}^2 - 3\dot{w}^2 + f_0 u^2 \dot{v}^2 + 4 v_0 u^2. \tag{5.7}$$

Then the Euler-Lagrange equations can be written as

$$\begin{aligned} 4\ddot{u} - 3 f_0 u \dot{v}^2 - 12 u v_0 &= 0, \\ 2 f_0 u^2 \dot{v} &= \text{constant}, \\ \ddot{w} &= 0. \end{aligned} \tag{5.8}$$

In the transformed Lagrangian (5.7), v and w are cyclic co-ordinates. So the corresponding conserved momentum are given by the 2nd and 3rd equations in (5.8). The first equation in (5.8) is the evolution equation for u. There is another equation here namely the scalar constraint equation i.e,

$$\frac{4}{3}\dot{u}^2 - 3\dot{w}^2 + f_0 u^2 \dot{v}^2 - 4v_0 u^2 = 0,$$

which is not presented in (5.8) as by differentiating this constraint equation one gets the evolution equation for u (i.e, the first equation of (5.8)). Hence essentially we have four equations in (5.8) with the new variables (u, v, w).

Solving this system of equations (5.8) we get

$$\begin{aligned}
u &= \left(\sin t + l_2^2\right)^{\frac{1}{2}}, \\
v &= \tan^{-1}\left(\frac{h\tan\frac{t}{2} + 2}{(h^2-1)^{\frac{1}{2}}}\right), \\
w &= c_1 t + c_2,
\end{aligned} \tag{5.9}$$

and the cosmological solution so obtained can be written as

$$\begin{aligned}
a &= \left[(l_1 t + l_2)^2 - \sin t - l_2^2\right]^{\frac{1}{3}}, \\
\phi &= \tanh^{-1}\left[\frac{\sqrt{\sin t + l_2^2}}{(l_1 t + l_2)}\right], \\
\psi &= \tan^{-1}\left[\frac{h\tan\frac{t}{2} + 2}{\sqrt{h^2-1}}\right], \\
V(\phi) &= V_0 \sinh^2\left[\tanh^{-1}\left\{\frac{\sqrt{\sin t + l_2^2}}{(l_1 t + l_2)}\right\}\right], \\
F(\phi) &= F_0 \sinh^2\left[\tanh^{-1}\left\{\frac{\sqrt{\sin t + l_2^2}}{(l_1 t + l_2)}\right\}\right],
\end{aligned} \tag{5.10}$$

where c_1, c_2, l_1, l_2 and h are integration constants.

The above solution represents an expanding model of the Universe starting from an accelerating phase to a decelerating era.

The graphical representation of the scale factor, Hubble parameter and $\frac{\ddot{a}}{a}$ have been shown in chapter 2 [173].

5.3 Conformal Symmetry and the present model

From physical point of view, the above geometrical features of the space can be applied to the Lagrangian. For physical systems described by the Lagrangian, the notion of conformal Lagrangian will be interesting. It has been shown in ref. [171], the equations of motion for two conformally related Lagrangian are covariantly transformed under the conformal transformation with the restriction that the total energy of the system (i.e, Hamiltonian) is zero. Also conversely, physical systems with zero total energy are conformally related and their equations of motion are covariantly invariant. Further, in the present context of Noether symmetry two conformally related physical systems are not identical due to distinct homothetic algebras for the systems (Noether symmetry follows homothetic algebra of the metric).

The present Lagrangian (given in equation (5.2)) defines the kinetic metric as

$$ds_3^{(k)^2} = -6ada^2 + a^3 d\phi^2 + a^3 F(\phi) d\psi^2, \tag{5.11}$$

with effective potential $V_{\text{eff}} = a^3 V(\phi)$.

It is well known that if the Lagrangian L is of the form $L = T - V_{\text{eff}}$ with T as the K.E. then the Noether point symmetries are generated by the elements of the homothetic group of the kinetic metric $ds_3^{(k)^2}$. Further, the above kinetic line element (5.11) can be rewritten as

$$ds_3^{(k)^2} = a^3 \left[-\frac{6}{a^2} da^2 + d\phi^2 + F(\phi) d\psi^2 \right], \tag{5.12}$$

which is conformal to the $(1+2)$ decomposable metric

$$ds_3^2 = -\frac{6}{a^2} da^2 + d\phi^2 + F(\phi) d\psi^2. \tag{5.13}$$

In general, the 3D kinetic metric (5.11) admits gradient homothetic vector (HV); $H_V = \frac{2}{3} a\partial_a$ with $\psi H_V = 1$, which does not generate a Noether point symmetry for the

Lagrangian (5.2). Further, the 2D metric in (ϕ,ψ)-plane (a space of constant curvature) admits three killing vectors spanning the $SO(3)$ group. Now, using the transformation:

$$A = \sqrt{\frac{8}{3}} a^{\frac{3}{2}},$$

the Lagrangian (5.2) becomes

$$L = -\frac{1}{2}\dot{A}^2 + \frac{1}{2}A^2\dot{\Phi}^2 + \frac{A^2}{2}F_0(\Phi)\dot{\Psi}^2 - A^2 U(\Phi), \tag{5.14}$$

with $\Phi = \frac{3}{4}\phi$, $\Psi = \frac{3}{4}\psi$, $F_0(\Phi) = F(\phi)$, $U(\Phi) = \frac{3}{8}V(\phi)$.

Thus the kinetic metric in these new variables becomes

$$ds_3^{(k)^2} = -dA^2 + A^2\left(d\Phi^2 + F_0(\Phi)d\Psi^2\right). \tag{5.15}$$

Due to conformally flat nature of this metric there are seven dimensional ($\frac{n(n+1)}{2} + 1$ for $n = 3$ gives 7) homothetic Lie algebra as follows:

(a) The gradient homothetic vector field: $H_V = A\frac{\partial}{\partial A}$ with $\psi.H_V = 1$.

(b) Three gradient killing vectors K^a, $a = 1,\ 2,\ 3$ (translational).

$$\begin{aligned} \overrightarrow{K}^{(1)} &= -\frac{1}{2}\left(e^{\Phi}\left(1+\Psi^2\right) + e^{-\Phi}\right)\partial_A + \frac{1}{2A}\left(e^{\Phi}\left(1+\Psi^2\right) - e^{-\Phi}\right)\partial_\Phi + \frac{1}{A}\Psi e^{-\Phi}\partial_\Psi, \\ \overrightarrow{K}^{(2)} &= -\frac{1}{2}\left(e^{\Phi}\left(1-\Psi^2\right) - e^{-\Phi}\right)\partial_A + \frac{1}{2A}\left(e^{\Phi}\left(1-\Psi^2\right) + e^{-\Phi}\right)\partial_\Phi + \frac{1}{A}\Psi e^{-\Phi}\partial_\Psi, \\ \text{and} \quad \overrightarrow{K}^{(3)} &= -\Psi e^{\Phi}\partial_A + \frac{1}{A}\Psi e^{\Phi}\partial_\Phi + \frac{1}{A}e^{-\Phi}\partial_\Psi, \end{aligned} \tag{5.16}$$

with the corresponding gradient killing functions:

$$\begin{aligned} f_1 &= \frac{A}{2}\left[F_0(\Phi)\left(1+\Psi^2\right) + F_0^{-1}(\Phi)\right], \\ f_2 &= \frac{A}{2}\left[F_0(\Phi)\left(1-\Psi^2\right) - F_0^{-1}(\Phi)\right], \\ \text{and} \quad f_3 &= A\Psi F_0(\Phi). \end{aligned}$$

(c) Three non-gradient (rotational) killing vectors which generate the $SO(3)$ algebra:

$$\overrightarrow{d}_{12} = \partial_\Psi, \quad \overrightarrow{d}_{23} = \partial_\Phi + \Psi\partial_\Psi \quad \text{and} \quad \overrightarrow{d}_{31} = \Psi\partial_\Phi + \frac{1}{2}\left(\Psi^2 - F_0^2(\Phi)\right)\partial_\Phi. \tag{5.17}$$

5.4 Quantum Cosmology: Wave Function of the Universe

In the present problem, the minisuperspace is a 3D manifold described by the augmented space. As the Hamiltonian is derived from the Lagrangian so it is preferable to use the simplified Lagrangian (5.7) and hence the transformation (5.6) is suitable to deal with quantum cosmology. So the momenta conjugate to the variables are

$$p_u = \frac{\partial L}{\partial \dot{u}} = \frac{8}{3}\dot{u}, \quad p_v = 2f_0 u^2 \dot{v} \quad \text{and} \quad p_w = -6\dot{w}. \tag{5.18}$$

As v and w are cyclic co-ordinates so the corresponding momenta are conserved i.e, $p_v = \Sigma_v$ and $p_w = \Sigma_w$ are constants of motion (Σ_v and Σ_w are termed as conserved charges). Now the Hamiltonian of the system is

$$H = \frac{3}{16}p_u^2 + \frac{1}{4f_0u^2}p_v^2 - \frac{1}{12}p_w^2 - 4v_0u^2. \tag{5.19}$$

So in quantum cosmology one has the WD equation: $\hat{\mathcal{H}}\psi = 0$ i.e,

$$\left[-\frac{3}{16}\frac{\partial^2}{\partial u^2} - \frac{1}{4f_0u^2}\frac{\partial^2}{\partial v^2} + \frac{1}{12}\frac{\partial^2}{\partial w^2} - 4v_0u^2\right]\psi(u,v,w) = 0. \tag{5.20}$$

Note that in the construction of the above WD equation there is no problem of operator ordering as none of the configuration variables and the corresponding momenta do not appear in product form. Further, in the context of quantum cosmology for probability measure, $\exists$ a conserved current for hyperbolic type of partial differential equation as

$$\overrightarrow{J} = \frac{i}{2}\left(\psi^\star \overrightarrow{\nabla}\psi - \psi\overrightarrow{\nabla}\psi^\star\right), \tag{5.21}$$

