Master of Science Examination, 2017

(1st Year, 1st Semester)

MATHEMATICS

Unit - 1.3

(Complex Analysis)

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

Symbols / Notations have their usual meanings.

Answer Question No. 7 and any *four* questions from the rest.

1. (a) Let $f(z)$ be holomorphic in a simply-connected open region D and Γ be a positively oriented rectifiable closed Jordan curve such that $I(\Gamma) \cup \Gamma \subseteq D$. Prove that the values of the holomorphic function $f(z)$ on Γ uniquely determine its value at any point in $I(\Gamma)$.

   (b) Let $f(z)$ and $g(z)$ be holomorphic at $z = z_0$ and $f(z_0) = g(z_0) = 0$. If $g'(z_0) \neq 0$, prove that $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$. 5+3

2. (a) Let $D \subseteq C$ be an open region and Γ be a rectifiable curve such that D and Γ are disjoint point sets. If $f(z)$ is continuous on Γ, then show that the function

$$F(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)\, d\zeta}{\zeta - z} \text{ for } z \in D$$

is holomorphic in *D*.

(b) Let Γ and Γ' be two rectifiable closed Jordan curve both oriented in the same sense such that $\Gamma' \subset I(\Gamma) \subset D \subset C$. If $f(z)$ is holomorphic in the open region *D*, then prove that

$$\oint_{\Gamma} f(z)\,dz = \oint_{\Gamma'} f(z)\,dz.$$ 5+3=8

3. (a) Define singular point or singularity of a complex valued function $f(z)$ defined in an open region $D \subseteq C$.

(b) Let $z_0 \in D \subseteq C$ be an isolated singularity of a complex valued function $f(z)$ defined in an open region $D \subseteq C$. With the help of the Laurent series expansion of $f(z)$ in some deleted neighbourhood $N'(z_0; \rho) \subseteq D\, (\rho > 0)$, define (i) a pole of order *m* at $z = z_0$, where *m* is a positive integer, and (ii) an essential singularity at $z = z_0$.

(c) Prove that $f(z)$ has a pole of order m at $z = z_0$ if and only if $\frac{1}{f(z)}$ has a zero of order m at $z = z_0$.

(d) If $f(z)$ is holomorphic and bounded for $|z| > R$, then show that the Laurent series expansion of $f(z)$ for $|z| > R$ is of the form $a_0 + \sum_{n=1}^{\infty} a_{-n} z^{-n}$. 1+2+2+3=8

4. (a) Using Lioville's theorem prove the Fundamental theorem of algebra.

(b) Let $f(z)$ be holomorphic in an open region $D \subseteq C$ and E be the set of all zeros of $f(z)$ in D. If z_0 is a limit point of E such that $z_0 \in D$ then show that $f(z)$ is identically zero in D.

(c) Let $f(z)$ be holomorphic in an open region $D \subseteq C$ and E be the set of all zeros of $f(z)$ in D. If $f(z)$ is not identically zero in D, then show that every point of E is an isolated point of E. 4+2+2=8

5. (a) State and prove Maximum Modulus Principle.

[*Turn over*]

(b) If z_0 is a simple pole of $f(z)$, then prove that

$Res(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z)$ 5+3=8

6. (a) Use Rouche's theorem to find the number of roots of the equation $z^8 - z^5 + 5z^2 + 1 = 0$ within the circle $|z| = 1$.

(b) Let $w = f(z) = u(x, y) + iv(x, y)$ be defined in an open region $D \subseteq C$ such that the partial derivatives of u and v are continuous in D and the Jacobian $\frac{\partial(u, v)}{\partial(x, y)} \neq 0$ in D. If the mapping $w = f(z)$ is conformal in *D*, then prove that $f(z)$ is holomorphic in *D*. 3+5=8

7. Prove the following results with the help of the method of the contour integration (any *three*) : 6×3=18

(a) $\int_0^\infty \frac{\sin x}{\sqrt{x}} dx = \int_0^\infty \frac{\cos x}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2}}$,

(b) $\int_0^{2\pi} \frac{1}{(a + b\cos\phi)^3} d\phi = \frac{\pi(2a^2 + b^2)}{(a^2 - b^2)^{5/2}} (a > b > 0)$,

(c) $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$,

(d) $\int_0^\infty \frac{\ln x}{x^2 + 1} dx = 0$.