ON SOME WARPED PRODUCT MANIFOLDS

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**On some warped product manifolds**" submitted by Sri **Nandan Bhunia** who got his name registered on 5th September, 2018 (Index No: 150/18/Maths./26) for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon his own work under the supervision of **Prof. Arindam Bhattacharyya** and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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Dedicated to

my parents

Umakanta Bhunia, Kananbala Bhunia

and

my beloved wife

Purbasha

for their patience, support & love.

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Preface

The aim of this doctoral thesis is to study on some warped product manifolds. The thesis consists of five chapters. After the introductory chapter, the second chapter is devoted to study the geometry of pseudo-projective curvature tensor on warped product manifolds. We study the generalized Robertson-Walker space-times and standard static space-times admitting pseudo-projective curvature tensor respectively.

The third chapter is to study the biwarped product submanifolds in metallic Riemannian manifold and locally nearly metallic Riemannian manifold. It describes the nature of biwarped product generalized J-induced submanifold of first order with an example. We find out necessary and sufficient conditions for the biwarped product generalized J-induced submanifold of first order to be locally trivial. The inequalities for the second fundamental form in metallic Riemannian manifold and locally nearly metallic Riemannian manifold have been established.

The fourth chapter is based on some space-times as an application of warped product manifolds. It discusses the generalized Friedmann-Robertson-Walker space-time in a new way with some examples of generalized black hole solutions. This chapter is also focused on hyper-generalized quasi Einstein warped product spaces with non positive scalar curvature. We investigate some geometric and physical properties of it. The last part conveys the behaviour of general relativistic viscous fluid space-time admitting vanishing and divergence free T-curvature tensor respectively.

In the last chapter, we introduce a new notion of gradient h-almost η-Ricci soliton and study Riemann soliton in the frame of warped product Kenmotsu manifold. Then Riemann soliton has been studied on warped product Kenmotsu manifold to deduce some conditions for its existence admitting W_2-curvature tensor, projective curvature tensor and Weyl-conformal curvature tensor. Ricci soliton and gradient Ricci soliton have been discussed with pointwise bi-slant submanifolds of trans-Sasakian manifolds to establish that the pointwise bi-slant submanifolds of trans-Sasakian manifold is Einstein manifolds under certain conditions. Lastly, we show the existence of the gradient h-almost η-Ricci soliton warped product. The nature of h-almost η-Ricci soliton and gradient h-almost η-Ricci soliton has been investigated admitting a concurrent vector field.

CHAPTERWISE PUBLICATION SUMMARY

Serial No.	Title of the Paper	Authors	Journal of Publication	Chapter
1.	Pseudo-projective curvature tensor on warped product manifolds and its applications in space-times	Nandan Bhunia, Sampa Pahan and Arindam Bhattacharyya	SUT Journal of Mathematics, 57(2), pp. 93-107, (2021)	2
2.	Biwarped product submanifolds in metallic Riemannian manifold	Nandan Bhunia, Sampa Pahan and Arindam Bhattacharyya	Communicated	3
3.	Biwarped product submanifolds in some structures of metallic Riemannian manifold	Nandan Bhunia, Sampa Pahan and Arindam Bhattacharyya	Afrika Matematika, Published Online, (2022)	3
4.	Application of hyper-generalized quasi-Einstein spacetimes in general relativity	Nandan Bhunia, Sampa Pahan and Arindam Bhattacharyya	Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci., 91(2), pp. 297-307, (2021)	4
5.	A new way to study on generalized Friedmann-Robertson-Walker spacetime	Nandan Bhunia, Buddhadev Pal and Arindam Bhattacharyya	Indian Journal of Physics, Published Online, (2022)	4

Serial No.	Title of the Paper	Authors	Journal of Publication	Chapter
6.	Application of τ-curvature tensor in spacetimes	Nandan Bhunia, Sampa Pahan and Arindam Bhattacharyya	Jordan Journal of Mathematics and Statistics, 15 (3B), pp. 629-641, (2022)	4
7.	Ricci and Riemann soliton on warped product space forms	Nandan Bhunia, Sampa Pahan and Arindam Bhattacharyya	Communicated	5
8.	Introduction to gradient h-almost η-Ricci soliton on warped product spaces	Nandan Bhunia, Sampa Pahan, Arindam Bhattacharyya and Sanjib Kumar Datta	Communicated	5

Contents

CHAPTER 1

Introduction

1.1 Riemannian manifold

Historically, Riemann geometry was a development of the differential geometry of surfaces in E^3. The crucial point of this development was initiated by Gauss in 1827. But due to the lack of necessary mathematical tools available at that time, the Gauss ideas developed very slowly. The independent approach of non-Euclidean geometry was also due to Lobachevski (1829) and Bolyai (1831). The ideas of Gauss were taken up again by Riemann in 1854 and generalized the idea of Gaussian curvature. Riemann was motivated by the fundamental question implicit in the development of non-Euclidean geometries, namely, the relationship between physics and geometry. The formalization of Riemann's work appeared explicitly in 1913 in the work of H. Weyl and the application of these ideas was made to the theory of relativity in 1916. Another fundamental step was the introduction of the parallelism of Levi-Civita in 1917. Two fundamental concepts of Riemannian geometry are geodesics and curvature. These geodesics are analogous to straight lines in Euclidean geometry and these geodesics are locally length minimizing, but this may fail in the global sense. Riemannian geometry is the study of manifolds which are equipped with some additional structure that permits measurements. For example, nowhere in the definition of a piecewise smooth curve is there anything that would enable us to measure the length of the curve? And given intersecting curves, how could we measure the angle they make at the point of intersection? The additional structure that is needed is a metric tensor which gives rise to the Levi-Civita connection or Riemannian connection. We give the formal definition of this metric tensor and Riemannian manifolds.

Definition 1.1.1 (Riemannian metric)**.** *Let M be a smooth manifold of dimension n. Then a Riemannian metric g on M is covariant tensor field of degree 2 i.e., of type (0,2) which satisfies the following conditions :*

(1) *g is symmetric, i.e., $g(X,Y) = g(Y,X), \ \forall X,Y \in \mathfrak{X}(M)$,*

(2) *g is positive definite, i.e., $g(X,X) \geq 0, \ \forall X \in \mathfrak{X}(M)$ and $g(X,X) = 0$ iff $X = 0$.*

Definition 1.1.2 (Riemannian manifold)**.** *A smooth manifold with a Riemannian metric is said to be a Riemannian manifold. It is denoted by (M^n, g) or (M, g) or simply by M where M, g are the smooth manifold and Riemaannian metric respectively.*

Example 1.1.3. *Every Euclidean space E^n is a Riemannian manifold, where the components of g are given by*

$$g_{ij} = \delta_{ij} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

The following deals with a connection on a Riemannian manifold *M* with the help of the Riemannian metric.

Definition 1.1.4 (Metric-compatible connection)**.** *Let (M, g) be an $n-$dimensional Riemannian manifold and ∇ be an affine connection on M. If*

$$\nabla g = 0, \tag{1.1.1}$$

$$i.e., \ (\nabla_U g)(V,W) = 0 \tag{1.1.2}$$

$\forall\, U,V,W \in \mathfrak{X}(M)$, *then* ∇ *is called a metric-compatible connection or simply metric connection on* (M,g).

Since ∇ is defined as an affine connection or linear connection on the Riemannian manifold M, it satisfies the following properties

$$(1)\ \nabla_{\alpha X+\beta Y}Z = \alpha\nabla_X Z + \beta\nabla_Y Z,$$
$$(2)\ \nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z,$$
$$(3)\ \nabla_X(fY+gZ) = f\nabla_X Y + (Xf)Y + g\nabla_X Z + (Xg)Z,$$

for all $\alpha,\beta \in \mathbb{R}; f,g \in C^\infty(M); X,Y,Z \in \mathfrak{X}(M)$. For ∇ to be a metric connection, it also satisfies the relation (1.1.1) i.e., ∇ parallelizes g. Thus for a metric connection on M, it follows from (1.1.2) that

$$\nabla_X g(Y,Z) = g(\nabla_X Y,Z) + g(Y,\nabla_X Z) \tag{1.1.3}$$
$$i.e.,\ Xg(Y,Z) = g(\nabla_X Y,Z) + g(Y,\nabla_X Z) \tag{1.1.4}$$

$\forall\, X,Y,Z \in \mathfrak{X}(M)$.

Definition 1.1.5 (Riemannian connection)**.** *Let* (M,g) *be a Riemannian manifold of dimension n with an affine connection* ∇. *Then the affine connection* ∇ *on M is said to be Levi-Civita connection or Riemannian connection if it satisfies the following :*

(1) ∇ *is symmetric or torsion free. i.e.,* $\nabla_X Y - \nabla_Y X = [X,Y]$

(2) ∇ *is a metric compatible or metric connection. i.e.,* $(\nabla_X g)(Y,Z) = 0$

$\forall\, X,Y,Z \in \mathfrak{X}(M)$, *then* ∇ *is called a metric-compatible connection or simply metric connection on* (M,g).

Formula 1.1.6 (Koszul)**.** *Let* (M,g) *be a Riemannian manifold of dimension n with an affine connection* ∇. *Then*

$$\begin{aligned}2g(\nabla_X Y,Z) =& Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z)\\ &- g([Y,Z],X) + g([Z,X],Y),\end{aligned} \tag{1.1.5}$$

for all $X,Y,Z \in \mathfrak{X}(M)$.

It is observed that a Riemannian connection ∇ on a Riemannian manifold M always satisfies the Koszul's formula.

Now the question arises about the existence of a Levi-Civita connection on a Riemannian manifold. In other words whether a Riemannian manifold always admits a Levi-Civita connection or not? The following theorem will give the answer to this question and is known as Levi-Civita Theorem or Fundamental theorem of Riemannian geometry.

Theorem 1.1.7 (Fundamental theorem of Riemannian geometry)**.** *Every Riemannian manifold (M,g) of dimension n admits a unique torsion-free metric connection.*

Definition 1.1.8 (Riemannian curvature tensor)**.** *Let (M,g) be a Riemannian manifold of dimension n with a Riemannian connection ∇. Then the Riemannian curvature tensor field R of type (1,3) of the connection ∇ is defined by the mapping $R:$ $\mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ given by $R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z$ for all $X,Y,Z\in\mathfrak{X}(M)$.*

We state some important identities on a Riemannian manifold.

Theorem 1.1.9 (First and Second Bianchi identity)**.** *If ∇ is a Levi-Civita connection on a Riemannian manifold (M,g) then $\forall\, X,Y,Z\in\mathfrak{X}(M)$, we have*

$$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0, \tag{1.1.6}$$

$$(\nabla_XR)(Y,Z)+(\nabla_YR)(Z,X)+(\nabla_ZR)(X,Y)=0. \tag{1.1.7}$$

Theorem 1.1.10. *If R is the Riemannian curvature tensor of a Riemannian manifold (M,g), then*

$$g(R(X,Y)Z,U)=-\,g(R(X,Y)U,Z), \tag{1.1.8}$$

$$g(R(X,Y)Z,U)=g(R(Z,U)X,Y), \tag{1.1.9}$$

for all $X,Y,Z\in\mathfrak{X}(M)$.

Theorem 1.1.11. *If $\tilde{R}$ is the Riemannian curvature tensor of type (0,4) of a Riemannian manifold (M,g), then for all $X,Y,Z,U,V \in \mathfrak{X}(M)$, we have*

$$\tilde{R}(X,Y,Z,U) = -\tilde{R}(Y,X,Z,U) = -\tilde{R}(X,Y,U,Z) = \tilde{R}(Z,U,X,Y), \tag{1.1.10}$$

$$\tilde{R}(X,Y,Z,U) + \tilde{R}(Y,Z,X,U) + \tilde{R}(Z,X,Y,U) = 0, \tag{1.1.11}$$

$$(\nabla_X \tilde{R})(Y,Z,U,V) + (\nabla_Y \tilde{R})(Z,X,U,V) + (\nabla_Z \tilde{R})(X,Y,U,V) = 0, \tag{1.1.12}$$

where $g(R(X,Y)Z,U) = \tilde{R}(X,Y,Z,U)$ for all $X,Y,Z \in \mathfrak{X}(M)$.

Definition 1.1.12 (Ricci tensor)**.** *Let (M,g) be a Riemannian manifold of dimension n with a Riemannian connection ∇. Then the Ricci tensor field S is the covariant tensor field of degree 2 defined as* $\mathrm{Ric}(Y,Z) = S(Y,Z) =$ *Trace of the linear map $X \to R(X,Y)Z$ for all $X,Y,Z \in \mathfrak{X}(M)$.*

Definition 1.1.13 (Ricci operator)**.** *If Q is the symmetric endomorphism of $T_pM \to T_pM, p \in M$ and we write $S(X,Y) = g(QX,Y)$, then Q is the (1,1)-Ricci tensor, sometimes Q is called the Ricci operator.*

Definition 1.1.14 (Scalar curvature)**.** *Let M be a Riemannian manifold with the Levi-Civita connection ∇. Then the scalar curvature r of the manifold is a scalar function defined as the trace of the (1,1)-Ricci tensor Q. Thus $r = \mathrm{Tr.}(Q)$, where $S(X,Y) = g(QX,Y)$.*

Definition 1.1.15 (Divergence)**.** *Let (M,g) be an n-dimensional Riemannian manifold and X is any vector field on M. Then the divergence of the vector field X, denoted by* $\mathrm{div}X$ *and is defined as* $\mathrm{div}X = \sum_{i=1}^{n} g(\nabla_{e_i}X, e_i)$, *where $\{e_i\}$ is an orthonormal basis of the tangent space T_pM at any point $p \in M$.*

Definition 1.1.16 (Gradient vector field)**.** *A vector field Z on a Riemannian manifold (M,g) is said to be a gradient vector field if there exists a function $f \in C^\infty(M)$ such that $g(\mathrm{grad}f, Y) = g(Z,Y) = \mathrm{df}(Y)$ for all $Y \in \mathfrak{X}(M)$.*

Definition 1.1.17 (Hessian)**.** *The Hessian of a function $f \in C^\infty(M)$ is defined as its second covariant differential $H^f = \nabla(\nabla f)$, where ∇ is the Levi-Civita connection* *on the Riemannian manifold M. Then it can be easily seen that the Hessian H^f of f is a symmetric (0,2)-type tensor field satisfying*

$$H^f(X,Y) = X(Yf) - (\nabla_X Y)f = g(\nabla_X(\mathrm{grad} f), Y) \tag{1.1.13}$$

for all $X, Y \in \mathfrak{X}(M)$.

Definition 1.1.18 (Laplacian)**.** *The Laplacian Δf of a function $f \in C^\infty(M)$ is the divergence of its gradient. i.e., $\Delta f = \mathrm{div}(\mathrm{grad} f) \in C^\infty(M)$.*

Definition 1.1.19 (Sectional curvature)**.** *Let (M,g) be a Riemannian manifold of dimension n. Let π be a 2-dimensional subspace of the tangent space T_pM for any point $p \in M$ and X, Y be any two linear independent vectors in π. Then*

$$K_p(\pi) = -\frac{\tilde{R}(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2} = -\frac{\tilde{R}(X,Y,X,Y)}{G(X,Y,X,Y)} \tag{1.1.14}$$

is a function of π and is independent of the choice of X and Y in π and is called the sectional curvature of M at (p,π). Sometimes we say $K_p(\pi)$ is the sectional curvature of the plane $\pi \subset T_pM$ at p.

Now we state the definition of some Einstein manifolds which are very important for further study.

Definition 1.1.20 (Einstein manifold)**.** *An n-dimensional ($n > 2$) Riemannian manifold is said to be Einstein if its Ricci tensor S of type (0,2) is of the form $S = \alpha g$, where α is a smooth function and g is the metric tensor.*

It turns into $S = \frac{r}{n}g$, r being the scalar curvature of the manifold. The above equation is also called the Einstein metric condition [9].

The notion of quasi-Einstein manifold has been developed by Chaki and Maity [24] and also in other form by R. Deszcz [38].

Definition 1.1.21 (Quasi-Einstein manifold)**.** *A Riemannian manifold (M^n, g), $(n > 2)$ is said to be a quasi Einstein manifold if its non zero Ricci tensor S of type $(0,2)$ satisfies the following condition*

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y), \tag{1.1.15}$$

on M, where α *and* β *are real valued, non zero scalar functions on* (M^n, g)*. A is a non zero* 1*-form such that*

$$g(X,U) = A(X), \; g(U,U) = 1. \tag{1.1.16}$$

A is known as an associated 1*-form and U is known as a generator of* (M^n, g)*. This kind of manifold of dimension n is denoted by* $(QE)_n$*. If* $\beta = 0$ *in (1.1.15), then* $(QE)_n$ *turns into an Einstein manifold.*

Then the notion of generalized quasi-Einstein manifold has been introduced by Chaki [26].

Definition 1.1.22 (Generalized quasi-Einstein manifold)**.** *A Riemannian manifold* (M^n, g)*, (*$n \geq 3$*) is said to be a generalized quasi-Einstein manifold denoted by* $G(QE)_n$ *if its non zero Ricci tensor S of type* $(0,2)$ *satisfies the following condition*

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)], \tag{1.1.17}$$

on M, where α*,* β *and* γ *are real valued, non zero scalar functions on* (M^n, g) *in which* $\beta \neq 0$*,* $\gamma \neq 0$*. A and B are two non zero* 1*-forms such that*

$$g(X,U) = A(X), g(X,V) = B(X), g(U,V) = 0, g(U,U) = 1, g(V,V) = 1. \tag{1.1.18}$$

Here α*,* β *and* γ *are known as associated scalars. A and B are called associated* 1*-forms. U and V are generators of this manifold.*

Shaikh et al.[109] introduced the notion of hyper-generalized quasi Einstein $(HGQE)_n$ manifold.

Definition 1.1.23 (Hyper-generalized quasi-Einstein manifold)**.** *A Riemannian manifold* (M^n, g)*,* $(n > 2)$ *is said to be a hyper-generalized quasi Einstein manifold if its Ricci tensor S of type* $(0,2)$ *is non zero and the following condition*

$$\begin{aligned} S(X,Y) = &\alpha g(X,Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] \\ &+ \delta[A(X)D(Y) + A(Y)D(X)], \end{aligned} \tag{1.1.19}$$

for all $X,Y \in \chi(M)$, is satisfied. Here α, β, γ and δ are real valued, non zero scalar functions on (M^n,g). A, B and D are non zero 1-forms such that

$$g(X,U) = A(X), g(X,V) = B(X), g(X,W) = D(X), \tag{1.1.20}$$

U, V and W are the mutually orthogonal unit vector fields, i.e.,

$$g(U,V) = g(V,W) = g(U,W) = 0; g(U,U) = g(V,V) = g(W,W) = 1. \tag{1.1.21}$$

α, β, γ and δ are called associated scalars. A, B and D are called associated 1-forms. U, V and W are called generators of this manifold. This manifold of dimension n is denoted by $(HGQE)_n$.

Kim et al. [75] studied compact Einstein warped product spaces with non positive scalar curvature. G*ü*ler and Demirbağ [56] dealt with some Ricci conditions on hyper-generalized quasi-Einstein manifolds. Pahan et al. [91] worked on multiply warped products quasi-Einstein manifolds with quarter-symmetric connection and they have discussed on compact super quasi-Einstein warped product with non positive scalar curvature. Motivated by these works, presently we study about hyper-generalized quasi Einstein warped product spaces with non positive scalar curvature. Later we apply our results on some physical properties of hyper-generalized quasi Einstein manifold.

Let $\{e_i : i = 1,2,3,\ldots,n\}$ be an orthogonal frame field at any point of the manifold. Then by putting $X = Y = e_i$ in (1.1.19) and taking summation over i $(1 \leq i \leq n)$, we get

$$r = n\alpha + \beta, \tag{1.1.22}$$

where r is the scalar curvature of the manifold.

It is considered that U as the timelike velocity vector field, V as the heat flux vector field and W as the stress vector field. i.e.,

$$g(U,U) = -1, g(V,V) = 1, g(W,W) = 1. \tag{1.1.23}$$

Many geometers worked with various types of curvature tensors in differential geometry. Tripathi [120] improved Chen-Ricci inequality for curvature like tensors and its applications. Chen and Yano [32] introduced the notion of quasi-constant curvature.

Definition 1.1.24 (Quasi constant curvature)**.** *A Riemannian manifold (M^n, g), $(n \geq 3)$ is said to be a quasi constant curvature if its curvature tensor R of type $(0,4)$ satisfies the following condition*

$$\begin{aligned} R(X,Y,Z,N) =& a_1[g(Y,Z)g(X,N) - g(X,Z)g(Y,N)] \\ &+ a_2[g(Y,Z)A(X)A(N) - g(X,Z)A(Y)A(N)] \\ &+ g(X,N)A(Y)A(Z) - g(Y,N)A(X)A(Z)], \end{aligned}$$

where A is a 1-form and a_1, a_2 are both non zero scalars.

Motivated by the definition of quasi constant curvature we define hyper-generalized quasi-constant curvature. It is defined as follows.

Definition 1.1.25 (Hyper-generalized quasi-constant curvature)**.** *A Riemannian manifold (M^n, g), $(n \geq 3)$ is called of hyper-generalized quasi-constant curvature if its curvature tensor has the following form*

$$\begin{aligned} R(X,Y,Z,N) =& b_1[g(Y,Z)g(X,N) - g(X,Z)g(Y,N)] \\ &+ b_2[g(Y,Z)A(X)A(N) + g(X,N)A(Y)A(Z) \\ &- g(X,Z)A(Y)A(N) - g(Y,N)A(X)A(Z)] \\ &+ b_3[g(Y,Z)\{A(X)B(N) + A(N)B(X)\} \\ &+ g(X,N)\{A(Y)B(Z) + A(Z)B(Y)\} \\ &- g(X,Z)\{A(Y)B(N) + A(N)B(Y)\} \\ &- g(Y,N)\{A(X)B(Z) + A(Z)B(X)\}] \\ &+ b_4[g(Y,Z)\{A(X)D(N) + A(N)D(X)\} \\ &+ g(X,N)\{A(Y)D(Z) + A(Z)D(Y)\} \\ &- g(X,Z)\{A(Y)D(N) + A(N)D(Y)\} \\ &- g(Y,N)\{A(X)D(Z) + A(Z)D(X)\}], \end{aligned} \tag{1.1.24}$$

where A, B, D are 1-forms and b_1, b_2, b_3, b_4 are non zero scalars.

Definition 1.1.26 (Almost contact manifold)**.** *[14] Let M be a* $(2n+1)$ *dimensional smooth manifold and* ϕ, ξ, η *be a tensor field of type* $(1,1)$, *a vector field, a 1-form on M respectively. If* ϕ, ξ *and* η *satisfies the conditions*

$$\eta(\xi) = 1,$$
$$\phi^2(X) = -X + \eta(X)\xi$$

for any vector field X on M, *then* M *is said to have an almost contact structure* (ϕ, ξ, η). *The manifold M equipped with the almost contact structure* (ϕ, ξ, η) *is called an almost contact manifold.*

We now state that every almost contact manifold admits a Riemannian metric tensor field which plays an analogous role to an almost Hermitian metric tensor field.

Theorem 1.1.27. *Every almost contact manifold M admits a Riemannian metric tensor field g such that*

$$\eta(X) = g(X, \xi), \tag{1.1.25}$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{1.1.26}$$
$$g(\phi X, Y) + g(X, \phi Y) = 0, \tag{1.1.27}$$

for all vector field X and Y.

The equation (1.1.27) means that ϕ is skew-symmetric with respect to g. We call the metric tensor g as an associated Riemannian metric of the given almost contact structure (ϕ, ξ, η). The metric g is also called a compatible metric.

Definition 1.1.28 (Almost contact metric manifold)**.** *If M admits a structure* (ϕ, ξ, η, g), *g being an associated Riemannian metric of an almost contact structure* (ϕ, ξ, η), *then M is said to have an almost contact metric structure* (ϕ, ξ, η, g) *and the manifold equipped with this structure is called an almost contact metric manifold.*

Definition 1.1.29 (Kenmotsu manifold)**.** *An almost contact metric manifold* (M^{2n+1}, g) *is said to be a Kenmotsu manifold [73] if it satisfies*

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X. \tag{1.1.28}$$

In a Kenmotsu manifold the following relations hold.

$$(i)\ \nabla_X \xi = X - \eta(X)\xi, \tag{1.1.29}$$

$$(ii)\ (\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{1.1.30}$$

$$(iii)\ R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{1.1.31}$$

$$(iv)\ S(X,\xi) = -2n\eta(X), \tag{1.1.32}$$

$$(v)\ Q\xi = -2n\xi, \tag{1.1.33}$$

for $X, Y \in \mathfrak{X}(M)$ and where ∇, R, S, Q are the Levi-Civita connection, curvature tensor, Ricci tensor and Ricci operator respectively.

The notion of trans-Sasakian manifold was introduced by J. A. Oubina [85] in 1985. Then, J. C. Marrero [78] characterized the local structure of trans-Sasakian manifolds of dimension ≥ 5.

Definition 1.1.30 (Trans-Sasakian manifold)**.** *An almost contact metric manifold $\tilde{M}$ is called a trans-Sasakian manifold if it satisfies the following condition*

$$(\tilde{\nabla}_X \phi)(Y) = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\phi X,Y)\xi - \eta(Y)\phi X\}, \tag{1.1.34}$$

for some smooth functions α, β on $\tilde{M}$ and we say that the trans-Sasakian structure is of type (α, β).

For trans-Sasakian manifold, we have from the equation (1.1.34) that

$$\tilde{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{1.1.35}$$

$$(\tilde{\nabla}_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{1.1.36}$$

For 3-dimensional trans-Sasakian manifold, we have

$$\begin{aligned}
\tilde{R}(X,Y)Z = &\left[\frac{\tilde{r}}{2} - 2(\alpha^2 - \beta^2 - \xi\beta)\right][g(Y,Z)X - g(X,Z)Y] \\
&- \left[\frac{\tilde{r}}{2} - 3(\alpha^2 - \beta^2) + \xi\beta\right][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \\
&+ [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)][\phi \operatorname{grad} \alpha - \operatorname{grad} \beta] \\
&- \left[\frac{\tilde{r}}{2} - 3(\alpha^2 - \beta^2) + \xi\beta\right]\eta(Z)[\eta(Y)X - \eta(X)Y]
\end{aligned}$$

$$
\begin{aligned}
&- [Z\beta + (\phi Z)\alpha]\eta(Z)[\eta(Y)X - \eta(X)Y] - [X\beta + (\phi X)\alpha] \\
&\times [g(Y,Z)\xi - \eta(Z)Y] - [Y\beta + (\phi Y)\alpha][g(X,Z)\xi - \eta(Z)X],
\end{aligned}
$$

$$
\begin{aligned}
\tilde{S}(X,Y) = &\left[\frac{\tilde{r}}{2} - (\alpha^2 - \beta^2 - \xi\beta)\right] g(X,Y) - \left[\frac{\tilde{r}}{2} - 3(\alpha^2 - \beta^2) + \xi\beta\right] \eta(X)\eta(Y) \\
&- [Y\beta + (\phi Y)\alpha]\eta(X) - [X\beta + (\phi X)\alpha]\eta(Y),
\end{aligned}
$$

$\tilde{r}$ being the scalar curvature of $\tilde{M}$.