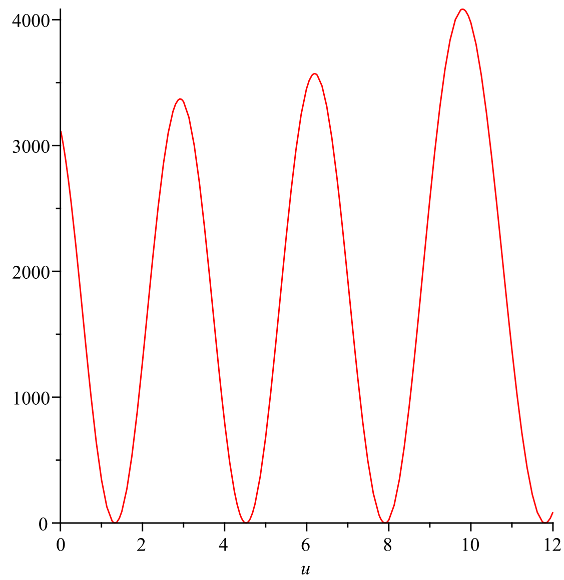

Figure 5.1: Graphical representation of the wave function for $k = 0$

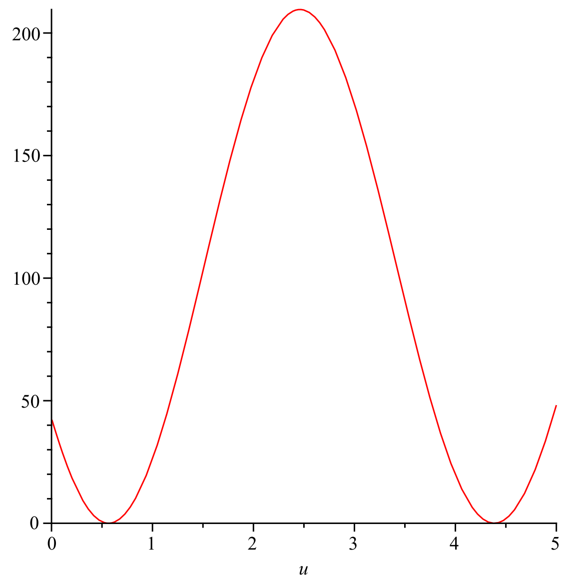

Figure 5.2: Representation of the wave function for $k < 0$

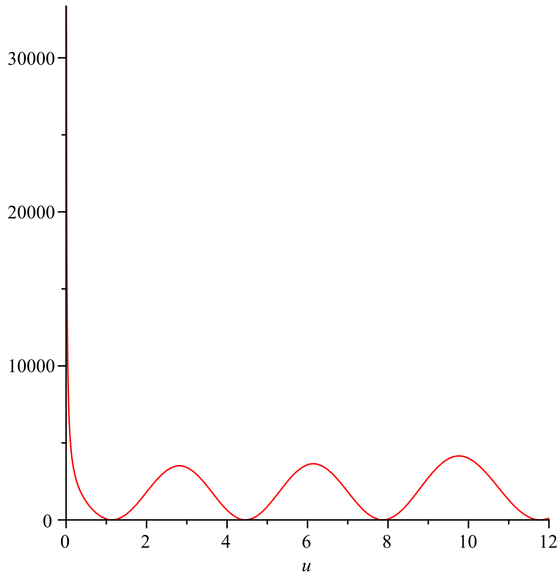

Figure 5.3: Shows the wave function for $k > 0$

and $\overrightarrow{J}$ satisfies the relation $\overrightarrow{\nabla}.\overrightarrow{J} = 0$. So the probability measure on the minisuperspace can be written as

$$dp = |\psi(q^\alpha)|^2 dV,$$

where dV denotes a volume element on minisuperspace.

Also the conserved momenta equations in operator form restrict the wave function as

$$-i\frac{\partial\psi}{\partial v} = \Sigma_v \psi, \tag{5.22}$$

and

$$-i\frac{\partial\psi}{\partial w} = \Sigma_w \psi, \tag{5.23}$$

i.e,

$$\psi(u,v,w) = A(u)e^{(i\Sigma_v.v + i\Sigma_w.w)}, \tag{5.24}$$

where $A(u)$ satisfies the 2nd order differential equation

$$-\frac{3}{16}\frac{d^2A}{du^2} + \frac{\Sigma_v^2 A}{4f_0u^2} - \frac{\Sigma_w^2 A}{12} - 4v_0u^2A = 0,$$

i.e,

$$\frac{d^2A}{du^2} + \frac{16}{3}\left[\frac{\Sigma_w^2}{12} + 4v_0u^2 - \frac{\Sigma_v^2}{4f_0u^2}\right]A = 0, \tag{5.25}$$

which has the solution

$$A(u) = \frac{1}{\sqrt{u}}\left[c_1 M_{-\frac{\frac{1}{4}il}{\sqrt{m}},\frac{1}{4}\sqrt{1+4k}}\left(i\sqrt{m}u^2\right) + c_2 W_{-\frac{\frac{1}{4}il}{\sqrt{m}},\frac{1}{4}\sqrt{1+4k}}\left(i\sqrt{m}u^2\right)\right], \tag{5.26}$$

with $l = \frac{4\Sigma_w^2}{9}$, $m = \frac{64v_0}{3}$, $k = \frac{4\Sigma_v^2}{3f_0}$.

Here M and W are the usual Whittaker functions of 1st and 2nd kind. For probability measure on the minisuperspace $|\psi|^2$ have been plotted in Figure (5.1-5.3) . Note that for $k > 0$ both the scalar fields are canonical in nature while for $k < 0$ the model has an analogy with quintom dark energy model. Here $k = 0$ corresponds to the situation where conserved charge corresponding to the transformed variable v will vanish. The figures show that quantum description avoids Big-Bang singularity except for $k > 0$ (probability measure become infinite as $u \to 0$).

5.5 Brief Conclusion

This chapter is an extension of chapter 2 [173] related to Noether symmetry approach to the double scalar field cosmological model. In chapter 2 only Noether symmetry is imposed on the physical Lagrangian. Then by using the symmetry vector a transformation in the augmented space is performed so that at least one of the variables becomes cyclic and consequently the Lagrangian as well as the evolution equations become much simpler in form. As a result classical cosmological solutions have been obtained and are analyzed from cosmological point of view. The present work is an extension of the earlier one in two ways (i) a detailed symmetry analysis has been done by considering the conformal symmetry of the physical space and (ii) quantum cosmology has been studied by determining the wave function of the Universe as a solution of the Wheeler-DeWitt equation.

Using the kinetic line element (which is conformal to a (1+2) decomposable metric) it is found that there are seven gradient homothetic vectors of which three are gradient killing vectors and three are non-gradient (i.e, rotational) killing vectors. It is to be noted that all of them do not generate Noether point symmetry to the Lagrangian. In the context of quantum cosmology, Noether point symmetry identifies the oscillatory part of the solution of the WD equation. The wave function has oscillatory behaviour along those two transformed augmented variables along which there are associated conserved charges (i.e, conserved momenta). As a result, the WD equation becomes an ordinary differential equation having solution in the form of Whittaker functions. A major issue in quantum cosmology, namely the operator ordering problem, does not arise in the present problem due to simplified form of the Hamiltonian constraint in the transformed variables. Further to have an idea about probability measure on minisuperspace $|\psi|^2$ has been plotted against u in Figure (5.1-5.3) for different choices for k. It is found that except $k > 0$, $|\psi|^2$ has a finite non-zero value at $u = 0$ i.e, $a = 0$. Therefore, it is reasonable to consider the present model as an alternative to Big-Bang singularity, considering quantum effects at small volume.

CHAPTER 6

GEOMETRIC SYMMETRIES OF THE PHYSICAL SPACE OF THE EINSTEIN-SKYRME MODEL AND QUANTUM COSMOLOGY: A NOETHER SYMMETRY ANALYSIS

6.1 Prelude

The geometrical symmetries to the space-time are very useful in studying physical problems. In particular, the Noether point symmetry has an extra advantage, namely the conserved Noether charge (associated with Noether symmetry) which can be considered as a selection criterion to distinguish different similar physical processes. In addition, the Noether integral can be used either to simplify a given system of differential equations or to determine the integrability of the system. Further, self consistency of any phenomenological physical model and (or) physical parameters are constrained by the analysis of Noether symmetry.

The physical model that we shall consider in this chapter is the Skyrme model which is a nonlinear theory of point masses with baryons as topological solutions [174]. There exist several cosmological models which have been explored in modified gravity theory (see the references [175, 176, 177, 178]). In this connection, the Skyrme model was introduced in the early 60s having applicability in various branches of physics [50, 51, 52, 53, 54, 55, 56]. In the context of cosmology this model gives an anisotropic fluid having no heat flux component but its equation of state parameter (ω_s) is constrained as $|\omega_s| \leq \frac{1}{3}$. This type of fluid is interesting in astrophysics as one can find with this fluid static black hole (BH) solutions having regular event horizon and they are also asymptotically Schwarzschild BH. Further, this type of BH solutions do not obey the "no-hair" conjecture [179, 180, 181, 182]. In the cosmological context, due to the complicated nature of the coupled field equations, it is hard to find exact analytic solutions. So there are numerical solutions and dynamical system analysis with this model. However, recently, exact cosmological solution has been obtained [170] using Noether symmetry analysis. Numerical solutions have been obtained for anisotropic Bianchi I and KS space-time [183] along with some restrictions on the cosmological constant and the Skyrme coupling. In the recent past, dynamical system analysis for Skyrme fluid with a constant radial profile has been studied in the Kantowski-Sachs space-time [184]. By global characterization of the critical points it is found that some equilibrium points correspond to the static model while there are critical points which describe an accelerated expansion of the Universe. Also the flow on the Poincare sphere has been shown and consequently the behaviour at infinity has been presented. This chapter studies the symmetry aspects of the physical space in the context of Noether symmetry and subsequently quantum cosmology has been developed with reference to Noether symmetry.

