Bachelor of Science Examination, 2017

(2nd Year, 1st Semester)

MATHEMATICS (Honours)

Paper - 3.2

(Differential Equation - II)

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

(Notations and Symbols have their usual meanings.)

Part - I

(30 marks)

Answer any *three* questions.

1. (a) Find the series solution of

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - 1\right) y = 0$$

about the point $x = 0$. 7

[*Turn over*]

(b) Show that $J_n'(x)J_{-n}(x) - J_n(x)J_{-n}'(x) = \frac{c}{x}$, where c is a constant. 3

2. (a) Prove that $\exp\left\{\frac{x}{2}\left(z - \frac{1}{z}\right)\right\} = \sum_{-\infty}^{\infty} z^n J_n(x)$ and hence deduce that

$$\cos(x\sin\phi) = J_0 + 2\cos 2\phi \,.\, J_2 + 2\cos 4\phi \,.\, J_4 + \ldots$$

(b) Show that

$$\int_{-1}^{1} x^2 P_{n-1}(x) P_{n+1}(x)\, dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

6+4

3. (a) If ϕ_m and ϕ_n are characteristic functions corresponding to two distinct characteristic roots λ_m and λ_n of Sturm-Liouville problem, then show that ϕ_m and ϕ_n are orthogonal with respect to some weight function.

(b) Show that,

$$\frac{d}{dx}\left(J_n^2 + J_{n+1}^2\right) = 2\left(\frac{n}{x}J_n^2 - \frac{n+1}{x}J_{n+1}^2\right)$$ 6+4

[*Turn over*]

4. (a) Find all the eigen values and eigen functions of the Strum-Lionville problem

$$\frac{d^2y}{dx^2}+\lambda y=0 \text{ with } \begin{matrix} y(0)+y'(0)=0 \\ y(1)+y'(1)=0 \end{matrix}$$

(b) Prove that,

$$P_n(x)=\frac{1}{\pi}\int_0^{\pi}\left[x\pm\sqrt{(x^2-1)}\ \cos\phi\right]^n d\phi$$

where n is a positive integer. 5+5

5. (a) Show that $\int_{-1}^{1} P_m(x)P_n(x)dx=0$, if $m\neq n$

$$=\frac{2}{2n+1}, \text{ if } m=n$$

(b) Show that

$$\frac{d}{dx}F(\alpha,\ \beta;\ \gamma;\ x)=\frac{\alpha\beta}{\gamma}F(\alpha+1,\ \beta+1;\ \gamma+1;\ x)$$

[Turn over]

(c) Show that

$$\sin^{-1} x = xF\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right)$$ 6+2+2

Part - II

(20 marks)

Answer any *two* questions.

6. (a) State and prove the existence theorem on Laplace transform.

(b) Find the Laplace transform of the function

$$f(t) = te^{-t} \sin 3t.$$

(c) If $F(p)$ be the Laplace transform of the function $f(t)$, show that for any constant $a > 0$,

$$L\{f(at)\} = \frac{1}{a} F(p/a).$$

Given that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{p}$, use the above result to find $L\left\{\frac{\sin at}{t}\right\}$. 5+2+3

[*Turn over*]

7. (a) If $L\{f(t)\} = F(p)$, then show that

$$\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty F(p)\,dp.$$

Hence evaluate the integral

$$\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\,dt.$$

(b) Find $L^{-1}\left\{\frac{1}{p}\log\left(1+\frac{1}{p}\right)\right\}$.

(c) Evaluate the integral

$$\int_0^t J_0(t)\,J_0(t-\tau)\,d\tau.$$

5+3+2

8. (a) Find the solution of the following initial value problem :

$$y''(t) + a^2 y(t) = f(t)$$

$$y(0) = 1,\ y'(0) = 2.$$

[*Turn over*]

(b) Solve the following simultaneous differential equations :

$$\frac{dx}{dt} = 2x - 3y$$

$$\frac{dy}{dt} = y - 2x$$

subject to the conditions :

$x(0) = 8$ and $y(0) = 3$. 4+6