When α and β are constants, the above equations give

$$
\tilde{Q}X = \left[\frac{\tilde{r}}{2} - (\alpha^2 - \beta^2)\right] X - \left[\frac{\tilde{r}}{2} - 3(\alpha^2 - \beta^2)\right] \eta(X)\xi, \tag{1.1.37}
$$

$$
\tilde{R}(X,Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y). \tag{1.1.38}
$$

In general, trans-Sasakian manifold of type $(0,0)$, $(\alpha,0)$, $(0,\beta)$ are called cosymplectic, α-Sasakian and β-Kenmotsu manifold, respectively.

Definition 1.1.31. *Let M and N be smooth manifolds with* $\dim M = m, \dim N = n$, *$f : M \to N$ be a smooth map and $f_{*p} : T_pM \to T_{f(p)}N$ be the tangential map at $p \in M$. Then*

(i) *f is said to be an immersion if f_{*p} is injective for each $p \in M$,*

(ii) *f is said to be an submersion if f_{*p} is surjective for all $p \in M$,*

(iii) *f is said to be a local diffeomorphism at $p \in M$ if f_{*p} is injective and surjective.*

(iv) *The pair (M, f) is called a submanifold of N if f is one to one and an immersion. If the inclusion map of M in N is a one to one immersion, then we say that M is a submanifold of N.*

(v) *f is said to be an imbedding if f is a one to one immersion on M.*

Let M be a submanifold of an almost contact manifold $\tilde{M}$ with induced metric g. Let ∇ and $\nabla^{\perp}$ be the induced connections on the tangent bundle TM and normal bundle $T^{\perp}M$ of M respectively. Let $\mathscr{F}$ denote the algebra of smooth functions on

M and $\Gamma(TM)$ denotes the $\mathscr{F}$-module of smooth sections of TM over M. Then the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{1.1.39}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \tag{1.1.40}$$

for each $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively, for the immersion of M into $\tilde{M}$. They are related as

$$g(h(X,Y),N) = g(A_N X, Y), \tag{1.1.41}$$

where g denotes the Riemannian metric on $\tilde{M}$ as well as the one induced on M.

For any $X \in \Gamma(TM)$,

$$\phi X = PX + FX, \tag{1.1.42}$$

where PX is the tangential component and FX is the normal component of ϕX.
For any $N \in \Gamma(T^{\perp}M)$,

$$\phi N = BN + CN, \tag{1.1.43}$$

where BN is the tangential component and CN is the normal component of ϕN.

Definition 1.1.32 (Almost contact metric manifold)**.** *A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be invariant if F is identically zero, that is $\phi X \in \Gamma(TM)$ and anti-invariant if P is identically zero, that is $\phi X \in \Gamma(T^{\perp}M)$, for any $X \in \Gamma(TM)$.*

Definition 1.1.33 (Slant submanifold)**.** *A slant submanifold is defined in [31] as a submanifold of (M, g, J) such that, for any nonzero vector $X \in T_pN$, the angle $\theta(X)$ between JX and the tangent space T_pN is a constant (which is independent of the choice of the point $p \in N$ and the choice of the tangent vector X in the tangent plane T_pN).*

We recall the following result which was obtained by Cabreizo et al. [20] for a slant submanifold of an almost contact metric manifold.

Theorem 1.1.34. *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$, such that $\xi \in TM$. Then, M is slant if and only if $\exists$ a constant $\lambda \in [0,1]$ such that*

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{1.1.44}$$

Again, if θ is slant angle of M, then $\lambda = \cos^2\theta$.

The following relations are straightforward consequences of the equation (1.1.44):

$$g(PX, PY) = \cos^2\theta[g(X,Y) - \eta(X)\eta(Y)], \tag{1.1.45}$$

$$g(FX, FY) = \sin^2\theta[g(X,Y) - \eta(X)\eta(Y)], \tag{1.1.46}$$

for any $X, Y \in \Gamma(TM)$.

For a pointwise slant submanifold of almost Hermitian manifold it is similarly derived in [79]

$$BFX = -X\sin^2\theta, \quad CFX = -FPX, \tag{1.1.47}$$

for all $X \in \Gamma(TM)$.

The mean curvature H of M is given by $H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i)$, where m is the dimension of M and $\{e_1, e_2 ......, e_m\}$ is a local orthonormal frame of vector fields on M.

Definition 1.1.35. *A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be totally umbilical if the second fundamental form satisfies $h(X,Y) = g(X,Y)H$, for all $X, Y \in \Gamma(TM)$.*

Definition 1.1.36. *A submanifold M is said to be totally geodesic if $h(X,Y) = 0$, for all $X, Y \in \Gamma(TM)$ and minimal if $H = 0$.*

Now, we explain the brief introduction of pointwise bi-slant submanifold of an almost contact metric manifold $\tilde{M}$.

Definition 1.1.37. *[20, 99] A submanifold M of an almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$ is said to be a pointwise bi-slant submanifold if there exists a pair of* *orthogonal distributions $\mathscr{D}_1$ and $\mathscr{D}_2$ on M such that:*

(i) *TM admits the orthogonal direct decomposition i.e.,* $TM = \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \langle \xi \rangle$, *where* $\langle \xi \rangle$ *is the one dimensional distribution spanned by the structure vector field* ξ.

(ii) $\phi(\mathscr{D}_1) \perp \mathscr{D}_2$ *and* $\phi(\mathscr{D}_2) \perp \mathscr{D}_1$ *that implies* $P(\mathscr{D}_i) \subset \mathscr{D}_i$, $i = 1,2$.

(iii) *The distribution* $\mathscr{D}_1$ *and* $\mathscr{D}_2$ *are pointwise slant with slant angles* θ_1 *and* θ_2 *respectively.*

Definition 1.1.38. *A pointwise bi-slant submanifold is called proper if its bi-slant angles* θ_1, θ_2 *satisfy* $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$ *and* θ_1, θ_2 *are not constants on M.*

For a pointwise bi-slant submanifold, we take

$$X = T_1X + T_2X, \ \ \forall X \in TM, \tag{1.1.48}$$

where T_i is the projection from TM onto D_i. So, T_iX are the components of X in D_i, $i = 1,2$.

If we put $P_i = T_i \circ P$, then from the equation (1.1.48) we get

$$\phi X = P_1X + P_2X + FX, \ \ \forall X \in TM. \tag{1.1.49}$$

$$P^2 = \cos^2\theta_i(-I + \eta \otimes \xi), \ \ i = 1,2. \tag{1.1.50}$$

Now we give the following definition for proving some theorems in Chapter 5.

Definition 1.1.39. *[110] A vector field ς on a Riemannian manifold M which satisfies $\nabla_X \varsigma = X$, for any vector field X is called a concurrent vector field. ς is called gradient if there is a function u defined on M such that $\varsigma = \nabla u$.*

1.2 Warped product

One of the most fruitful generalizations of the notion of Cartesian or direct products is the notion of warped products defined in [11]. The concept of warped products appeared in the mathematical and physical literature before [11]. For instance, warped product spaces were called semi-reducible spaces in [77].

Many exact solutions of the Einstein field equations and modified field equations are warped products. For instance, the Schwarzschild solution and Robertson-Walker models are warped products. While the Robertson-Walker models describes a simply-connected homogeneous isotropic expanding or contracting universe, the Schwarzschild solution is the best relativistic model that describes the outer space around a massive star or a black hole. The Schwarzschild model laid the groundwork for the description of the final stages of gravitational collapse and the objects known today as black holes. Twisted products and convolution manifolds are two natural extensions of warped product manifolds.

Let B and F be two pseudo-Riemannian manifolds of positive dimensions equipped with pseudo-Riemannian metrics g_B and g_F, respectively, and let $f : B \to (0,\infty)$ be a positive smooth function on B.

Consider the product manifold $B \times F$ with its natural projection $\pi : B \times F \to B$ and $\eta : B \times F \to F$.

Definition 1.2.1 (Warped product). *The warped product $M = B \times_f F$ is the manifold $B \times F$ equipped with the pseudo-Riemannian structure such that*

$$\langle X, X\rangle = \langle \pi^*(X), \pi^*(X)\rangle + f^2(\pi(X))\langle \eta^*(X), \eta^*(X)\rangle,$$

for any tangent vector $X \in TM$.

Thus we have $g = g_B + f^2 g_F$. The function f is called the warping function of the warped product.

A warped product $B \times_f F$ is called trivial if f is a constant. In this case, $B \times_f F$ is the Riemannian product $B \times F_f$, where F_f is the manifold F equipped with the metric $f^2 g_F$, which is homothetic to g_F.

Though in the Riemannian geometry, the class of warped products which have a non-constant warping functions serve a rich class of examples, Kim et al. [75] showed it there hardly exists a compact Einstein warped product having non-constant warping function in condition of non-positiveness of scalar curvature. Additionaly, they noticed that one warped product would be an Einstein manifold if its base is a quasi-Einstein metric. It should be focused that some paradigms of expanding quasi - Einstein manifolds with an arbitrary Einstein manifold as a fiber and steady quasi-Einstein manifolds having fiber of non-negative scalar curvature which were developed in Besse [9]. In recent times, Barros, Batista and Ribeiro [7] served few volume estimations of Einstein warped products which are similar to a classical result because of Yau [130] and Calabi [21] for complete Riemannian manifolds which have non-negative Ricci curvature. Their approach is with quasi-Einstein manifold. They also showed a hindrance for the existence of such a class of manifolds. In this regard, we want to mention He, Petersen and Wylie's [61] work relating Einstein warped product manifolds. As it is an elongation of Case, Shu and Wei's [23] work and some erstwhile works of Kim et al. [75], the result of [61] is that the base may have non-void boundary.

For a warped product $B \times_f F$, B is called the base of the warped product and F the fiber. The leaves $B \times \{q\} = \eta^{-1}(q)$ and the fibers $\{p\} \times F = \pi^{-1}(p)$ are pseudo-Riemannian submanifolds of M. Vectors tangent to leaves are called horizontal and those tangent to fibers are called vertical. We denote by $\mathscr{H}$ the orthogonal projection of $T_{(p,q)}M$ onto its horizontal subspace $T_{(p,q)}(B \times \{q\})$ and by $\mathscr{V}$ the projection onto the vertical subspace $T_{(p,q)}(\{p\} \times F)$.

If $u \in T_pB,\ p \in B$ and $q \in F$, then the lift $\bar{u}$ of u to (p,q) is the unique vector in $T_{(p,q)}M$ such that $\pi_*(\bar{u}) = u$. For a vector field $X \in \mathfrak{X}(B)$, the lift of X to M is the vector field $\overline{X}$ whose value at each (p,q) is the lift of X_p to (p,q). The set of all horizontal lifts is denoted by $\mathscr{L}(B)$. Similarly, we denote by $\mathscr{L}(F)$ the set of all vertical lifts.

For $\overline{X}, \overline{Y} \in \mathscr{L}(B)$ and $\overline{V}, \overline{W} \in \mathscr{L}(F)$, we have

$$[\overline{X}, \overline{Y}] = [\overline{X}, \overline{Y}]^- \in \mathscr{B}, \tag{1.2.1}$$

$$[\overline{V}, \overline{W}] = [\overline{V}, \overline{W}]^- \in \mathscr{F}, \tag{1.2.2}$$

$$[\overline{X}, \overline{V}] = 0, \tag{1.2.3}$$

where $[\overline{X}, \overline{Y}]^-$ denotes the lift of $[\overline{X}, \overline{Y}]$.

The Levi-Civita connection ∇ of $M = B \times_f F$ is related with the Levi-Civita connections of B and F as follows.

Proposition 1.2.2. *[84] For $X,Y \in \mathscr{B}$ and $V,W \in \mathscr{F}$, we have on $B \times_f F$ that*

(1) $\nabla_X Y \in \mathscr{B}$ *is the lift of* $\nabla_X Y$ *on* B;

(2) $\nabla_X V = \nabla_V X = (X \ln f)V$;

(3) $\operatorname{nor}(\nabla_V W) = \sigma(V,W) = -\frac{\langle V,W \rangle}{f} \nabla f$;

(4) $\tan(\nabla_V W) \in \mathscr{F}$ *is the lift of* ∇'_V *on F, where* ∇' *is the Levi-Civita connection of F.*

The next results provide the curvature of a warped product $M = B \times_f F$ in terms of its warping function f and the curvature tensors R^B and R^F of B and F.

Proposition 1.2.3. *[84] Let $M = B \times_f F$ be a warped product with Riemannian curvature tensor R. If $X,Y,Z \in \mathfrak{X}(B)$ and $U,V,W \in \mathfrak{X}(F)$, then*

$$
\begin{aligned}
&(1) \quad R(X,Y)Z = R^B(X,Y)Z, \\
&(2) \quad R(V,X)Y = \frac{H^f(X,Y)}{f} V, \\
&(3) \quad R(X,Y)V = R(V,W)X = 0, \\
&(4) \quad R(X,V)W = \frac{g(V,W)}{f} D^1_X(\nabla f), \\
&(5) \quad R(V,W)U = R^F(V,W)U + \frac{\|\nabla f\|^2}{f^2}\left[g(W,U)V - g(V,U)W\right].
\end{aligned}
$$

Proposition 1.2.4. *[84] On the warped product $M = B \times_f F$ with $\dim(F) = d > 1$, let $X,Y \in \mathfrak{X}(B)$ and $V,W \in \mathfrak{X}(F)$. Then the Ricci tensor S_M of M are given by*

$$
\begin{aligned}
&(1) \quad S_M(X,Y) = S_B(X,Y) - \frac{d}{f} H^f(X,Y), \\
&(2) \quad S_M(X,V) = 0, \\
&(3) \quad S_M(V,W) = S_F(V,W) - g(V,W) f^{\#}, \quad f^{\#} = \frac{\Delta f}{f} + \frac{d-1}{f^2}\|\nabla f\|^2,
\end{aligned}
$$

where $\Delta f = tr\left(H^f\right)$ and H^f are respectively the Laplacian and the Hessian of f on B.

Proposition 1.2.5. *[84] Let $M = B \times_f F$ be a semi-Riemannian warped product furnished with the metric $g_M = g_B \oplus f^2 g_F$. Then the scalar curvature τ of M admits the following relation*

$$\tau = \tau_B + \frac{\tau_F}{f^2} - 2s\frac{\Delta_B(f)}{f} - s(s-1)\frac{\|\mathrm{grad}_B f\|_B^2}{f^2},$$

where $r = \dim(B)$ and $s = \dim(F)$.

Multiply warped products is the generalization of warped products.

Definition 1.2.6. *[126] A multiply warped product is the product manifold $\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2 ... \times_{h_m} F_m$ endowed with the metric tensor $\overline{g} = g_B \oplus h_1^2 g_{F_1} \oplus h_2^2 g_{F_2} \oplus h_3^2 g_{F_3} \oplus .... \oplus h_m^2 g_{F_m}$ defined by*

$$\overline{g} = \pi^*(g_B) \oplus (h_1 \circ \pi)^2 \sigma_1^*(g_{F_1}) \oplus ... \oplus (h_m \circ \pi)^2 \sigma_m^*(g_{F_m}),$$

where π and σ_i $(i = 1,2,...,m)$ are the natural projections of $B \times F_1 \times F_2 ..... \times F_m$ onto $B, F_1, F_2, ..., F_{m-1}$ and F_m respectively. For each $i \in \{1,2,...,m\}$ the function $h_i : B \to (0,\infty)$ is smooth and (F_i, g_{F_i}) is a pseudo-Riemannian manifold.

Note 1.2.7. *In particular, when $B = (c,d)$ equipped with the negative definite metric $g_B = -dt^2$, where $c < d$ and (F_i, g_{F_i}) is a Riemannian manifold for each $i \in \{1,2,...,m\}$, then we call $(\overline{M}, \overline{g})$ as the generalized Robertson-Walker spacetimes.*

Let $M = M_0 \times_{f_1} M_1 \times_{f_2} M_2$ be a biwarped product submanifold. Letting $\mathscr{D}^T = TM_T$, $\mathscr{D}^\perp = TM_\perp$, $\mathscr{D}^\theta = TM_\theta$ and $N =_{f_1} M_1 \times_{f_2} M_2$, we obtain [29, 123]

$$\nabla_X Z = \sum_{i=1}^{2} (X(\ln f_i)) Z^i, \tag{1.2.4}$$

where $Z \in \Gamma(TN)$, $X \in \mathscr{D}^T$, ∇ is the Levi-Civita connection of M and M_i-component of Z is Z^i $(i = 1,2)$.

1.3 Ricci and Riemann soliton

Ricci solitons are the generalization of Einstein manifolds. Hamilton [59] developed this idea at the beginning of 80's.

Definition 1.3.1. *One complete Riemannian manifold M furnished with a metric g is said to be a Ricci soliton if it satisfies the following relation*

$$\mathrm{Ric} + \frac{1}{2}\pounds_X g = \lambda g, \tag{1.3.1}$$

where λ being a scalar quantity and X being a vector field of M.

The above equation (1.3.1) is known as the fundamental equation. Ricci solitons are of three types. They are shrinking, expanding and steady. These classifications depend on the value of λ. If $\lambda > 0$, $\lambda < 0$ and $\lambda = 0$, then a Ricci soliton will be shrinking, expanding and steady respectively. Moreover, If we take $X = \nabla\psi$ in (1.3.1), where ψ being a smooth function on M, then we denote the gradient Ricci soliton as $(M, g, \nabla\psi, \lambda)$. Hence the equation (1.3.1) becomes

$$\mathrm{Ric} + \nabla^2\psi = \lambda g, \tag{1.3.2}$$

where Hessian of $\psi = \nabla^2\psi$. To know more see [22, 59]. If λ is a smooth function then a Ricci soliton is called almost Ricci soliton.

J. N. Gomes, Q. Wang and C. Xia introduced a new kind of Ricci soliton, called h-almost Ricci soliton in [58]. They have given the following definition.

Definition 1.3.2 (h-almost Ricci soliton)**.** *An h-almost Ricci soliton is a complete Riemannian manifold (M^n, g) which are smooth and satisfy the equation*

$$\mathrm{Ric} + \frac{h}{2}\pounds_X g = \lambda g,$$

where $X \in \mathfrak{X}(M)$, $\lambda : M \to R$ is a soliton function and $h : M \to R$ is a function. Then (M^n, g, X, h, λ) is called an h-almost Ricci soliton.

Definition 1.3.3 (η-Ricci soliton)**.** *[34] Let (M, ϕ, ξ, η, g) be an almost paracontact metric manifold. Consider the equation*

$$\pounds_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0,$$

where $\pounds_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, and λ and μ are real constants. Writing $\pounds_\xi g$ in terms of the Levi-Civita connection ∇, we obtain:

$$2S(X,Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y),$$

for any $X,Y \in \mathfrak{X}(M)$. The data (g,ξ,λ,μ) which satisfy the above equation is said to be an η-Ricci soliton on M.

We introduce a new notion of "h-almost η-Ricci soliton" as follows.

Definition 1.3.4 (h-almost η-Ricci soliton)**.** *A complete Riemannian manifold (M^n,g) furnished with a metric g is said to be an h-almost η-Ricci soliton if it satisfies the following relation*

$$\mathrm{Ric}+\frac{h}{2}\pounds_X g=\lambda g+\mu(\eta\otimes\eta), \tag{1.3.3}$$

where λ being a scalar quantity, X being a vector field belonging to M, $h: M\to R$ is a smooth function and η is a 1-form.

Moreover, if we put $X=\nabla\psi$ in (1.3.3), then we obtain an another definiton as follows.

Definition 1.3.5 (Gradient h-almost η-Ricci soliton)**.** *A complete Riemannian manifold (M^n,g) furnished with a metric g is said to be a gradient h-almost η-Ricci soliton if it satisfies the following relation*

$$\mathrm{Ric}+h\nabla^2\psi=\lambda g+\mu(\eta\otimes\eta), \tag{1.3.4}$$

where ψ being a smooth function on M and Hessian of $\psi=\nabla^2\psi$, then it is said to be a gradient h-almost η- Ricci soliton and we denote it as $(M,g,\nabla\psi,h,\eta,\lambda)$ for convenience.

Hamilton [60] developed the idea of Ricci flow in 1982. The Ricci flow is a special case of Riemann flow [125]. Hiriča and Udriste [63] introduced and studied Riemann soliton as a comparison of Ricci soliton. This arises as a self-similar solution of Riemann flow

$$\frac{\partial}{\partial t}G(t)=-2R(g(t));\quad t\in[0,I], \tag{1.3.5}$$

where $G=\frac{1}{2}(g\wedge g)$, R is the Riemann curvature tensor with respect to the metric tensor g and $\wedge$ is the Kulkarni-Nomizu product. These are the natural extensions because some results in Riemann flow resemble Ricci flow. Riemann flow verifies the uniqueness and short time existence.

Definition 1.3.6 (Kulkarni-Nomizu product)**.** *The Kulkarni-Nomizu product $\wedge$ of two $(0,2)$-type tensors A and B is defined by*

$$\begin{aligned}(A\wedge B)(X,Y,Z,W) =& A(X,Z)B(Y,W)+A(Y,W)B(X,Z)\\ &-A(X,W)B(Y,Z)-A(Y,Z)B(X,W),\end{aligned} \tag{1.3.6}$$

Definition 1.3.7 (Riemann soliton)**.** *A smooth manifold M furnished with the Riemannian metric tensor g is said to be a Riemann soliton [39] if it satisfies*

$$2R+\alpha(g\wedge g)+(g\wedge \pounds_V g)=0, \tag{1.3.7}$$

where $\pounds_V$ is the Lie derivative with respect to the potential vector field V and α is a constant.

Riemann soliton corresponds as a fixed point of Riemann flow and they are viewed as a dynamical system on space of Riemannian metric modulo diffeomorphism. It is noted that the concept of Riemann soliton generalizes a space of constant sectional curvature. That is, $R = c(g\wedge g)$, where c is a constant. Moreover, a Riemann soliton is said to be expanding, steady and shrinking if $\alpha > 0$, $\alpha = 0$ and $\alpha < 0$ respectively. If $V = \nabla u$, where ∇u denotes the gradient of the potential function u, then we obtain the concept of gradient Riemann soliton. For this case, the equation (1.3.2) becomes

$$R+\frac{\alpha}{2}(g\wedge g)+(g\wedge H^u)=0, \tag{1.3.8}$$

where H^u is the Hessian of the smooth function u. According to Perelman [96], we know that a Ricci soliton on a compact manifold is a gradient Ricci soliton. If the potential vector field V vanishes identically, then a Riemann soliton is said to be trivial. For the trivial case, the manifold is of constant sectional curvature.

Ramesh Sharma [113], Mukut Mani Tripathi [119], Cornelia Livia Bejan and Mircea Crasmareanu [19], S. Pahan [86, 87, 90], etc studied Ricci soliton on various types of contact metric manifolds.

1.4 Spacetimes

Definition 1.4.1. *Let (M^n, g) be a semi-Riemannian manifold of dimension n. Then G is said to be an Einstein gravitational tensor field of M if it satisfies the relation*

$$G(X,Y) = \mathrm{Ric}(X,Y) - \frac{1}{2}Sg(X,Y)$$

for every $X,Y \in \mathfrak{X}(M)$, where S is the scalar curvature on M.

Therefore the Einstein field equations can be written in the form

$$\mathrm{Ric}(X,Y) - \frac{1}{2}Sg(X,Y) + \kappa g(X,Y) = \lambda T(X,Y),$$

where T is the stress-energy tensor, κ is the cosmological constant and λ is the Einstein gravitational constant. The basic solutions of the Einstein field equations have been studied in Lorentzian geometry and general theory of relativity and they can be expressed in terms of the warped products [8]. In Lorentzian geometry some well-known solutions of the Einstein field equations such as Schwarzschild and Friedmann-Robertson-Walker metrics can be expressed in terms of the warped products. The generalized Friedmann-Robertson-Walker metric and solutions of the Einstein field equations can be expressed in terms of the Lorentzian warped products. Different models like the general relativistic model of gravitation and cosmological model provided the importance to find the Einstein equations. The warped product geometry is used to solve the partial differential equations since we can easily use the method of separation of variables. In five dimensional warped product geometry [101], the world has been considered as a higher dimensional universe expressed in terms of warped product geometry. Albert Einstein provided a static solution of the field equations and introduced the cosmological constant [47]. Recently, the cosmological constants were studied by many authors on various spaces [54, 51, 5, 93].

Many authors studied the warped product manifolds and locally conformally flat manifolds, see [16, 17]. There are several studies correlating the warped product Einstein manifolds under various conditions on the curvature and symmetry, see

[28, 61, 62, 83]. It is well-known that the Einstein condition on warped geometries requires that the fibers must be necessarily Einstein [9]. In 2000, B. Ünal [126] derived the covariant derivative formulas for multiply warped products and also studied the geodesic equations for such type of spaces. In 2000, J. Choi [35] investigated the curvature of a multiply warped product with C^0-warping functions and represented the interior Schwarzschild spacetime as a multiply warped product spacetime with warping functions. In 2005, F. Dobarro and B. Ünal [41] studied the Ricci-flat and Einstein-Lorentzian multiply warped products and provided some results on the generalized Kasner spacetimes. In 2005 [18], authors obtained the necessary and sufficient conditions for a static spacetime to be locally conformally flat. In 2016, D. Dumitru [46] calculated the warping functions for multiply generalized Robertson-Walker space-time to be an Einstein manifold when all fibers are Ricci flat. In 2017, F. Gholami, F. Darabi and A. Haji-Badali [54] studied the multiply warped product metrics and reduced the Einstein equations for generalized Friedmann-Robrtson-Walker spacetime. In 2017, Sousa and Pina [114] studied the warped product semi-Riemannian Einstein manifolds under consideration that the base is conformal to an n-dimensional pseudo-Euclidean space and invariant under the action of an $(n-1)$-dimensional group. More recently, in [94], the authors generalized the work of Sousa and Pina for multiply warped product semi-Riemannian Einstein manifolds.

So, there are several studies correlating the warped product manifolds, multiply warped product manifolds, Einstein-Lorentzian multiply warped product manifolds, generalized Kasner spacetimes, static spacetime with conformal condition and generalized Friedmann-Robrtson-Walker spacetime etc. It is well-known that the generalized Friedmann-Robertson-Walker metric and solutions of the Einstein field equations can be expressed in terms of the Lorentzian warped products. The multiply warped product $(\overline{M}, \overline{g})$ is a Lorentzian multiply warped product when it satisfies Note 1.2.7. Then the Lorentzian multiply warped product $(\overline{M}, \overline{g})$ is called a generalized Robertson-Walker spacetime. In this literature we consider a multiply warped product metric of the generalized Friedmann-Robertson-Walker spacetime of type $\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2$ with $\dim(B) = 1$, the warping functions h_1, h_2 associated to the submanifolds F_1, F_2 with dimensions n_1, n_2 respectively and the submanifold F_1 is conformal to $(\mathbb{R}^{n_1}, g)$, a pseudo-Euclidean space. A new way to study on generalized Friedmann-Robertson-Walker spacetime means we discuss the Einstein gravitational field tensors and the cosmological constant in generalized Friedmann-Robertson-Walker spacetime $(\overline{M}, \overline{g})$ of type $\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2$ equipped with the metric $\overline{g} = g_B \oplus {h_1}^2 g_1 \oplus {h_2}^2 g_2$, where $g_1 = \frac{g}{\varphi^2}$, g being the pseudo-Euclidean metric on $\mathbb{R}^{n_1}$ with respect to the co-ordinates $x = (x_1, x_2, ..., x_{n_1})$, $g_{ij} = \delta_{ij}\varepsilon_i$ and $\varphi : \mathbb{R}^{n_1} \to \mathbb{R}$ is a smooth function.

