# Bachelor of Science Examination, 2017

**(1st Year, 1st Semester)**

**MATHEMATICS (Honours)**

**Paper - 1.3**

**[ Algebra - I ]**

Full Marks : 50 Time : Two Hours

*The figures in the margin indicate full marks.*

Use a separate Answer-Script for each Part.

(Notations and Symbols have their usual meanings)

## Part - I

Answer any *five* questions.

Let $\mathbb{R}$ denotes the field of all real numbers.

1. Define a *vector space*. Let $V = \mathbb{R} \times \mathbb{R}$. For $(a, b), (c, d) \in V$ and $c \in \mathbb{R}$, define

   $$(a, b) + (c, d) = (a + 2c, b + 3d) \text{ and } c(a, b) = (ca, cb)$$

   Is $V$ a vector space over $\mathbb{R}$ under the above addition and scalar multiplication ? Justify your answer. 5

[*Turn over*]

2. Define a *subspace* of a vector space. Let $W_1$ and $W_2$ be two subspaces of a vector space $V$. Show that $W = W_1 + W_2 = \{u + v | u \in W_1, v \in W_2\}$ is the smallest subspace of $V$ containing both $W_1$ and $W_2$. 5

3. Define *linearly dependent* and *linearly independent* set of vectors in a vector space over $\mathbb{R}$. Let

$$S = \{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4), (-1, 0, 1, 0)\} \subset \mathbb{R}^4.$$

Determine whether $S$ is linearly dependent or independent set of vectors in $\mathbb{R}^4$ over $\mathbb{R}$. Find the dimension of the subspace of $\mathbb{R}^4$ generated by $S$. 5

4. Define a *basis* of a vector space. Let $W$ be the subspace generated by the vectors $(2, -3, 4, -5, 2)$, $(-6, 9, -12, 15, -6)$, $(3, -2, 7, -9, 1)$, $(2, -8, 2, -2, 6)$, $(-1, 1, 2, 1, -3)$, $(0, -3, -18, 9, 12)$, $(1, 0, -2, 3, -2)$ and $(2, -1, 1, -9, 7)$. Find a basis of $W$ over $\mathbb{R}$ that is a subset of the set containing the above vectors. 5

5. Let $V$ and $W$ be two vector spaces over $\mathbb{R}$. Define a *linear transformation* $T$ from $V$ into $W$. Define *nullity and rank* of $T$. If $V$ is finite dimensional with dimension $n$ over $\mathbb{R}$, then show that rank ($T$) + nullity ($T$) = $n$. 5

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

[*Turn over*]

$$T(x, y, z) = (4x + z, 2x + 3y + 2z, x + 4z)$$

Find the matrix representation $A$ to $T$ with respect to the standard ordered basis of $\mathbb{R}^3$. Determine whether $A$ is diagonalizable. 5

7. Define the *characteristic equation, eigenvalues* and corresponding *eigenspaces* of a square matrix over $\mathbb{R}$ and find those for the following matrix : 5

$$\begin{pmatrix} 7 & -4 & 10 \\ 4 & -3 & 8 \\ -2 & 1 & -2 \end{pmatrix}$$

**Part - II**

Answer any *five* questions. 5×5=25

8. (i) Let $A$ and $B$ be two square matrices of the same order $n$ such that $AB = I_n$. Show that $BA = I_n$. 2

(ii) Let $A$ be a nilpotent matrix of order $n$ and index $k$. Show that both $I_n - A$ and $I_n + A$ are invertible. Is $A$ invertible ? Justify your answer. 2+1

9. (i) (a) Let $A$ and $B$ be two orthogonal matrices of the same order. Show that $AB$ is also an orthogonal matrix. 1

[*Turn over*]

(b) Let $A$ be a real skew symmetric matrix of order $n$ such that $A^2 + I_n = 0$. Show that $A$ is an orthogonal matrix. 2

(ii) Show that an invertible skew symmetric matrix must be of even order. 2

10. (i) Let $A$ be a square matrix of order $n$. Show that $A(adj\ A) = (\det A) I_n$. Hence show that $\det(adj\ A) = (\det A)^{n-1}$ if $\det A \neq 0$. 2

(ii) By using Laplace's expansion, show that

$$\begin{vmatrix} a & -b & -a & b \\ b & a & -b & -a \\ c & -d & c & -d \\ d & c & d & c \end{vmatrix} = 4\left(a^2 + b^2\right)\left(c^2 + d^2\right).$$ 3

11. (i) (a) Let $A$ be a non-singular matrix of order $n$. What is the rank of $A^3$ ? Justify your answer. 1

(b) Let $A$ be a square matrix of order 3 such that $A$ is not a symmetric matrix. Show that rank of the matrix $A - A^T$ is 2. 2

(ii) If the system of equations $x = cy + bz$, $y = az + cx$, $z = bx + ay$ has a non-zero solution then show that $a^2 + b^2 + c^2 + 2abc = 1$. 2

[*Turn over*]

12. (i) Show that two similar matrices have same eigen values. Is the converse true ? Justify your answer. 2+1

(ii) Let $A$ be a real square matrix of order 2 with trace 5 and determinant value 6. Find the eigen values of the matrix $B = A^2 - 2A + I_2$. 2

13. (i) What is the multiplicity of the root $x = 1$ of the equation $x^n - nx + (n-1) = 0$ $(n > 1)$ ? Justify your answer. 2

(ii) (a) If the roots of the equation $8x^3 - 12x^2 - Kx + 3 = 0$ are in A.P. then find the value of $K$. 1

(b) Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of the equation $x^n - 1 = 0$. Show that

$(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{n-1}) = n$. 2

14. (i) Apply Descartes' rule of signs to find the nature of the roots of the equation $x^6 + x^4 + x^2 + x + 3 = 0$. 2

(ii) Solve the equation $x^3 - 12x + 65 = 0$ by Cardan's method. 3

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