6.2 Noether Symmetry in Einstein-Skyrme model: A review

In Einstein-Skyrme (ES) model the action in general can be written as [170]

$$\mathcal{A} = \int d^4x\sqrt{-g}(R - 2\Lambda) + \mathcal{A}_k, \tag{6.1}$$

with the Skyrme-field action

$$\mathcal{A}_k = \frac{k}{2}\int d^4x\sqrt{-g}\left(\frac{1}{2}R_\alpha R^\alpha + \frac{\lambda}{16}F_{\alpha\beta}F^{\alpha\beta}\right). \tag{6.2}$$

Here $R_\alpha = U^{-1}U_\alpha$, $F_{\alpha\beta} = [R_\alpha, R_\beta]$, $U(x^\alpha)$ is the SU(2)- valued scalar and k, λ are positive parameters [174, 185, 186]. By variation of the action with respect to the metric tensor, the field equations of the model can be obtained as Einstein field equations

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}, \tag{6.3}$$

while varying with respect to the Skyrme field $U(x^\alpha)$ equation (6.1) gives the matter conservation law (i.e, $T^{\alpha\beta}_{;\beta} = 0$) as

$$g^{\alpha\beta}\left(R_{\alpha;\beta} + \frac{\lambda}{4}\left([R_\sigma F^{\sigma\nu}]\right)_{;\nu}\right) = 0, \tag{6.4}$$

where κ is the usual gravitational constant.

The space-time geometry is characterized by the locally rotational symmetric freedom with a 4D killing algebra characterized by the Lie group SO(3) and in addition a killing vector $\dfrac{\partial}{\partial r}$ (i.e, along the r co-ordinate line). The line element for this anisotropic Kantowski-Sachs (KS) space-time can be written as

$$ds^2 = -N^2dt^2 + a^2(t)dr^2 + b^2(t)d\Omega_2^2, \tag{6.5}$$

where $N = N(t)$ is the Lapse function, $a(t)$, $b(t)$ are the scale factors and $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$, is the metric on the unit 2-sphere. Then the matter density and thermodynamic pressure for the Skyrme fluid, respectively, take the forms

$$\rho_s = \frac{1}{b^2}\left(\bar{\kappa} + \frac{m}{2b^2}\right),$$

$$\text{and} \qquad p_s = -\frac{1}{3b^2}\left(\bar{\kappa} - \frac{m}{2b^2}\right), \tag{6.6}$$

with $\omega_s = \dfrac{p_s}{\rho_s} = -\dfrac{1}{3}\left(\dfrac{\bar{\kappa}b^2 - m}{\bar{\kappa}b^2 + m}\right)$, $\bar{\kappa} = \kappa k$ and $m = \bar{\kappa}\lambda$.

Clearly, $\omega_s \simeq -\dfrac{1}{3}$ for $\bar{\kappa}b^2 \gg m$ and then the Skyrme fluid acts as a curvature-like component. On the otherhand, for $\bar{\kappa}b^2 \ll m$, $\omega_s \simeq \frac{1}{3}$ i.e, the fluid behaves as radiation field and so one may consider $-\frac{1}{3} \leq \omega_s \leq \frac{1}{3}$. Note that in absence of Skyrme fluid the model corresponds to Einstein gravity with a cosmological constant.

The explicit form of the Lagrangian for the present Einstein-Skyrme model in the background of KS space-time is given by [170]

$$L(a, \dot{a}, b, \dot{b}) = \frac{ab^2}{N}\left(2\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{b}^2}{b^2}\right) + Nab^2\left(\Lambda + \frac{(\bar{\kappa}-1)}{b^2} + \frac{\bar{\kappa}\lambda}{2b^4}\right). \tag{6.7}$$

The form of the above Lagrangian shows that one can interpret the present model as equivalent to a non-minimally coupled scalar field $\psi = b(t)$ having the potential $V(\psi) = b^2\left[\Lambda + \frac{\bar{\kappa}-1}{b^2} + \frac{\bar{\kappa}\lambda}{2b^4}\right]$ in the FLRW space-time with scale factor 'a'. The explicit field equations representing this model are given by

$$2\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{N^2}{b^2} - \Lambda N^2 = \frac{N^2}{b^2}\left(\bar{\kappa} + \frac{m}{2b^2}\right), \tag{6.8}$$

$$2\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{N^2}{b^2} - \Lambda N^2 = \frac{N^2}{b^2}\left(\bar{\kappa} + \frac{m}{2b^2}\right), \tag{6.9}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} - \Lambda N^2 = -\frac{mN^2}{2b^4}. \tag{6.10}$$

According to Noether, if the Lagrangian of a physical system is invariant with respect to the Lie derivative along a vector field in the augmented space then there should be some conserved quantities associated with the system. For n configuration variables the symmetry condition $\mathcal{L}_{\overrightarrow{X}}L = 0$ gives $\left\{\frac{n(n+1)}{2} + 1\right\}$ number of partial differential equations whose solutions assign the symmetry. In the present problem the augmented space is a 3D space (N, a, b) and the Noether symmetry vector is [102, 103, 187]

$$\overrightarrow{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial b} + \gamma\frac{\partial}{\partial N} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{b}}, \tag{6.11}$$

with $\alpha = \alpha_0 a$, $\beta = -2\alpha_0 b$, $\frac{\gamma}{N} = -3\alpha_0$, $\bar{\kappa} = 1$, $\lambda = 3b^4\Lambda$, α_0, being a constant of integration (for details see [170]). Note that $\bar{\kappa} = 1$ is simply a choice for mathematical simplicity.

In the context of field theory in curved space, there is no well defined notion of energy. However, while considering Noether symmetry one has energy-momentum tensor as the conserved quantity (As a particular case, if there is a time-like killing vector field then there is a conserved charge). Further, in analogy with the point-like Lagrangian (due to no explicit time dependence) one has an energy (Hamiltonian) which is conserved (note that, in general, the Hamiltonian is not conserved). Thus in the present Noether symmetry analysis one gets two conserved quantities [104, 135] , namely conserved charge

$$Q = -\frac{2\alpha_0}{N} a b^2 \left(2\frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right), \tag{6.12}$$

and conserved energy

$$E = 2b\frac{\dot{a}\dot{b}}{N} + \frac{a\dot{b}^2}{N} - 4N\Lambda a b^2. \tag{6.13}$$

(In general, Noether symmetry analysis gives a conserved vector known as Noether current. By integrating its time component over spatial volume gives a conserved charge (Noether charge).)

To obtain the classical cosmological solution, it is desirable to make a transformation in the augmented space $(N, a, b) \to (u, v, w)$ so that one of the transformed variables becomes cyclic and consequently the evolution equations become much simpler to solve. The restriction on the above transformation to have a cyclic co-ordinate gives

$$i_{\vec{X}} du = 1, \quad i_{\vec{X}} dv = 0, \quad i_{\vec{X}} dw = 0, \tag{6.14}$$

where $i_{\vec{X}}$ stands for the inner product with the vector field $\vec{X}$. The restriction implies that the symmetry vector is along the normal to the surface $u =$ constant while it lies in the surfaces $v =$ constant and $w =$ constant. As a result the transformation gives [165]

$$e^u = a, \quad e^v = a^2 b, \quad e^w = \frac{ab^2}{N}, \tag{6.15}$$

with the simplified form of the transformed Lagrangian,

$$L_T = e^w \left[\dot{v}^2 - 2\dot{u}\dot{v} + \frac{2}{3}\lambda e^{-2w} e^{2u} \right]. \tag{6.16}$$

The conserved charge and energy in the new variables become

$$
\begin{aligned}
Q &= -2\alpha_0 e^{w} e^{u} \dot{v} - 4\alpha_0 e^{-2u} e^{v} e^{w} \dot{v} + 4\alpha_0 e^{-2u} e^{v} e^{w} \dot{v}, \\
E &= e^{w} \left[\dot{v}^2 - 2\dot{v}\dot{u} - \frac{2}{3}\lambda e^{-2w} e^{2u} \right].
\end{aligned} \tag{6.17}
$$

The Euler-Lagrange equations

$$
\begin{aligned}
\ddot{v} + \frac{2}{3}\lambda e^{-2w} e^{2u} &= 0, \\
\frac{d}{dt}\left\{ e^{w} \left(2\dot{v} - 2\dot{u} \right) \right\} &= 0,
\end{aligned}
$$

corresponding to the Lagrangian (6.16) can be solved and using (6.15) one may obtain the following set of explicit solution for the present cosmological model which can be compactly written as

$$
\begin{aligned}
a &= \frac{a_0}{\cosh(\beta t)}, \\
b &= b_0 e^{\alpha t} \cosh(\beta t),
\end{aligned} \tag{6.18}
$$

where a_0, b_0, α and β are integration constants. The above solution is an example of a singularity free model of the Universe. The bouncing nature of the present model has been shown in Figure 6.1 for the geometric volume ($V = ab^2$). Also the variation of the Hubble parameter $H = \frac{1}{3}\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)$ has been presented in Figure 6.2. The graph of deceleration parameter $q = -\left(1 + \frac{\dot{H}}{H^2}\right)$ is given in Figure 6.3 and it shows that the present model describes the late time acceleration era of evolution. Further, for stability analysis of the cosmological solution, the graphs for the cosmological parameters namely 'V', 'H' and 'q' are drawn in Figures (6.1-6.3) for four sets of closeby values of the parameters involved. These parameter sets differ among themselves by a small amount and it is found that the nature of the solutions remains unaltered. Hence it is reasonable to claim that the cosmological solution presented above (in equation (6.18)) is a stable one.

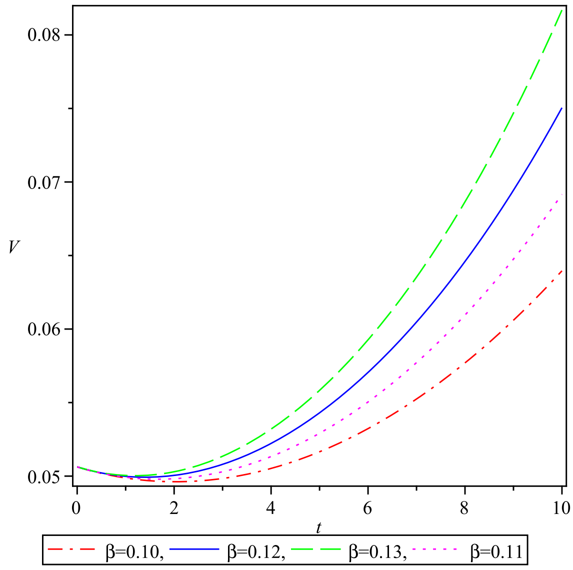

Figure 6.1: Volume vs t for the fixed parameters $a_0 = 1$, $b_0 = 0.225$, $\alpha = -0.01$ for four values of β as color coded above.