This literature deals with some investigations in the theory of general relativity with respect to the coordinate vanishing method in differential geometry. In this type of study a spacetime of general relativity is considered like a connected pseudo-Riemannian manifold of dimension four equipped with the Lorentzian metric g having signature (–, +, +, +). The field equation of Einstein [84] follows that the energy momentum tensor is of divergence free. If the energy momentum tensor is covariant constant then this demand is fulfilled. Chaki and Roy [25] had proved that a general relativistic spacetime admitting the covariant constant energy momentum tensor is Ricci symmetric. Many authors [57, 131, 89, 87] had studied spacetimes in different ways on different manifolds and different curvature tensors.

Definition 1.4.2 (Einstein spacetime)**.** *A spacetime is called an Einstein spacetime if the Ricci tensor S of type $(0,2)$ satisfies the relation $S = \frac{r}{n}g$, $n > 2$ on M where r is the scalar curvature of (M^n, g).*

Definition 1.4.3 (Spacetime with constant curvature)**.** *A spacetime is called a spacetime with constant curvature if the curvature tensor satisfies the relation $R(X,Y,Z,W) = k[g(X,Z)g(Y,W) - g(X,W)g(Y,Z)]$ on M for any $X,Y,Z,W \in \mathfrak{X}(M)$.*

Definition 1.4.4 (Killing vector field)**.** *The vector field ξ is said to be a Killing vector field if it satisfies the relation $\left(\pounds_\xi g\right)(X,Y) = 0$ where $X,Y \in \mathfrak{X}(M)$.*

Definition 1.4.5 (Conformal Killing vector field)**.** *The vector field ξ is said to be a conformal Killing vector field if it satisfies the relation $\left(\pounds_\xi g\right)(X,Y) = 2\phi g(X,Y)$ where $X,Y \in \mathfrak{X}(M)$ and ϕ is being a scalar.*

The aim of this doctoral thesis is to study on some warped product manifolds. The thesis consists of five chapters.

After this introductory chapter, the second chapter is devoted to study the geometry of pseudo-projective curvature tensor on warped product manifolds. This chapter is divided into five units. Firstly, there is an introductory part. The next unit "preliminaries" is to present some basic definitions and useful results on pseudo-projective curvature tensor and pseudo-Riemannian manifold briefly. Then in the third unit the nature of pseudo-projective curvature tensor on warped product manifolds in terms of its base and fiber manifolds has been investigated. Some interesting results describing the geometry of base and fiber manifolds for a pseudo-projectively flat warped product manifold are obtained as well. The last two units deal with the generalized Robertson-Walker space-times and standard static space-times admitting pseudo-projective curvature tensor respectively.

The third chapter is devoted to the study of biwarped product submanifolds in metallic Riemannian manifold and locally nearly metallic Riemannian manifold. The third chapter consists of eight units. After the "introduction" part, the "preliminaries" unit is given to recall some important results for further study. Then the third unit describes the nature of biwarped product generalized J-induced submanifold of first order. The fourth unit gives illustration to ensure the existence of biwarped product generalized J-induced submanifold of first order in metallic Riemannian manifold. Then we find out necessary and sufficient conditions for the biwarped product generalized J-induced submanifold of first order of type $M_T \times_f M_\perp \times_\sigma M_\theta$ to be locally trivial. The sixth unit establishes an inequality for the second fundamental form in metallic Riemannian manifold. Next biwarped product submanifolds of a locally nearly metallic Riemannian manifold has been studied. The eighth unit yields a sharp inequality for the second fundamental form in locally nearly metallic Riemannian manifold.

The fourth chapter is based on some spacetimes as an application of warped product manifolds. It contains fourteen sections. After the "introduction" part, there is "preliminaries" unit to remind some significant facts regarding this. Then the third section discusses the generalized Friedmann-Robertson-Walker spacetime in a new way. The fourth section represents some examples of generalized black hole solutions. The fifth section is focused on hyper-generalized quasi Einstein warped product spaces with non positive scalar curvature. Then consecutively four sections are used to investigate some geometric and physical properties of $(HGQE)_n$ manifolds. The tenth section illuminates the general relativistic viscous fluid $(HGQE)_4$ spacetimes with some physical applications. Then a non trivial example has been set up to ensure the existence of $(HGQE)_4$ spacetimes. Twelfth section deals with a spacetime admitting vanishing $\mathscr{T}$-curvature tensor. The last two sections convey the behaviour of general relativistic viscous fluid spacetime admitting vanishing and divergence free $\mathscr{T}$-curvature tensor respectively.

In the last chapter, we introduce a new notion of gradient h-almost η-Ricci soliton and study Riemann soliton in the frame of warped product Kenmotsu manifold. This chapter is divided into six units. The first one is introductory unit. Some basic definitions, ideas and results related to it belong to the preliminaries unit. Then Riemann soliton has been studied on warped product Kenmotsu manifold to deduce some conditions for its existence admitting W_2-curvature tensor, projective curvature tensor and Weyl-conformal curvature tensor. The fourth unit is added to ensure the existence of Riemann soliton on 5-dimensional warped product Kenmotsu manifold by constructing an example. In the fifth unit, Ricci soliton and gradient Ricci soliton have been discussed with pointwise bi-slant submanifolds of trans-Sasakian manifolds to establish that the pointwise bi-slant submanifolds of trans-Sasakian manifold is Einstein manifolds under certain conditions. The last unit is dealt with the existence of the gradient h-almost η-Ricci soliton warped product. The nature of h-almost η-Ricci soliton and gradient h-almost η-Ricci soliton has been investigated admitting a concurrent vector field.

CHAPTER 2

Pseudo-projective curvature tensor on warped product manifolds

2.1 Introduction

B. Prasad [100] developed the notion of pseudo-projective curvature tensor. It includes the projective curvature tensor. Many authors [45, 101, 81, 82] studied the pseudo-projective curvature tensor in different ways. It has been studied in mathematics as well as physics as a research topic. Shenawy and Ünal [111] studied the W_2-curvature tensor on warped product manifolds.

The second chapter is devoted to study the geometry of pseudo-projective curvature tensor on warped product manifolds. Moreover, this chapter discusses its applications in generalized Robertson-Walker space-times and standard static space-times respectively. The pseudo-projective curvature tensor provides a way to frame the main results on warped product manifolds in generalized Robertson-Walker space-times and standard static space-times respectively.

This chapter is divided into five units. Firstly, there is an introductory part. The next unit “preliminaries” is to present some basic definitions and useful results on pseudo-projective curvature tensor and pseudo-Riemannian manifold briefly. Then in the third unit the nature of pseudo-projective curvature tensor on warped product manifolds in terms of its base and fiber manifolds has been investigated. Some interesting results describing the geometry of base and fiber manifolds for a pseudo-projectively flat warped product manifold are obtained as well. The last two units deal with the generalized Robertson-Walker space-times and standard static space-times admitting pseudo-projective curvature tensor respectively.

2.2 Preliminaries

In this unit some basic ideas related to pseudo-projective curvature tensor and pseudo-Riemannian manifold have been highlighted shortly. B. Prasad defined the pseudo-projective curvature tensor as follows.

Definition 2.2.1 (Pseudo-projective curvature tensor)**.** *[100] The pseudo-projective curvature tensor $\bar{P}^*$ on a pseudo-Riemannian manifold is defined by*

$$\begin{aligned}\bar{P}^*(X,Y,Z,W) =& a_1\bar{R}(X,Y,Z,W)+a_2[S(Y,Z)g(X,W)-S(X,Z)g(Y,W)]\\ &-\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)], \quad (2.2.1)\end{aligned}$$

where a_1 and a_2 $(\neq 0)$ are two constants, S is the Ricci tensor of $(0,2)$-type, τ is the scalar curvature of the manifold, $\bar{P}^*(X,Y,Z,W)=g(P^*(X,Y)Z,W)$, $\bar{R}(X,Y,Z,W)=g(R(X,Y)Z,W)$ and R is the Riemannian curvature tensor.

If $a_1=1$ and $a_2=-\frac{1}{n-1}$, then (2.2.1) reduces to the projective curvature tensor. Moreover, if $P^*=0$ for $n>3$, then a pseudo-Riemannian manifold is called pseudo-projectively flat.

It clearly follows from (2.2.1) that

$$\begin{aligned}P^*(X,Y)Z =& a_1R(X,Y)Z+a_2\left[S(Y,Z)X-S(X,Z)Y\right]\\ &-\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\left[g(Y,Z)X-g(X,Z)Y\right]. \quad (2.2.2)\end{aligned}$$

Remark 2.2.2. *Suppose M is a pseudo-Riemannian manifold. Then*

$$P^*(X,Y)Z+P^*(Y,Z)X+P^*(Z,X)Y=0 \text{ for } X,Y,Z\in\mathfrak{X}(M).$$

Proposition 2.2.3. *Suppose M is a pseudo-Riemannian manifold. Then the pseudo-projective curvature tensor vanishes if and only if the tensor P^* vanishes.*

Definition 2.2.4 (Hessian type metric)**.** *A Riemannian metric g is said to be of Hessian type metric if $H^{f_1}=f_2g$ for any two smooth functions f_1 and f_2, where H^{f_1} denotes the Hessian of the function f_1.*

2.3 Pseudo-projective curvature tensor on warped product manifolds

This unit is to give a new concept of pseudo-projective curvature tensor on warped product manifolds. We consider the warped product $M=M_1\times_f M_2$ where $\dim(M)=n$, $\dim(M_1)=n_1$ and $\dim(M_2)=n_2$ such that $n=n_1+n_2$, $n_i\neq 1$ for $i=1,2$. We denote R, R^i as curvature tensor and S, S^i as Ricci tensor on M, M_i respectively. On the other hand, ∇f, Δf and H^f are respectively the gradient, Laplacian and Hessian of f on M_1. D, D^i indicate the Levi-Civita connection with respect to the metric g, g_i for $i=1$, 2 respectively. Throughout our entire study we use the relation $f^{\#}=\frac{\Delta f}{f}+\frac{n_2-1}{f^2}\|\nabla f\|^2$. Last of all, we denote the pseudo-projective curvature tensor on M and M_i by $\bar{P}^*$ and $\bar{P}_i^*$ respectively. We also indicate the tensor P^* on M and P_i^* on M_i respectively.

Now the following theorems have been proved for the pseudo-projective curvature tensor on warped product manifolds. These theorems describe the warped geometry in terms of its base and fiber manifolds.

Theorem 2.3.1. *Let $M=M_1\times_f M_2$ be a warped product manifold furnished with the metric $g=g_1\oplus f^2g_2$. If $X_i,Y_i,Z_i\in\mathfrak{X}(M_i)$ for $i=1,2$, then*

$$P^*(X_1,Y_1)Z_1 = P_1^*(X_1,Y_1)Z_1+\tau\left[\frac{n_2(n+n_1-1)}{nn_1(n-1)(n_1-1)}a_1+\frac{n_2}{nn_1}a_2\right]$$

$$
\begin{aligned}
&\times [g_1(Y_1,Z_1)X_1 - g_1(X_1,Z_1)Y_1] \\
&+ \frac{a_2 n_2}{f}\left[H^f(X_1,Z_1)Y_1 - H^f(Y_1,Z_1)X_1\right],
\end{aligned}
$$

$$
\begin{aligned}
P^*(X_1,Y_1)Z_2 =& P^*(X_2,Y_2)Z_1 = 0,\\
P^*(X_1,Y_2)Z_1 =& \left(\frac{a_2 n_2 - a_1}{f}\right) H^f(X_1,Z_1)Y_2 - a_2 S^1(X_1,Z_1)Y_2\\
&+ \frac{\tau}{n}\left(\frac{a_1}{n-1} + a_2\right) g_1(X_1,Z_1)Y_2,\\
P^*(X_1,Y_2)Z_2 =& a_1 f g_2(Y_2,Z_2) D^1_{X_1}\nabla f + a_2 S^2(Y_2,Z_2)X_1\\
&- f^2\left[a_2 f^{\#} + \frac{\tau}{n}\left(\frac{a_1}{n-1} + a_2\right)\right] g_2(Y_2,Z_2)X_1,\\
P^*(X_2,Y_2)Z_2 =& P_2^*(X_2,Y_2)Z_2 + \left[\left(\frac{n^2 - n - n_2^2 f^2 + n_2 f^2}{n n_2 (n-1)(n_2-1)}\right) a_1 \tau\right.\\
&\left. + \left(\frac{n - n_2 f^2}{n n_2}\right)\tau a_2 - a_2 f^2 f^{\#} + a_1 \|\nabla f\|^2\right]\\
&\times [g_2(Y_2,Z_2)X_2 - g_2(X_2,Z_2)Y_2].
\end{aligned}
$$

Proof. Let $M = M_1 \times_f M_2$ be a warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. Let $\dim(M) = n$, $\dim(M_i) = n_i$ for $i = 1,2$ and $n = n_1 + n_2$. Let $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$ for $i = 1,2$. Then, we obtain

$$
\begin{aligned}
P^*(X_1,Y_1)Z_1 =& a_1 R(X_1,Y_1)Z_1 + a_2 [S(Y_1,Z_1)X_1 - S(X_1,Z_1)Y_1]\\
&- \frac{\tau}{n}\left(\frac{a_1}{n-1} + a_2\right)[g(Y_1,Z_1)X_1 - g(X_1,Z_1)Y_1]\\
=& a_1 R^1(X_1,Y_1)Z_1 + a_2\left[\left\{S^1(Y_1,Z_1) - \frac{n_2}{f}H^f(Y_1,Z_1)\right\}X_1\right.\\
&\left. - \left\{S^1(X_1,Z_1) - \frac{n_2}{f}H^f(X_1,Z_1)\right\}Y_1\right]\\
&- \frac{\tau}{n}\left(\frac{a_1}{n-1} + a_2\right)[g_1(Y_1,Z_1)X_1 - g_1(X_1,Z_1)Y_1]\\
=& a_1 R^1(X_1,Y_1)Z_1 + a_2[S^1(Y_1,Z_1)X_1 - S^1(X_1,Z_1)Y_1]\\
&- \frac{\tau}{n_1}\left(\frac{a_1}{n_1-1} + a_2\right)[g_1(Y_1,Z_1)X_1 - g_1(X_1,Z_1)Y_1]\\
&+ \left[\frac{\tau}{n_1}\left(\frac{a_1}{n_1-1} + a_2\right) - \frac{\tau}{n}\left(\frac{a_1}{n-1} + a_2\right)\right]\\
&\times [g_1(Y_1,Z_1)X_1 - g_1(X_1,Z_1)Y_1]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{a_2 n_2}{f}\left[H^f(X_1,Z_1)Y_1 - H^f(Y_1,Z_1)X_1\right] \\
=&P_1^*(X_1,Y_1)Z_1 + \tau\left[\frac{n_2(n+n_1-1)}{nn_1(n-1)(n_1-1)}a_1 + \frac{n_2}{nn_1}a_2\right] \\
&\times\left[g_1(Y_1,Z_1)X_1 - g_1(X_1,Z_1)Y_1\right] \\
&+\frac{a_2 n_2}{f}\left[H^f(X_1,Z_1)Y_1 - H^f(Y_1,Z_1)X_1\right], \\
P^*(X_1,Y_1)Z_2 =&a_1 R(X_1,Y_1)Z_2 + a_2\left[S(Y_1,Z_2)X_1 - S(X_1,Z_2)Y_1\right] \\
&-\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\left[g(Y_1,Z_2)X_1 - g(X_1,Z_2)Y_1\right] \\
=&0, \\
P^*(X_1,Y_2)Z_1 =&a_1 R(X_1,Y_2)Z_1 + a_2\left[S(Y_2,Z_1)X_1 - S(X_1,Z_1)Y_2\right] \\
&-\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\left[g(Y_2,Z_1)X_1 - g(X_1,Z_1)Y_2\right] \\
=&-\left(\frac{a_1}{f}\right)H^f(X_1,Z_1)Y_2 - a_2\Bigg[S^1(X_1,Z_1)Y_2 \\
&-\frac{n_2}{f}H^f(X_1,Z_1)Y_2\Bigg] + \frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)g_1(X_1,Z_1)Y_2 \\
=&\left(\frac{a_2 n_2 - a_1}{f}\right)H^f(X_1,Z_1)Y_2 - a_2 S^1(X_1,Z_1)Y_2 \\
&+\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)g_1(X_1,Z_1)Y_2, \\
P^*(X_1,Y_2)Z_2 =&a_1 R(X_1,Y_2)Z_2 + a_2\left[S(Y_2,Z_2)X_1 - S(X_1,Z_2)Y_2\right] \\
&-\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\left[g(Y_2,Z_2)X_1 - g(X_1,Z_2)Y_2\right] \\
=&\left(\frac{a_1}{f}\right)g(Y_2,Z_2)D^1_{X_1}\nabla f + a_2\big[S^2(Y_2,Z_2)X_1 \\
&-f^{\#}g(Y_2,Z_2)X_1\big] - \frac{\tau f^2}{n}\left(\frac{a_1}{n-1}+a_2\right)g_2(Y_2,Z_2)X_1 \\
=&a_1 f g_2(Y_2,Z_2)D^1_{X_1}\nabla f + a_2 S^2(Y_2,Z_2)X_1 \\
&-f^2\left[a_2 f^{\#} + \frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\right]g_2(Y_2,Z_2)X_1, \\
P^*(X_2,Y_2)Z_1 =&a_1 R(X_2,Y_2)Z_1 + a_2\left[S(Y_2,Z_1)X_2 - S(X_2,Z_1)Y_2\right] \\
&-\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\left[g(Y_2,Z_1)X_2 - g(X_2,Z_1)Y_2\right] \\
=&0,
\end{aligned}
$$

$$
\begin{aligned}
P^*(X_2,Y_2)Z_2 =& a_1R(X_2,Y_2)Z_2+a_2\left[S(Y_2,Z_2)X_2-S(X_2,Z_2)Y_2\right]\\
&-\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\left[g(Y_2,Z_2)X_2-g(X_2,Z_2)Y_2\right]\\
=& a_1\left[R^2(X_2,Y_2)Z_2+\frac{\|\nabla f\|^2}{f^2}\{g(Y_2,Z_2)X_2-g(X_2,Z_2)Y_2\}\right]\\
&+a_2[\{S^2(Y_2,Z_2)X_2-f^{\#}g(Y_2,Z_2)X_2\}\\
&-\{S^2(X_2,Z_2)Y_2-f^{\#}g(X_2,Z_2)Y_2\}]\\
&-\frac{\tau f^2}{n}\left(\frac{a_1}{n-1}+a_2\right)\left[g_2(Y_2,Z_2)X_2-g_2(X_2,Z_2)Y_2\right]\\
=& a_1R^2(X_2,Y_2)Z_2+a_2\left[S^2(Y_2,Z_2)X_2-S^2(X_2,Z_2)Y_2\right]\\
&-\frac{\tau}{n_2}\left(\frac{a_1}{n_2-1}+a_2\right)\left[g_2(Y_2,Z_2)X_2-g_2(X_2,Z_2)Y_2\right]\\
&+\left[\frac{\tau}{n_2}\left(\frac{a_1}{n_2-1}+a_2\right)-\frac{\tau f^2}{n}\left(\frac{a_1}{n-1}+a_2\right)\right.\\
&\left.-a_2f^2f^{\#}+a_1\|\nabla f\|^2\right]\left[g_2(Y_2,Z_2)X_2-g_2(X_2,Z_2)Y_2\right]\\
=& P_2^*(X_2,Y_2)Z_2+\left[\left(\frac{n^2-n-n_2^2f^2+n_2f^2}{nn_2(n-1)(n_2-1)}\right)a_1\tau\right.\\
&\left.+\left(\frac{n-n_2f^2}{nn_2}\right)\tau a_2-a_2f^2f^{\#}+a_1\|\nabla f\|^2\right]\\
&\times\left[g_2(Y_2,Z_2)X_2-g_2(X_2,Z_2)Y_2\right].
\end{aligned}
$$

This completes the proof. □

Corollary 2.3.2. *Let $M=M_1\times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g=g_1\oplus f^2g_2$. Then*

$$
\begin{aligned}
\bar{P}_1^*(X_1,Y_1,Z_1,W_1)=&\tau\left[\frac{n_2(n+n_1-1)}{nn_1(n-1)(n_1-1)}a_1+\frac{n_2}{nn_1}a_2\right]\\
&\times\left[g_1(X_1,Z_1)g_1(Y_1,W_1)-g_1(Y_1,Z_1)g_1(X_1,W_1)\right]\\
&+\frac{a_2n_2}{f}\left[H^f(Y_1,Z_1)g_1(X_1,W_1)-H^f(X_1,Z_1)g_1(Y_1,W_1)\right],
\end{aligned}
$$

for $X_1,Y_1,Z_1,W_1\in\mathfrak{X}(M_1)$.

Proof. Let us assume that $M=M_1\times_f M_2$ be a pseudo-projectively flat warped product manifold. Therefore, in view of Theorem 2.3.1, we obtain

$$\begin{aligned}P_1^*(X_1,Y_1)Z_1 =&\tau\left[\frac{n_2(n+n_1-1)}{nn_1(n-1)(n_1-1)}a_1+\frac{n_2}{nn_1}a_2\right]\\&\times[g_1(X_1,Z_1)Y_1-g_1(Y_1,Z_1)X_1]\\&+\frac{a_2n_2}{f}\left[H^f(Y_1,Z_1)X_1-H^f(X_1,Z_1)Y_1\right].\end{aligned}$$

Therefore, we derive

$$\begin{aligned}\bar{P}_1^*(X_1,Y_1,Z_1,W_1) =&g_1\left(P_1^*(X_1,Y_1)Z_1,W_1\right)\\=&\tau\left[\frac{n_2(n+n_1-1)}{nn_1(n-1)(n_1-1)}a_1+\frac{n_2}{nn_1}a_2\right]\\&\times[g_1(X_1,Z_1)g_1(Y_1,W_1)-g_1(Y_1,Z_1)g_1(X_1,W_1)]\\&+\frac{a_2n_2}{f}\big[H^f(Y_1,Z_1)g_1(X_1,W_1)\\&-H^f(X_1,Z_1)g_1(Y_1,W_1)\big].\end{aligned}$$

This completes the proof. □

Corollary 2.3.3. *Let $M=M_1\times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g=g_1\oplus f^2g_2$. Then the base manifold M_1 is pseudo-projectively flat if and only if*

$$\begin{aligned}&\tau\left[\frac{n_2(n+n_1-1)}{nn_1(n-1)(n_1-1)}a_1+\frac{n_2}{nn_1}a_2\right]\\&\times[g_1(X_1,Z_1)g_1(Y_1,W_1)-g_1(Y_1,Z_1)g_1(X_1,W_1)]\\&+\frac{a_2n_2}{f}\left[H^f(Y_1,Z_1)g_1(X_1,W_1)-H^f(X_1,Z_1)g_1(Y_1,W_1)\right]=0,\end{aligned}$$

for $X_1,Y_1,Z_1,W_1\in\mathfrak{X}(M_1)$.

Proof. Let the base manifold M_1 be pseudo-projectively flat. Then

$$\bar{P}_1^*(X_1,Y_1,Z_1,W_1)=0.$$

Clearly, the proof follows from Corollary 2.3.2. □

Theorem 2.3.4. *Let $M=M_1\times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g=g_1\oplus f^2g_2$. Then the scalar curvature τ_1 of M_1 is given by*

$$\tau_1=\frac{1}{a_2}\left[\left(\frac{a_2n_2-a_1}{f}\right)\Delta f+\frac{\tau n_1}{n}\left(\frac{a_1}{n-1}+a_2\right)\right].$$

Proof. Let us assume that $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold. Then Theorem 2.3.1 implies that

$$S^1(X_1,Z_1) = \frac{1}{a_2}\left[\left(\frac{a_2n_2 - a_1}{f}\right)H^f(X_1,Z_1) + \frac{\tau}{n}\left(\frac{a_1}{n-1} + a_2\right)g_1(X_1,Z_1)\right].$$

Taking contraction over X_1 and Z_1, we gain

$$\tau_1 = \frac{1}{a_2}\left[\left(\frac{a_2n_2 - a_1}{f}\right)\Delta f + \frac{\tau n_1}{n}\left(\frac{a_1}{n-1} + a_2\right)\right].$$

This completes the proof. □

Remark 2.3.5. *Proposition 1.2.5 [41] and Theorem 2.3.4 jointly imply that the scalar curvature τ_2 of (M_2, g_2) is a constant since the left hand side of the equation in Theorem 2.3.4 depends only on the base manifold (M_1, g_1).*

Theorem 2.3.6. *Let $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. Then the pseudo-projective curvature tensor of M_2 is given by*

$$\begin{aligned}\bar{P}_2^*(X_2,Y_2,Z_2,W_2) = &\left[\left(\frac{n^2 - n - n_2^2f^2 + n_2f^2}{nn_2(n-1)(n_2-1)}\right)a_1\tau + \left(\frac{n - n_2f^2}{nn_2}\right)\tau a_2\right. \\ &\left. - a_2f^2f^\# + a_1\|\nabla f\|^2\right] \times [g_2(X_2,Z_2)g_2(Y_2,W_2) \\ &- g_2(Y_2,Z_2)g_2(X_2,W_2)],\end{aligned}$$

for $X_2, Y_2, Z_2, W_2 \in \mathfrak{X}(M_2)$.

