Ex/FM/5.3/43/2017

BACHELOR OF SCIENCE EXAMINATION, 2017

(Final Year, 1st Semester)

MATHEMATICS (Honours)

Paper - 5.3

(Algebra III)

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

Notations have their usual meanings.

Answer any *five* questions.

1. (i) For which real number x, the vectors x and 1 are linearly independent in $\mathbb{R}(\mathbb{Q})$?

 (ii) Under what conditions on a the vectors $(1-a,\ 1+a)$ and $(1+a,\ 1-a)$ are linearly independent in $\mathbb{R}^2(\mathbb{Q})$?

 (iii) Do there exist two bases in $\mathbb{C}^4$ such that the only vectors common to them are (0, 0, 1, 1) and (1, 1, 0, 0) ?

[*Turn over*]

(iv) Prove that the set V defined as follows

$$V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\ a + b + c + d = 0 \right\}$$

is a subspace of $M_{2\times2}(R)$. Find a basis for the space.

2+2+2+4

2. (i) Find the kernel of the differential operator

$D : P_4(\mathbb{R}) \to P_3(\mathbb{R})$ defined as $D(p(x)) = p'(x)$. 4

(ii) Determine the eigenvalues and their algebraic and geometric multiplicities for the following matrix :

$$\begin{pmatrix} 2 & 6 & -1 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}$$

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3. (i) If A is a matrix of rank one then show that trace A is an eigenvalue of A. Hence determine the eigenvalues of the $n \times n$ matrix $A = (a_{ij})$, with $a_{ij} = 1$ for all i and j.

5

[*Turn over*]

(ii) Let N be a 2×2 matrix such that $N^2 = 0$. Then show that either $N = 0$ or N is similar to a matrix of the form

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. 5

4. (i) Show that if λ is an eigenvalue of A and $p(x)$ is any polynomial, then $p(\lambda)$ is an eigenvalue of $p(A)$. Hence show that a non-zero nilpotent matrix can't have a non-zero eigenvalue. 3+2

(ii) Show that if a square matrix A is diagonalizable then the minimal polynomial can be factored into distinct linear factors. 5

5. (i) Let T be a linear operator on a vector space V of dimension 5 and the minimal polynomial of T is $(x-1)(x-2)^2(x-3)^2$. Then find the direct sum decomposition of the vector space V and hence find the canonical form of T. 5

(ii) Starting from the basis $\{(0,1,1),(1,0,1),(1,1,0)\}$ of $\mathbb{R}^3(R)$ construct an orthonormal basis of $\mathbb{R}^3(R)$ using Gram-Schmidt orthogonalisation process. 5

[*Turn over*]

6. (i) If x and y are two vectors in an inner product space V, then prove that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 .$$ 3

(ii) Let T be a diagonalizable linear operator on a finite dimensional vector space V with distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$. Then show that there exist linear operators $E_1,\ E_2,\ ...,\ E_k$ on V such that

(i) $E_i^2 = E_i,\ R(E_i) = N(T - \lambda_i I),\ \forall i = 1, 2, ..., k$

(ii) $E_i E_j = 0,\ (i \neq j)$

(iii) $E_1 + E_2 + ... + E_k = I$

(iv) $T = \lambda_1 E_1 + \lambda_2 E_2 + ... + \lambda_k E_k$. 7

7. (i) Let T be a linear operator on a finite dimensional inner product space V over the complex field. Show that there exists an orthonormal basis B w.r.t. which the matrix of T is upper triangular. 5

[*Turn over*]

(ii) Let λ, μ be distinct eigenvalues of a normal operator T and x, y are corresponding eigenvectors. Then prove that $\langle x, y \rangle = 0$. 3

(iii) If λ is an eigenvalue of a normal operator T then show that $\bar{\lambda}$ is an eigenvalue of T^*. 2