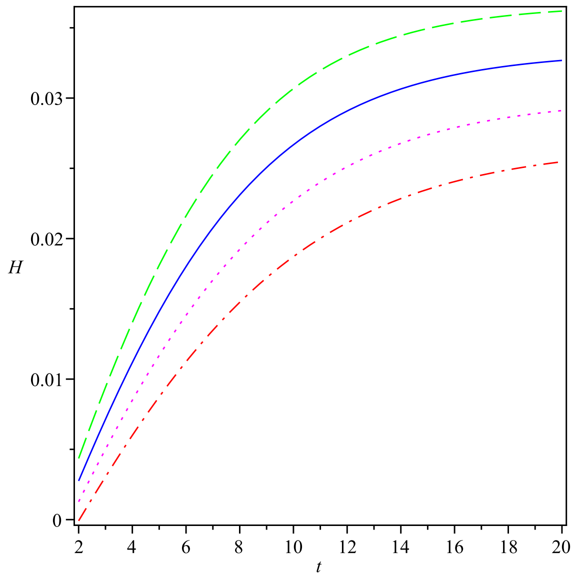

Figure 6.2: H vs t (Parameters and color coding same as in Figure 6.1)

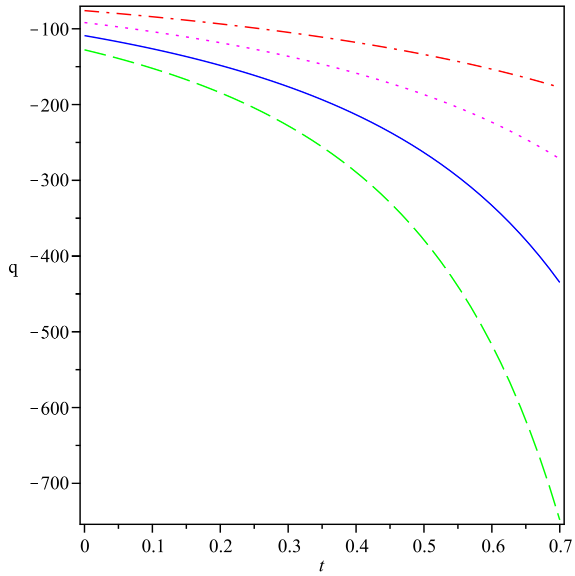

Figure 6.3: Deceleration parameter(q) vs t (Parameters and color coding same as in Figure 6.1)

6.3 Symmetry of the physical space

According to Tsamparlis etal. [92] if the Lagrangian of a physical system can be written as $L = T - V$ then the Noether point symmetries correspond to the elements of the homothetic group of the kinetic metric. In the present cosmological model as the Lagrangian (given by equation (6.7)) has the desired form, it is possible to analyze the Noether point symmetry by homothetic group of the kinetic metric (obtained from the kinetic part of the Lagrangian (6.7)) given by

$$ds^2_{(2k)} = 2adb^2 + 4bdadb, \tag{6.19}$$

with an effective potential $V_{eff}(b) = b^2 \left[\Lambda + \frac{\lambda}{2b^4}\right]$.

The above metric can be rewritten (by a suitable transformation) as

$$ds^2_{(2k)} = \frac{u^2}{a}\left[-\frac{da^2}{a^2} + \frac{du^2}{u^2}\right], \tag{6.20}$$

with $u = ab$. Clearly, the above metric is conformal to the $1+1$ decomposable metric

$$ds^2_{2k} = -d\alpha^2 + d\beta^2, \tag{6.21}$$

with $\alpha = \ln a$, $\beta = \ln u$. If the Lagrangian is rewritten with the above transformation of variables then one gets

$$L = \frac{e^{2u}}{a}\left[-\frac{\dot{a}^2}{a^2} + \dot{u}^2 + V(u)\right]. \tag{6.22}$$

So the present cosmological model is equivalent to a minimally coupled scalar field in the background of FLRW model with scale factor 'a'.

The 2D metric (6.20) admits the gradient homothetic vector (HV): $H_V = a\partial_a$ with $uH_V = 1$. However, this HV does not generate a Noether point symmetry for the Lagrangian. Further, as the 2D metric (6.20) is conformal to the flat metric (6.21) so it also admits three killing vectors which span the $E(2)$ group. Also the metric admits a 4D homothetic Lie algebra with the following vectors:

(a) Two gradient translation killing vectors: $X^{(1)} = -\partial_\alpha$, $\quad X^{(2)} = \frac{1}{\alpha}\partial_\beta$
with the corresponding gradient killing functions: $k^{(1)} = u$, $\quad k^{(2)} = 0$.
(b) One non-gradient killing vector (rotation): $X^{(3)} = \partial_\beta$.
(c) The gradient homothetic vector: $X^{(4)} = \alpha\partial_\alpha$.

6.4 Quantum Cosmology and Noether Point Symmetry: The Wave Function of the Universe

In quantum cosmology, the WD equation is essentially a partial differential equation of second order and is hyperbolic in nature. However, in the context of minisuperspace this WD equation reduces to the Klein-Gordon (KG) equation corresponding to conformal Laplacian operator.

In WKB approximation, the wave function is written as $\psi(x^k) \sim e^{is(x^k)}$ so that the WD equation reduces to the Hamiltonian-Jacobi equation as

$$\frac{1}{4}\left(\frac{\partial s}{\partial u}\right)^2 + \frac{1}{2}\left(\frac{\partial s}{\partial u}\right)\left(\frac{\partial s}{\partial v}\right) + \frac{2}{3}\lambda e^{2u} = 0. \tag{6.23}$$

Moreover, the constants of motion along the symmetry directions help to reduce the WD equation further. Also it is nice to investigate whether the wave function can be associated with the evolution of the dynamical variables. It is generally speculated that a quantum cosmological model is considered to be consistent if it has distinct non-singular solution at early eras while at a later time it matches with the classical solution.

For the present cosmological model the simplified form of the Lagrangian is

$$L = e^w \left[\dot{v}^2 - 2\dot{u}\dot{v} + \frac{2}{3}\lambda e^{-2w} e^{2u}\right],$$

and so the canonically conjugate momenta corresponding to the variables are

$$\begin{aligned} p_u &= -2e^w \dot{v}, \\ p_v &= e^w (2\dot{v} - 2\dot{u}) = \text{Conserved(as } v \text{ is cyclic).} \end{aligned}$$

Hence the Hamiltonian of the system takes the form

$$H = -\frac{1}{4}e^{-w}p_u^2 - \frac{1}{2}e^{-w}p_u p_v - \frac{2}{3}\lambda e^{-w}e^{2u}. \tag{6.24}$$

Now, in the process of quantization, the fundamental equation is the WD equation $\hat{H}\psi(u,v) = 0$, where $\hat{H}$ is the operator version of the above Hamiltonian in equation (6.24) and $\psi(u,v)$ is the wave function of the Universe. In the process of quantization of a physical system an important problem which arises, is the issue of operator ordering relating the order of a dynamical variable and its conjugate momentum. In the present model there is no such product of the variable and the corresponding momentum and so such an ordering problem does not arise. Now with the usual operator conversion:

$p_u \rightarrow -i\dfrac{\partial}{\partial u}$ and $p_v \rightarrow -i\dfrac{\partial}{\partial v}$, the explicit form of the WD equation is

$$\frac{1}{4}\frac{\partial^2\psi}{\partial u^2}+\frac{1}{2}\frac{\partial^2\psi}{\partial u\partial v}-\frac{2}{3}\lambda e^{2u}\psi=0. \tag{6.25}$$

The wave function is constrained by the (operator version) conserved charge as

$$-i\frac{\partial\psi}{\partial v}=\Sigma_0\psi. \tag{6.26}$$

Now using separation of variables for the wave function as $\psi(u,v)=A(u)B(v)$, the restriction (6.26) gives

$$B(v)=B_0\exp(i\Sigma_0 v).$$

Hence the differential equation for A takes the form :

$$\frac{1}{4}\frac{d^2A}{du^2}+\frac{1}{2}i\Sigma_0\frac{dA}{du}-\frac{2}{3}\lambda e^{2u}A=0. \tag{6.27}$$

So the wave function of the Universe for the Einstein-Skyrme model can be written as

$$\psi(u,v)=e^{i\Sigma_0(v-u)}\left[EJ_{i\Sigma_0}\left(\frac{2}{3}i\sqrt{6}\sqrt{\lambda}e^{u}\right)+FY_{i\Sigma_0}\left(\frac{2}{3}i\sqrt{6}\sqrt{\lambda}e^{u}\right)\right], \tag{6.28}$$

where E and F are arbitrary constants and J and Y are Bessel functions of the 1st and 2nd kinds having complex orders. The corresponding graphical representation of the wave function has been presented in Figure 6.4. The figure shows that the Universe may have finite non-zero probability in the limit of zero volume.

6.5 Brief Summary

In this chapter, Noether symmetry has been extensively used both in classical and quantum cosmology. Using a transformation in the augmented space with the help of the Noether symmetry vector it is possible to simplify the Lagrangian as one of the variables becomes cyclic. As a result, classical solutions are obtained and are presented graphically in Figures (6.1-6.3). The solution describes a singularity free bouncing model of the Universe. Also due to the negativity of the deceleration parameter, the present model may be considered as a dark energy model.