Proof. Let $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold. From Theorem 2.3.1, it follows that

$$\begin{aligned}0 = &P_2^*(X_2,Y_2)Z_2 + \left[\left(\frac{n^2 - n - n_2^2f^2 + n_2f^2}{nn_2(n-1)(n_2-1)}\right)a_1\tau + \left(\frac{n - n_2f^2}{nn_2}\right)\tau a_2\right. \\ &\left. - a_2f^2f^\# + a_1\|\nabla f\|^2\right][g_2(Y_2,Z_2)X_2 - g_2(X_2,Z_2)Y_2].\end{aligned}$$

Therefore,

$$
\begin{aligned}
\bar{P}_2^*(X_2,Y_2,Z_2,W_2) =& g_2\left(P_2^*(X_2,Y_2)Z_2,W_2\right)\\
=& \Bigg[\left(\frac{n^2-n-n_2^2f^2+n_2f^2}{nn_2(n-1)(n_2-1)}\right)a_1\tau+\left(\frac{n-n_2f^2}{nn_2}\right)\tau a_2\\
&-a_2f^2f^\#+a_1\|\nabla f\|^2\Bigg][g_2(X_2,Z_2)g_2(Y_2,W_2)\\
&-g_2(Y_2,Z_2)g_2(X_2,W_2)].
\end{aligned}
$$

This completes the proof. □

Theorem 2.3.7. *Let $M=M_1\times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g=g_1\oplus f^2g_2$. If the fiber manifold M_2 is Ricci flat, then the base manifold M_1 is of Hessian type.*

Proof. Let $M=M_1\times_f M_2$ be a pseudo-projectively flat warped product manifold. Then from Theorem 2.3.1, we derive

$$
\begin{aligned}
0 =& a_1 f g_2(Y_2,Z_2)D^1_{X_1}\nabla f+a_2S^2(Y_2,Z_2)X_1\\
&-f^2\left[a_2f^\#+\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\right]g_2(Y_2,Z_2)X_1.
\end{aligned}
$$

Suppose that M_2 is Ricci flat. Then $S^2(X_2,Y_2)=0$ for any $X_2,Y_2\in\mathfrak{X}(M_2)$. Hence, we obtain from the above relation

$$
D^1_{X_1}\nabla f=\frac{f}{a_1}\left[a_2f^\#+\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\right]X_1.
$$

This implies that

$$
H^f=\frac{f}{a_1}\left[a_2f^\#+\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\right]g_1.
$$

Hence, M_1 is of Hessian type. This completes the proof. □

Theorem 2.3.8. *Let $M=M_1\times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g=g_1\oplus f^2g_2$. If the fiber manifold M_2 is Ricci flat, then the pointwise constant sectional curvature τ_2 of M_2 is given by*

$$
\begin{aligned}
\tau_2 =& \frac{1}{a_1}\Bigg[-\left(\frac{n^2-n-n_2^2f^2+n_2f^2}{nn_2(n-1)(n_2-1)}\right)a_1\tau-\left(\frac{n-n_2f^2}{nn_2}\right)\tau a_2+a_2f^2f^\#\\
&-a_1\|\nabla f\|^2+\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\Bigg].
\end{aligned}
$$

Proof. Let M_2 be Ricci flat. From (2.2.1), we have

$$\begin{aligned}\bar{R}^2(X_2,Y_2,Z_2,W_2) =& \frac{1}{a_1}\Bigg[\bar{P}_2^*(X_2,Y_2,Z_2,W_2) + \frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right) \\ & \times \{g_2(Y_2,Z_2)g_2(X_2,W_2) - g_2(X_2,Z_2)g_2(Y_2,W_2)\}\Bigg].\end{aligned}$$

In view of Theorem 2.3.1, we derive from the above relation that

$$\begin{aligned}\bar{R}^2(X_2,Y_2,Z_2,W_2) =& \frac{1}{a_1}\Bigg[-\left(\frac{n^2-n-n_2^2f^2+n_2f^2}{nn_2(n-1)(n_2-1)}\right)a_1\tau - \left(\frac{n-n_2f^2}{nn_2}\right)\tau a_2 \\ & + a_2f^2f^{\#} - a_1\|\nabla f\|^2 + \frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\Bigg] \\ & \times \{g_2(Y_2,Z_2)g_2(X_2,W_2) - g_2(X_2,Z_2)g_2(Y_2,W_2)\}.\end{aligned}$$

This implies that M_2 has a pointwise constant sectional curvature and this curvature is given by

$$\begin{aligned}\tau_2 =& \frac{1}{a_1}\Bigg[-\left(\frac{n^2-n-n_2^2f^2+n_2f^2}{nn_2(n-1)(n_2-1)}\right)a_1\tau - \left(\frac{n-n_2f^2}{nn_2}\right)\tau a_2 + a_2f^2f^{\#} \\ & - a_1\|\nabla f\|^2 + \frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\Bigg].\end{aligned}$$

This completes the proof. □

Theorem 2.3.9. *Let $M = M_1 \times_f M_2$ be a warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. If $H^f = 0$, $\Delta f = 0$ and M is pseudo-projectively flat, then M_2 is an Einstein manifold.*

Proof. Let M be pseudo-projectively flat. Therefore, M_1 is flat in view of Corollary 2.3.2. Furthermore, from Theorem 2.3.1, we obtain

$$\begin{aligned}0 =& a_1 f g_2(Y_2,Z_2) D^1_{X_1}\nabla f + a_2 S^2(Y_2,Z_2)X_1 \\ & - f^2\left[a_2 f^{\#} + \frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\right] g_2(Y_2,Z_2)X_1. \end{aligned} \tag{2.3.1}$$

Since $H^f(X_1,Y_1) = 0$ and $\Delta f = 0$. Therefore, we derive from (2.3.1) that

$$S^2(Y_2,Z_2) = \left[(n_2-1)\|\nabla f\|^2 + \frac{\tau f^2}{a_2 n}\left(\frac{a_1}{n-1}+a_2\right)\right] g_2(Y_2,Z_2).$$

This implies that M_2 is an Einstein manifold. This completes the proof. □

2.4 Pseudo-projective curvature tensor on generalized Robertson-Walker space-times

Let (M,g) be a Riemannian manifold of dimension n. The function $f : I \to (0,\infty)$ is a smooth function where I is a connected and open subinterval of $\mathbb{R}$. Then the warped product manifold $\breve{M} = I \times_f M$ of dimension $(n+1)$ equipped with the metric $\breve{g} = -dt^2 \oplus f^2 g$ is known as generalized Robertson-Walker space-time. Here dt^2 is the Euclidean metric on I. This structure is the generalization of Robertson-Walker space-times [53, 106, 107, 112]. We use ∂_t instead of $\frac{\partial}{\partial t} \in \mathfrak{X}(I)$ for simplicity in the following results.

With the help of Proposition 1.2.3, Proposition 1.2.4 and (2.2.2), the following theorems are obtained after some calculations.

Theorem 2.4.1. *Let $\breve{M} = I \times_f M$ be a generalized Robertson-Walker space-time furnished with the metric $\breve{g} = -dt^2 \oplus f^2 g$. Then for $X,Y,Z \in \mathfrak{X}(M)$ and $\partial_t \in \mathfrak{X}(I)$ the curvature tensor $\breve{P}^*$ on $\breve{M}$ is given by*

$$
\begin{aligned}
\breve{P}^*(\partial_t,\partial_t)\partial_t =& \breve{P}^*(\partial_t,\partial_t)X = \breve{P}^*(X,Y)\partial_t = 0,\\
\breve{P}^*(\partial_t,X)\partial_t =& \left[\left(\frac{na_2-a_1}{f}\right)\ddot{f} - \frac{\tau}{n+1}\left(\frac{a_1}{n}+a_2\right)\right]X,\\
\breve{P}^*(X,\partial_t)Y =& \Bigg[\Bigg\{-(a_1+a_2)f\ddot{f}-(n-1)a_2\dot{f}^2\\
&+\frac{\tau f^2}{n+1}\left(\frac{a_1}{n}+a_2\right)\Bigg\}g(X,Y)-a_2S(X,Y)\Bigg]\partial_t,\\
\breve{P}^*(X,Y)Z =& a_1R(X,Y)Z + a_2\left[S(Y,Z)X - S(X,Z)Y\right]\\
&+\left[-a_1\dot{f}^2 + a_2 f\ddot{f} + a_2(n-1)\dot{f}^2 - \frac{\tau f^2}{n+1}\left(\frac{a_1}{n}+a_2\right)\right]\\
&\times\left[g(Y,Z)X - g(X,Z)Y\right].
\end{aligned}
$$

Theorem 2.4.2. *Let $\breve{M} = I \times_f M$ be a generalized Robertson-Walker space-time furnished with the metric $\breve{g} = -dt^2 \oplus f^2 g$. If $\breve{M}$ is pseudo-projectively flat, then the* *warping function f is given by*

$$f = \begin{cases} c_1 e^{\mu t} + c_2 e^{-\mu t}, & \textit{if } \mu^2 > 0 \\ c_1 + c_2 t, & \textit{if } \mu^2 = 0 \\ c_1 \cos \mu t + c_2 \sin \mu t, & \textit{if } \mu^2 < 0 \end{cases}$$

where $\mu^2 = \frac{\tau(a_1+na_2)}{n(n+1)(na_2-a_1)}$ *and* c_1, c_2 *are two arbitrary constants.*

Proof. Let $\breve{M}$ be pseudo-projectively flat. Then from the second relation of Theorem 2.4.1, we have

$$\ddot{f} - \mu^2 f = 0.$$

Hence, by solving the above differential equation the warping function f is obtained and it is given by

$$f = \begin{cases} c_1 e^{\mu t} + c_2 e^{-\mu t}, & \text{if } \mu^2 > 0 \\ c_1 + c_2 t, & \text{if } \mu^2 = 0 \\ c_1 \cos \mu t + c_2 \sin \mu t, & \text{if } \mu^2 < 0 \end{cases}$$

where c_1, c_2 are two arbitrary constants. This completes the proof. □

Theorem 2.4.3. *Let $\breve{M} = I \times_f M$ be a generalized Robertson-Walker space-time furnished with the metric $\breve{g} = -dt^2 \oplus f^2 g$. If $\breve{M}$ is pseudo-projectively flat, then M is an Einstein manifold.*

Proof. Let $\breve{M}$ be pseudo-projectively flat. Then from the third relation of Theorem 2.4.1, we have

$$S(X,Y) = \frac{1}{a_2}\left[-(a_1+a_2) f\ddot{f} - (n-1)a_2 \dot{f}^2 + \frac{\tau f^2}{n+1}\left(\frac{a_1}{n} + a_2\right)\right] g(X,Y).$$

Hence, M is an Einstein manifold. This completes the proof. □

2.5 Pseudo-projective curvature tensor on standard static space-times

Let (M,g) be a Riemannian manifold of dimension n. The function $f : M \to (0,\infty)$ is a smooth function. Then the warped product manifold $\breve{M} = I \times_f M$ of dimension

$(n+1)$ equipped with the metric $\breve{g} = -f^2 dt^2 \oplus g$ is known as standard static space-time. Here I is the connected, open subinterval of $\mathbb{R}$ and dt^2 is the Euclidean metric on I. This structure is the generalization of Einstein static universe [1, 2, 3, 10]. We write ∂_t instead of $\frac{\partial}{\partial t} \in \mathfrak{X}(I)$ to express the following results in simpler way.

In view of Proposition 1.2.3, Proposition 1.2.4 and (2.2.2), the following theorems are obtained after some calculations.

Theorem 2.5.1. *Let $\breve{M} = I \times_f M$ be a standard static space-time furnished with the metric $\breve{g} = -f^2 dt^2 \oplus g$. Then the curvature tensor $\breve{P}^*$ on $\breve{M}$ is given by*

$$
\begin{aligned}
\breve{P}^*(\partial_t,\partial_t)\partial_t =& \breve{P}^*(\partial_t,\partial_t)X = \breve{P}^*(X,Y)\partial_t = 0,\\
\breve{P}^*(\partial_t,X)\partial_t =& f\left[a_1 D^1_X \nabla f - a_2 \Delta f X - \frac{\tau f}{n+1}\left(\frac{a_1}{n} + a_2\right) X\right],\\
\breve{P}^*(\partial_t,X)Y =& \left[\left(\frac{a_1 - a_2}{f}\right) H^f(X,Y) + a_2 S(X,Y)\right.\\
&\left. - \frac{\tau}{n+1}\left(\frac{a_1}{n} + a_2\right) g(X,Y)\right]\partial_t,\\
\breve{P}^*(X,Y)Z =& a_1 R(X,Y)Z + a_2\left[S(Y,Z)X - S(X,Z)Y\right]\\
& - \frac{a_2}{f}\left[H^f(Y,Z)X - H^f(X,Z)Y\right]\\
& - \frac{\tau}{n+1}\left(\frac{a_1}{n} + a_2\right)\left[g(Y,Z)X - g(X,Z)Y\right],
\end{aligned}
$$

for $X,Y,Z \in \mathfrak{X}(M)$ and $\partial_t \in \mathfrak{X}(I)$.

Theorem 2.5.2. *Let $\breve{M} = I \times_f M$ be a standard static space-time furnished with the metric $\breve{g} = -f^2 dt^2 \oplus g$. If $\breve{M}$ is pseudo-projectively flat, then $H^f = \frac{\Delta f}{n} g$.*

Proof. Let $\breve{M} = I \times_f M$ be pseudo-projectively flat. Then from the second relation of Theorem 2.5.1, we have

$$
\begin{aligned}
D^1_X \nabla f &= \frac{1}{a_1}\left[a_2 \Delta f + \frac{\tau f}{n+1}\left(\frac{a_1}{n} + a_2\right)\right] X\\
\text{i.e., } H^f &= \frac{1}{a_1}\left[a_2 \Delta f + \frac{\tau f}{n+1}\left(\frac{a_1}{n} + a_2\right)\right] g. \qquad (2.5.1)
\end{aligned}
$$

Taking trace on both sides, we obtain

$$
\Delta f = \frac{n f \tau}{(n+1)(a_1 - n a_2)}\left(\frac{a_1}{n} + a_2\right). \qquad (2.5.2)
$$

Using (2.5.2) in (2.5.1), we derive $H^f = \frac{\Delta f}{n} g$. $\square$

Theorem 2.5.3. *Let $\breve{M} = I \times_f M$ be a standard static space-time furnished with the metric $\breve{g} = -f^2 dt^2 \oplus g$. If $\breve{M}$ is pseudo-projectively flat, then M is an Einstein manifold.*

Proof. Let $\breve{M} = I \times_f M$ be pseudo-projectively flat. We derive from the third relation of Theorem 2.5.1 by using Theorem 2.5.2 and (2.5.2) that

$$S(X,Y) = \frac{(1-n)\Delta f}{nf} g(X,Y).$$

This implies that M is an Einstein manifold. This completes the proof. □

CHAPTER 3

Biwarped product submanifolds of some Riemannian manifolds

3.1 Introduction

Hretcanu et al. [67, 64] introduced the notion of metallic Riemannian manifolds and their submanifolds to generalize the golden Riemannian manifolds [37, 68]. They also added some important properties of invariant, anti-invariant, slant [69], hemi slant [66] and semi slant submanifolds [13] of golden and metallic Riemannian manifolds. They discussed some integrability conditions of some distributions involved in such types of submanifolds. Furthermore, they described some properties of golden and metallic Riemannian manifolds in [67, 12].

Two roots of the quadratic equation $x^2 - ax - b = 0$ are $\frac{a+\sqrt{a^2+4b}}{2}$ and $\frac{a-\sqrt{a^2+4b}}{2}$ where a and b are positive integers. Out of these two roots one is positive and the other is negative. The positive root $\lambda_{a,b} = \frac{a+\sqrt{a^2+4b}}{2}$ is called the metallic number [49]. Metallic structure [115, 55] is a special case of the polynomial structure.

Recently, Taştan [117] studied the biwarped product submanifolds in Kähler structure. Then biwarped product submanifolds have been studying in different kind of structures, for example in nearly Kaehlerian structures, see [124]. Motivated by these works [118, 88, 74], we wish to study biwarped product submanifolds in metallic Riemannian manifold and locally nearly metallic Riemannian manifold.

The third chapter consists of eight units. After the "introduction" part, the "preliminaries" unit is given to recall some important results for further study. Then the third unit describes the nature of biwarped product generalized J-induced submanifold of first order. The fourth unit gives illustration to ensure the existence of biwarped product generalized J-induced submanifold of first order in metallic Riemannian manifold. Then we find out a necessary and sufficient condition for the biwarped product generalized J-induced submanifold of first order of type $M_T \times_f M_\perp \times_\sigma M_\theta$ to be locally trivial. The sixth unit establishes an inequality for the second fundamental form in metallic Riemannian manifold. Next biwarped product submanifolds of a locally nearly metallic Riemannian manifold has been studied. The eighth unit yields a sharp inequality for the second fundamental form in locally nearly metallic Riemannian manifold.

3.2 Preliminaries

This unit is focused to present the concept and some significant results on submanifold of Riemannian manifold, metallic Riemannian manifold and locally nearly metallic Riemannian manifold respectively.

3.2.1 Submanifold of Riemannian manifold :

The geometry of submanifolds plays a very important role in differential geometry. Suppose M is an isometrically immersed submanifold in a Riemannian manifold $(\breve{M}, g)$. We consider $\breve{\nabla}$ is the Levi-Civita connection on $\breve{M}$ equipped with the metric g. The induced and induced normal connections of M are respectively ∇ and $\nabla^\perp$. Hence, $\forall X, Y \in TM$ and $\forall Z \in T^\perp M$, the Gauss and Weingarten formulas can be stated respectively as follows

$$\breve{\nabla}_X Y = \nabla_X Y + h(X,Y), \ \ \breve{\nabla}_X Z = -A_Z X + \nabla^{\perp}_X Z, \tag{3.2.1}$$

where TM and $T^{\perp}M$ are respectively the tangent and normal bundles of M in $\breve{M}$, the second fundamental form h and the shape operator A_Z satisfy

$$g(h(X,Y),Z) = g(A_Z X,Y). \tag{3.2.2}$$

Let H be the mean curvature of M. This H can be calculated from $H = \frac{\text{trace(h)}}{\text{dim}(M)}$. If $h = 0, H = 0$, then M is called totally geodesic and minimal in $\breve{M}$ respectively. On the other hand, M is said to be totally umbilical if $h(X,Y) = g(X,Y)H$; $\forall X,Y \in TM$. M is called spherical if $g(\breve{\nabla}_X H, Z) = 0$.

For any two distributions $\mathscr{D}^1$ and $\mathscr{D}^2$ of M, M is said to be $\mathscr{D}^1$-geodesic if $h(X,Y) = 0$, $\forall X,Y \in \mathscr{D}^1$ and $(\mathscr{D}^1, \mathscr{D}^2)$-mixed geodesic if $h(Y,W) = 0$, $\forall Y \in \mathscr{D}^1$ and $W \in \mathscr{D}^2$. $\mathscr{D}^1$ is said to be $\mathscr{D}^2$-parallel if $\nabla_W Y \in \mathscr{D}^1$, $\forall Y \in \mathscr{D}^1$ and $W \in \mathscr{D}^2$. When $\mathscr{D}^1$ is $\mathscr{D}^1$-parallel, then $\mathscr{D}^1$ is called auto parallel. By using the Gauss formula, we can conclude that M will be totally geodesic if M has an autoparallel distribution.

3.2.2 Submanifold of metallic Riemannian manifold :

Definition 3.2.1 (Metallic structure)**.** *Let $\breve{M}$ be a manifold of n-dimension furnished with a $(1,1)$-type tensor field J. J is said to be a metallic structure if*

$$J^2 = aJ + bI, \tag{3.2.3}$$

holds for J, where a,b are positive integers and I is the identity operator in $T\breve{M}$.

If $\forall X,Y \in T\breve{M}$, $g(JX,Y) = g(X,JY)$ holds for a Riemannian metric g in $\breve{M}$, then $(\breve{M}, J, g)$ is said to be a metallic Riemannian manifold. The metric g also satisfies

$$g(JX,JY) = g(J^2X,Y) = ag(JX,Y) + bg(X,Y), \ \ \forall X,Y \in T\breve{M}. \tag{3.2.4}$$

For the case of $a = b = 1$, we get the golden structure J that verifies

$$J^2 = J + I. \tag{3.2.5}$$

Definition 3.2.2 (Locally metallic Riemannian manifold)**.** *A metallic Riemannian manifold $(\breve{M}, J, g)$ is said to be locally metallic if J is parallel with respect to $\breve{\nabla}$, i.e.,*

$$(\breve{\nabla}_X J)Y = 0, \ \ \forall X, Y \in T\breve{M}. \tag{3.2.6}$$

Let M be an isometrically immersed submanifold in a metallic Riemannian manifold $(\breve{M}, J, g)$. M is said to be a pointwise submanifold [33, 50] if for any point $z \in M$, Wirtinger angle $\theta(Z)$ between JZ and tangent space T_zM of M at z is independent of the choice of the non zero vector $Z \in T_zM$. Here, θ can be considered as a function on M and it is known as the slant function. Now, M will be a proper pointwise slant submanifold if neither $\cos\theta(z) = 0$ nor $\sin\theta(z) = 0$ at any point $z \in M$. By decomposition, the tangent space $T_z\breve{M}$ of $\breve{M}$ at the point $z \in M$ can be expressed as a direct summand $T_z\breve{M} = T_zM \oplus T_z^{\perp}M$, $\forall z \in M$, where the normal space of M is $T_z^{\perp}M$ at the point z. Consider the differential i_* of an immersion $i : M \to \breve{M}$ defined by $g(X,Y) = \breve{g}(i_*X, i_*Y)$, $\forall X, Y \in TM$.

Suppose that $TZ = (JZ)^T$ and $PZ = (JZ)^{\perp}$ are respectively the tangential and normal components of JZ, for $Z \in TM$ and $tW = (JW)^T$ and $pW = (JW)^{\perp}$ are respectively the tangential and normal components of JW, for $W \in T^{\perp}M$. Hence, we gain

$$JZ = TZ + PZ, \ JW = tW + pW, \ \forall Z \in TM, \forall W \in T^{\perp}M \tag{3.2.7}$$

Therefore, M is a pointwise slant submanifold of $\breve{M}$ if and only if

$$T^2X = \cos^2\theta(aT + bI)X, \ \forall X \in TM. \tag{3.2.8}$$

Also, we obtain

$$tPX = \sin^2\theta(aT + bI)X, \forall X \in TM. \tag{3.2.9}$$

Two maps T and p are g-symmetric. i.e.,

$$g(TX,Y) = g(X,TY), \ \forall X, Y \in TM \tag{3.2.10}$$

$$g(pV,W) = g(V,pW), \ \forall V, W \in T^{\perp}M \tag{3.2.11}$$

$$g(PX,V) = g(X,tV), \ \forall X \in TM, \ \forall V \in T^{\perp}M. \tag{3.2.12}$$

We also get the following relations [65] as well

$$T^2X = aTX + bX - tPX,\ aPX = PTX + pPX, \tag{3.2.13}$$

$$p^2V = apV + bV - PtV,\ atV = TtV + tpV, \tag{3.2.14}$$

for $X \in TM,\ V \in T^{\perp}M$.

In view of (3.2.11) and (3.2.13) and metallic structure, one can get the following relations

$$g(TX,TY) = \cos^2\theta[ag(TX,Y) + bg(X,Y)], \tag{3.2.15}$$

$$g(PX,PY) = \sin^2\theta[ag(TX,Y) + bg(X,Y)], \tag{3.2.16}$$

for $X,Y \in TM$.

Definition 3.2.3 (Slant submanifold)**.** *Let M be a pointwise slant submanifold of a metallic Riemannian manifold $(\breve{M},J,g)$ with respect to the slant function θ. M is said to be a slant submanifold [30] if θ is a constant function.*

Definition 3.2.4 (Holomorphic submanifold)**.** *M is said to be a holomorphic submanifold of $\breve{M}$ [128] if $\theta = 0$. For this case, T_zM is invariant with the metallic structure J at any point $z \in M$, i. e., $J(T_zM) \subseteq T_zM$.*

Definition 3.2.5 (Totally real submanifold)**.** *M is said to be a totally real submanifold of $\breve{M}$ [128] if $\theta = \frac{\pi}{2}$. In this case, T_zM is anti-invariant with the metallic structure J at any point $z \in M$, i. e., $J(T_zM) \subseteq T_z^{\perp}M$.*

3.2.3 Submanifold of a locally nearly metallic Riemannian manifold :

Definition 3.2.6 (Locally nearly metallic Riemannian manifold)**.** *A differentiable manifold N_k of even dimensional furnished by Riemannian metric g and metallic structure J is said to be a locally nearly metallic Riemannian manifold denoted by $(\bar{M},J,g)$ if*

$$g(JX,JY) = ag(JX,Y) + bg(X,Y),\ g(JX,Y) = g(X,JY),$$
$$(\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X = 0, \tag{3.2.17}$$

for all $X,Y \in \Gamma(TN_k)$ and a,b are positive integers.

If we consider $a = b = 1$ in (3.2.17), then the manifold N_k becomes a locally nearly golden Riemannian manifold.

Let M be a submanifold of dimension n of an almost Hermitian manifold $\bar{M}$ of dimension $2m$. We consider a local orthonormal frame field $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m}\}$ restricted to M, $e_1, ..., e_n$ and $e_{n+1}, ..., e_{2m}$ are respectively tangent and normal to M. Let h^r_{ij}, $1 \leq i, j \leq n$, $n+1 \leq r \leq 2m$ be the coefficients of the second fundamental form h in view of the local frame field. Hence, we obtain

$$h^r_{ij} = g(h(e_i, e_j), e_r) = g(A_{e_r} e_i, e_j), \ \|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \quad (3.2.18)$$

3.3 Biwarped product generalized J-induced submanifold of metallic Riemannian manifold

Let $(\breve{M}, J, g)$ be a metallic Riemannian manifold and M be its submanifold. Then, for each $z \in M$ and $X, Y \in T_z M$, we obtain by using (3.2.8) and (3.2.11)

$$g(TX, Y) = g(X, TY). \quad (3.3.1)$$

Therefore, it also implies that

$$g(T^2 X, Y) = g(T^2 Y, X). \quad (3.3.2)$$

Clearly, it is seen from (3.3.1) and (3.3.2) that the operators T and T^2 are both symmetric operator in $T_z M$ for each $z \in M$.

Definition 3.3.1 (Generalized J-induced submanifold)**.** *[103, 117, 129] Let $(\breve{M}, J, g)$ be a metallic Riemannian manifold and M be its submanifold. Then we say that M is a generalized J-induced submanifold if the tangent bundle TM of M has the following form*

$$TM = \mathscr{D}^T \oplus \mathscr{D}^\perp \oplus \mathscr{D}^{\theta_1} \oplus ... \oplus \mathscr{D}^{\theta_s},$$

where $\mathscr{D}^T$ and $\mathscr{D}^\perp$ are respectively holomorphic and totally real. $\mathscr{D}^{\theta_i}$ are pointwise distribution in M and all $\mathscr{D}^{\theta_i}$ are different for $i \in \{1,\ 2,\ ...\ ,\ s\}$.

As a special case of it i.e., for $s=1$, we can state the following.

Definition 3.3.2. *Let $(\breve{M},J,g)$ be a metallic Riemannian manifold and M be its submanifold. Then we say that M is a biwarped product generalized J-induced submanifold of first order if the tangent bundle TM of M has the following form*

$$TM=\mathscr{D}^T\oplus\mathscr{D}^\perp\oplus\mathscr{D}^\theta, \tag{3.3.3}$$

where $\mathscr{D}^T$ and $\mathscr{D}^\perp$ are respectively holomorphic and totally real. $\mathscr{D}^\theta$ is a pointwise slant distribution in M.

In this regard, the normal bundle $T^\perp M$ of M can be decomposed as follows.

$$T^\perp M=J(\mathscr{D}^\perp)\oplus P(\mathscr{D}^\theta)\oplus\bar{\mathscr{D}}^T, \tag{3.3.4}$$

$\bar{\mathscr{D}}^T$ is a orthogonal complementary distribution of $J(\mathscr{D}^\perp)\oplus P(\mathscr{D}^\theta)$ on $T^\perp M$. This is also an invariant subbundle of $T^\perp M$ with J.

A generalized J-induced submanifold of first order is said to be proper if $\mathscr{D}^T\neq\{0\}$, $\mathscr{D}^\perp\neq\{0\}$ and $\theta\in(0,\frac{\pi}{2})$.

For our study, we state and prove the following two lemmas.

Lemma 3.3.3. *Let $(\breve{M},J,g)$ be a metallic Riemannian manifold and M be a biwarped product generalized J-induced submanifold of first order. Then, we obtain*

$$b\sin^2\theta g(\nabla_Y Z,U)=g(a\cos^2\theta A_{TU}Z+A_{PTU}Z+A_{PU}JZ-aA_UJZ,Y), \tag{3.3.5}$$

$$b\sin^2\theta g(\nabla_U V,Z)=g(aA_VJZ-a\cos^2\theta A_{TV}Z-A_{PTV}Z-A_{PV}JZ,U), \tag{3.3.6}$$

where $Y,Z\in\mathscr{D}^T$ and $U,V\in\mathscr{D}^\theta$.

