Ex/INT/M/3.3/27/2017

BACHELOR OF SCIENCE EXAMINATION, 2017

(2nd Year, 1st Semester)

MATHEMATICS (Honours)

Unit - 3.3

(Analysis - I)

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

(Notations/Symbols have their usual meanings.)

Answer any *five* questions.

1. (a) Define a *countable* set. Prove that every subset of a countable set is countable. 5

   (b) Prove that a countable union of countable sets is countable. 5

2. (a) Define an *uncountable* set. Let A be the set of all sequences whose elements are the digits 0 and 1. Show that A is uncountable. 5

[*Turn over*]

(b) Let $\mathbb{N}$ be the set of all natural numbers, $\mathbb{R}$ be the set of all real numbers, $n \in \mathbb{N}$ and $|X|$ denote the cardinality of a set *X*. Prove that $|\mathbb{R}| = |\mathbb{R}^n| = |\mathbb{R}^{\mathbb{N}}|$. 5

3. (a) Define the least upper bound property of a linearly ordered set. Prove that the set of all rational numbers does not have the least upper bound property. 5

(b) Define the *derived set* S' of a subset *S* of $\mathbb{R}$. Find S', where $S = \left\{ \frac{1}{m} + \frac{1}{n} \mid m, n \in \mathbb{N} \right\}$. 5

4. (a) Define a *limit point* of a subset of $\mathbb{R}$. Prove that a finite subset of $\mathbb{R}$ has no limit points. 5

(b) Define an *open subset* and a *closed subset* of $\mathbb{R}$. Prove that a subset *S* of $\mathbb{R}$ is open if and only if $\mathbb{R} \setminus S$ is closed. 5

5. (a) Define the *closure* $\overline{S}$ of a subset *S* of $\mathbb{R}$. Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$, where *A* and *B* are subsets of $\mathbb{R}$. 5

(b) Define a *perfect subset* of $\mathbb{R}$ and the *Cantor set*. Prove that the Cantor set is perfect. 5

[*Turn over*]

6. (a) Define a *covering* $\mathcal{F}$ of a subset of $\mathbb{R}$ and a *subcovering* of $\mathcal{F}$. Let $S \subseteq \mathbb{R}$. Prove that every covering of S by open intervals has a countable subcovering. 5

(b) Let $T = S'$, the derived set of $S \subseteq \mathbb{R}$. In the following, either prove that T is compact or find a covering of T by open intervals which has no finite subcovering :

(i) $S = \left\{ \frac{1}{m} + \frac{1}{n} \mid m,\ n \in \mathbb{N} \right\}$,

(ii) $S = \{ x \in \mathbb{R} \mid 0 < x < 1 \}$.

7. (a) Define a *conditionally convergent infinite series* of real numbers. Show that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

is conditionally convergent. 5

[*Turn over*]

(b) Define a *uniformly continuous* real function on a subset of $\mathbb{R}$. Prove that $f(x)=\frac{1}{x}$ is not uniformly continuous on (0, 1) but is uniformly continuous on $\left[\frac{1}{2},\ 1\right)$. 5