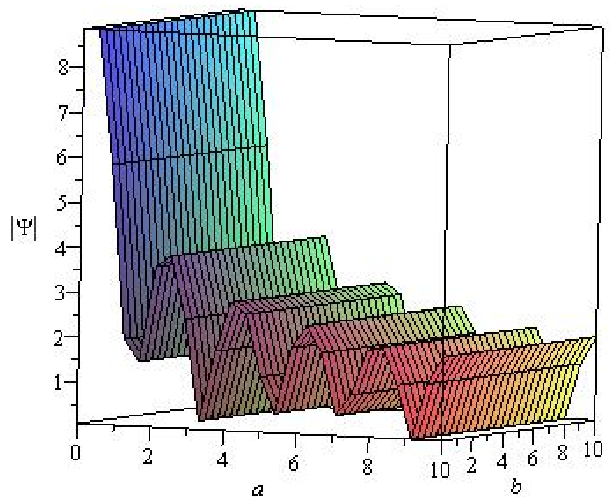

Figure 6.4: The graphical representation of $|\psi|$
(for $E = 10$, $F = .001$, $\Sigma_0 = 0$, $\lambda = -1$)

As the Lagrangian of the problem is in the form of a point particle and so the elements of the homothetic group of the kinetic metric generate the Noether point symmetries. Here the kinetic metric is conformal to the flat 2D metric and so it has 3 killing vector fields as well as 4D homothetic vector fields. In the context of quantum cosmology one has an infinite dimensional superspace. But one may restrict to a minisuperspace for practical purpose. In the present problem the minisuperspace is a 2D Lorentzian manifold on which the WD equation is constructed. Due to the conserved charge, the wave function has periodic nature along one direction in the minisuperspace. The graphical representation of the wave function shows that with proper choice of the parameters it is possible to have finite non-zero probability at zero volume i.e, Big-Bang singularity may be avoided in the present model by using quantum cosmology.

In quantum cosmology, symmetry analysis (particularly Noether symmetry) has been widely used in the minisuperspace. The operator version (with suitable operator ordering) of the Hamiltonian constraint (known as Wheeler-DeWitt (WD) equation) is related to the existence of Lie point symmetries. Further, Noether symmetries give a subset of the general solution of the WD equation having oscillatory behaviour with relevant physical meaning [103, 126]. Moreover, according to Hartle [188] the Noether symmetries are associated with the classical trajectories (in minisuperspace) which are solutions to the classical field equations i.e, Noether symmetries act as a bridge to correlate quantum cosmology and classically observable Universe.

CHAPTER 7

STUDY OF NOETHER SYMMETRY ANALYSIS FOR A COSMOLOGICAL MODEL WITH VARIABLE G AND Λ GRAVITY THEORY

7.1 Prelude

Based on the series of observational data for the last fifteen to twenty years cosmologists have a unanimous opinion about concordance cosmological model [17] - a cosmological paradigm based on general Relativity with a cosmological constant Λ. This model not only describe the early formation of large-scale structures but also the present era of accelerated expansion. At present two different types of unknown matter are well known from observational point of view and they constitute more than 95% of the matter energy around us. These matter components are termed as dark matter (DM) and dark energy (DE). Dark matter is invisible but attractive in nature as visible matter. It is speculated that it is present in between the galaxies and successfully describes rotational curves in spiral galaxies [189] (for an alternative view point see ref. [190]). Dark energy, (Λ being the simplest choice) on the other hand is supposed to be the major ingredient of the cosmic matter [191] to account for the present accelerating era. Although there are lot of physical dark energy models in the literature, still none of them is suitable both from theoretical as well as observational point of view.

As there is no strong basis (both theoretical and experimental) for these hidden matter parts so there are several alternative ways to accommodate the above Cosmological and Astrophysical issues. One such possible physical theory deals with variable cosmological constant as well as variable gravitational coupling. Such physical theory describes cosmological dynamics by analyzing renormalization group induced quantum version [192, 193, 194, 195, 196, 197, 198, 199, 200] of the theory with non-perturbative renormalization due to non-Gaussianity–the quantum Einstein Gravity [201]. In cosmological context, the inherent infrared divergence in the above quantum Einstein Gravity implies dynamical nature of the cosmological constant [202].

The basic aim of this chapter is to determine analytic solutions of the above variable G, Λ cosmological model using the Noether symmetries of the field equations. The idea of using Noether symmetry to cosmological models is not at all new, rather there are lot of works in the literature [101, 203]. The key idea of this approach is geometric [204, 205] and in particular the homothetic vectors of the kinetic metric for a first order Lagrangian are the Noether point symmetries, i.e, determination of Noether point symmetries reduces to a problem of differential geometry. Also mathematically, the first integral in Noether symmetry can be considered as a tool to simplify a system of differential equations or to determine the integrability of the system.

In the context of quantum cosmology, symmetry analysis has a great role to determine the solution of the Wheeler-DeWitt (WD) equation. Noether symmetries play the role of a bridge between quantum cosmology and classically observable Universe. In fact Noether symmetries provide a subset of the general solutions of the WD equation, giving oscillatory behaviours with suitable physical meaning [103, 126, 127]. Moreover, the criterion due to Hartle are associated with Noether symmetries to identify typical classical trajectories satisfying the cosmological evolution equations [103, 128].

7.2 Basic Equations of the Model

In Quantum Einstein gravity, for homogeneous and isotropic space-time geometry if G is treated as independent dynamical variable then at classical level it reduces to metric-scalar gravity. On the other hand for both G and Λ to be independent variables leads to a pathological situation: the momentum conjugate to Λ vanishes. Then the preservation of this primary constraint leads to vanishing Lapse function i.e, a collapse of space-time geometry. Hence it is reasonable to assume a generic functional dependence $\Lambda = \Lambda(G)$. In the present model the point like Lagrangian has the explicit form as [206]

$$L = -\frac{3a\dot{a}^2}{8\pi G} - \frac{a^3\Lambda(G)}{8\pi G} + \frac{\mu a^3\dot{G}^2}{16\pi G^3} - Da^{-3(\gamma-1)}, \tag{7.1}$$

where dot indicates the derivative with respect to the cosmic time t, G is a function of t and $\Lambda = \Lambda(G)$ while μ is a non-vanishing interaction parameter. L_m, the Lagrangian for the matter field, is chosen for a perfect fluid as $-Da^{-3(\gamma-1)}$. Where γ (index of state parameter) is a constant and D is a suitable integration constant. Here lapse function $N = 1$ and shift vector $N^i = 0$.

So the Euler-Lagrange equations for a and G take the form as:

$$\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{2a^2} - \frac{\Lambda}{2} - \frac{\dot{a}\dot{G}}{aG} + \frac{\mu\dot{G}^2}{4G^2} = 0, \tag{7.2}$$

$$\mu\ddot{G} - \frac{3}{2}\mu\frac{\dot{G}^2}{G} + 3\mu\frac{\dot{a}}{a}\dot{G} + \frac{G}{2}\left(-6\frac{\dot{a}^2}{a^2} - 2\Lambda + 2G\frac{d\Lambda}{dG}\right) = 0. \tag{7.3}$$

The Hamiltonian constraint is given by

$$\frac{\dot{a}^2}{a^2} - \frac{\Lambda}{3} - \frac{\mu}{6}\frac{\dot{G}^2}{G^2} - \frac{8\pi G}{3}Da^{-3} = 0. \tag{7.4}$$

This is equivalent to the constraint on the energy function associated with the Lagrangian.

The kinetic metric corresponding to the Lagrangian (7.1) is given by

$$ds^2_{(k)} = -\frac{3ada^2}{8\pi G} + \frac{\mu a^3 dG^2}{16\pi G^3}, \tag{7.5}$$

with effective potential,

$$V_{eff} = a^3\left[\frac{\Lambda(G)}{8\pi G} + Da^{-3\gamma}\right]. \tag{7.6}$$

It is to be noted that the Noether point symmetries of the system are generated by the elements of the homothetic group of the above kinetic metric as the Lagrangian is in the form of a point particle.

Further, the above kinetic metric can be rewritten as

$$\begin{aligned} ds^2_k &= \frac{a^3}{16\pi G}\left[-\frac{6}{a^2}da^2 + \frac{\mu dG^2}{G^2}\right] \\ &= \frac{e^{\left(\sqrt{\frac{3}{2}}u - \frac{v}{\sqrt{\mu}}\right)}}{16\pi}\left[-du^2 + dv^2\right], \end{aligned} \tag{7.7}$$

with $a = e^{\frac{u}{\sqrt{6}}}$, $G = e^{\frac{v}{\sqrt{\mu}}}$. Hence the kinetic metric is conformal to the flat $2D$ Minkowskian geometry, having four dimensional homothetic Lie Algebra with elements

(a) the gradient homothetic vector $l_1 = a\partial_a$
(b) three killing vectors which span the $E(2)$ group.

7.3 Noether Symmetry and Classical Cosmological Solutions

Noether's first theorem states that every differentiable symmetry of the action of a physical system with conservative forces has a corresponding conservation law. For solving the field equations we use Noether symmetry approach. In the present model the point like Lagrangian is given by (7.1).

Now the existence of Noether symmetry demands that there exist a vector valued function $F(t,a,G)$ such that

$$X^{[1]}L + LD_t\zeta = D_tF, \tag{7.8}$$

where $X^{[1]}$ is the first prolongation vector defined by

$$X^{[1]} = X + (D_t\alpha - \dot{a}D_t\zeta)\frac{\partial}{\partial \dot{a}} + (D_t\beta - \dot{G}D_t\zeta)\frac{\partial}{\partial \dot{G}}, \tag{7.9}$$

and the vector field X is defined as

$$X = \zeta\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial G}. \tag{7.10}$$

D_t, the total derivative operator, is given by

$$D_t = \frac{\partial}{\partial t} + \dot{a}\frac{\partial}{\partial a} + \dot{G}\frac{\partial}{\partial G}. \tag{7.11}$$

From (7.8) using (7.1), (7.9), (7.10) and (7.11) we get the following set of partial differential equations.

$$\begin{aligned}
-\frac{3\alpha a^2\Lambda(G)}{8\pi G} + 3D(\gamma-1)\alpha a^{-3\gamma+2} + \frac{\beta a^3\Lambda(G)}{8\pi G^2} - \frac{\beta a^3\Lambda'(G)}{8\pi G} & \\
-\frac{a^3\zeta_t\Lambda(G)}{8\pi G} - D\zeta_t a^{-3(\gamma-1)} &= F_t, &(7.12)\\
-\frac{6a\alpha_t}{8\pi G} - \frac{a^3\Lambda(G)\zeta_a}{8\pi G} - D\zeta_a a^{-3(\gamma-1)} &= F_a, &(7.13)\\
\frac{\mu a^3\beta_t}{8\pi G^3} - \frac{a^3\Lambda(G)\zeta_G}{8\pi G} - Da^{-3(\gamma-1)}\zeta_G &= F_G, &(7.14)\\
-\frac{3\alpha}{8\pi G} + \frac{3a\beta}{8\pi G^2} - \frac{6a\alpha_a}{8\pi G} + \frac{3a\zeta_t}{8\pi G} &= 0, &(7.15)\\
\frac{3\mu a^2\alpha}{16\pi G^3} - \frac{3\mu a^3\beta}{16\pi G^4} + \frac{\mu a^3\beta_G}{8\pi G^3} - \frac{\mu a^3\zeta_t}{8\pi G^3} + \frac{\mu a^3\zeta_t}{16\pi G^3} &= 0, &(7.16)\\
-\frac{6a\alpha_G}{8\pi G} + \frac{\mu a^3\beta_a}{8\pi G^3} &= 0. &(7.17)
\end{aligned}$$

Now if one consider ζ as a function of t only then ζ takes the form $\zeta(t) = c_1t + c_2$ and α, β are independent of t.