Proof. With the help of (3.2.4), (3.2.8), (3.2.10), (3.2.11) and (3.2.13), we gain

$$\begin{aligned}g(\nabla_Y Z,U)=&\frac{1}{b}[g(\breve{\nabla}_Y JZ,JU)-ag(\breve{\nabla}_Y JZ,U)]\\=&\frac{1}{b}[g(\breve{\nabla}_Y JZ,TU)+g(\breve{\nabla}_Y JZ,PU)-ag(\breve{\nabla}_Y JZ,U)]\\=&\frac{1}{b}[g(\breve{\nabla}_Y Z,T^2U+PTU)+g(\breve{\nabla}_Y JZ,PU)-ag(\breve{\nabla}_Y JZ,U)]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{b}[g(\breve{\nabla}_Y Z, \cos^2\theta(aT+bI)U) + g(\breve{\nabla}_Y Z, PTU) + g(\breve{\nabla}_Y JZ, PU) \\
&\quad - ag(\breve{\nabla}_Y JZ, U)] \\
&= \cos^2\theta g(\nabla_Y Z, U) + \frac{1}{b}[(a\cos^2\theta)g(\breve{\nabla}_Y Z, TU) \\
&\quad + g(\breve{\nabla}_Y Z, PTU) + g(\breve{\nabla}_Y JZ, PU) - ag(\breve{\nabla}_Y JZ, U)]
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
\sin^2\theta g(\nabla_Y Z, U) = &\frac{1}{b}[(a\cos^2\theta)g(A_{TU}Z, Y) + g(A_{PTU}Z, Y) \\
&+ g(A_{PU}JZ, Y) - ag(A_U JZ, Y)]
\end{aligned}
$$

This implies (3.3.5).

Now, we prove (3.3.6). With the help of (3.2.4), (3.2.8), (3.2.10), (3.2.11) and (3.2.13), we get

$$
\begin{aligned}
g(\nabla_U V, Z) &= \frac{1}{b}[g(\breve{\nabla}_U JV, JZ) - ag(\breve{\nabla}_U JV, Z)] \\
&= \frac{1}{b}[-g(\breve{\nabla}_U JZ, TV) - g(\breve{\nabla}_U JZ, PV) + ag(\breve{\nabla}_U JZ, V)] \\
&= \frac{1}{b}[-g(\breve{\nabla}_U Z, T^2V + PTV) - g(\breve{\nabla}_U JZ, PV) + ag(\breve{\nabla}_U JZ, V)] \\
&= \frac{1}{b}[-g(\breve{\nabla}_U Z, \cos^2\theta(aT+bI)V) - g(\breve{\nabla}_U Z, PTV) \\
&\quad - g(\breve{\nabla}_U JZ, PV) + ag(\breve{\nabla}_U JZ, V)] \\
&= \cos^2\theta g(\nabla_U V, Z) + \frac{1}{b}[a\cos^2\theta g(\breve{\nabla}_Z TV, U) \\
&\quad + g(\breve{\nabla}_Z PTV, U) + g(\breve{\nabla}_{JZ} PV, U) - ag(\breve{\nabla}_{JZ} V, U)]
\end{aligned}
$$

That is,

$$
\begin{aligned}
\sin^2\theta g(\nabla_U V, Z) = &\frac{1}{b}[-a\cos^2\theta g(A_{TV}Z, U) - g(A_{PTV}Z, U) \\
&- g(A_{PV}JZ, U) + ag(A_V JZ, U)]
\end{aligned}
$$

This follows (3.3.6). □

Lemma 3.3.4. *Let $(\breve{M}, J, g)$ be a metallic Riemannian manifold and M be a bi-warped product generalized J-induced submanifold of first order. Then, we get*

$$
\begin{aligned}
g(\nabla_Y Z, X) =& \frac{1}{b}[g(A_{JX}Y - aA_X Y, JZ)], && (3.3.7)\\
g(\nabla_Y X, U) =& \frac{\sec^2\theta}{b}[g(aA_{JX}U - A_{PTU}X - A_{JX}TU \\
& - aA_X PU - a\sin^2\theta A_X TU, Y)], && (3.3.8)\\
g(\nabla_X W, Y) =& \frac{1}{b}[g(aA_W X - A_{JW}X, JY)], && (3.3.9)\\
g(\nabla_U V, W) =& \frac{\sec^2\theta}{b}[g(A_{JW}TV + a\sin^2\theta A_W TV \\
& + aA_W PV + A_{PTV}W + aA_{JV}W, U)], && (3.3.10)\\
g(\nabla_U W, Y) =& \frac{1}{b}[g(A_{JW}JY - aA_Y JW, U)], && (3.3.11)\\
g(\nabla_X W, U) =& -\frac{\sec^2\theta}{b}[g(A_{JW}X + a\sin^2\theta A_W X, TU) \\
& + g(aA_W X, PU) + g(A_{PTU}X + aA_{JX}U, W)] && (3.3.12)\\
g(\nabla_X Y, U) =& \frac{\csc^2\theta}{b}[g(a\cos^2\theta A_{TU}Y - aA_U JY \\
& + A_{PTU}Y + A_{PU}JY, X)], && (3.3.13)
\end{aligned}
$$

where $Y, Z \in \mathscr{D}^T$, $X, W \in \mathscr{D}^\perp$ and $U, V \in \mathscr{D}^\theta$.

Proof. For proof see [105]. □

With the help of Lemma 3.3.3 and Lemma 3.3.4, we obtain the following two theorems.

Theorem 3.3.5. *Let $(\breve{M}, J, g)$ be a metallic Riemannian manifold and M be a bi-warped product generalized J-induced submanifold of first order. Then, the holomorphic distribution $\mathscr{D}^T$ will be totally geodesic if and only if*

$$
\begin{aligned}
& g(A_{JX}Y - aA_X Y, JZ) = 0, && (3.3.14)\\
& g(a\cos^2\theta A_{TU}Z + A_{PTU}Z + A_{PU}JZ - aA_U JZ, Y) = 0, && (3.3.15)
\end{aligned}
$$

where $Y, Z \in \mathscr{D}^T$, $X \in \mathscr{D}^\perp$ and $U \in \mathscr{D}^\theta$.

Proof. Let $(\breve{M}, J, g)$ be a metallic Riemannian manifold and M be a biwarped product generalized J-induced submanifold of first order. We know that a holomorphic distribution $\mathscr{D}^T$ is totally geodesic if and only if $g(\nabla_Y Z, X) = 0$ and $g(\nabla_Y Z, U) = 0$, where $Y, Z \in \mathscr{D}^T$, $X \in \mathscr{D}^\perp$ and $U \in \mathscr{D}^\theta$. Thus, in view of (3.3.7) and (3.3.5), the proof is complete. $\square$

Theorem 3.3.6. *Let $(\breve{M}, J, g)$ be a metallic Riemannian manifold and M be a biwarped product generalized J-induced submanifold of first order. Then, the pointwise slant distribution $\mathscr{D}^\theta$ will be integrable if and only if*

$$\begin{aligned} &g(aA_U JZ - a\cos^2\theta A_{TU} Z - A_{PTU} Z - A_{PU} JZ, V) \\ =&g(aA_V JZ - a\cos^2\theta A_{TV} Z - A_{PTV} Z - A_{PV} JZ, U), \end{aligned} \tag{3.3.16}$$

$$\begin{aligned} &g(A_{JX} TV + a\sin^2\theta A_X TV + aA_X PV + A_{PTV} X + aA_{JV} X, U) \\ =&g(A_{JX} TU + a\sin^2\theta A_X TU + aA_X PU + A_{PTU} X + aA_{JU} X, V), \end{aligned} \tag{3.3.17}$$

where $Z \in \mathscr{D}^T$, $X \in \mathscr{D}^\perp$ and $U, V \in \mathscr{D}^\theta$.

Proof. Let $(\breve{M}, J, g)$ be a metallic Riemannian manifold and M be a biwarped product generalized J-induced submanifold of first order. We know that a pointwise slant distribution $\mathscr{D}^\theta$ is integrable if and only if $g([U,V], Z) = 0$ and $g([V,U], X) = 0$, where $Z \in \mathscr{D}^T$, $X \in \mathscr{D}^\perp$ and $U, V \in \mathscr{D}^\theta$. Thus, in view of (3.3.6) and (3.3.10), the proof is complete. $\square$

Remark 3.3.7. *[121] The totally real distribution $\mathscr{D}^\perp$ is always integrable.*

3.4 Example of first order biwarped product generalized J-induced submanifold of metallic Riemannian manifold

Let us consider a metallic Riemannian manifold $\mathbb{R}^{12}$ with respect to the metallic structure $J : \mathbb{R}^{12} \to \mathbb{R}^{12}$ defined by

$$\begin{aligned} &J(W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_8, W_9, W_{10}, W_{11}, W_{12}) \\ =&(\lambda W_1, \bar{\lambda} W_2, \lambda W_3, \bar{\lambda} W_4, \lambda W_5, \bar{\lambda} W_6, \lambda W_7, \bar{\lambda} W_8, \lambda W_9, \bar{\lambda} W_{10}, \lambda W_{11}, \bar{\lambda} W_{12}), \end{aligned}$$

where $\lambda = \lambda_{a,b} = \frac{a+\sqrt{a^2+4b}}{2}$ is the metalic number, a and b are positive integers and $\bar{\lambda} = a - \lambda$.

Let us consider a submanifold M in $\mathbb{R}^{12}$ with $(w_1, w_2, ..., w_{12})$ as natural coordinates of $\mathbb{R}^{12}$, where $w_1, w_2, ...\ , w_{12}$ are given by

$$w_1 = y\sin u,\ w_2 = z\sin u,\ w_3 = y\sin v,\ w_4 = z\sin v,$$
$$w_5 = y\cos x,\ w_6 = z\cos x,\ w_7 = y\sin x,\ w_8 = z\sin x,$$
$$w_9 = y\cos u,\ w_{10} = z\cos u,\ w_{11} = y\cos v,\ w_{12} = z\cos v,$$

where $y, z \neq 0, 1$ and $x, u, v \in (0, \frac{\pi}{2})$.

Now, the local frame of the tangent bundle TM of M are generated by

$$\begin{aligned}
Y =& \sin u\frac{\partial}{\partial w_1} + \sin v\frac{\partial}{\partial w_3} + \cos x\frac{\partial}{\partial w_5} + \sin x\frac{\partial}{\partial w_7}, \\
&+ \cos u\frac{\partial}{\partial w_9} + \cos v\frac{\partial}{\partial w_{11}} \\
Z =& \sin u\frac{\partial}{\partial w_2} + \sin v\frac{\partial}{\partial w_4} + \cos x\frac{\partial}{\partial w_6} + \sin x\frac{\partial}{\partial w_8}, \\
&+ \cos u\frac{\partial}{\partial w_{10}} + \cos v\frac{\partial}{\partial w_{12}} \\
X =& -y\sin x\frac{\partial}{\partial w_5} - z\sin x\frac{\partial}{\partial w_6} + y\cos x\frac{\partial}{\partial w_7} + z\cos x\frac{\partial}{\partial w_8}, \\
U =& y\cos u\frac{\partial}{\partial w_1} + z\cos u\frac{\partial}{\partial w_2} - y\sin u\frac{\partial}{\partial w_9} - z\sin u\frac{\partial}{\partial w_{10}}, \\
V =& y\cos v\frac{\partial}{\partial w_3} + z\cos v\frac{\partial}{\partial w_4} - y\sin v\frac{\partial}{\partial w_{11}} - z\sin v\frac{\partial}{\partial w_{12}},
\end{aligned}$$

Clearly, J satisfies $J^2W = (aJ + bI)W$ and $g(JW, L) = g(W, JL)$, for all $W, L \in \mathbb{R}^{12}$. We also get,

$$\begin{aligned}
&JU = \lambda y\cos u\frac{\partial}{\partial w_1} + \bar{\lambda} z\cos u\frac{\partial}{\partial w_2} - \lambda y\sin u\frac{\partial}{\partial w_9} - \bar{\lambda} z\sin u\frac{\partial}{\partial w_{10}}, \\
&JV = \lambda y\cos v\frac{\partial}{\partial w_3} + \bar{\lambda} z\cos v\frac{\partial}{\partial w_4} - \lambda y\sin v\frac{\partial}{\partial w_{11}} - \bar{\lambda} z\sin v\frac{\partial}{\partial w_{12}}, \\
&g(JU, U) = \lambda y^2\cos^2 u + \bar{\lambda} z^2\cos^2 u + \lambda y^2\sin^2 u + \bar{\lambda} z^2\sin^2 u, \\
&g(JV, V) = \lambda y^2\cos^2 v + \bar{\lambda} z^2\cos^2 v + \lambda y^2\sin^2 v + \bar{\lambda} z^2\sin^2 v, \\
&\|Y\| = \|Z\| = \sqrt{3},\ \|X\| = \|U\| = \|V\| = \sqrt{y^2 + z^2},
\end{aligned}$$

Thus $\mathscr{D}^T = \text{span}\{Y,Z\}$, $\mathscr{D}^\perp = \text{span}\{X\}$ and $\mathscr{D}^\theta = \text{span}\{U,V\}$ are respectively a holomorphic, totally real and proper pointwise slant distribution with respect to the slant function

$$\begin{aligned}\theta =& \cos^{-1}\left(\frac{g(JU,U)}{\|U\|\,\|JU\|}\right) = \cos^{-1}\left(\frac{g(JV,V)}{\|V\|\,\|JV\|}\right)\\ =& \cos^{-1}\left(\frac{\lambda y^2+\bar{\lambda}z^2}{\sqrt{y^2+z^2}\sqrt{\lambda^2y^2+\bar{\lambda}^2z^2}}\right).\end{aligned}$$

Consequently, M is a biwarped product generalized J-induced submanifold of first order in the metallic Riemannian manifold $(\mathbb{R}^{12},J,g)$. It is clearly seen that $\mathscr{D}^T$ is totally geodesic and $\mathscr{D}^\perp$ and $\mathscr{D}^\theta$ are integrable. We denote the integral submanifolds of $\mathscr{D}^T,\mathscr{D}^\perp$ and $\mathscr{D}^\theta$ by $M_T,M_\perp$ and M_θ respectively. Hence, the induced metric tensor of M is given by

$$\begin{aligned}ds^2 =& 3(dy^2+dz^2)+(y^2+z^2)dx^2+(y^2+z^2)(du^2+dv^2).\\ =& g_{M_T}+(y^2+z^2)g_{M_\perp}+(y^2+z^2)g_{M_\theta}\end{aligned}$$

Therefore, $M = M_T \times_f M_\perp \times_\sigma M_\theta$ is an example of a non trivial biwarped product generalized J-induced submanifold of first order in the metallic Riemannian manifold $(\mathbb{R}^{12},J,g)$, where two warping functions are respectively $f = \sqrt{y^2+z^2}$ and $\sigma = \sqrt{y^2+z^2}$.

3.5 Biwarped product generalized J-induced submanifold of metallic Riemannian manifold of type $M_T \times_f M_\perp \times_\sigma M_\theta$

In this section we give a necessary and sufficient condition for the biwarped product generalized J-induced submanifold of first order of type $M_T \times_f M_\perp \times_\sigma M_\theta$ to be locally trivial.

Definition 3.5.1. *[40] If the tangent bundle TM of M can be expressed as an orthogonal sum $TM = \mathscr{D}_0 \oplus \mathscr{D}_1 \oplus \ldots \oplus \mathscr{D}_s$, where each $\mathscr{D}_i$ is non trivial, spherical and* *its complement in TM is autoparallel for $i \in \{1,2,...,s\}$, then M is isometric to a multiply warped product in the form $M_0 \times_{f_1} M_1 \times_{f_2} ... \times_{f_s} M_s$.*

Now we prove a very interesting theorem of this section on biwarped product generalized J-induced submanifold of first order of type $M_T \times_f M_\perp \times_\sigma M_\theta$.

Theorem 3.5.2. *$(\breve{M},J,g)$ be a metallic Riemannian manifold and M be a biwarped product generalized J-induced submanifold of first order. Then, M is a locally biwarped submanifold in the form $M_T \times_f M_\perp \times_\sigma M_\theta$ if and only if*

$$A_{JX}Z - aA_XZ = -JZ(\eta)X, \tag{3.5.1}$$

$$aA_UJZ - a\cos^2\theta A_{TU}Z - A_{PTU}Z - A_{PU}JZ = b\sin^2\theta Z(\omega)U, \tag{3.5.2}$$

where $X(\eta) = U(\eta) = 0$ and $X(\omega) = U(\omega) = 0$, and

$$\begin{aligned} &g(A_{JW}X + a\sin^2\theta A_WX, TU) + g(aA_WX, PU) \\ &+g(A_{PTU}X + aA_{JX}U, W) = 0, \end{aligned} \tag{3.5.3}$$

$$g(A_{JW}TV + a\sin^2\theta A_WTV + aA_WPV + A_{PTV}W + aA_{JV}W, U) = 0, \tag{3.5.4}$$

where $Z \in \mathscr{D}^T$, $X,W \in \mathscr{D}^\perp$ and $U,V \in \mathscr{D}^\theta$.

Proof. $(\breve{M},J,g)$ be a metallic Riemannian manifold and M be a biwarped product generalized J-induced submanifold of first order of type $M_T \times_f M_\perp \times_\sigma M_\theta$. Now, for $Z \in \mathscr{D}^T$, $X \in \mathscr{D}^\perp$, $U,V \in \mathscr{D}^\theta$ and using (3.2.4), (3.2.5), (3.2.8) and (3.2.10), we obtain

$$\begin{aligned} g(A_{JX}Z - aA_XZ, Y) &= -g(\breve{\nabla}_Y JX, Z) + ag(\breve{\nabla}_Y X, Z) \\ &= -g(\breve{\nabla}_Y X, JZ) + ag(\breve{\nabla}_Y X, Z) \\ &= -g(\nabla_Y X, JZ) + ag(\nabla_Y X, Z). \end{aligned}$$

It is known from (3.2.2) that $\nabla_Y X = Y(\ln f)X$. Hence, we have

$$\begin{aligned} g(A_{JX}Z - aA_XZ, Y) &= -g(\nabla_Y X, JZ) + ag(\nabla_Y X, Z) \\ &= -Y(\ln f)g(X, JZ) + aY(\ln f)g(X, Z) \\ &= 0, \end{aligned} \tag{3.5.5}$$

since $g(X,JZ)=g(X,Z)=0$.

By a similar manner, we also have

$$\begin{aligned} g(A_{JX}Z-aA_XZ,U)=&-g(\breve{\nabla}_U JX,Z)+ag(\breve{\nabla}_U X,Z)\\ =&-g(\breve{\nabla}_U X,JZ)+ag(\breve{\nabla}_U X,Z)\\ =&-g(\nabla_U X,JZ)+ag(\nabla_U X,JZ). \end{aligned}$$

From (3.2.3), we see that $\nabla_U X=0$. Therefore, we get

$$g(A_{JX}Z-aA_XZ,U)=0. \tag{3.5.6}$$

Similarly, we have

$$\begin{aligned} g(A_{JX}Z-aA_XZ,W)=&-g(\breve{\nabla}_W JX,Z)+ag(\breve{\nabla}_W X,Z)\\ =&-g(\breve{\nabla}_W X,JZ)+ag(\breve{\nabla}_X Z,W)\\ =&-g(\nabla_{JZ} X,W)+ag(\nabla_X Z,W)\\ =&-g(\nabla_{JZ} X,W), \end{aligned}$$

since $\nabla_X Z=0$. In view of (3.2.2), we see that $\nabla_{JZ}X=JZ(\ln f)X$. Therefore, we get

$$g(A_{JX}Z-aA_XZ,W)=g(-JZ(\ln f)X,W). \tag{3.5.7}$$

Since f is only depending on points of M_T, therefore, $X(\ln f)=U(\ln f)=0$. Hence, we can say that $\eta=\ln f$. In view of (3.5.5), (3.5.6) and (3.5.7), it implies (3.5.1).

With the help of (3.2.4), (3.2.5), (3.2.8), (3.2.10), (3.2.11) and (3.2.13), we obtain

$$\begin{aligned} &g(aA_U JZ-a\cos^2\theta A_{TU}Z-A_{PTU}Z-A_{PU}JZ,Y)\\ =&ag(A_U JZ,Y)-a\cos^2\theta g(A_{TU}Z,Y)-g(A_{PTU}Z,Y)-g(A_{PU}JZ,Y)\\ =&ag(h(JZ,Y),U)-a\cos^2\theta g(h(Z,Y),TU)-g(h(Z,Y),PTU)\\ &-g(h(JZ,Y),PU)\\ =&ag(\breve{\nabla}_{JZ}Y,U)-a\cos^2\theta g(\breve{\nabla}_Z Y,TU)+g(\breve{\nabla}_Z PTU,Y)+g(\breve{\nabla}_{JZ}PU,Y)\\ =&aJZ(\ln\sigma)g(Y,U)-a\cos^2\theta Z(\ln\sigma)g(Y,TU)+g(\breve{\nabla}_Z JTU-T^2U,Y)\\ &+g(\breve{\nabla}_{JZ}(JU-TU),Y) \end{aligned}$$

$$
\begin{aligned}
&=g(\breve{\nabla}_Z JTU, Y) - g(\breve{\nabla}_Z T^2 U, Y) + g(\breve{\nabla}_{JZ} JU, Y) - g(\breve{\nabla}_{JZ} TU, Y) \\
&=g(\breve{\nabla}_Z TU, JY) - g(\breve{\nabla}_Z \cos^2\theta(aT + bI)U, Y) + g(\breve{\nabla}_{JZ} U, JY) - g(\breve{\nabla}_{JZ} TU, Y) \\
&=Z(\ln\sigma)g(TU, JY) - g(a\cos^2\theta\breve{\nabla}_Z TU + Z(a\cos^2\theta)TU + b\cos^2\theta\breve{\nabla}_Z U \\
&\quad + Z(b\cos^2\theta)U, Y) + JZ(\ln\sigma)g(U, JY) - JZ(\ln\sigma)g(TU, JY) \\
&= -a\cos^2\theta g(\breve{\nabla}_Z TU, Y) - Z(a\cos^2\theta)g(TU, Y) \\
&\quad - b\cos^2\theta g(\breve{\nabla}_Z U, Y) - Z(b\cos^2\theta)g(U, Y) \\
&= -a\cos^2\theta Z(\ln\sigma)g(TU, Y) - b\cos^2\theta Z(\ln\sigma)g(U, Y) \\
&=0,
\end{aligned}
$$

since $g(TU, JY) = g(U, Y) = g(U, JY) = g(TU, Y) = 0$. So, we obtain

$$
g(aA_U JZ - a\cos^2\theta A_{TU} Z - A_{PTU} Z - A_{PU} JZ, Y) = 0. \tag{3.5.8}
$$

By a similar manner, we also have

$$
\begin{aligned}
&g(aA_U JZ - a\cos^2\theta A_{TU} Z - A_{PTU} Z - A_{PU} JZ, X) \\
&=ag(A_U JZ, X) - a\cos^2\theta g(A_{TU} Z, X) - g(A_{PTU} Z, X) - g(A_{PU} JZ, X) \\
&=ag(h(JZ, X), U) - a\cos^2\theta g(h(Z, X), TU) - g(h(Z, X), PTU) \\
&\quad - g(h(JZ, X), PU) \\
&=ag(\breve{\nabla}_{JZ} X, U) - a\cos^2\theta g(\breve{\nabla}_Z X, TU) + g(\breve{\nabla}_X PTU, Z) + g(\breve{\nabla}_X PU, JZ) \\
&=g(\breve{\nabla}_X (JTU - T^2 U), Z) + g(\breve{\nabla}_X (JU - TU), JZ) \\
&=g(\breve{\nabla}_X JTU, Z) - g(\breve{\nabla}_X T^2 U, Z) + g(\breve{\nabla}_X JU, JZ) - g(\breve{\nabla}_X TU, JZ) \\
&=g(\breve{\nabla}_X TU, JZ) - g(\breve{\nabla}_X \cos^2\theta(aT + bI)U, Z) + ag(\breve{\nabla}_X JU, Z) \\
&\quad + bg(\breve{\nabla}_X U, Z) - g(\breve{\nabla}_X TU, JZ) \\
&=g(\breve{\nabla}_X TU, JZ) - g(a\cos^2\theta\breve{\nabla}_X TU + X(a\cos^2\theta)TU + b\cos^2\theta\breve{\nabla}_X U \\
&\quad + X(b\cos^2\theta)U, Z) + ag(\breve{\nabla}_X JU, Z) + bg(\breve{\nabla}_X U, Z) - g(\breve{\nabla}_X TU, JZ) \\
&= -a\cos^2\theta g(\breve{\nabla}_X TU, Z) - X(a\cos^2\theta)g(TU, Z) - b\cos^2\theta g(\breve{\nabla}_X U, Z) \\
&\quad - X(b\cos^2\theta)g(U, Z) + ag(\breve{\nabla}_X JU, Z) + bg(\breve{\nabla}_X U, Z) \\
&=0,
\end{aligned}
$$

since $\nabla_Z TU = \nabla_Z U = \nabla_X JU = \nabla_{JZ} X = \nabla_X U = \nabla_Z X = 0$. Hence, we obtain

$$g(aA_U JZ - a\cos^2\theta A_{TU} Z - A_{PTU} Z - A_{PU} JZ, X) = 0. \tag{3.5.9}$$

With the help of (3.3.6), it follows that

$$\begin{aligned}
&g(aA_U JZ - a\cos^2\theta A_{TU} Z - A_{PTU} Z - A_{PU} JZ, V)\\
=&b\sin^2\theta g(\nabla_V U, Z)\\
=&b\sin^2\theta g(\nabla_Z U, V)\\
=&g(b\sin^2\theta Z(\ln\sigma)U, V).
\end{aligned} \tag{3.5.10}$$

Since σ is only depending on points of M_T, therefore, $X(\ln\sigma) = U(\ln\sigma) = 0$. Hence, we can say that $\omega = \ln\sigma$. In view of (3.5.8), (3.5.9) and (3.5.10), it implies (3.5.2).

From (3.3.12) and (3.2.3), it follows that

$$\begin{aligned}
&g(A_{JW} X + a\sin^2\theta A_W X, TU) + g(aA_W X, PU) + g(A_{PTU} X + aA_{JX} U, W)\\
=&-b\cos^2\theta g(\nabla_X W, U)\\
=&b\cos^2\theta g(\nabla_X U, W)\\
=&0.
\end{aligned}$$

Therefore, (3.5.3) follows.

From (3.3.10) and (3.2.3), it follows that

$$\begin{aligned}
&g(A_{JW} TV + a\sin^2\theta A_W TV + aA_W PV + A_{PTV} W + aA_{JV} W, U)\\
=&b\cos^2\theta g(\nabla_U V, W)\\
=&-b\cos^2\theta g(\nabla_U W, V)\\
=&0
\end{aligned}$$

Hence, (3.5.4) follows.

