BACHELOR OF COMPUTER SCIENCE & ENGINEERING EXAMINATION, 2017

(1st Year, 2nd Semester)

MATHEMATICS - III

Full Marks : 100 Time : Three Hours

Answer Q. No. 10 and any six questions from Q. No. 1 to 9

1. (a) Define a *group*. Let $G = \{a \in \mathbb{R} \mid -1 < a < 1\}$. Define $a \star b = \frac{a+b}{1+ab}$ for all $a, b \in G$. Determine whether $(G, \star)$ is a group, where $\mathbb{R}$ is the set of all real numbers 8

   (b) Let $\beta = (1\ 3\ 5\ 7\ 9\ 8\ 6)(2\ 4\ 10) \in S_{10}$. What is the smallest positive integer n for which $\beta^n = \beta^{-5}$? 8

2. (a) Define a *subgroup* of a group G. Let G be a group and H be a nonempty subset of G. Show that H is a subgroup of G if and only if for all $a, b \in H$, $ab^{-1} \in H$. 8

   (b) Define a *cyclic group*. Let $\mathbb{Q}$ be the set of all rational numbers. Is the group $(\mathbb{Q}, +)$ cyclic? Justify your answer. 8

3. (a) Let G be a finite group and $a \in G$. Show that $\circ(a)$ divides $\circ(G)$. 8

   (b) Prove that every infinite cyclic group is isomorphic to the group $(\mathbb{Z}, +)$, where $\mathbb{Z}$ is the set of all integers. 8

4. (a) Define a *ring*. Let R be the set of even integers. Define addition as usual and multiplication by $a \cdot b = \frac{1}{2}ab$. Show that R is a ring. Is there an identity in R? 8

   (b) Define an *ideal* of a ring. Let I be an ideal of a ring R. Define

$$J = \{x \in R \mid rx \in I \text{ for all } r \in R\}.$$

   Show that J is an ideal of R. 8

5. (a) Define a *prime ideal* of a ring. Let R be a commutative ring with identity $1 \neq 0$. Then show that an ideal P of R is prime if and only if R/P is an integral domain. 8

   (b) Define a *field*. Prove that the number of elements of a finite field is a power of a prime number. 8

6. (a) Define a *subspace* of a vector space over a field. Let

$$W_1 = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 \mid a_2 = a_3 = 0\}, \ W_2 = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 \mid a_1 = a_4 = 0\}$$

be two subspaces of $\mathbb{R}^4$ over the field $\mathbb{R}$. Either prove that $W_1 \cup W_2$ is a subspace of $\mathbb{R}^4$ or find the smallest subspace of $\mathbb{R}^4$ containing W_1 and W_2. 8

(b) Define a *basis* of a vector space over a field. Let $\{u, v, w\}$ be a basis of a vector space V over a field $\mathbb{R}$. Then show that $\{u + v + w, u - v + w, -u + v + w\}$ is also a basis of V over $\mathbb{R}$. 8

7. (a) Let A and B be two $n \times n$ matrices over a field F. Prove that if $I_n - AB$ is invertible, then $I_n - BA$ is also invertible. Further show that AB and BA have the same eigen values. 8

(b) Define ***eigen values*** and ***eigen vectors*** of a square matrix over a field. Find eigen values and corresponding eigen vectors of the matrix A over $\mathbb{R}$. Also determine whether A is diagonalizable, where 8

$$A = \begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$$

8. (a) Find the nature of solutions of the following system of linear equations. Find solutions, if they exist, otherwise show that there is no solution of the system. 8

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 4x_4 &= 2 \\ -3x_1 + x_2 + 2x_3 + 5x_4 &= 1 \\ 5x_1 + 3x_2 - 8x_3 + 3x_4 &= 2 \\ -x_1 + 5x_2 - 4x_3 + 13x_4 &= 4 \end{aligned}$$

(b) Determine for which $Y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$ the equation $AX = Y$ has a solution for $X \in \mathbb{R}^4$, where 8

$$A = \begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix}$$

9. (a) Find the Jordan canonical form J of A. Find a Jordan canonical basis and a non-singular matrix Q such that $J = Q^{-1}AQ$, where

$$A = \begin{pmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{pmatrix}$$

Given that the characteristic polynomial of A is $(x-2)^2(x-4)^4$. 8

(b) Let $V = \mathbb{R}^5$ and W be a subspace of V, where

$$W = \left\{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 \mid a_1 + 2a_3 + a_5 = 0,\ 2a_2 + a_4 = 0\right\}.$$

Find the dimension and a basis of W over $\mathbb{R}$. Further extend this basis to a basis of V over $\mathbb{R}$. 8

10. Construct a finite field of order 25. 4