12. Verify Green's theorem in the plane for $\oint_C \{(x^2 + xy)dx + xdy\}$, where C is the curve enclosing the region bounded by $y = x^2$ and $y = x$. [10]

Ex/ME/5/Math/T/111/2017(Old)/(S)

Bachelor of Engineering in Mechanical Engineering (Evening) Examination, 2017

(1st Year, 1st Semester)

Mathematics - VM (Old)

Time : Three hours Full Marks : 100

Answer ***any ten*** questions.

1. a) Define linearly dependent and independent set of vectors of a vector space V over the field F.

   b) Prove that the set of vectors {(2, 1, 1), (1, 2, 2), (1,1,1)} is linearly dependent in $\mathbb{R}^3$. [5+5]

2. a) Define linear span and basis of a vector space.

   b) Prove that the set S={(1,0,1), (0,1,1), (1,1,0)} is a basis of $\mathbb{R}^3$. [5+5]

3. a) Show that the function $T : V_3(\mathbb{R}) \to V_2(\mathbb{R})$ defined by $T(a,b,c) = (a,b)\ \forall\ a,b,c \in \mathbb{R}$ is a linear transformation from $V_3(\mathbb{R})$ into $V_2(\mathbb{R})$.

   b) If α, β are vectors in an inner product space V, then prove that

   $$\|\alpha + \beta\| \le \|\alpha\| + \|\beta\|.$$ [5+5]

[Turn over

4. a) Show that the mapping $T: V_3(\mathbb{R}) \to V_2(\mathbb{R})$ defined as $T(a_1, a_2, a_3) = (3a_1 - 2a_2 + a_3,\ a_1 - 3a_2 - 2a_3)$ $\forall\ a_1, a_2, a_3 \in \mathbb{R}$ is a linear transformation from $V_3(\mathbb{R})$ into $V_2(\mathbb{R})$.

b) In an inner product space V(F), prove that $|(\alpha + \beta)| \leq \|\alpha\| \cdot \|\beta\|$. [5+5]

5. a) If $\vec{r} = 3t\,\hat{i} + 3t^2\hat{j} + 2t^3\hat{k}$, then find $\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}$.

b) If $\vec{r} = (5+3t)\hat{i} + (3-2t)\hat{j} + (4+t-16t^2)\hat{k}$, then show that $\frac{d^2\vec{r}}{dt^2} + 32\hat{k} = \vec{0}$. [5+5]

6. a) Find the curvature k of the space curve x = t, y = t^2, $z = \frac{2}{3}t^3$.

b) If $\phi(x, y, z) = 6x^3y^2z$, then find $\nabla \cdot \nabla\phi$ (or div grad ϕ). [5+5]

7. a) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$, then prove that $\nabla^2\left(\frac{1}{r}\right) = 0$.

b) Prove that $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$. [5+5]

8. a) Find the directional derivative of $\phi = xy^2z - 4xz^2$ at the point (2, 1, –1) in the direction $(2\hat{i} - 2\hat{j} + \hat{k})$.

b) Evaluate $\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$, where $\vec{r} = 2t^2\hat{i} + t\hat{j} - 3t^2\hat{k}$.

[5+5]

9. a) If $\vec{F} = 3xy\,\hat{i} - y^2\hat{j}$, then evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve in the xy-plane, given by $y = 2x^2$ from the point (0, 0) to (1, 2).

b) Evaluate the surface integral $\iint_S (yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}) \cdot d\vec{S}$, where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant. [5+5]

10. Verify Stoke's theorem for $\vec{F} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. [10]

11. Use divergence theorem to evaluate $\iint_S \vec{F} \cdot \hat{n}\,ds$, where $\vec{F} = 3xz\hat{i} + y^2\hat{j} - 3yz\hat{k}$ and S is the surface of the cube bounded by x = 0, y = 0, z = 0, x = 1, y = 1, z = 1. [10]