Master of Science Examination, 2017

(1st Year, 1st Semester)

MATHEMATICS

Unit - 1.1

(Algebra - I)

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

Unexplained symbols and notations have their usual meanings.

Group - A

(Marks - 26)

Answer Q. No. 1 and *two* from the rest.

1. Answer any *five* : 2×5=10

(a) Every Abelian simple group is a finite cyclic group of prime order — Justify !

(b) Obtain S_3 as a suitable semidirect product.

(c) Are the groups $\mathbb{Z}_{26} \oplus \mathbb{Z}_{42} \oplus \mathbb{Z}_{49} \oplus \mathbb{Z}_{200} \oplus \mathbb{Z}_{100}$ and $\mathbb{Z}_4 \oplus \mathbb{Z}_{91} \oplus \mathbb{Z}_{300} \oplus \mathbb{Z}_{56} \oplus \mathbb{Z}_{175}$ isomorphic ? Answer with reasons.

[*Turn over*]

(d) If p is a prime and G is a nonabelion group of order p^3 then $|Z(G)| = p$ and $G/Z(G) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$ — Explain !

(e) For $n \geq 5$, S_n is not solvable — Explain !

(f) There is no simple group of order 1989 — Justify !

(g) Any unique sylow subgroup is a characteristic subgroup — Explain !

2. (a) Prove that every nilpotent group is solvable. Justify the converse of this result. 4+1

(b) Prove that if n_p is the number of sylow p-subgroups of a group G then G has a subgroup of index n_p. 3

3. (a) Deduce orbit formula for group action. Use it to show that the number of elements in the conjugacy class of any element of a finite group divides the order of the group. 3+2

(b) Prove that if a group G contains a subgroup $(\neq G)$ of finite index, it contains a normal subgroup $(\neq G)$ of finite index. 3

4. (a) Suppose G is a group of order pqr where p, q, r are primes with $p<q<r$. Prove that G has a normal Sylow r-subgroup. 2

(b) Suppose G is a group with $|G|=60$ and G has more than one Sylow 5- subgroups. Prove that G is simple. 6

Group - B

(Marks - 14)

Answer Q. No. 5 and any *two* from the rest.

5. Answer any *four*. 1×4=4

(a) Suppose R is a commutative ring with identity $(\neq 0)$ such that $\{0\}$ is a maximal ideal. Then R is a field — Explain !

(b) Suppose I_1, I_2 and I_3 are three ideals of a ring. Then which of following are necessarily true ?

(i) $I_1+(I_2 \cap I_3) \subseteq (I_1+I_2) \cap (I_1+I_3)$

(ii) $I_1+(I_2 \cap I_3) \supseteq (I_1+I_2) \cap (I_1+I_3)$

[*Turn over*]

(iii) $I_1 \cap (I_2 + I_3) \subseteq (I_1 \cap I_2) + (I_1 \cap I_3)$

(iv) $I_1 \cap (I_2 + I_3) \supseteq (I_1 \cap I_2) + (I_1 \cap I_3)$

(c) Let a, b be elements of a commutative ring with identity. Then prove that a divides b if and only if $\langle b \rangle \subseteq \langle a \rangle$.

(d) $\mathbb{Z}$ is a Noetherian ring but not Artinian — Justify !

(e) If F is a field then $\langle x \rangle$ is a maximal ideal in $F[x]$, but it is not the only maximal ideal — Explain !

(f) $x^2 + 1$ has an infinite number of distinct roots in the division ring of real quaternions — Justify !

6. (a) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by $T(x_1, x_2, x_3, x_4) = (2x_1 + x_2, 2x_2, 3x_3, 3x_4)$. Prove that $\{f(x) \in \mathbb{R}[x] : f(T) = 0\}$ is a principal ideal of the ring $\mathbb{R}[x]$. Find the unique monic generator of this ideal.

(b) What is meant by the p-th cyclotomic polynomial over $\mathbb{Z}[x]$? Prove that it is irreducible. (2+1)+(1+1)

7. (a) Prove that the polynomial $8x^3 - 6x - 1 \in \mathbb{Z}[x]$ has no integral root. Hence prove that it is irreducible in $Q[x]$. Write the statement of the main result (general form) used to solve the above problem. 4

(b) Write the statement of the General form of the Chinese Remainder Theorem. 1

8. (a) Illustrate with an example that there exists of subring R of a ring S (having the same identity) such that there exists $a \in R$ which is an irreducible element in S but a reducible element in R. 2

(b) Suppose $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ and p is a prime.

Let $\bar{f}(x) = \sum_{i=0}^{n} \bar{a}_i x^i$ where $\bar{a}$ is the image of a under the canonical epimorphism $\mathbb{Z} \to \mathbb{Z}_p$, $a \mapsto \bar{a}$. Prove that if $f(x)$ is monic and $\bar{f}(x)$ is irreducible in $\mathbb{Z}_p[x]$ for some prime p, then $f(x)$ is irreducible in $\mathbb{Z}[x]$. 3

Group - C

(Marks : 10)

9. Justify any *four* of the following statements providing either example or proof (as appropriate). $1\times4=4$

   (a) In a module, a linearly independent set can be extended to a basis.

   (b) In a module, a spanning set may not be reduced to a basis.

   (c) For a simple R-module M, the ring $\mathrm{End}_R(M)$ is a division ring.

   (d) If $\mathbb{Z}_n$ is $\mathbb{Z}/m\mathbb{Z}$-module via the canonical action then n divides m.

   (e) For a left R-module M, the annihilator of M in R i.e., $Ann_R(M)$ is the Kernel of the induced ring homomorphism $R \to \mathrm{End}\ (M)$, $r \to f_r$, $f_r : M \to M$, $m \mapsto rm$.

   (f) A ring R considered as a left R-module may be simple but the ring R may not be simple.

10. Answer any *one* : $6\times1=6$

(a) Suppose T is a linear operator on the vector space $V=\mathbb{R}^7$ over $\mathbb{R}$. Then $\mathbb{R}^7$ is canonically an $\mathbb{R}[x]$ module via T.

(i) Prove that $\mathbb{R}^7$ is finitely generated as well as a torsion $\mathbb{R}[x]$-module. 3

(ii) By the structure theorem of finitely generated module over PID, prove that

$$\mathbb{R}^7 \simeq \mathbb{R}[x]/\langle f_1(x)\rangle \oplus \mathbb{R}[x]/\langle f_2(x)\rangle \oplus \ldots \oplus \mathbb{R}[x]/\langle f_k(x)\rangle$$

with $f_1(x)|f_2(x)|f_3(x)| \ldots\ldots |f_k(x)$. If the minimal polynomial of T is $(x-2)^2(x+3)^3$, find the possible values of k and $f_i(x)$'s in each case. 3

(b) (i) Give an example of a nonfree module with justification. 1

(ii) Illustrate with an example that a submodule need not be a direct summand of the module. 1

[*Turn over*]

(iii) Let M be a left R-module. Define $A_R(m)$ for any $m \in M$. 1

(iv) Every free module is torsion free but not vice versa — Justify ! 3