Ex/M.Sc/M/B-1.38/36/2017

MASTER OF SCIENCE EXAMINATION, 2017

(2nd Year, 1st Semester)

MATHEMATICS

Unit - 3.4 (B-1.38)

(Theory of Semigroups - I)

Full Marks : 50 Time : Two Hours

All questions carry equal marks.

Answer any *five* questions. 10×5=50

1. (a) Define a *left zero semigroup*, a *right zero semigroup* and a *rectangular band*. Prove that a semigroup is isomorphic to a rectangular band if and only if it is isomorphic to the direct product of a left zero semigroup and a right zero semigroup.

   (b) Define a *periodic semigroup*. Prove that a periodic semigroup contains at least one idempotent.

2. (a) Define the semigroup $(\mathcal{PT}(X), \circ)$ of *partial mappings* on a non-empty set X.

[*Turn over*]

Let $\phi, \psi \in \mathcal{PT}(X)$. Prove that

$$\text{dom}(\phi \circ \psi) = \left[\text{ran}(\phi) \cap \text{dom}(\psi)\phi^{-1}\right] \text{ and}$$

$$\text{ran}(\phi \circ \psi) = \left[\text{ran}(\phi) \cap \text{dom}(\psi)\right]\psi .$$

(b) Define a *congruence* ρ on a semigroup S. Let ρ and σ be two congruences on a semigroup S such that $\rho \subseteq \sigma$. Then prove that

$$\sigma/\rho = \{(x\rho,\ y\rho) \in S/\rho \times S/\rho \,|\, (x,\ y) \in \sigma\}$$

is a congruence on S/ρ and $(S/\rho)/(\sigma/\rho) \cong S/\sigma$.

3. (a) Define the free semigroup F_A for a non-empty set A. If S is a semigroup and $\phi : A \to S$ is an arbitrary mapping, then show that there exists a unique homomorphism $\psi : F_A \to S$ such that $\psi|_A = \phi$.

(b) Define an *ideal* of a semigroup. Let I and J be two ideals of a semigroup S such that $I \subseteq J$. Show that $(S/J) \cong (S/I)(J/I)$.

[*Turn over*]

4. (a) Define the *Green's equivalence relations* D and I on a semigroup. Prove that in a periodic semigroup, $\mathsf{D} = \mathsf{I}$.

   (b) Let a be an element of a regular D-class D in a semigroup S. If $b \in D$ is such that $R_a \cap L_b$ and $L_a \cap R_b$ contain idempotents e and f respectively, then show that H_b contains an inverse a^* of a such that $aa^* = e$ and $a^*a = f$.

5. (a) Define a *regular* semigroup. Define an *idempotent separating* congruence. If S is a regular semigroup, then show that a congruence ρ on S is idempotent separating if and only if $\rho \subseteq \mathsf{H}$.

   (b) Define the *full transformation semigroup* $\mathsf{T}(X)$ for a nonempty set X. Show that $\mathsf{T}(X)$ is a regular semigroup.

6. (a) Define a *completely simple* semigroup. If a semigroup S is a union of groups and is simple, then prove that S is completely simple.

[*Turn over*]

(b) Define a *Clifford* semigroup and a *semilattice of groups*. Show that a Clifford semigroup is a semilattice of groups.

7. (a) Define a *completely regular semigroup*. Prove that in a completely regular semigroup, $D = I$.

(b) Define an *orthodox semigroup S*. Define an *idempotent pure* congruence in a regular semigroup. Let ρ be an idempotent pure congruence on a completely regular semigroup such that S/ρ is orthodox. Show that *S* is also an orthodox semigroup.