Ex/M.Sc/M/B-1.30/36/2017

MASTER OF SCIENCE EXAMINATION, 2017

(2nd Year, 1st Semester)

MATHEMATICS

Unit - 3.4 (B-1.30)

(Operator Theory - I)

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

(Symbols and Notations have their usual meanings.)

Answer any *five* questions.

1. (a) Show that the spectrum of a bounded linear operator $T : X \to X$ on a complex Banach space X is bounded. Justify whether the result holds for unbounded linear operator or not. 4+2

(b) Let $T : l^{\infty} \to l^{\infty}$ be defined by

$$T(\xi_1, \xi_2, \ldots) = (\xi_2, \xi_3, \ldots).$$

If $|\lambda| > 1$ then show that $\lambda \in \rho(T)$ and if $|\lambda| \leq 1$ then $\lambda \in \sigma(T)$. 4

[*Turn over*]

2. (a) Let X be a complex Banach space, $T \in B(X)$ and $p(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$. Then show that $\sigma(p(T)) = p(\sigma(T))$. 7

(b) Let $\mathbb{A}$ be a complex Banach Algebra with identity. Then show that the set G of all invertible elements of $\mathbb{A}$ is an open subset of $\mathbb{A}$. 3

3. (a) Let $\mathbb{A}$ be a complex Banach Algebra with identity, S denotes the set of singular elements of $\mathbb{A}$ and Z denotes the set of topological divisor of zero. Then prove that $Z \subset S$ and $Bd\ S \subset Z$. 3+3

(b) Show that closure of the numerical range of a bounded linear operator on a complex Hilbert space H always contains the spectrum. 4

4. (a) Let $T : X \to Y$ be bounded linear operator. Then show that T is compact iff it maps every bounded sequence $\{x_n\}$ in X onto a sequence $\{Tx_n\}$ which has a convergent subsequence in Y. 4

(b) Let Y be a Banach space, X be a normed linear space and let $T_n : X \to Y$, $n = 1, 2, ...,$ be operators of finite rank. If $\{T_n\}$ is uniformly operator convergent, then show that the limit operator is compact. 6

[*Turn over*]

5. Let $T: X \to X$ be a compact linear operator on a normed space X. Then show that for every $\lambda \neq 0$ the range of $T - \lambda I$ is closed. Verify whether the result holds for $\lambda = 0$ or not. 8+2

6. (a) Let $T: X \to X$ be a compact linear operator on a normed space. If dim $X = \infty$, show that $0 \in \sigma(T)$. 4

   (b) Let $T: l^2 \to l^2$ be defined by $y = Tx$, $x = \{\xi_j\}$, $y = \{\eta_j\}$, $\eta_j = \alpha_j \xi_j$ where (α_j) is dense in [0, 1]. Find $\sigma_p(T)$, $\sigma_c(T)$, $\sigma_r(T)$ and hence show that T is not compact. 6

7. (a) Let $T: X \to X$ be a compact linear operator on a normed space X and let $\lambda \neq 0$. Then for $y \in X$ the equation $Tx - \lambda x = y$ has a solution x if and only if y is such that $f(y) = 0$ for all $f \in X'$ satisfying $T'f - \lambda f = 0$. 6

   (b) Let $T: X \to X$ be a compact linear operator on a Banach space X. Then prove that every spectral value $\lambda \neq 0$ of T (if it exists) is an eigenvalue of T. 4