Master of Science Examination, 2017

(2nd Year, 1st Semester)

MATHEMATICS

Unit - 3.3 (A1.1)

(Advanced Algebra - I)

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

Notations and Symbols have their usual meanings.

Answer any *five* questions. 10×5=50

(Throughout R denotes a commutative ring with identity.)

1. (a) Define comaximal ideals of a ring. Let A, B, C be three ideals of a ring R such that A, B are comaximal and A, C are comaximal. Show that (i) $AB = A \cap B$ (ii) A and BC are comaximal. 1+2+2

   (b) State Chinese Remainder Theorem for ring. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ be a prime factorization of the integer n. Show that $\mathbb{Z}_n \simeq \mathbb{Z}_{p_1}\alpha_1 \times \mathbb{Z}_{p_2}\alpha_2 \times \ldots \times \mathbb{Z}_{p_r}\alpha_r$ asrings by using Chinese Remainder theorem. 2+3

[*Turn over*]

2. (a) Let $J(R)$ and $N(R)$ be the Jacobson radical and nilradical of a ring R. Show that

(i) $J(R)$ has no non-zero idempotent element.

(ii) if every ideal of R not contained in $N(R)$ contains a non-zero idempotent then $N(R) = J(R)$. 2+3

(b) Let M be a maximal ideal of a ring R such that $1+m$ is a unit for all $m \in M$. Show that R is a local ring. By using this result show that R/M^n is a local ring for every positive integer n. 2+3

3. (a) Let S be a multiplicatively closed subset of a ring R and I be an ideal of R. Show that $S^{-1}I$ is an ideal of $S^{-1}R$. Hence show that $S^{-1}R/S^{-1}I \simeq S^{-1}(R/I)$. 2+3

(b) Let S be a multiplicatively closed subset of a ring R and M be an R-module. Define $S^{-1}M$, the module of fractions w.r.t. S. Let N_1 and N_2 be two submodules of M, show that

$$S^{-1}(N_1 \cap N_2) = S^{-1}(N_1) \cap S^{-1}(N_2).$$ 3+2

[*Turn over*]

4. (a) Define primary ideal of a ring. Let P be a primary ideal of a ring R. Show that $\sqrt{P}$ is the smallest prime ideal of R containing P. 2+3

(b) State and prove Lying-over Theorem for ring. 2+3

5. (a) Let M_1 and M_2 be two finitely generated submodules of an R-module M. Show that $M_1 + M_2$ is a finitely generated R-module. Let R be a ring with 10 elements and M be an R-module with 20 elements. Is M a finitely generated free R-module ? Justify your answer. 3+2

(b) Let M and N be two R-modules. Show that an R-homomorphism $f : M \to N$ is regular if and only if ker f is a direct summand of M and Imf is a direct summand of N. 5

6. (a) Let M, N be two R-modules and F be a free R-module. Show that every short exact sequence of R-modules and R-homomorphisms

$O \longrightarrow N \xrightarrow{f} M \xrightarrow{g} F \longrightarrow O$ is a split exact sequence. Is the short exact sequence

$O \longrightarrow F \xrightarrow{f'} M \xrightarrow{g'} N \longrightarrow O$ a split exact sequence ? Justify your answer. 3+2

[*Turn over*]

(b) Define projective module. Show that every free R-module is a projective R-module. 2+3

7. (a) Define divisible group. Show that an additive abelian group M is a divisible group if and only if $\mathbb{Z}$-module M is injective. 1+4

(b) Let R be a ring. Show that $R \otimes_R R \simeq R$ as modules. 5