BACHELOR OF SCIENCE EXAMINATION, 2017

(Final Year, 1st Semester)

MATHEMATICS (Honours)

Paper - 5.4

(Analysis - III)

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

Part - I

(25 Marks)

(Symbols / Notations have their usual meaning.)

Answer any *five* questions. 5×5=25

1. Let $(X,\ d)$ be a metric space and $C_b(X, \mathbb{R})$ denote the set of all continuous bounded real-valued functions defined on *X*, equipped with the uniform metric,

$$d(f, g) = \sup\left\{\left| f(x) - g(x) \right| : x \in X\right\}.$$

Show that $C_b(X, \mathbb{R})$ is a complete metric space. 5

[*Turn over*]

2. Show that a mapping $f:(X,\ d)\to(Y,\ \rho)$ is continuous on *X*, if and only if $f^{-1}(G)$ is open in *X* for all open subset *G* of *Y*. 5

3. Prove that every compact metric space is separable. 5

4. Suppose $T:(X,\ d)\to(X,\ d)$ be a contraction mapping of the complete metric space $(X,\ d)$. Then prove that *T* has a unique fixed point. 5

5. State and Prove Baire Category Theorem. 5

6. Show that every sequentially compact metric space $(X,\ d)$ is totally bounded. 5

7. (a) Let $\mathbb{N}$ denote the set of natural numbers. Define $d(m,\ n)=\left|\frac{1}{m}-\frac{1}{n}\right|$, $\forall\, m, n\in\mathbb{N}$. Then examine if $(\mathbb{N},\ d)$ is complete or not.

   (b) Let $\mathbb{Z}$ denote the set of integers in the metric space $(\mathbb{R},\ d)$, *d* denote the usual metric. Is $\mathbb{Z}$ compact ? — Justify. 2½+2½

[*Turn over*]

Part - II

(25 Marks)

(Notations have their usual meaning.)

Answer any *five* questions. 5×5=25

8. What do you mean by uniform convergence of an infinite series of functions ? If $\sum_n u_n(x)$ converges pointwise to the sum function $f(x)$ and each $u_n(x)$ is continuous then is $f(x)$ also continuous ? If not then give a counter example of it. State and prove the condition for which $f(x)$ will also be continuous. 1+2+2

9. State and prove Weierstrass's M-test for uniform convergence. 2+3

10. Let the power series $\sum_n a_n Z^n$ converges at $Z = Z_1$ and diverges for $Z = Z_2 \left(|Z_1| < |Z_2|\right)$. Show that $\exists$ a positive real no. 'r' such that the series converges absolutely for $|Z| < r$ and diverges if $|Z| > r$. 5

[*Turn over*]

11. Show that the error term in Taylor's formula can be expressed as

$$E_n(x) = \frac{1}{n!}\int_a^x f^{n+1}(t)(x-t)^n\, dt$$

Also state the condition for which the Taylor's series generated by $f(x)$ at a converges to $f(x)$ in some open interval $(a-r,\ a+r)$. 3+2

12. State and prove Bessel's inequality relating the Fourier coefficients of a Fourier series. 5

13. Obtain the Fourier sine series for the function $f(x) = x(\pi - x)$ in $(0,\ \pi)$. What is the sum of the series in $(\pi,\ 2\pi)$? Also show that

(i) $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \ldots \infty = \pi^3/32$ and

(ii) $\frac{1}{1^6} - \frac{1}{3^6} + \ldots \infty = \pi^6/960$. 5

14. Starting from the power series expansion for $\frac{1}{1+x^2}\left(|x|<1\right)$, derive the power series expansion of $\tan^{-1} x$. Hence obtain the sum of the infinite series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$ 5