For the converse part, let $(\breve{M}, J, g)$ be a metallic Riemannian manifold and M be a bi-warped product generalized J-induced submanifold of first order satisfying (3.5.1), (3.5.2), (3.5.3) and (3.5.4). (3.3.14) and (3.3.15) are satisfied respectively with respect to the (3.5.1) and (3.5.2). Therefore, by Theorem 3.3.5, the holomorphic distribution $\mathscr{D}^T$ is totally geodesic and hence it is integrable. (3.3.15) and (3.3.16) are satisfied respectively with respect to the (3.5.3) and (3.5.4). Therefore, by Theorem 3.3.6, the pointwise slant distribution $\mathscr{D}^\theta$ is integrable. The totally real distribution $\mathscr{D}^\perp$ is always integrable by Remark 3.3.7. We consider the integral manifolds M_T, $M_\perp$ and M_θ of $\mathscr{D}^T$, $\mathscr{D}^\perp$ and $\mathscr{D}^\theta$ respectively. Let $h^\perp$ be the second fundamental form of $M_\perp$ in M. From (3.2.4), (3.3.12) and (3.5.3), we obtain for $X, W \in \mathscr{D}^\perp$ and $U \in \mathscr{D}^\theta$

$$g(h^\perp(X,W),U) = g(\nabla_X W,U) = 0. \tag{3.5.11}$$

For all $X, W \in \mathscr{D}^\perp$ and $Z \in \mathscr{D}^T$, from (3.2.4), (3.3.9) and (3.5.1), we obtain

$$g(h^\perp(X,W),Z) = g(\nabla_X W,Z) = -\frac{1}{b}[A_{JW}X - aA_W X, JZ] = -Z(\eta)g(X,W).$$

After some steps, we have

$$g(h^\perp(X,W),Z) = g(-g(X,W)\nabla\eta, W), \tag{3.5.12}$$

whereas $\nabla\eta = \text{grad}(\eta)$. From (3.5.11) and (3.5.12), we see that

$$h^\perp(X,W) = -g(X,W)\nabla\eta.$$

Therefore, $M_\perp$ is totally umbilic in M with mean curvature $-\nabla\eta$. Now, we prove that $-\nabla\eta$ is parallel. For this we are to show $g(\nabla_X\nabla\eta, E) = 0$ for $X \in \mathscr{D}^\perp$ and $E \in (\mathscr{D}^\perp)^\perp = \mathscr{D}^T \oplus \mathscr{D}^\theta$. Thus, we can write $E = Z + U$, whereas $Z \in \mathscr{D}^T$ and $U \in \mathscr{D}^\theta$. So, we obtain

$$\begin{aligned}
g(\nabla_X\nabla\eta, E) =& Xg(\nabla\eta, E) - g(\nabla\eta, \nabla_X E)\\
=& X(E(\eta)) - [X,E]\eta - g(\nabla\eta, \nabla_E X)\\
=& [X,E]\eta + E(X(\eta)) - [X,E]\eta - g(\nabla\eta, \nabla_E X)\\
=& -g(\nabla\eta, \nabla_Z X) - g(\nabla\eta, \nabla_U X),
\end{aligned}$$

since $X(\eta) = 0$. Since M_T is totally geodesic in M, so $g(\nabla_Z X, Y) = -g(\nabla_Z Y, X)$ $= 0$ for all $Y \in \mathscr{D}^T$. Therefore, either $\nabla_Z X \in \mathscr{D}^\perp$ or $\nabla_Z X \in \mathscr{D}^\theta$. For both cases

$$g(\nabla\eta, \nabla_Z X) = 0. \tag{3.5.13}$$

From (3.3.13) and (3.5.2), we obtain $g(\nabla_U X, Y) = 0$. Hence, either $\nabla_U X \in \mathscr{D}^\perp$ or $\nabla_U X \in \mathscr{D}^\theta$. For both cases, we deduce

$$g(\nabla \eta, \nabla_U X) = 0. \tag{3.5.14}$$

From (3.5.14) and (3.5.15), we see

$$g(\nabla_X \nabla \eta, E) = 0.$$

Hence, $M_\perp$ is spherical as it is totally umbilic. So, $\mathscr{D}^\perp$ is spherical.

Now, we wish to show that $\mathscr{D}^\theta$ is spherical. Let h^θ be the second fundamental form of M_θ in M. From (3.2.4), (3.3.10) and (3.5.4), we obtain for $U, V \in \mathscr{D}^\theta$ and $X \in \mathscr{D}^\perp$

$$g(h^\theta(U,V), X) = g(\nabla_U V, X) = 0. \tag{3.5.15}$$

From (3.2.4) and (3.3.6), we obtain for all $Z \in \mathscr{D}^T$

$$\begin{aligned} g(h^\theta(U,V), Z) =& g(\nabla_U V, Z) \\ =& \frac{\csc^2\theta}{b} g(aA_V JZ - a\cos^2\theta A_{TV} Z - A_{PTV} Z - A_{PV} JZ, U). \end{aligned}$$

From (3.5.2), we get

$$g(h^\theta(U,V), Z) = b\sin^2\theta Z(\omega) g(U,V).$$

After simplification, we have

$$g(h^\theta(U,V), Z) = g(g(U,V)(b\sin^2\theta)\nabla\omega, Z), \tag{3.5.16}$$

where $\nabla\omega = \mathrm{grad}(\omega)$. From (3.5.16) and (3.5.17), we gain

$$g(h^\theta(U,V), Z) = g(U,V) b\sin^2\theta \nabla\omega,$$

Thus, M_θ is totally umbilic in M with mean curvature $b\sin^2\theta\nabla\omega$. Now, we prove that $b\sin^2\theta\nabla\omega$ is parallel. So, we are to satisfy that $g(\nabla_U(b\sin^2\theta\nabla\omega), E) = 0$ for all $U \in \mathscr{D}^\theta$ and $E \in (\mathscr{D}^\theta)^\perp = \mathscr{D}^T \oplus \mathscr{D}^\perp$. Thus, we can write $E = Z + X$ for all $Z \in \mathscr{D}^T$ and $X \in \mathscr{D}^\perp$.

$$\begin{aligned}
g(\nabla_U(b\sin^2\theta\nabla\omega),E) =& b\sin^2\theta g(\nabla_U\nabla\omega,E) + g(U(b\sin^2\theta)\nabla\omega,E)\\
=& b\sin^2\theta\{Ug(\nabla\omega,E) - g(\nabla\omega,\nabla_U E)\}\\
=& b\sin^2\theta\{U(E(\omega)) - [U,E]\omega - g(\nabla\omega,\nabla_E U)\}\\
=& b\sin^2\theta\{[U,E]\eta + E(U(\omega)) - [U,E]\omega - g(\nabla\omega,\nabla_E U)\}\\
=& b\sin^2\theta\{-g(\nabla\omega,\nabla_Z U) - g(\nabla\eta,\nabla_X U)\},
\end{aligned}$$

since $U(\omega) = 0$.

From (3.3.13) and (3.5.2), it implies that $g(\nabla_X U,Y) = 0$. Hence, either $\nabla_X U \in \mathscr{D}^\perp$ or $\nabla_X U \in \mathscr{D}^\theta$. Thus,

$$g(\nabla\omega,\nabla_X U) = 0, \tag{3.5.17}$$

since $\nabla\omega \in \mathscr{D}^\perp$. Since M_T is totally geodesic in M, so

$$g(\nabla_Z U,Y) = -g(\nabla_Z Y,U) = 0.$$

Therefore, either $\nabla_Z U \in \mathscr{D}^T$ or $\nabla_Z U \in \mathscr{D}^\theta$. Hence, we deduce

$$g(\nabla\omega,\nabla_Z U) = 0. \tag{3.5.18}$$

From (3.5.18) and (3.5.19), we obtain

$$g(\nabla_U(b\sin^2\theta\nabla\omega),E) = 0.$$

Finally, we show that $(\mathscr{D}^\perp)^\perp = \mathscr{D}^T \oplus \mathscr{D}^\theta$ and $(\mathscr{D}^\theta)^\perp = \mathscr{D}^T \oplus \mathscr{D}^\perp$ are auto parallel. Clearly, $\mathscr{D}^T \oplus \mathscr{D}^\theta$ will be auto parallel iff $\nabla_Y Z$, $\nabla_Y U$, $\nabla_U Y$ and $\nabla_U V$ belong to $\mathscr{D}^T \oplus \mathscr{D}^\theta$ for all $Y,Z \in \mathscr{D}^T$ and $U,V \in \mathscr{D}^\theta$. That is $g(\nabla_Y Z,X)$, $g(\nabla_Y U,X)$, $g(\nabla_U Y,X)$ and $g(\nabla_U V,X)$ vanish for $X \in \mathscr{D}^\perp$. From (3.3.7) and (3.5.1), it follows that

$$g(\nabla_Y Z,X) = g(\nabla_U Y,X) = 0.$$

From (3.3.8), (3.3.10) and (3.5.3), it implies that

$$g(\nabla_Y U, X) = g(\nabla_U V, X) = 0.$$

Hence, $\mathscr{D}^T \oplus \mathscr{D}^\theta$ is auto parallel.

Now, $\mathscr{D}^T \oplus \mathscr{D}^\perp$ will be auto parallel iff $g(\nabla_Y Z, U)$, $g(\nabla_Y X, U)$, $g(\nabla_X Y, U)$ and $g(\nabla_X W, U)$ vanish for $Y, Z \in \mathscr{D}^T$, $X, W \in \mathscr{D}^\perp$ and $U \in \mathscr{D}^\theta$. At first, from above we have $g(\nabla_Y X, U) = 0$. From (3.3.5), (3.3.13) and (3.5.2), we obtain

$$g(\nabla_Y Z, U) = g(\nabla_X Y, U) = 0.$$

From (3.3.12) and (3.5.3), we see

$$g(\nabla_X W, U) = 0.$$

Hence, $\mathscr{D}^T \oplus \mathscr{D}^\perp$ is auto parallel. So, by Definition 3.5.1, M becomes a locally biwarped product submanifold in the form $M_T \times_f M_\perp \times_\sigma M_\theta$. $\square$

Now we prove the following Lemmas to establish the Theorem 3.5.5.

Lemma 3.5.3. *Let $(\breve{M}, J, g)$ be a metallic Riemannian manifold and M be a biwarped product generalized J-induced submanifold of first order of type $M_T \times_f M_\perp \times_\sigma M_\theta$. Then, we obtain*

$$g(h(Y,Z), JX) = 0, \tag{3.5.19}$$

$$g(h(Z,X), JW) = -JZ(\ln f) g(X,W), \tag{3.5.20}$$

$$g(h(Z,U), JX) = 0, \tag{3.5.21}$$

where h is the second fundamental form of M in $\breve{M}$ and $Y, Z \in \mathscr{D}^T$, $X, W \in \mathscr{D}^\perp$ and $U \in \mathscr{D}^\theta$.

Proof. From (3.2.4), (3.2.8) and (3.2.10), it follows that

$$g(h(Y,Z), JX) = g(\breve{\nabla}_Y Z, JX) = -g(\breve{\nabla}_Y JX, Z) = -g(\breve{\nabla}_Y X, JZ),$$

where $Y, Z \in \mathscr{D}^T$ and $X \in \mathscr{D}^\perp$. By using (3.2.4), it implies that $g(h(Y,Z), JX) = g(\nabla_Y X, JZ)$. Also, from (3.2.2), it is known that $\nabla_Y X = Y(\ln f) X$. Hence, we gain $g(h(Y,Z),JX) = Y(\ln f)g(X,JZ) = 0$, since $g(X,JZ) = 0$. Hence, (3.5.19) follows.

From (3.2.4), (3.2.8) and (3.2.10), it follows that

$$g(h(Z,X),JW) = g(\breve{\nabla}_X JZ,W) = -g(\nabla_X JZ,W),$$

for $Z \in \mathscr{D}^T$ and $X,W \in \mathscr{D}^\perp$. Also, from (3.2.2), we see that $\nabla_X JZ = JZ(\ln f)X$. Hence, we obtain

$$g(h(Z,X),JW) = -g(JZ(\ln f)X,W) = -JZ(\ln f)g(X,W),$$

Thus, (3.5.20) follows.

Similarly, (3.5.21) can be proved. □

Lemma 3.5.4. *Let $(\breve{M},J,g)$ be a metallic Riemannian manifold and M be a bi-warped product generalized J-induced submanifold of first order of type $M_T \times_f M_\perp \times_\sigma M_\theta$. Then, we obtain*

$$g(h(Y,Z),PU) = 0, \tag{3.5.22}$$

$$g(h(Z,X),PU) = 0, \tag{3.5.23}$$

$$g(h(Z,U),PV) = -JZ(\ln\sigma)g(U,V) + Z(\ln\sigma)g(U,TV), \tag{3.5.24}$$

where h is the second fundamental form of M in $\breve{M}$ and $Y,Z \in \mathscr{D}^T$, $X \in \mathscr{D}^\perp$ and $U,V \in \mathscr{D}^\theta$.

Proof. From (3.2.4), (3.2.8) and (3.2.10), it follows that

$$g(h(Y,Z),PU) = g(\breve{\nabla}_Y Z,PU) = g(\breve{\nabla}_Y Z,JU) - g(\breve{\nabla}_Y Z,TU),$$

where $Y,Z \in \mathscr{D}^T$ and $U \in \mathscr{D}^\theta$.

After some steps, we have

$$g(h(Y,Z),PU) = g(\nabla_Y U,JZ) - g(\nabla_Y TU,Z).$$

From (3.2.2), we see that

$$\nabla_Y U = Y(\ln\sigma)U, \ \ \nabla_Y TU = Y(\ln\sigma)TU.$$

Therefore, we have

$$\begin{aligned}g(h(Y,Z),PU) =& g(Y(\ln\sigma)U,JZ) - g(Y(\ln\sigma)TU,Z)\\ =& Y(\ln\sigma)g(U,JZ) - Y(\ln\sigma)g(TU,Z\\ =& 0, \quad \text{since } g(U,JZ) = g(TU,Z) = 0.\end{aligned}$$

Therefore, (3.5.22) follows. Similarly, (3.5.23) can be proved.

From (3.2.4), (3.2.8) and (3.2.10), it follows that

$$g(h(Z,U),PV) = -g(\nabla_{JZ}U,V) + g(\nabla_Z U,TV).$$

From (3.2.2), we see that

$$\nabla_{JZ}U = JZ(\ln\sigma)U, \;\; \nabla_Z U = Z(\ln\sigma)U.$$

Hence, we obtain

$$\begin{aligned}g(h(Z,U),PV) =& -g(JZ(\ln\sigma)U,V) + g(Z(\ln\sigma)U,TV)\\ =& -JZ(\ln\sigma)g(U,V) + Z(\ln\sigma)g(U,TV).\end{aligned}$$

Thus, (3.5.24) follows. □

Theorem 3.5.5. *Let $(\breve{M},J,g)$ be a metallic Riemannian manifold and M be a bi-warped product generalized J-induced submanifold of first order of type $M_T \times_f M_\perp \times_\sigma M_\theta$ such that invariant normal subbundle $\bar{\mathscr{D}} = \{0\}$. Then, M will be locally trivial iff M is $(\mathscr{D}^T,\mathscr{D}^\perp)$ and $(\mathscr{D}^T,\mathscr{D}^\theta)$-mixed geodesic.*

Proof. Let $(\breve{M},J,g)$ be a metallic Riemannian manifold and M be a biwarped product generalized J-induced submanifold of first order of type $M_T \times_f M_\perp \times_\sigma M_\theta$ such that invariant normal subbundle $\bar{\mathscr{D}} = \{0\}$. If M becomes locally trivial, then f and σ are constants. Since $JZ(\ln f) = 0$, so using (3.5.20), we obtain $g(h(Z,X),JW) = 0$ for $Z \in \mathscr{D}^T$ and $X,W \in \mathscr{D}^\perp$. From (3.3.4) and (3.5.23) of Lemma 3.5.4, it implies that $h(Z,X) = 0$. Thus, M is $(\mathscr{D}^T,\mathscr{D}^\perp)$-mixed geodesic.

Since $JZ(\ln\sigma) = 0$ and $Z(\ln\sigma) = 0$, so using (3.5.24) of Lemma 3.5.4, we obtain $g(h(Z,U),PV) = 0$ for $Z \in \mathscr{D}^T$ and $U,V \in \mathscr{D}^\theta$. From (3.3.4) and (3.5.21), it implies that $h(Z,X) = 0$. Consequently, M is $(\mathscr{D}^T,\mathscr{D}^\theta)$-mixed geodesic.

For the converse part, let M be $(\mathscr{D}^T, \mathscr{D}^\perp)$ and $(\mathscr{D}^T, \mathscr{D}^\theta)$-mixed geodesic. Since M is $(\mathscr{D}^T, \mathscr{D}^\perp)$-mixed geodesic, so using (3.5.20), we obtain $JZ(\ln f) = 0$ for $Z \in \mathscr{D}^T$. This implies f is constant. Since M is $(\mathscr{D}^T, \mathscr{D}^\theta)$-mixed geodesic, so using (3.5.24), we obtain for $Z \in \mathscr{D}^T$ and $U, V \in \mathscr{D}^\theta$

$$-JZ(\ln\sigma)g(U,V) + Z(\ln\sigma)g(U,TV) = 0. \tag{3.5.25}$$

Putting $Z = JZ$ in (3.5.25), we have

$$\begin{aligned} &-J^2Z(\ln\sigma)g(U,V) + JZ(\ln\sigma)g(U,TV) = 0.\\ i.e., \ &-aJZ(\ln\sigma)g(U,V) - bZ(\ln\sigma)g(U,V) + JZ(\ln\sigma)g(U,TV)\\ &= 0. \end{aligned} \tag{3.5.26}$$

Putting $V = TV$ in (3.5.26) and using (3.2.13) and (3.5.25) we have

$$\begin{aligned} &-aJZ(\ln\sigma)g(U,TV) - bZ(\ln\sigma)g(U,TV) + JZ(\ln\sigma)g(U,T^2V) = 0.\\ i.e., \ &-aJZ(\ln\sigma)g(U,TV) - bZ(\ln\sigma)g(U,TV)\\ &+ JZ(\ln\sigma)g(U,\cos^2\theta(aT+bI)V) = 0.\\ i.e., \ &-aJZ(\ln\sigma)g(U,TV) - bZ(\ln\sigma)g(U,TV) + a\cos^2\theta JZ(\ln\sigma)g(U,TV)\\ &+ b\cos^2\theta JZ(\ln\sigma)g(U,V) = 0.\\ i.e., \ &-a\sin^2\theta JZ(\ln\sigma)g(U,TV) - b\sin^2\theta JZ(\ln\sigma)g(U,V) = 0.\\ i.e., \ &\sin^2\theta[aJZ(\ln\sigma)g(U,TV) + bJZ(\ln\sigma)g(U,V)] = 0. \end{aligned} \tag{3.5.27}$$

As M is proper, $\sin\theta \neq 0$. Hence, from (3.5.27) it follows that $JZ(\ln\sigma) = 0$. This implies that σ is constant. Consequently, M is locally trivial since f and σ are constants. This completes the proof. $\square$

Remark 3.5.6. *From Theorem 3.5.5, we can conclude that a proper biwarped product generalized J-induced submanifold of first order of type $M_T \times_f M_\perp \times_\sigma M_\theta$ of a metallic Riemannian manifold is neither $(\mathscr{D}^T, \mathscr{D}^\perp)$-mixed geodesic nor $(\mathscr{D}^T, \mathscr{D}^\theta)$-mixed geodesic.*

3.6 An inequality for the second fundamental form in metallic Riemannian manifold

In this section, we set up an inequality for the second fundamental form for the biwarped product generalized J-induced submanifold of first order of type $M_T \times_f M_\perp \times_\sigma M_\theta$, where M_T, $M_\perp$ and M_θ are respectively a holomorphic, totally real and pointwise slant submanifolds of a metallic Riemannian manifold $(\breve{M}, J, g)$.

Let $(\breve{M}, J, g)$ be a metallic Riemannian manifold and M be a biwarped product generalized J-induced submanifold of first order in the form $M_T \times_f M_\perp \times_\sigma M_\theta$ of dimension $(k+n+m)$. We consider an orthogonal basis $\{e_1, ..., e_k, \tilde{e}_1, ..., \tilde{e}_n, \bar{e}_1, ..., \bar{e}_m, e_1^*, ..., e_m^*, J\tilde{e}_1, ..., J\tilde{e}_n, \hat{e}_1, ..., \hat{e}_l\}$ of $\breve{M}$ such that $g(J\tilde{e}_i, \tilde{e}_j) = 0$ for $i \neq j$, where $\{e_1, ..., e_k\}$ is an orthonormal basis of $\mathscr{D}^T$, $\{\tilde{e}_1, ..., \tilde{e}_n\}$ is an orthonormal basis of $\mathscr{D}^\perp$, $\{\bar{e}_1, ..., \bar{e}_m\}$ is an orthonormal basis of $\mathscr{D}^\theta$, $\{J\tilde{e}_1, ..., J\tilde{e}_n\}$ is an orthogonal basis of $J\mathscr{D}^\perp$, $\{e_1^*, ..., e_m^*\}$ is an orthonormal basis of $P\mathscr{D}^\theta$ and $\{\hat{e}_1, ..., \hat{e}_l\}$ is an orthonormal basis of $\bar{\mathscr{D}}^T$. Here, $k = \dim(\mathscr{D}^T)$, $n = \dim(\mathscr{D}^\perp)$, $m = \dim(\mathscr{D}^\theta)$ and $l = \dim(\bar{\mathscr{D}}^T)$.

Remark 3.6.1. *From (3.2.4), we see that $\{Je_1, ..., Je_k\}$ is an orthogonal basis of $\mathscr{D}^T$ with respect to the condition $g(Je_i, e_j) = 0$ for $i \neq j$. On the other side, by virtue of (3.2.15) and (3.2.16) we observe that $\{\sec\theta T\bar{e}_1, ..., \sec\theta T\bar{e}_m\}$ and $\{\csc\theta P\bar{e}_1, ..., \csc\theta P\bar{e}_m\}$ are respectively the orthogonal bases of $\mathscr{D}^\theta$ and $P\mathscr{D}^\theta$ with respect to the condition $g(T\bar{e}_i, \bar{e}_j) = 0$ for $i \neq j$.*

Theorem 3.6.2. *Let $(\breve{M}, J, g)$ be a metallic Riemannian manifold and M be a biwarped product generalized J-induced submanifold of first order of the type $M_T \times_f M_\perp \times_\sigma M_\theta$. Then the length of the second fundamental form h of M satisfies*

$$\begin{aligned} \|h\|^2 \geq & 2bn\|\nabla(\ln f)\|^2 + 2[bm + ax\cos^2\theta + bm\cos^2\theta]\|\nabla(\ln\sigma)\|^2 \\ & + 2[an + am - 2x]g(J\nabla(\ln\sigma), \nabla(\ln\sigma)), \end{aligned} \tag{3.6.1}$$

where $n = \dim(M_\perp)$, $m = \dim(M_\theta)$ and $x = \sum_{r=1}^m g(T\bar{e}_r, \bar{e}_r)$. The equality occurs if and only if

(i) M_T is totally geodesic in $\breve{M}$.

(ii) *$M_\perp$ and M_θ are totally umbilic in $\breve{M}$, where $-\nabla(\ln f)$ and $-\nabla(\ln\sigma)$ are respectively the mean curvatures of $M_\perp$ and M_θ*

(iii) *M is minimal in $\breve{M}$.*

(iv) *M is $(\mathscr{D}^\perp,\mathscr{D}^\theta)$-mixed geodesic.*

Proof. From (3.3.3), it follows that

$$\begin{aligned}\|h\|^2 =& \|h(\mathscr{D}^T,\mathscr{D}^T)\|^2+\|h(\mathscr{D}^\perp,\mathscr{D}^\perp)\|^2+\|h(\mathscr{D}^\theta,\mathscr{D}^\theta)\|^2\\ &+2\{\|h(\mathscr{D}^T,\mathscr{D}^\perp)\|^2+\|h(\mathscr{D}^T,\mathscr{D}^\theta)\|^2+\|h(\mathscr{D}^\perp,\mathscr{D}^\theta)\|^2\}.\end{aligned}\tag{3.6.2}$$

With the help of (3.3.4), (3.5.19), (3.5.20), (3.5.21), (3.5.22), (3.5.23) and (3.5.24), one can explicitly write as follows

$$\begin{aligned}\|h\|^2 =& \sum_{p,q,r=1}^{n} g^2(h(\tilde{e}_p,\tilde{e}_q),J\tilde{e}_r)+\sum_{p,q=1}^{n}\sum_{r=1}^{m} g^2(h(\tilde{e}_p,\tilde{e}_q),e_r^*)\\ &+\sum_{p,q=1}^{m}\sum_{r=1}^{n} g^2(h(\bar{e}_p,\bar{e}_q),J\tilde{e}_r)+\sum_{p,q,r=1}^{m} g^2(h(\bar{e}_p,\bar{e}_q),e_r^*)\\ &+2\sum_{p=1}^{k}\sum_{q,r=1}^{n} g^2(h(e_p,\tilde{e}_q),J\tilde{e}_r)+2\sum_{p=1}^{k}\sum_{q,r=1}^{m} g^2(h(e_p,\bar{e}_q),e_r^*)\\ &+\sum_{p,q=1}^{k+n+m}\sum_{r=1}^{l} g^2(h(e_p,e_q),\hat{e}_r).\end{aligned}\tag{3.6.3}$$

Thus, we obtain

$$\begin{aligned}\|h\|^2 \geq& 2\sum_{p=1}^{k}\sum_{q,r=1}^{n} g^2(h(e_p,\tilde{e}_q),J\tilde{e}_r)+2\sum_{p=1}^{k}\sum_{q,r=1}^{m} g^2(h(e_p,\bar{e}_q),e_r^*)\\ =& 2\sum_{p=1}^{k}\sum_{q,r=1}^{n} g^2(h(e_p,\tilde{e}_q),J\tilde{e}_r)+2\sum_{p=1}^{k}\sum_{q,r=1}^{m} g^2(h(e_p,\bar{e}_q),\csc\theta P\bar{e}_r)\\ =& 2\sum_{p=1}^{k}\sum_{q,r=1}^{n}[-Je_p(\ln f)g(\tilde{e}_q,\tilde{e}_r)]^2\\ &+2\sum_{p=1}^{k}\sum_{q,r=1}^{m}[-Je_p(\ln\sigma)g(\bar{e}_q,\bar{e}_r)+e_p(\ln\sigma)g(\bar{e}_q,T\bar{e}_r)]^2\\ =& 2n\sum_{p=1}^{k}[Je_p(\ln f)]^2+2m\sum_{p=1}^{k}[Je_p(\ln\sigma)]^2+2\sum_{p=1}^{k}\sum_{r=1}^{m}[e_p(\ln\sigma)]^2 g(T\bar{e}_r,T\bar{e}_r)\\ &-4\sum_{p=1}^{k}\sum_{r=1}^{m}[Je_p(\ln\sigma)e_p(\ln\sigma)]g(T\bar{e}_r,\bar{e}_r)\end{aligned}$$

$$
\begin{aligned}
=&2n[ag(J\nabla(\ln f),\nabla(\ln f))+b\|\nabla(\ln f)\|^2]+2m[ag(J\nabla(\ln\sigma),\nabla(\ln\sigma))\\
&+b\|\nabla(\ln\sigma)\|^2]+2\|\nabla(\ln\sigma)\|^2[a\cos^2\theta\sum_{r=1}^{m}g(T\bar{e}_r,\bar{e}_r)+bm\cos^2\theta]\\
&-4g(J\nabla(\ln\sigma),\nabla(\ln\sigma))\sum_{r=1}^{m}g(T\bar{e}_r,\bar{e}_r)\\
=&2bn\|\nabla(\ln f)\|^2+2[bm+ax\cos^2\theta+bm\cos^2\theta]\|\nabla(\ln\sigma)\|^2\\
&+2[an+am-2x]g(J\nabla(\ln\sigma),\nabla(\ln\sigma)),
\end{aligned}
\tag{3.6.4}
$$

where $x=\sum_{r=1}^{m}g(T\bar{e}_r,\bar{e}_r)$.

