MASTER OF SCIENCE EXAMINATION, 2017

(2nd Year, 1st Semester)

MATHEMATICS

Unit - 3.3 (A1.5)

(Mathematical Modelling of Biological Systems - I)

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

(Symbols/Notations have their usual meanings.)

Answer question no. 6 and any *three* from the rest.

1. (a) Characterize the bifurcation with respect to the parameter $\mu \in R$ for the system of equations

$$\dot{x} = y$$

$$\dot{y} = \mu x - x^2 - y \cdot$$

(b) State conditions for the existence of a Hopf bifurcation. Does the system $\ddot{x} + \left(x^2 + x - \mu\right)\dot{x} + x = 0$, $\mu \in R$ exhibit a Hopf bifurcation ? Justify your answer. 8+8

[*Turn over*]

2. (a) Use cobwebing method to sketch the solutions of the equation $x_{n+1} = \dfrac{r x_n^2}{x_n^2 + A}$, $r > 0,\ A > 0$ with initial condition $x_0 > 0$.

(b) Suppose a predator-prey system satisfies the following equations :

$$\dot{x}_{n+1} = x_n \left(a - x_n - y_n\right),$$

$$\dot{y}_{n+1} = y_n \left(b + x_n\right),$$

where x and y represent the prey and predator populations at the n-th generation, respectively and $a > 0,\ 0 < b < 1$.

(i) Find all fixed points of this system.

(ii) Use Jury conditions to show that the interior fixed point is locally asymptotically stable if $2 < a + b < 3$.

8+8

3. (a) Find the equilibrium points of the nonlinear difference equation

$$x_{n+1} = r x_n \left(1 - x_n\right),\ r > 0.$$

[*Turn over*]

Find the two cycles of this equation and hence comment on its stability.

(b) Calculate the Lyapunov exponent at the equilibria of the system

$$x_{n+1} = \frac{x_n + x_n^2}{2}.$$ 12+4

4. (a) Consider the nonlinear autonomous system $\dot{x} = f(x), \quad x \in R^n$ with an isolated equilibrium point $\bar{x}$. Define Lyapunov stability, quasi asymptotic stability, asymptotic stability and global stability of the equilibrium point.

(b) Draw the phase diagram of the following system :

$$\dot{x} = -y + x\left(1 - x^2 - y^2\right),$$

$$\dot{y} = x + y\left(1 - x^2 - y^2\right).$$

Determine its α and ω limit sets.

[*Turn over*]

(c) Consider the system

$$\dot{x} = -y^3,$$

$$\dot{y} = x^3.$$

Show that (0, 0) is a nonhyperbolic equilibrium point. Construct a suitable Lyapunov function to show that (0, 0) is stable. 6+6+4

5. (a) State Poincare-Bendixson theorem. Use this theorem to show that the following system has at least one periodic orbit :

$$\dot{x} = x - y - x\left(x^2 + \frac{3}{2}y^2\right),$$

$$\dot{y} = x + y - y\left(x^2 + \frac{1}{2}y^2\right).$$

(b) Discuss the behaviour of the solution to the second order difference equation

$$\dot{x}_{n+2} + c(1+\vartheta)x_{n+1} + c\vartheta x_n = 0,$$

$$0 < c < 1, \quad \vartheta > 0.$$

[*Turn over*]

Give a rough sketch of the operating diagram on $c-\vartheta$ parametric plane. 8+8

6. Define Bendixson-Dulac criterion. 2