For solving the set of differential equations we use the method of separation of variables i.e,

$$\alpha = \alpha_1(a)\alpha_2(G),$$
$$\beta = \beta_1(a)\beta_2(G), \tag{7.18}$$

with $\alpha_2(G) = \frac{\beta_2(G)}{G}$ and $\beta_1(a) = \frac{\alpha_1(a)}{a}$. Then we get

$$\alpha = \alpha_0\beta_0 G^{-1}, \tag{7.19}$$
$$\beta = \alpha_0\beta_0 a^{-1}, \tag{7.20}$$

where α_0 and β_0 are arbitrary constants.

We also get $\mu = 6$; and the value of γ is either 1 or 0.

Putting $\gamma = 1$ in (7.12) we get

$$\Lambda(G) = \Lambda_0 G^{-2}, \tag{7.21}$$

where Λ_0 is a strictly positive integration constant.

Similarly putting $\gamma = 0$ in (7.12) we get

$$\Lambda(G) = -8\pi DG + \Lambda' G^{-2}, \tag{7.22}$$

where Λ' is positive integration constant.

Now we want to make a point transformation $(a, G) \to (u, v)$ in such a way that u becomes a cyclic coordinate. So, the symmetry vector X should satisfy the following equations

$$i_X du = 1, \tag{7.23}$$
$$i_X dv = 0, \tag{7.24}$$

where i_X is the inner product operator of X. Solving these equations we get

$$u = \frac{1}{2}aG, \tag{7.25}$$
$$\text{and} \quad v = \ln\frac{a}{G}. \tag{7.26}$$

Case:I: $\gamma = 1$

The transformed Lagrangian takes the form

$$L_1 = -\frac{3}{4\pi}e^{2v}\dot{u}\dot{v} - \frac{1}{8\pi}\Lambda_0 e^{3v} - D. \tag{7.27}$$

Using Euler-Lagrangian equation we get

$$\frac{d}{dt}\left(-\frac{3}{4\pi}e^{2v}\dot{v}\right) = 0, \tag{7.28}$$
$$\text{and} \quad e^{2v}\ddot{u} = 3\frac{\Lambda_0}{6}e^{3v}. \tag{7.29}$$

Solving equations (7.28) and (7.29) we obtain

$$u = \frac{2\Lambda_0}{15A^2}(At+B)^{\frac{5}{2}} + Ct + E \tag{7.30}$$
$$\text{and} \quad v = \frac{1}{2}\ln(At+B) \tag{7.31}$$

where A, B, C and E are arbitrary constants.

Hence the explicit cosmological solutions are of the form,

$$a^2(t) = a_1(t-t_1)^{\frac{1}{2}}\left\{(t-t_1)^{\frac{5}{2}} + c_1 t + d_1\right\}, \tag{7.32}$$
$$G^2(t) = G_1(t-t_1)^{-\frac{1}{2}}\left\{(t-t_1)^{\frac{5}{2}} + c_1 t + d_1\right\}, \tag{7.33}$$

where a_1, t_1, c_1 and d_1 are (arbitrary) constants constructed out of the integration constants A, B, C and E. Here $t = t_1$ is the Big-Bang singularity (assuming $c_1 t_1 + d_1 > 0$). So the above solution describes an early era of evolution and it is supported by the choice $\gamma = 1$ i.e, stiff fluid. The Figures (7.1-7.3) shows the graphical representation of the evolution history. From the figure, it is found that the scale factor gradually increases from the Big-Bang singularity and it becomes enormously large as $t \to \infty$. The Hubble parameter gradually decreases and the Universe is in accelerated era of expansion throughout the evolution. The gravitational parameter blows up both at the Big-Bang and as $t \to \infty$ and it reaches a finite minimum at some finite time. The equation (7.21) shows that the variable cosmological constant has zero value at the Big-Bang, then it increases to a maximum and finally it approaches to zero again. However, throughout the evolution ΛG^2 remains a finite constant.

Case:II: $\gamma = 0$

The transformed Lagrangian takes the form

$$L_2 = -\frac{3}{4\pi} e^{2v} \dot{u} \dot{v} - \frac{\Lambda'}{8\pi} e^{3v}. \tag{7.34}$$

Using Euler-Lagrangian equation we get

$$\frac{d}{dt}\left(-\frac{3}{4\pi} e^{2v} \dot{v}\right) = 0, \tag{7.35}$$

$$e^{2v} \ddot{u} = 3\frac{\Lambda'}{6} e^{3v}. \tag{7.36}$$

Solving equation (7.35) and (7.36) we obtain

$$u = \frac{2\Lambda'}{15A'^2}(A't + B')^{\frac{5}{2}} + C't + E', \tag{7.37}$$

$$v = \frac{1}{2}\ln(A't + B'), \tag{7.38}$$

where A', B', C' and E' are arbitrary constants.

As $\gamma = 0$ corresponds to dust era of evolution so by choosing the above constants suitable one may write the explicit cosmological solution as

$$a^2(t) = a_0(t + t_0)^{\frac{1}{2}} \left\{(t + t_0)^{\frac{5}{2}} + c_0 t + d_0\right\}, \tag{7.39}$$

$$G^2(t) = G_0(t + t_0)^{-\frac{1}{2}} \left\{(t + t_0)^{\frac{5}{2}} + c_0 t + d_0\right\}, \tag{7.40}$$

where the evolution has started from a time $t > -t_0$ (with $-c_0 t_0 + d_0 > 0$) with a finite value of the scale factor and the scale factor blows up at $t \to \infty$. The Figures

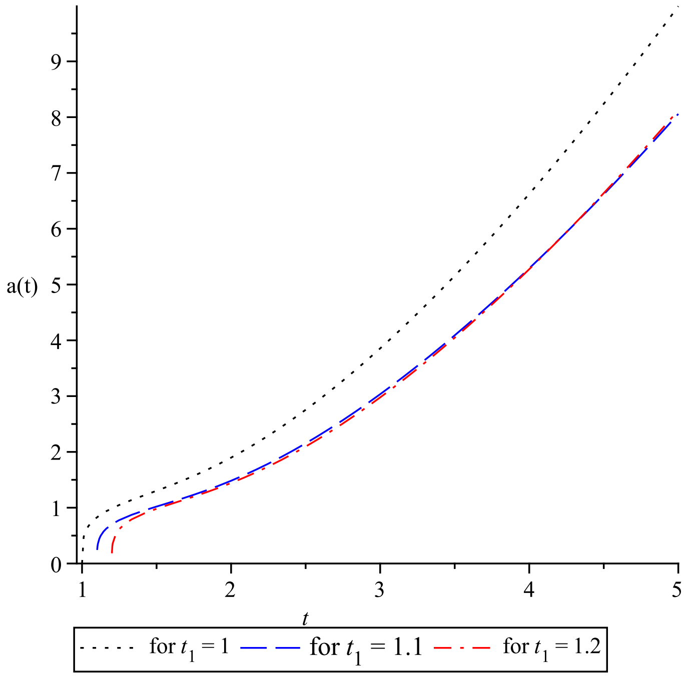

Figure 7.1: Graphical representation of variation of the scale factor a vs t for $\gamma = 1$

(7.4-7.6) show the diagrammatic representation of the evolution of the Universe. The solution represents the evolution of the Universe from matter dominated era to the present late time accelerated evolution. Thus the present model may be considered as an alternative to the dark energy model. Now due to equation (7.22) both Λ and G have similar behaviour namely both of them have finite value at the beginning and then they gradually blows up to infinity.

Further, the above expressions for the scale factor show that the improper integral in the expression for the particle horizon does not converge. Hence the particle horizon does not exist for the present model both for $\gamma = 0$, 1.

7.4 The Minisuperspace Approach in Quantum Cosmology

The minisuperspace approach has already been described in section 3.4.