Using (3.5.19), (3.5.20), (3.5.21), (3.5.22), (3.5.23), (3.5.24) and (3.6.3) we observe that the equality occurs if and only if

$$h(\mathscr{D}^T,\mathscr{D}^T)=\{0\},\ h(\mathscr{D}^\perp,\mathscr{D}^\perp)=\{0\},\ h(\mathscr{D}^\theta,\mathscr{D}^\theta)=\{0\}, \tag{3.6.5}$$

$$h(\mathscr{D}^\perp,\mathscr{D}^\theta)=\{0\}. \tag{3.6.6}$$

Since M_T is totally geodesic in M, from (3.6.5) it implies that M_T is also totally geodesic in $\breve{M}$. Hence, (i) follows.

We denote $h^\perp$ as the second fundamental form of $M_\perp$ in M. From [83], it follows that $h^\perp(\mathscr{D}^\perp,\mathscr{D}^\perp)\subseteq\mathscr{D}^T$. Then, $g(h^\perp(X,W))=g(\nabla_X W,Z)$, where $Z\in\mathscr{D}^T$ and $X,W\in\mathscr{D}^\perp$. Using Proposition 1.2.2, we see that $\nabla_X W=\nabla^\perp_X W-g(X,W)\nabla(\ln f)$, where $\nabla^\perp$ is the induced connection on $M_\perp$. Thus,

$$g(h^\perp(X,W),Z)=-Z(\ln f)g(X,W)=-g(g(X,W)\nabla(\ln f),Z).$$

$$\text{Hence,}h^\perp(X,W)=-g(X,W)\nabla(\ln f). \tag{3.6.7}$$

In view of (3.6.5) and (3.6.7), one can conclude that $M_\perp$ is totally umbilic in $\breve{M}$ with mean curvature $-\nabla(\ln f)$. By a similar fashion, we derive that M_θ is totally umbilic in $\breve{M}$ with mean curvature $-\nabla(\ln\sigma)$. Hence, (ii) follows.

Assertions (iii) and (iv) follow respectively from (3.6.5) and (3.6.6). This completes the proof. □

3.7 Biwarped product submanifold of locally nearly metallic Riemannian manifold

In this section, we study the biwarped product submanifolds of a locally nearly metallic Riemannian manifold $\bar{M}$ in the form $M_T \times_f M_\perp \times_\sigma M_\theta$, where M_T, $M_\perp$ and M_θ are respectively the holomorphic, totally real and proper slant submanifolds. If we consider $\mathscr{D}^T = TM_T$, $\mathscr{D}^\perp = TM_\perp$ and $\mathscr{D}^\theta = TM_\theta$, then the tangent and normal bundles of M can be respectively decomposed as

$$TM = \mathscr{D}^T \oplus \mathscr{D}^\perp \oplus \mathscr{D}^\theta, \ \ T^\perp M = J\mathscr{D}^T \oplus P\mathscr{D}^\perp \oplus \delta,$$

where δ is the j-invariant subbundle of $T^\perp M$.

We state the following two Lemmas for later use.

Lemma 3.7.1. *Let $M = M_T \times_f M_\perp \times_\sigma M_\theta$ be a biwarped product submanifold of a locally nearly metallic Riemannian manifold $\bar{M}$. Then we derive*

$$(i)\ g(h(U,V),JX) = 0,$$
$$(ii)\ g(h(U,V),PZ) = 0,$$
$$(iii)\ g(h(U,X),JY) = \frac{1}{3}JU(\ln f)g(X,Y),$$

where $U,V \in \Gamma(\mathscr{D}^T)$, $X,Y \in \Gamma(\mathscr{D}^\perp)$ and $Z \in \Gamma(\mathscr{D}^\theta)$.

Proof. For all $U,V \in \Gamma(\mathscr{D}^T)$ and $X \in \Gamma(\mathscr{D}^\perp)$, we obtain

$$g(h(U,V),JX) = g(\bar{\nabla}_U V,JX) = g(\bar{\nabla}_U JV,X) - g((\bar{\nabla}_U J)V,X).$$

From (1.2.4), it follows that

$$g(h(U,V),JX) = g(\bar{\nabla}_U V,JX) = U(\ln f)g(JV,X) - g((\bar{\nabla}_U J)V,X).$$

Since $g(JV,X) = 0$, we find

$$g(h(U,V),JX) = -g((\bar{\nabla}_U J)V,X). \tag{3.7.1}$$

Replacing U and V by V and U respectively in (3.7.1), we derive

$$g(h(U,V),JX) = -g((\bar{\nabla}_V J)U,X). \tag{3.7.2}$$

By adding (3.7.1), (3.7.2) and using (3.2.17), we see

$$g(h(U,V),JX) = 0.$$

Hence, (i) follows.

By a similar manner, we can prove (ii).

Now, we wish to prove the third assertion of the Lemma. For all $U \in \Gamma(\mathscr{D}^T)$ and $X,Y \in \Gamma(\mathscr{D}^\perp)$, we obtain

$$g(h(U,X),JY) = g(\bar{\nabla}_X U,JY) = g(\bar{\nabla}_X JU,Y) - g((\bar{\nabla}_X J)U,Y).$$

From (1.2.4) and (3.2.17), it implies that

$$\begin{aligned} g(h(U,X),JY) =& JU(\ln f)g(X,Y) + g((\bar{\nabla}_U J)X,Y). \\ =& JU(\ln f)g(X,Y) + g(\bar{\nabla}_U JX,Y) - g(\bar{\nabla}_U X,JY) \end{aligned}$$

From (3.2.1), (3.2.2) and (3.2.17), we find

$$2g(h(U,X),JY) = JU(\ln f)g(X,Y) - g(h(U,Y),JX). \tag{3.7.3}$$

Putting $X = Y$ and $Y = X$, we obtain

$$2g(h(U,Y),JX) = JU(ln\ f)g(X,Y) - g(h(U,X),JY). \tag{3.7.4}$$

From (3.7.3) and (3.7.4), it follows that

$$\begin{aligned} 2g(h(U,X),JY) &= JU(\ln f)g(X,Y) - \frac{1}{2}[JU(\ln f)g(X,Y) - g(h(U,X),JY)] \\ i.e.,\ g(h(U,X),JY) &= \frac{1}{3}JU(\ln f)g(X,Y). \end{aligned}$$

Hence, (iii) follows. This completes the proof. □

Lemma 3.7.2. *Let $M = M_T \times_f M_\perp \times_\sigma M_\theta$ be a biwarped product submanifold of a locally nearly metallic Riemannian manifold $\bar{M}$. Then we derive*

$$(i)\ g(h(U,X),PZ) = -\frac{1}{2}g(h(U,Z),JX) = 0,$$
$$(ii)\ g(h(U,Z),PW) = \frac{1}{3}[JU(\ln\sigma)g(Z,W) - U(\ln\sigma)g(TZ,W)],$$

where $U \in \Gamma(\mathscr{D}^T)$, $X \in \Gamma(\mathscr{D}^\perp)$ and $Z,W \in \Gamma(\mathscr{D}^\theta)$.

Proof. For all $U \in \Gamma(\mathscr{D}^T)$, $X \in \Gamma(\mathscr{D}^\perp)$ and $Z \in \Gamma(\mathscr{D}^\theta)$, we get

$$\begin{aligned} g(h(U,X),PZ) =& g(\bar{\nabla}_X U, PZ) \\ =& g(\bar{\nabla}_X U, JZ) - g(\bar{\nabla}_X U, TZ) \\ =& g(\bar{\nabla}_X JU, Z) - g((\bar{\nabla}_X J)U, Z) - g(\bar{\nabla}_X U, TZ). \end{aligned}$$

In view of (3.2.17), (1.2.4) and the condition of orthogonality of two vector fields, we derive

$$\begin{aligned} g(h(U,X),PZ) =& -g((\bar{\nabla}_X J)U,Z) = g((\bar{\nabla}_U J)X,Z) \\ =& g(\bar{\nabla}_U JX, Z) - g(\bar{\nabla}_U X, JZ) \\ =& -g(\bar{\nabla}_U Z, JX) - g(\bar{\nabla}_U X, TZ) - g(\bar{\nabla}_U X, PZ) \\ =& -g(\bar{\nabla}_U Z, JX) - g(\bar{\nabla}_U X, PZ) \\ =& -g(h(U,Z),JX) - g(h(U,X),PZ). \end{aligned}$$

This implies that

$$g(h(U,X),PZ) = -\frac{1}{2}g(h(U,Z),JX), \tag{3.7.5}$$

which is the first equality of the first assertion of the Lemma. Also, we find

$$g(h(U,Z),JX) = g(\bar{\nabla}_Z U, JX) = g(\bar{\nabla}_Z JU, X) - g((\bar{\nabla}_Z J)U, X).$$

In view of (3.2.17), (1.2.4) and the condition of orthogonality of two vector fields, we derive

$$\begin{aligned} g(h(U,Z),JX) =& -g((\bar{\nabla}_Z J)U,X) = g((\bar{\nabla}_U J)Z,X) \\ =& g(\bar{\nabla}_U JZ, X) - g(\bar{\nabla}_U Z, JX) \\ =& g(\bar{\nabla}_U TZ, X) + g(\bar{\nabla}_U PZ, X) - g(\bar{\nabla}_U Z, JX). \end{aligned}$$

Since $g(\bar{\nabla}_U TZ, X) = 0$, thus by using (3.2.1) and (3.2.2), we find

$$\begin{aligned} g(h(U,Z),JX) =& g(\bar{\nabla}_U PZ, X) - g(\bar{\nabla}_U Z, JX) \\ =& - g(h(U,X),PZ) - g(h(U,Z),JX). \end{aligned}$$

This implies that

$$g(h(U,Z),JX) = -\frac{1}{2} g(h(U,X),PZ). \tag{3.7.6}$$

From (3.7.5) and (3.7.6), we obtain

$$g(h(U,X),PZ) = 0.$$

Hence, the second equality of the first assertion of the Lemma is proved.
Now, we wish to prove the second assertion of the Lemma. For all $U \in \Gamma(\mathscr{D}^T)$ and $Z, W \in \Gamma(\mathscr{D}^\theta)$, we have

$$\begin{aligned} g(h(U,Z),PW) =& g(\bar{\nabla}_Z U, PW). \\ =& g(\bar{\nabla}_Z U, JW) - g(\bar{\nabla}_Z U, TW) \\ =& g(\bar{\nabla}_Z JU, W) - g((\bar{\nabla}_Z J)U, W) - g(\bar{\nabla}_Z U, TW) \\ =& JU(\ln\sigma) g(Z,W) + g((\bar{\nabla}_U J)Z, W) - U(\ln\sigma) g(Z, TW) \\ =& JU(\ln\sigma) g(Z,W) + g(\bar{\nabla}_U JZ, W) - g(\bar{\nabla}_U Z, JW) \\ & - U(\ln\sigma) g(Z, TW) \\ =& JU(\ln\sigma) g(Z,W) + g(\bar{\nabla}_U TZ, W) + g(\bar{\nabla}_U PZ, W) \\ & - g(\bar{\nabla}_U Z, TW) - g(\bar{\nabla}_U Z, PW) - U(\ln\sigma) g(Z, TW) \end{aligned}$$

From (1.2.4), (3.2.1) and (3.2.2), we have

$$\begin{aligned} g(h(U,Z),PW) =& JU(\ln\sigma) g(Z,W) - U(\ln\sigma) g(Z, TW) \\ & - g(\bar{\nabla}_U W, PZ) - g(\bar{\nabla}_U Z, PW). \\ =& JU(\ln\sigma) g(Z,W) - U(\ln\sigma) g(Z, TW) \\ & - g(h(U,W),PZ) - g(h(U,Z),PW). \end{aligned}$$

This implies that

$$\begin{aligned} 2g(h(U,Z),PW) =& JU(\ln\sigma) g(Z,W) - U(\ln\sigma) g(Z, TW) \\ & - g(h(U,W),PZ). \end{aligned} \tag{3.7.7}$$

Interchanging Z by W, we have

$$\begin{aligned}2g(h(U,W),PZ) =& JU(\ln\sigma)g(Z,W) - U(\ln\sigma)g(Z,TW)\\ & - g(h(U,Z),PW). \end{aligned} \tag{3.7.8}$$

Using (3.7.7) and (3.7.8), we derive

$$g(h(U,Z),PW) = \frac{1}{3}[JU(\ln\sigma)g(Z,W) - U(\ln\sigma)g(TZ,W)],$$

Hence, the second part is proved. This completes the proof of the Lemma. □

Putting $W = TW$ in the second part of the Lemma 3.7.2, we obtain

$$\begin{aligned} g(h(U,Z),PTW) =& \frac{1}{3}[JU(\ln\sigma)g(Z,TW) - U(\ln\sigma)g(TZ,TW)]\\ =& \frac{1}{3}[JU(\ln\sigma)g(Z,TW)\\ & - U(\ln\sigma)\cos^2\theta\{ag(TZ,W) + bg(Z,W)\}]\\ =& \frac{1}{3}[JU(\ln\sigma)g(Z,TW) - a\cos^2\theta U(\ln\sigma)g(TZ,W)\\ & - b\cos^2\theta U(\ln\sigma)g(Z,W)]. \end{aligned} \tag{3.7.9}$$

Now, we give a necessary and sufficient conditions for such submanifolds to be locally trivial.

Theorem 3.7.3. *Let M be a biwarped product submanifold of type $M_T \times_f M_\perp \times_\sigma M_\theta$ of a locally nearly metallic Riemannian manifold $(\bar{M}, J, g)$ such that the invariant normal subbundle $\delta = \{0\}$. Then M is locally trivial if and only if M is $(\mathscr{D}^T, \mathscr{D}^\perp)$ and $(\mathscr{D}^T, \mathscr{D}^\theta)$-mixed geodesic.*

Proof. Let M be a biwarped product submanifold of type $M_T \times_f M_\perp \times_\sigma M_\theta$ of a locally nearly metallic Riemannian manifold $(\bar{M}, J, g)$ such that the invariant normal subbundle $\delta = \{0\}$. Let M be locally trivial. Then both the warping functions f and σ are constants. Since f is constant, so $JU(\ln f) = 0$. Therefore, by Lemma 3.7.1, we see that $g(h(U,X),JY) = 0$ for any $U \in \mathscr{D}^T$ and $X,Y \in \mathscr{D}^\perp$. Also, from Lemma 3.7.2 and the decomposition of the normal bundles of M, we gain $h(U,X) = 0$. Consequently, it implies that M is $(\mathscr{D}^T, \mathscr{D}^\perp)$-mixed geodesic. On the other side, since the function σ is constant, so $JU(\ln\sigma) = 0$ and $U(\ln\sigma) = 0$. Therefore, from Lemma 3.7.2, we find $g(h(U,Z),PW) = 0$ for $U \in \mathscr{D}^T$ and $Z,W \in \mathscr{D}^\theta$. Also, from Lemma 3.7.2 and the decomposition of the normal bundles of M, we gain $h(U,Z) = 0$. Consequently, it implies that M is $(\mathscr{D}^T,\mathscr{D}^\theta)$-mixed geodesic.

For the converse part of the theorem, let M be $(\mathscr{D}^T,\mathscr{D}^\perp)$ and $(\mathscr{D}^T,\mathscr{D}^\theta)$-mixed geodesic. If M is $(\mathscr{D}^T,\mathscr{D}^\perp)$-mixed geodesic, then $h(U,X) = 0$ for any $U \in \mathscr{D}^T$ and $X \in \mathscr{D}^\perp$. Hence, from Lemma 3.7.1, we see $JU(\ln f) = 0$. Therefore, f is a constant function. On the other side, if M is $(\mathscr{D}^T,\mathscr{D}^\theta)$-mixed geodesic, then $h(U,Z) = 0$ for any $U \in \mathscr{D}^T$ and $Z \in \mathscr{D}^\theta$. Hence, from Lemma 3.7.2, we obtain

$$JU(\ln\sigma)g(Z,W) - U(\ln\sigma)g(TZ,W) = 0. \tag{3.7.10}$$

Putting $U = JU$ in (3.7.10), we get

$$\begin{aligned}
&J^2U(\ln\sigma)g(Z,W) - JU(\ln\sigma)g(TZ,W) = 0\\
&\textit{i.e.}, (aJ+bI)U(\ln\sigma)g(Z,W) - JU(\ln\sigma)g(TZ,W) = 0\\
&\textit{i.e.}, aJU(\ln\sigma)g(Z,W) + bU(\ln\sigma)g(Z,W)\\
&\quad - JU(\ln\sigma)g(TZ,W) = 0.
\end{aligned} \tag{3.7.11}$$

Putting $Z = TZ$ in (3.7.11) and using (3.7.10), we have

$$\begin{aligned}
&aJU(\ln\sigma)g(TZ,W) + bU(\ln\sigma)g(TZ,W) - JU(\ln\sigma)g(T^2Z,W) = 0\\
&\textit{i.e.}, aJU(\ln\sigma)g(TZ,W) + bU(\ln\sigma)g(TZ,W)\\
&\quad - JU(\ln\sigma)[a\cos^2\theta g(TZ,W) + b\cos^2\theta g(Z,W)] = 0\\
&\textit{i.e.}, a(1-\cos^2\theta)JU(\ln\sigma)g(TZ,W) + b(1-\cos^2\theta)JU(\ln\sigma)g(Z,W) = 0\\
&\textit{i.e.}, a\sin^2\theta JU(\ln\sigma)g(TZ,W) + b\sin^2\theta JU(\ln\sigma)g(Z,W) = 0.\\
&\textit{i.e.}, \sin^2\theta JU(\ln\sigma)[ag(TZ,W) + bg(Z,W)] = 0.
\end{aligned} \tag{3.7.12}$$

Since M is a proper biwarped product submanifold of type $M_T \times_f M_\perp \times_\sigma M_\theta$ of a locally nearly metallic Riemannian manifold $(\bar{M},J,g)$, $\sin\theta \neq 0$. Also, since a,b are positive integers, $g(TZ,W) \neq 0$ and $g(Z,W) \neq 0$ for $Z,W \in \mathscr{D}^\theta$, hence $ag(TZ,W) + bg(Z,W) \neq 0$. Therefore, from (3.7.12) we can conclude that $JU(\ln\sigma) = 0$. Consequently, σ is a constant function. Therefore, M is locally trivial. This completes the proof. $\square$

Remark 3.7.4. *From Theorem 3.7.3, it follows that a proper biwarped product submanifold $M = M_T \times_f M_\perp \times_\sigma M_\theta$ in a locally nearly metallic Riemannian manifold is neither $(\mathscr{D}^T, \mathscr{D}^\perp)$-mixed geodesic nor $(\mathscr{D}^T, \mathscr{D}^\theta)$-mixed geodesic.*

3.8 Inequality for the second fundamental form in locally nearly metallic Riemannian manifold

In this section, we give a sharp inequality for the second fundamental form with respect to some conditions. We also investigate its equality case.

Let $M = M_T \times_f M_\perp \times_\sigma M_\theta$ be a proper biwarped product submanifold of a locally nearly metallic Riemannian manifold $(\bar{M}, J, g)$ of dimension $2m$. We choose a local orthogonal basis $\{e_1, ..., e_n\}$ of the tangent bundle TM in such a manner that $g(Je_i, e_j) = g(Te_i, e_j) = 0$ for $i \neq j$ and

$$\begin{aligned}
\mathscr{D}^T &= \text{span}\{e_1, ..., e_t, e_{t+1} = Je_1, ..., e_{2t} = Je_t\},\\
\mathscr{D}^\perp &= \text{span}\{e_{2t+1} = \hat{e}_1, ..., e_{2t+p} = \hat{e}_p\},\\
\mathscr{D}^\theta &= \text{span}\{e_{2t+p+1} = e_1^*, ..., e_{2t+p+q} = e_q^*, e_{2t+p+q+1} = \sec\theta e_1^*, ..., e_n = \sec\theta e_q^*\},
\end{aligned}$$

in which $\{e_1, ..., e_t\}$, $\{\hat{e}_1, ..., \hat{e}_p\}$ and $\{e_1^*, ..., e_q^*\}$ are three orthonormal set of vectors. Therefore, $\dim M_T = 2t$, $\dim M_\perp = p$ and $\dim M_\theta = 2q$. Furthermore, the orthonormal basis $\{E_1, ...E_{2m-n-p-2q}\}$ of the normal bundle $T^\perp M$ are given by

$$\begin{aligned}
J\mathscr{D}^\perp =& \text{span}\{E_1 = J\hat{e}_1, ..., E_p = J\hat{e}_p\},\\
P\mathscr{D}^\theta =& \text{span}\{E_{p+1} = \csc\theta P e_1^*, ..., E_{p+q} = \csc\theta P e_q^*,\\
& E_{p+q+1} = \csc\theta\sec\theta PTe_1^*, ..., E_{p+2q} = \csc\theta\sec\theta PTe_q^*\},\\
\delta =& \text{span}\{E_{p+2q+1}, ..., E_{2m-n-p-2q}\}.
\end{aligned}$$

Theorem 3.8.1. *Let M be a biwarped product submanifold of type $M_T \times_f M_\perp \times_\sigma M_\theta$ of a locally nearly metallic Riemannian manifold $(\bar{M}, J, g)$. Then the second* *fundamental form h satisfies*

$$\begin{aligned}\|h\|^2 \geq & \frac{2bp}{9}\|\nabla(\ln f)\|^2 + \frac{2}{9}\big[bq\csc^2\theta + ax\cot^2\theta + bq\cot^2\theta + abx\csc^2\theta \\ & + b^2q\csc^2\theta + a^3x\cot^2\theta\cos^2\theta + a^2bq\cot^2\theta\cos^2\theta + b^2q\cot^2\theta \\ & + 2abx\cot^2\theta\big]\|\nabla(\ln\sigma)\|^2 + \frac{2}{9}\big[ap + aq\csc^2\theta - 2x\csc^2\theta \\ & + a^2x\csc^2\theta + abq\csc^2\theta - 2a^2x\cot^2\theta - 2abq\cot^2\theta \\ & - 2bx\csc^2\theta\big]g(J\nabla(\ln\sigma), \nabla(\ln\sigma)),\end{aligned} \tag{3.8.1}$$

whereas $\dim M_\perp = p$*,* $\dim M_\theta = 2q$ *and* $x = \sum_{r=1}^{q} g(Te_r^*, e_r^*)$*.*

The equality occurs in (3.8.1) when M_T is totally geodesic in $\bar{M}$ and $M_\perp$, M_θ are totally umbilical in $\bar{M}$. Furthermore, M is neither $(\mathscr{D}^T, \mathscr{D}^\perp)$-mixed geodesic nor $(\mathscr{D}^T, \mathscr{D}^\theta)$-mixed geodesic in $\bar{M}$.

Proof. From the definition of the second fundamental form h, we have

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) = \sum_{r=1}^{2m-n-p-2q}\sum_{i,j=1}^{n} g^2(h(e_i, e_j), E_r). \tag{3.8.2}$$

Now, by decomposing (3.8.2) for the normal subbundles $T^\perp M$ of M as follows

$$\begin{aligned}\|h\|^2 = & \sum_{r=1}^{p}\sum_{i,j=1}^{n} g^2(h(e_i, e_j), J\hat{e}_r) + \sum_{r=p+1}^{p+2q}\sum_{i,j=1}^{n} g^2(h(e_i, e_j), E_r) \\ & + \sum_{r=p+2q+1}^{2m-n-p-2q}\sum_{i,j=1}^{n} g^2(h(e_i, e_j), E_r).\end{aligned} \tag{3.8.3}$$

We omit the last δ-components terms in (3.8.3) and by using the orthonormal bases of TM and $T^\perp M$, we have

$$\begin{aligned}\|h\|^2 \geq & \sum_{r=1}^{p}\sum_{i,j=1}^{2t} g^2(h(e_i, e_j), J\hat{e}_r) + 2\sum_{r=1}^{p}\sum_{i=1}^{2t}\sum_{j=1}^{p} g^2(h(e_i, \hat{e}_j), J\hat{e}_r) \\ & + \sum_{r=1}^{p}\sum_{i,j=1}^{p} g^2(h(\hat{e}_i, \hat{e}_j), J\hat{e}_r) + 2\sum_{r=1}^{p}\sum_{i=1}^{2t}\sum_{j=1}^{2q} g^2(h(e_i, e_j^*), J\hat{e}_r) \\ & + \sum_{r=1}^{p}\sum_{i,j=1}^{2q} g^2(h(e_i^*, e_j^*), J\hat{e}_r) + 2\sum_{r=1}^{p}\sum_{i=1}^{2q}\sum_{j=1}^{p} g^2(h(e_i^*, \hat{e}_j), J\hat{e}_r)\end{aligned}$$

$$
\begin{aligned}
&+\csc^2\theta\sum_{r=1}^{q}\sum_{i,j=1}^{2t}\left[g^2(h(e_i,e_j),Pe_r^*)+\sec^2\theta g^2(h(e_i,e_j),PTe_r^*)\right]\\
&+2\csc^2\theta\sum_{r=1}^{q}\sum_{i=1}^{2t}\sum_{j=1}^{p}\left[g^2(h(e_i,\hat{e}_j),Pe_r^*)+\sec^2\theta g^2(h(e_i,\hat{e}_j),PTe_r^*)\right]\\
&+\csc^2\theta\sum_{r=1}^{q}\sum_{i,j=1}^{p}\left[g^2(h(\hat{e}_i,\hat{e}_j),Pe_r^*)+\sec^2\theta g^2(h(\hat{e}_i,\hat{e}_j),PTe_r^*)\right]\\
&+2\csc^2\theta\sum_{r=1}^{q}\sum_{i=1}^{p}\sum_{j=1}^{2q}\left[g^2(h(\hat{e}_i,e_j^*),Pe_r^*)+\sec^2\theta g^2(h(\hat{e}_i,e_j^*),PTe_r^*)\right]\\
&+\csc^2\theta\sum_{r=1}^{q}\sum_{i,j=1}^{2q}\left[g^2(h(e_i^*,e_j^*),Pe_r^*)+\sec^2\theta g^2(h(e_i^*,e_j^*),PTe_r^*)\right]\\
&+2\csc^2\theta\sum_{r=1}^{q}\sum_{i=1}^{2t}\sum_{j=1}^{2q}\left[g^2(h(e_i,e_j^*),Pe_r^*)\right.\\
&\left.+\sec^2\theta g^2(h(e_i,e_j^*),PTe_r^*)\right]. \qquad (3.8.4)
\end{aligned}
$$

