Ref. No.: **Ex/CSE/T/224A/2017**

B.E. Computer Science and Engineering
2^{nd} Year, 2^{nd} Semester Examination, 2017
Graph Theory and Combinatorics

Full Marks: 100 Time : 3 Hr

Answer Five Questions : Q1 (Compulsory) and any four from the rest.
Write answers to the point and state all the assumptions (wherever required). Make assumptions wherever necessary.

ALL PARTS OF THE QUESTION SHOULD BE ANSWERED TOGETHER

Q 1) The cube graph Q_n is defined as: the vertices of Q_n are all sequences of length n with entries from $\{0, 1\}$ and two sequences are joined by an edge if they differ in exactly one position. $(4 \times 5 = 20)$

(a) How many edges does Q_n have?

(b) Which cube graphs Q_n have an Euler tour?

(c) Which cube graphs Q_n have an Hamiltonian Cycle?

(d) Show that the cube graph Q_n is bipartite.

(e) Sketch the cube in three dimensions, Q_3 , and find a 2-colour vertex colouring of the cube.

Q 2) (a) In a village there are three schools with n students in each of them. Every student from any of the schools is on speaking terms with at least $n + 1$ students from the other two schools. Show that we can find three students, no two from the same school, who are on speaking terms with each other. (7)

(b) Prove: "Every graph with at least 2 vertices contains 2 vertices of the same degree." (6)

(c) Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group (7)

Q 3) (a) Does there exist an Eulerian graph with $(4 + 4 = 8)$

(i) an even number of vertices and an odd number of edges,

(ii) and odd number of vertices and an even number of edges.

Draw such a graph if it exists.

(b) Prove that a map G is 2-colourable if and only if G is an Eulerian graph. (6)

(c) Let G^* be the dual of an Eulerian graph G. What are the implications of [Ques 3b] in G^* ? (6)

Q 4) (a) Let G be a bipartite, planar graph. $(4 + 4 = 8)$

(i) Use Euler's formula to prove that $|E| \leq 2|V| - 4$, if $|V| \geq 3$, where $|E|$ is the number of edges and $|V|$ is the number of vertices

(ii) Prove that G has a vertex v with $degree(v) \leq 3$

(b) In the Konigsberg Bridge Problem, prove or disprove that it is possible to take a closed walk from any point. Under what condition can the Konigsberg Bridge Problem be modified, such that previous statement can be falsified. (6)

(c) Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities? (6)

Q 5) (a) If A is the adjacency matrix for the graph G, show that the (j, j) entry of A^2 is the degree of v_j (6)

(b) Show that Prim's algorithm produces a minimum spanning tree of a connected weighted graph. (8)

(c) During a month with 30 days, a football team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games. (6)

Q 6) (a) Show that the Petersen graph is not Hamiltonian, but does have a Hamiltonian path. (5)

(b) Suppose there are seven coins, all with the same weight, and a counterfeit coin that weighs less than the others. How many weighings are necessary using a balance scale to determine which of the eight coins is the counterfeit one? Give an algorithm for finding this counterfeit coin. (7)

(c) Suppose we have an urn containing m balls, labelled from 1 to m and we draw, with replacement, k balls. What is the probability of drawing the same ball twice (This is referred to as a collision.) ? For what value of of k, there will be 50% of collision ? What will be the value of k for m=365. (8)