Using WKB approximation one may write the wave function as $\psi(x^k) \sim e^{is(x^k)}$ and consequently, the WD equation transforms to first order Hamilton-Jacobi equation. In

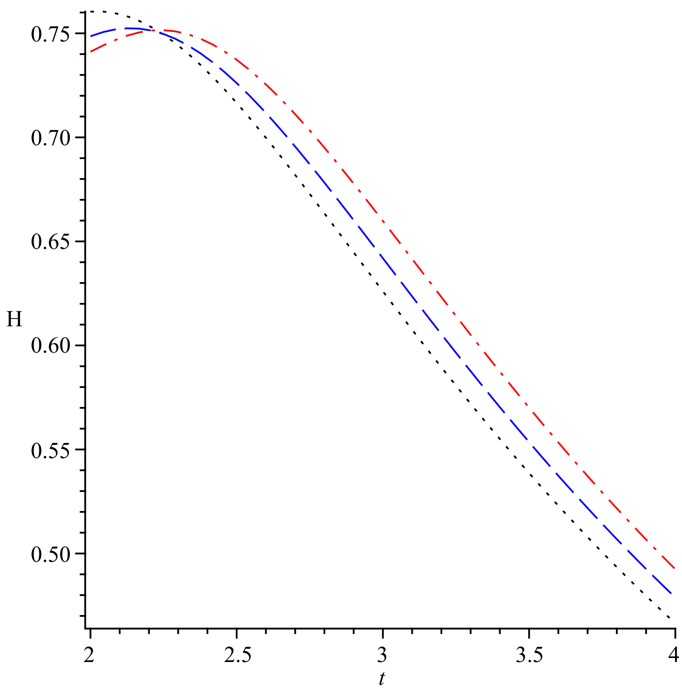

Figure 7.2: Graphical representation of variation of the Hubble parameter H vs t for $\gamma = 1$

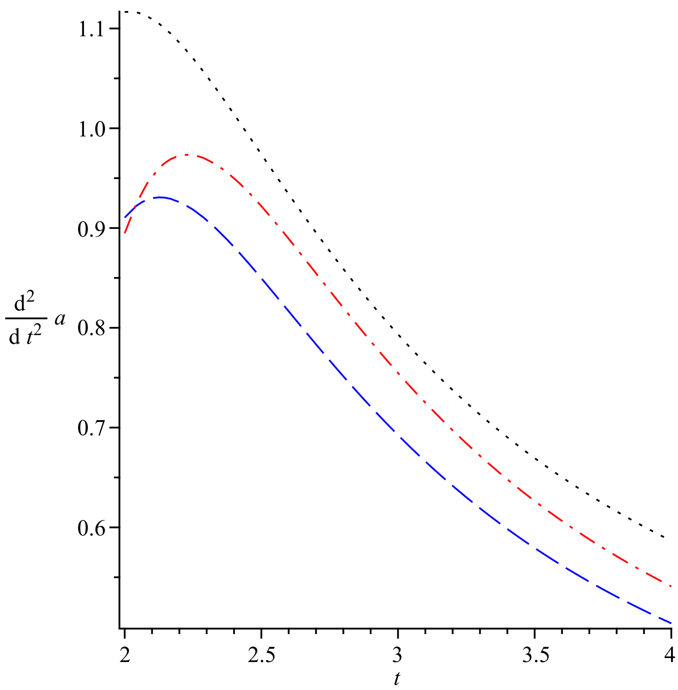

Figure 7.3: Represents the variation of acceleration parameter vs t for $\gamma = 1$

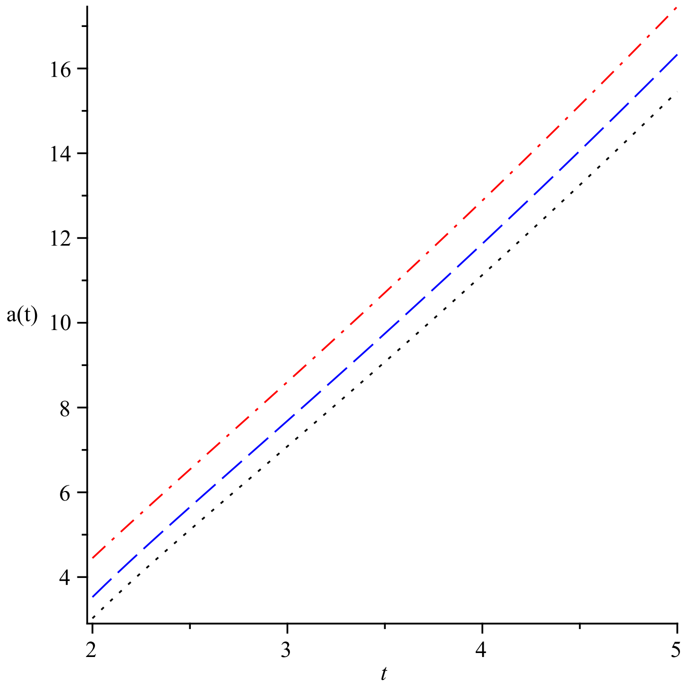

Figure 7.4: Graphical representation of variation of the scale factor a vs t for $\gamma = 0$

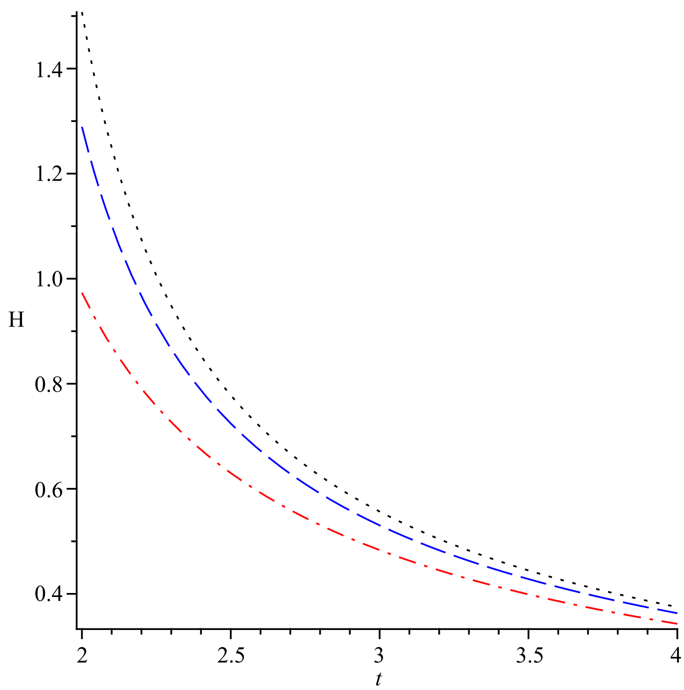

Figure 7.5: Graphical representation of variation of the Hubble parameter H vs t for $\gamma = 0$

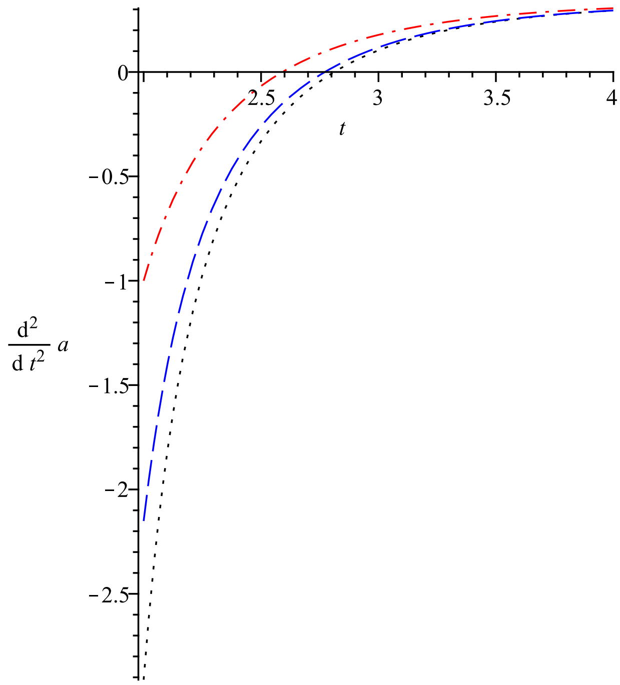

Figure 7.6: Represents the variation of acceleration parameter vs t for $\gamma = 0$

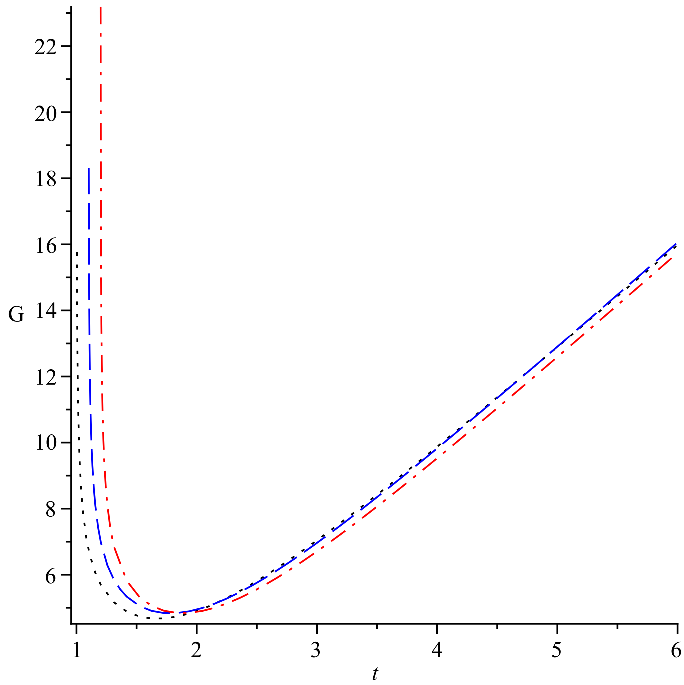

Figure 7.7: Graphical representation of G vs t for $\gamma = 1$

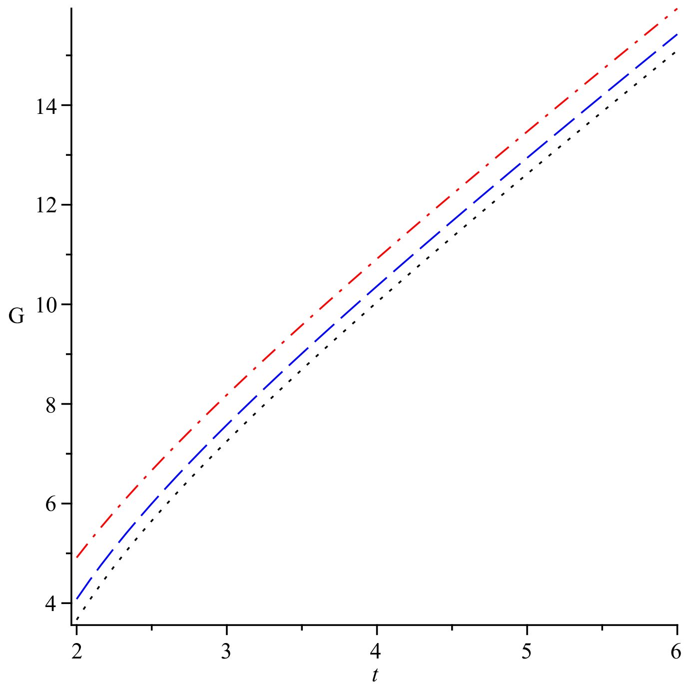

Figure 7.8: Graphical representation of G vs t for $\gamma = 0$

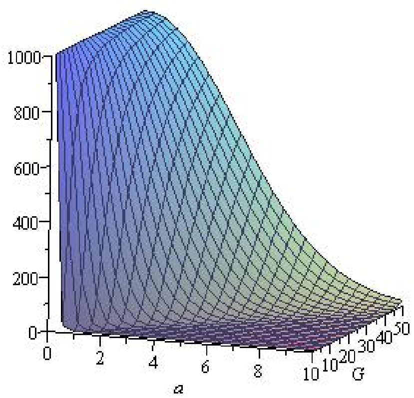

Figure 7.9: Represents wave function with respect to a and G.

the following we shall consider the two cases (for $\gamma = 0, 1$) separately.

case:I: $\gamma = 1$

From (7.27), the momentum associated with the variables u and v can be written as

$$\begin{aligned} p_u &= \frac{\partial L}{\partial \dot{u}} = -\frac{3}{4\pi} e^{2v}\dot{v} = \text{conserved} = p_0, \\ p_v &= \frac{\partial L}{\partial \dot{v}} = -\frac{3}{4\pi} e^{2v}\dot{u}. \end{aligned} \tag{7.41}$$

Then the Hamiltonian takes the form

$$H_1 = -\frac{4\pi}{3} e^{-2v} p_u p_v + \frac{1}{8\pi}\Lambda_0 e^{3v} + D. \tag{7.42}$$

Thus the WD equation takes the form

$$\left[\frac{4\pi}{3} e^{-2v} \frac{\partial^2}{\partial u \partial v} + \frac{1}{8\pi}\Lambda_0 e^{3v} + D\right] \psi(u, v) = 0. \tag{7.43}$$

The operator version of the conserved momentum can be written as

$$i \frac{\partial \psi(u, v)}{\partial u} = p_0 \psi(u, v). \tag{7.44}$$

Now writing $\psi = \psi_1(u)\psi_2(v)$, one has

$$\begin{aligned} i\frac{d\psi_1}{du} &= p_0\psi_1, \\ i.e, \quad \psi_1(u) &= \psi_0 e^{-ip_0 u}, \end{aligned} \tag{7.45}$$

with ψ_0, a constant of integration.

Now from the WD equation we get

$$\begin{aligned} -ip_0 \frac{4\pi}{3} e^{-2v} \frac{d\psi_2}{dv} + \frac{\Lambda_0}{8\pi} e^{3v}\psi_2 + D\psi_2 &= 0, \\ i.e, \quad e^{-2v}\frac{d\psi_2}{dv} &= (k_1 + k_2 e^{3v})\psi_2, \end{aligned} \tag{7.46}$$

where $k_1 = \dfrac{3D}{i4\pi p_0}$ and $k_2 = \dfrac{3\Lambda_0}{i32\pi^2 p_0}$.

Solving equation (7.46), we get,

$$\psi_2(v) = \psi_0' e^{\left(\frac{k_1}{2}e^{2v}+\frac{k_2}{5}e^{5v}\right)}, \tag{7.47}$$

with ψ_0', an integration constant. Thus the resulting wave function of the Universe in terms of the old variables takes the form

$$\psi(a,G) = \psi_{01} e^{\frac{i}{2}p_1 aG} exp\left[\frac{k_1}{2}\frac{a^2}{G^2} + \frac{k_2}{5}\frac{a^5}{G^5}\right], \tag{7.48}$$

with ψ_{01}, a constant.

From the solutions (7.32) and (7.33) we see that as the Universe approaches the Big-Bang singularity $\frac{a}{G} \to 0$ while $aG \to$ a finite non-zero constant. So near the Big-Bang singularity the above wave function has purely oscillatory in nature with finite amplitude and frequency. Thus there is finite probability to have the Big-Bang singularity for the present model with stiff fluid. The graphical representation of the wave function has been shown in Figure 7.9.

As the probability is constant near the Big-Bang singularity so no boundary proposal will not be valid there. Further, it is to be noted that there is no (curvature) divergence even at the singularity due to the running of the Newton's constant.

case:II: $\gamma = 0$

From (7.34), the momentum can be written as

$$\begin{aligned} p_u &= \frac{\partial L}{\partial \dot{u}} = -\frac{3}{4\pi}e^{2v}\dot{v} = \text{conserved} = p_1, \\ p_v &= \frac{\partial L}{\partial \dot{v}} = -\frac{3}{4\pi}e^{2v}\dot{u}. \end{aligned} \tag{7.49}$$

Then the Hamiltonian takes the form

$$H_2 = -\frac{4\pi}{3}e^{-2v}p_u p_v + \frac{\Lambda'}{8\pi}e^{3v}. \tag{7.50}$$

Thus the WD equation takes the form

$$\left[\frac{4\pi}{3}e^{-2v}\frac{\partial^2}{\partial u\partial v}+\frac{\Lambda'}{8\pi}e^{3v}\right]\phi(u,v)=0. \tag{7.51}$$

The operator version of the conserved momentum can be written as

$$i\frac{\partial\phi(u,v)}{\partial u}=p_1\phi(u,v). \tag{7.52}$$

Now writing $\phi=\phi_1(u)\phi_2(v)$, one has

$$\begin{aligned} i\frac{d\phi_1}{du} &= p_1\phi_1, \\ i.e, \quad \phi_1(u) &= \phi_0 e^{-ip_1 u}, \end{aligned} \tag{7.53}$$

with ϕ_0, a constant of integration.

Now from the WD equation we get

$$\begin{aligned} -ip_0\frac{4\pi}{3}e^{-2v}\frac{d\phi_2}{dv}+\frac{\Lambda'}{8\pi}e^{3v}\phi_2 &= 0, \\ i.e, \quad e^{-2v}\frac{d\phi_2}{dv} &= k_3 e^{3v}\phi_2, \end{aligned} \tag{7.54}$$

where $k_3=\dfrac{3\Lambda'}{i32\pi^2 p_1}$.

Solving equation (7.54), we get,

$$\phi_2(v)=\phi_0' e^{\frac{k_3}{5}e^{5v}}, \tag{7.55}$$

ϕ_0' is a constant of integration.

The wave function of the Universe has similar form as in the previous case. As in this case the model does not correspond to early epoch of evolution so quantum cosmology will not be interesting.

7.5 Brief Summary

The chapter deals with a complicated cosmological model where Newtonian gravitational coupling is no longer a constant and the cosmological constant is a function of this coupling parameter. Due to non-linear coupled field equations it is hard to infer any physical prediction from the model. As a result, Noether symmetry analysis has been used to the model. By obtaining Noether symmetry vector, it is possible to have a transformation in the augmented space so that one of the variables become cyclic and the Lagrangian is so simplified that one obtains classical cosmological solutions for the choices $\gamma = 1$ (stiff fluid) and $\gamma = 0$ (dust). The relevant cosmological parameters have been presented graphically (in Figures (7.1-7.3)) for $\gamma = 1$ and (in Figures (7.4-7.6)) for $\gamma = 0$ and are analyzed from observational point of view. The variation of G with the cosmic time has been presented graphically in Figures (7.7 and 7.8) for $\gamma = 1$ and $\gamma = 0$ respectively. For the present work, in both the models there does not exist a fixed point for which G vanishes at small 't' and Λ diverges but the product ΛG approaches a constant. Further, the Noether symmetry analysis has a very crucial role in studying quantum cosmology. The associated conserved charge in Noether symmetry identifies is the oscillatory part of the wave function and consequently the WD equation simplifies to a great extend. The graphical representation of the wave function (shown in Figure 7.9) shows that the quantum cosmological description favours the occurrence of Big-Bang singularity at the beginning.

CHAPTER 8

BRIEF SUMMARY AND FUTURE PROSPECT

In this chapter, we shall summarize this thesis. Also future prospects will be discussed.

In the second chapter, a cosmological model has been considered. There are two scalar fields minimally coupled to gravity in this model. The model is characterized by the coupling function and the potential function and they both depend on one of the scalar fields. Using Noether symmetry approach, we have evaluated these two functions instead of choosing phenomenologically. Finally, the classical cosmological solutions are analyzed from cosmological view point.

Third chapter deals with a cosmological model where a scalar field is non-minimally coupled to cold dark matter. This model is known as Chameleon field cosmological model. Here both classical and quantum cosmology have been studied. Constructing the Wheeler-DeWitt (WD) equation we have found the wave function of the Universe for this model.

In the framework of $f(T)$-gravity theory, classical and quantum cosmology have been studied in the fourth chapter. A specific functional form of $f(T)$ has been determined using Noether symmetry approach. Making a suitable point transformation we get a transformed Lagrangian where one variable is cyclic. Then the field equations become simplified and we have solved them. Finally, WD equation is constructed for this model and the solution has been evaluated.

Fifth chapter is actually the extension of the second chapter. Using symmetries of the 3D augmented space, homothetic and killing vectors are determined. Lastly, quantum cosmology has been formulated and wave function of the Universe has been evaluated.

In chpter six, we have discussed about anisotropic Skyrme fluid with zero heat flux in the background of homogeneous Kantowski-Sachs (KS) space-time geometry in the framework of Einstein gravity. In the context of quantum cosmology, WD equation is constructed and wave function of the Universe is evaluted using conserved Noether charge for this model. This model shows a bouncing nature of evolution that is a singularity free model.

In chapter seven, we have discussed about the solution of the matter-dominated cosmological equations using the expressions $\Lambda = \Lambda(G)$. Using Noether symmetry analysis, classical cosmological solutions for this model are obtained. WD equation is constructed and finally we have analyzed the solution of WD equation in the cosmological point of view.

Now we will discuss about the future prospects of Noether symmetry analysis. Symmetry analysis can be applied to the extended theories of gravity. Also one may analyze how specific Lagrangian multiplier related to symmetries can be used to reduce the dynamics and exact analytical solutions may be easily obtained.

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