Clearly, there is no connection for warped products for the third, fifth, sixth, ninth, tenth and eleventh terms in (3.8.4). Hence, we omit these positive terms. With the help of Lemma 3.7.1, Lemma 3.7.2 and (3.7.9), we see that

$$
\begin{aligned}
\|h\|^2 \geq& 2\sum_{r=1}^{p}\sum_{i=1}^{2t}\sum_{j=1}^{p}\left[\frac{1}{3}Je_i(\ln f)g(\hat{e}_j,\hat{e}_r)\right]^2\\
&+2\csc^2\theta\sum_{r=1}^{q}\sum_{i=1}^{2t}\sum_{j=1}^{2q}\left[\frac{1}{3}\{Je_i(\ln\sigma)g(e_j^*,e_r^*)-e_i(\ln\sigma)g(Te_j^*,e_r^*)\}\right]^2\\
&+2\csc^2\theta\sec^2\theta\sum_{r=1}^{q}\sum_{i=1}^{2t}\sum_{j=1}^{2q}\left[\frac{1}{3}\{Je_i(\ln\sigma)g(e_j^*,Te_r^*)\right.\\
&\left.-a\cos^2\theta e_i(\ln\sigma)g(Te_j^*,e_r^*)-b\cos^2\theta e_i(\ln\sigma)g(e_j^*,e_r^*)\}\right]^2\\
=&\frac{2p}{9}\sum_{i=1}^{2t}\left[Je_i(\ln f)\right]^2+\frac{2q\csc^2\theta}{9}\sum_{i=1}^{2t}\left[Je_i(\ln\sigma)\right]^2\\
&+\frac{2\csc^2\theta}{9}\sum_{i=1}^{2t}\sum_{r=1}^{q}\left[e_i(\ln\sigma)\right]^2g(Te_r^*,Te_r^*)\\
&-\frac{4\csc^2\theta}{9}\sum_{i=1}^{2t}\sum_{r=1}^{q}\left[Je_i(\ln\sigma)e_i(\ln\sigma)\right]g(Te_r^*,e_r^*)\\
&+\frac{2\csc^2\theta\sec^2\theta}{9}\sum_{i=1}^{2t}\sum_{r=1}^{q}\left[Je_i(\ln\sigma)\right]^2g(Te_r^*,Te_r^*)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2a^2\cot^2\theta}{9}\sum_{i=1}^{2t}\sum_{r=1}^{q}\big[e_i(\ln\sigma)\big]^2 g(Te_r^*,Te_r^*)+\frac{2b^2q\cot^2\theta}{9}\sum_{i=1}^{2t}[e_i(\ln\sigma)]^2\\
&-\frac{4a\csc^2\theta}{9}\sum_{i=1}^{2t}\sum_{r=1}^{q}\big[Je_i(\ln\sigma)e_i(\ln\sigma)\big]g(Te_r^*,Te_r^*)\\
&-\frac{4b\csc^2\theta}{9}\sum_{i=1}^{2t}\sum_{r=1}^{q}\big[Je_i(\ln\sigma)e_i(\ln\sigma)\big]g(Te_r^*,e_r^*)\\
&+\frac{4ab\cot^2\theta}{9}\sum_{i=1}^{2t}\sum_{r=1}^{q}\big[e_i(\ln\sigma)\big]^2 g(Te_r^*,e_r^*)\\
=&\frac{2p}{9}\big[ag(J\nabla(\ln f),\nabla(\ln f))+b\|\nabla(\ln f)\|^2\big]\\
&+\frac{2q\csc^2\theta}{9}\big[ag(J\nabla(\ln\sigma),\nabla(\ln\sigma))+b\|\nabla(\ln\sigma)\|^2\big]\\
&+\frac{2\csc^2\theta}{9}\|\nabla(\ln\sigma)\|^2\big[a\cos^2\theta\sum_{r=1}^{q}g(Te_r^*,e_r^*)+bq\cos^2\theta\big]\\
&-\frac{4\csc^2\theta}{9}g(J\nabla(\ln\sigma),\nabla(\ln\sigma))\sum_{r=1}^{q}g(Te_r^*,e_r^*)\\
&+\frac{2\csc^2\theta\sec^2\theta}{9}\big[ag(J\nabla(\ln\sigma),\nabla(\ln\sigma))+b\|\nabla(\ln\sigma)\|^2\big]\\
&\times\big[a\cos^2\theta\sum_{r=1}^{q}g(Te_r^*,Te_r^*)+bq\cos^2\theta\big]\\
&+\frac{2a^2\cot^2\theta}{9}\|\nabla(\ln\sigma)\|^2\big[a\cos^2\theta\sum_{r=1}^{q}g(Te_r^*,e_r^*)+bq\cos^2\theta\big]\\
&+\frac{2b^2q\cot^2\theta}{9}\|\nabla(\ln\sigma)\|^2\\
&-\frac{4a\csc^2\theta}{9}g(J\nabla(\ln\sigma),\nabla(\ln\sigma))\big[a\cos^2\theta\sum_{r=1}^{q}g(Te_r^*,e_r^*)+bq\cos^2\theta\big]\\
&-\frac{4b\csc^2\theta}{9}g(J\nabla(\ln\sigma),\nabla(\ln\sigma))\sum_{r=1}^{q}g(Te_r^*,e_r^*)\\
&+\frac{4ab\cot^2\theta}{9}\|\nabla(\ln\sigma)\|^2\sum_{r=1}^{q}g(Te_r^*,e_r^*)\\
=&\frac{2bp}{9}\|\nabla(\ln f)\|^2+\frac{2}{9}\big[bq\csc^2\theta+ax\cot^2\theta+bq\cot^2\theta+abx\csc^2\theta\\
&+b^2q\csc^2\theta+a^3x\cot^2\theta\cos^2\theta+a^2bq\cot^2\theta\cos^2\theta+b^2q\cot^2\theta\\
&+2abx\cot^2\theta\big]\|\nabla(\ln\sigma)\|^2+\frac{2}{9}\big[ap+aq\csc^2\theta-2x\csc^2\theta
\end{aligned}
$$

$$+a^2x\csc^2\theta+abq\csc^2\theta-2a^2x\cot^2\theta-2abq\cot^2\theta$$
$$-2bx\csc^2\theta\big]g(J\nabla(\ln\sigma),\nabla(\ln\sigma)),$$

where $x=\sum_{r=1}^{q}g(Te_r^*,e_r^*)$. Thus we obtain the inequality.

Now, we wish to consider the equality case. We obtain by omitting the third term in (3.8.3) that

$$h(TM,TM)\perp\delta. \tag{3.8.5}$$

By vanishing the first term and omitting the seventh term in (3.8.4), we see

$$h(\mathscr{D}^T,\mathscr{D}^T)\perp J\mathscr{D}^\perp \textit{ and } h(\mathscr{D}^T,\mathscr{D}^T)\perp P\mathscr{D}^\theta. \tag{3.8.6}$$

From (3.8.5) and (3.8.6), it follows that

$$h(\mathscr{D}^T,\mathscr{D}^T)=0. \tag{3.8.7}$$

Also, by leaving the third and ninth terms in (3.8.4), we find

$$h(\mathscr{D}^\perp,\mathscr{D}^\perp)\perp J\mathscr{D}^\perp \textit{ and } h(\mathscr{D}^\perp,\mathscr{D}^\perp)\perp P\mathscr{D}^\theta. \tag{3.8.8}$$

Hence, we can conclude from (3.8.5) and (3.8.8) that

$$h(\mathscr{D}^\perp,\mathscr{D}^\perp)=0. \tag{3.8.9}$$

On the other side, by omitting the fifth and eleventh terms in (3.8.4), we derive

$$h(\mathscr{D}^\theta,\mathscr{D}^\theta)\perp J\mathscr{D}^\perp \textit{ and } h(\mathscr{D}^\theta,\mathscr{D}^\theta)\perp P\mathscr{D}^\theta. \tag{3.8.10}$$

Therefore, we have from (3.8.5) and (3.8.10) that

$$h(\mathscr{D}^\theta,\mathscr{D}^\theta)=0. \tag{3.8.11}$$

Furthermore, from leaving the sixth and tenth terms in (3.8.4), we have

$$h(\mathscr{D}^\perp,\mathscr{D}^\theta)\perp J\mathscr{D}^\perp \textit{ and } h(\mathscr{D}^\perp,\mathscr{D}^\theta)\perp P\mathscr{D}^\theta. \tag{3.8.12}$$

Thus, from (3.8.5) and (3.8.12) that

$$h(\mathscr{D}^\perp,\mathscr{D}^\theta)=0. \tag{3.8.13}$$

By vanishing the eighth term in (3.8.4) with (3.8.5), we derive

$$h(\mathscr{D}^T, \mathscr{D}^\perp) \subset J\mathscr{D}^\perp. \tag{3.8.14}$$

By a similar fashion, vanishing the forth term in (3.8.4)with (3.8.5), we find

$$h(\mathscr{D}^T, \mathscr{D}^\theta) \subset P\mathscr{D}^\theta. \tag{3.8.15}$$

Since M_T is totally geodesic in M, hence by using (3.8.7), (3.8.9) and (3.8.13), we conclude that M_T is totally geodesic in $\bar{M}$. On the other hand, since $M_\perp$ and M_θ are totally umbilical in M, hence by using (3.8.9), (3.8.11), (3.8.14) and (3.8.15), we can say that $M_\perp$ and M_θ are both totally umbilical in $\bar{M}$. Moreover, from Remark 3.7.4, (3.8.14) and (3.8.15), it follows that M is neither $(\mathscr{D}^T, \mathscr{D}^\perp)$-mixed geodesic nor $(\mathscr{D}^T, \mathscr{D}^\theta)$-mixed geodesic in $\bar{M}$. This completes the proof. $\square$

CHAPTER 4

Some spacetimes as an application of warped product manifolds

4.1 Introduction

This chapter is based on some spacetimes as an application of warped product manifolds. It brings out the significance of the generalized Friedmann-Robertson-Walker spacetime, hyper-generalized quasi Einstein spacetime and $\mathscr{T}$-flat spacetime. A new way to study on generalized Friedmann-Robertson-Walker spacetime means we discuss the Einstein gravitational field tensors and the cosmological constant in generalized Friedmann-Robertson-Walker spacetime $(\overline{M},\overline{g})$ of type $\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2$ equipped with the metric $\overline{g} = g_B \oplus {h_1}^2 g_1 \oplus {h_2}^2 g_2$, where $g_1 = \frac{g}{\varphi^2}$, g being the pseudo-Euclidean metric on $\mathbb{R}^{n_1}$ with respect to the co-ordinates $x = (x_1, x_2, ..., x_{n_1})$, $g_{ij} = \delta_{ij}\varepsilon_i$ and $\varphi : \mathbb{R}^{n_1} \to \mathbb{R}$ is a smooth function.

The fourth chapter contains fourteen sections. After the "introduction" part, there is "preliminaries" unit to remind some significant facts. Then the third section discusses the generalized Friedmann-Robertson-Walker spacetime in a new way. The fourth section represents some examples of generalized black hole solutions. The fifth section is focused on hyper-generalized quasi Einstein warped product spaces with non positive scalar curvature. Then consecutively four sections are used to investigate some geometric and physical properties of $(HGQE)_n$ manifolds. The tenth section illuminates the general relativistic viscous fluid $(HGQE)_4$ spacetimes with some physical applications. Then a non trivial example has been set up to ensure the existence of $(HGQE)_4$ spacetimes. Twelfth section deals with a spacetime admitting vanishing $\mathscr{T}$-curvature tensor. The last two sections convey the behaviour of general relativistic viscous fluid spacetime admitting vanishing and divergence free $\mathscr{T}$-curvature tensor respectively.

4.2 Preliminaries

This section recalls some basic results for multiply warped product manifolds [41] which will be needed throughout the current work. Let f be a smooth function on a semi-Riemannian manifold (M,g) of dimension n. Then the Hessian of f is defined by $H^f(X,Y) = X(Yf) - (\nabla_X Y)f$ and Laplacian of f is defined by $\Delta f = \text{trace}_g(H^f)$, or $\Delta = \text{div}(\text{grad})$, where grad, div and ∇ are the gradient, divergence and covariant derivative operators respectively.

Proposition 4.2.1. *[41] Let $M = B \times_{f_1} M_1 \times ... \times_{f_m} M_m$ be a pseudo-Riemannian multiply warped product endowed with the metric tensor $g = g_B \oplus f_1^2 g_{M_1} \oplus f_2^2 g_{M_2} \oplus ... \oplus f_m^2 g_{M_m}$ and also let $X,Y,Z \in \mathscr{L}(B)$ and $V \in \mathscr{L}(M_i)$, $W \in \mathscr{L}(M_j)$. Then*

$$\text{Ric}(X,Y) = \text{Ric}^B(X,Y) - \sum_{i=1}^{m} \left(\frac{n_i}{f_i}\right) H_B^{f_i}(X,Y), \tag{4.2.1}$$

$$\text{Ric}(V,X) = 0, \tag{4.2.2}$$

$$\text{Ric}(V,W) = 0; for\ i \neq j, \tag{4.2.3}$$

$$\begin{aligned}\text{Ric}(V,W) =& \text{Ric}^{M_i}(V,W) - \left[\frac{\Delta_B f_i}{f_i} + (n_i - 1)\frac{|\text{grad}_B f_i|_B^2}{f_i^2}\right. \\ &\left. + \sum_{k=1,k\neq i}^{m} n_k \frac{g_B(\text{grad}_B f_i, \text{grad}_B f_k)}{f_i f_k}\right] g(V,W); for\ i = j,\end{aligned} \tag{4.2.4}$$

where $\text{Ric}, \text{Ric}^{\text{B}}$ *and* $\text{Ric}^{\text{M}_\text{i}}$ *are the Ricci curvature tensors of the metrics g, g_B and g_{M_i} respectively.*

Proposition 4.2.2. *[41] Let $M = B \times_{f_1} M_1 \times ... \times_{f_m} M_m$ be a pseudo-Riemannian multiply warped product with the metric tensor $g = g_B \oplus f_1^2 g_{M_1} \oplus f_2^2 g_{M_2} \oplus ... \oplus f_m^2 g_{M_m}$. Then the scalar curvature S of (M,g) admits the following expressions*

$$\begin{aligned} S =& S^B - 2\sum_{i=1}^{m} n_i \frac{\Delta_B f_i}{f_i} + \sum_{i=1}^{m} \frac{S^{M_i}}{f_i^2} - \sum_{i=1}^{m} n_i(n_i-1)\frac{|\mathrm{grad}_B f_i|_B^2}{f_i^2} \\ & - \sum_{i=1}^{m} \sum_{k=1, k\neq i}^{m} n_i n_k \frac{g_B(\mathrm{grad}_B f_i, \mathrm{grad}_B f_k)}{f_i f_k}, \end{aligned} \tag{4.2.5}$$

where S^B and S^{M_i} are the scalar curvatures of the metrics g_B and g_{M_i} respectively.

Tripathi and Gupta [122] developed the notion of $\mathscr{T}$- curvature tensor in pseudo-Riemannian manifolds. They defined $\mathscr{T}$- curvature tensor as follows.

Definition 4.2.3 ($\mathscr{T}$- curvature tensor of type (1,3))**.** *In an n-dimensional pseudo-Riemannian manifold (M,g), a $\mathscr{T}$- curvature tensor is a tensor of type $(1,3)$ defined by*

$$\begin{aligned} \mathscr{T}(X,Y)Z =& c_0 R(X,Y)Z + c_1 S(Y,Z)X + c_2 S(X,Z)Y \\ & + c_3 S(X,Y)Z + c_4 g(Y,Z)QX + c_5 g(X,Z)QY \\ & + c_6 g(X,Y)QZ + r c_7 \left[g(Y,Z)X - g(X,Z)Y \right], \end{aligned} \tag{4.2.6}$$

where $X,Y,Z \in \mathfrak{X}(M)$; $c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7$ are smooth functions on M; S, Q, R, r, g are respectively the Ricci tensor, Ricci operator, curvature tensor, scalar curvature and pseudo-Riemannian metric tensor.

Note that $\mathscr{T}$-curvature tensor reduces to many other curvature tensors for different values of $c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7$.

Definition 4.2.4 ($\mathscr{T}$- curvature tensor of type (0,4))**.** *A $\mathscr{T}$-curvature tensor of type $(0,4)$ is defined by*

$$\begin{aligned} \tilde{\mathscr{T}}(X,Y,Z,W) =& c_0 R(X,Y,Z,W) + c_1 S(Y,Z) g(X,W) + c_2 S(X,Z) g(Y,W) \\ & + c_3 S(X,Y) g(Z,W) + c_4 g(Y,Z) S(X,W) + c_5 g(X,Z) S(Y,W) \\ & + c_6 g(X,Y) S(Z,W) + r c_7 [g(Y,Z) g(X,W) \\ & - g(X,Z) g(Y,W)], \end{aligned} \tag{4.2.7}$$

where $X,Y,Z,W \in \mathfrak{X}(M)$, R is the Riemannian curvature tensor, S is the Ricci tensor, g is the pseudo-Riemannian metric tensor and $\tilde{\mathscr{T}}(X,Y,Z,W) = g(\mathscr{T}(X,Y)Z,W)$.

Definition 4.2.5 ($\mathscr{T}$-flat spacetime)**.** *A spacetime is called $\mathscr{T}$-flat if the $\mathscr{T}$-curvature tensor of type $(0,4)$ satisfies the relation $\tilde{\mathscr{T}}(X,Y,Z,W) = 0$ on M for any $X,Y,Z,W \in \mathfrak{X}(M)$.*

Definition 4.2.6 (Curvature collineation)**.** *If a spacetime M admits a symmetry then it is said to be a curvature collineation (CC) [72, 42, 43] if*

$$\left(\pounds_\xi R\right)(X,Y)Z = 0, \tag{4.2.8}$$

where R is the Riemannian curvature tensor.

Definition 4.2.7 ($\mathscr{T}$-conservative spacetime)**.** *A spacetime is called $\mathscr{T}$-conservative if $(\mathrm{div}\mathscr{T})(X,Y,Z) = 0$.*

Definition 4.2.8 (Codazzi type tensor)**.** *A $(0,2)$-type symmetric tensor field F in a pseudo-Riemannian manifold (M^n, g) is called Codazzi type if $(\nabla_X F)(Y,Z) = (\nabla_Y F)(X,Z)$ for $X,Y,Z \in \mathfrak{X}(M)$.*

4.3 Generalized Friedmann-Robertson-Walker spacetime

The Friedmann-Robertson-Walker metric is an exact solution of the Einstein's field equations in four dimensional spacetime. It describes an isotropic, homogeneous, contracting or expanding universe which may be simply or multiply connected. This metric can be written in the following general form

$$\overline{g}(x^\alpha) = \varepsilon dt^2 + f^2(t) g_{ab}(x) dx^a dx^b, \tag{4.3.1}$$

where $a,b \in \{1,2,3\}$.

Definition 4.3.1. *Let (F_1, g_1) and (F_2, g_2) be two Riemannian manifolds and B be a manifold of dimension one. Also, let $h_i : B \to (0,\infty)$, $i \in \{1,2\}$ be smooth functions. The Lorentzian multiply warped product is the product manifold $\overline{M} = B \times F_1 \times F_2$ equipped with the metric $\overline{g}$ on $\overline{M}$ given by*

$$\overline{g}(x^\alpha) = \varepsilon dt^2 + {h_1}^2(t) g_{ab}(x^\mu) dx^a dx^b + {h_2}^2(t) g_{ij}(x^k) dx^i dx^j \tag{4.3.2}$$

with the local components

$$\begin{aligned}\overline{g}_{00} =& \overline{g}(\partial_t, \partial_t) = \varepsilon, \ \ \overline{g}_{ab} = {h_1}^2(t) g_{1ab}(x^\mu), \\ \overline{g}_{ij} =& {h_2}^2(t) g_{2ij}(x^k), \ \ \overline{g}_{ia} = 0, \ \ \overline{g}_{0i} = 0,\end{aligned} \tag{4.3.3}$$

where $\varepsilon^2 = 1$, $(x^\mu), (x^k)$ and t are the co-ordinate systems on F_1, F_2 and B respectively. It is also noted that $a, b \in \{1, 2, ..., n_1\}$, $i, j \in \{n_1 + 1, ..., n_1 + n_2\}$ and $\alpha \in \{1, ..., n_1 + n_2\}$. We use $\partial_t = \frac{\partial}{\partial t}, \partial_i = \frac{\partial}{\partial x^i}, \partial_a = \frac{\partial}{\partial x^a}$. We consider $h_1' = \frac{dh_1}{dt}, h_2' = \frac{dh_2}{dt}, A_1 = \frac{2h_1'}{h_1}, A_2 = \frac{2h_2'}{h_2}$.

Now we obtain the following results in terms of the Ricci tensor and scalar curvature of generalized Friedmann-Robertson-Walker spacetime $(\overline{M}, \overline{g})$ of type $\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2$ equipped with the metric $\overline{g} = g_B \oplus {h_1}^2 g_1 \oplus {h_2}^2 g_2$, where $g_1 = \frac{g}{\varphi^2}$, g being the pseudo-Euclidean metric on $\mathbb{R}^{n_1}$.

Proposition 4.3.2. *Let $(\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2, \overline{g})$ be a generalized Friedmann-Robertson-Walker spacetime. Then we have*

$$\begin{aligned}
\overline{\mathrm{Ric}}(\partial_t, \partial_t) =& -n_1 \left(\frac{{A_1}^2}{4} + \frac{A_1'}{2} \right) - n_2 \left(\frac{{A_2}^2}{4} + \frac{A_2'}{2} \right), &(4.3.4)\\
\overline{\mathrm{Ric}}(\partial_a, \partial_b) =& \frac{1}{\varphi}(n_1 - 2) H_g^\phi(\partial_a, \partial_b) - \overline{g}_{ab} \left[\varepsilon \left(\frac{{A_1}^2}{4} + \frac{A_1'}{2} \right) \right. \\
& \left. + (n_1 - 1)\varepsilon \frac{{A_1}^2}{4} + n_2 \varepsilon \frac{A_1 A_2}{4} \right]; \ \ a \neq b, &(4.3.5)\\
\overline{\mathrm{Ric}}(\partial_a, \partial_b) =& \frac{1}{\varphi}(n_1 - 2) H_g^\phi(\partial_a, \partial_a) + \frac{1}{\varphi} \varepsilon_a \Delta_g \varphi \\
& - \frac{1}{\varphi^2}(n_1 - 1)\varepsilon_a |\nabla_g \varphi|^2 - \overline{g}_{ab} \left[\varepsilon \left(\frac{{A_1}^2}{4} + \frac{A_1'}{2} \right) \right. \\
& \left. + (n_1 - 1)\varepsilon \frac{{A_1}^2}{4} + n_2 \varepsilon \frac{A_1 A_2}{4} \right]; \ \ a = b, &(4.3.6)
\end{aligned}$$

$$
\begin{aligned}
\overline{\mathrm{Ric}}(\partial_i,\partial_j) =& \mathrm{Ric}^{F_2}(\partial_i,\partial_j) \\
& -\overline{g}_{ij}\left[\varepsilon\left(\frac{A_2{}^2}{4}+\frac{A_1'}{2}\right)+(n_2-1)\varepsilon\frac{A_2{}^2}{4}+n_1\varepsilon\frac{A_1A_2}{4}\right], \qquad (4.3.7)\\
\overline{\mathrm{Ric}}(\partial_t,\partial_a) =& 0, \qquad (4.3.8)\\
\overline{\mathrm{Ric}}(\partial_a,\partial_i) =& 0, \qquad (4.3.9)
\end{aligned}
$$

where local components of the Ricci tensor on (F_2,g_2) *is* $\mathrm{Ric}^{F_2}(\partial_i,\partial_j)$.

Proof. Here $(\overline{M}=B\times_{h_1}F_1\times_{h_2}F_2,\overline{g})$ be a generalized Friedmann-Robertson-Walker spacetime equipped with the metric $\overline{g}=g_B\oplus h_1{}^2g_1\oplus h_2{}^2g_2$, where $g_1=\frac{g}{\varphi^2}$, g being the pseudo-Euclidean metric on $\mathbb{R}^{n_1}$. In view of Proposition 4.2.1, we obtain

$$
\begin{aligned}
\overline{\mathrm{Ric}}(\partial_t,\partial_t) =& \mathrm{Ric}^B(\partial_t,\partial_t)-\sum_{i=1}^{2}\left(\frac{n_i}{h_i}\right)H_B^{h_i}(\partial_t,\partial_t)\\
=& -\left[\left(\frac{n_1}{h_1}\right)H_B^{h_1}(\partial_t,\partial_t)+\left(\frac{n_2}{h_2}\right)H_B^{h_2}(\partial_t,\partial_t)\right]\\
=& -\left[\left(\frac{n_1}{h_1}\right)\ddot{h}_1+\left(\frac{n_2}{h_2}\right)\ddot{h}_2\right];\ \text{since } H_B^{h_i}=\ddot{h}_i\\
=& -n_1\left(\frac{A_1{}^2}{4}+\frac{A_1'}{2}\right)-n_2\left(\frac{A_2{}^2}{4}+\frac{A_2'}{2}\right), \qquad (4.3.10)\\
\overline{\mathrm{Ric}}(\partial_a,\partial_b) =& \mathrm{Ric}^{F_1}(\partial_a,\partial_b)-\left[\frac{\Delta_B h_1}{h_1}+(n_1-1)\frac{|\mathrm{grad}_B h_1|_B^2}{h_1{}^2}\right.\\
& \left.+n_2\frac{g_B(\mathrm{grad}_B h_1,\mathrm{grad}_B h_2)}{h_1h_2}\right]\overline{g}(\partial_a,\partial_b)\\
=& \frac{1}{\varphi}(n_1-2)H_g^{\phi}(\partial_a,\partial_b)-\overline{g}_{ab}\left[\varepsilon\left(\frac{A_1{}^2}{4}+\frac{A_1'}{2}\right)\right.\\
& \left.+(n_1-1)\varepsilon\frac{A_1{}^2}{4}+n_2\varepsilon\frac{A_1A_2}{4}\right];\ \ a\neq b, \qquad (4.3.11)\\
\overline{\mathrm{Ric}}(\partial_a,\partial_b) =& \mathrm{Ric}^{F_1}(\partial_a,\partial_a)-\left[\frac{\Delta_B h_1}{h_1}+(n_1-1)\frac{|\mathrm{grad}_B h_1|_B^2}{h_1{}^2}\right.\\
& \left.+n_2\frac{g_B(\mathrm{grad}_B h_1,\mathrm{grad}_B h_2)}{h_1h_2}\right]\overline{g}(\partial_a,\partial_a)\\
=& \frac{1}{\varphi}(n_1-2)H_g^{\phi}(\partial_a,\partial_a)+\frac{1}{\varphi}\varepsilon_a\Delta_g\varphi\\
& -\frac{1}{\varphi^2}(n_1-1)\varepsilon_a|\nabla_g\varphi|^2-\overline{g}_{aa}\left[\varepsilon\left(\frac{A_1{}^2}{4}+\frac{A_1'}{2}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+(n_1-1)\varepsilon\frac{{A_1}^2}{4}+n_2\varepsilon\frac{A_1A_2}{4}\Bigg]; \;\; a=b, && (4.3.12)\\
\overline{\mathrm{Ric}}(\partial_i,\partial_j)=&\mathrm{Ric}^{F_2}(\partial_i,\partial_j)-\Bigg[\frac{\Delta_B h_2}{h_2}+(n_2-1)\frac{|\mathrm{grad}_B h_2|_B^2}{{h_2}^2}\\
&+n_1\frac{g_B(\mathrm{grad}_B h_1,\mathrm{grad}_B h_2)}{h_1h_2}\Bigg]\overline{g}(\partial_i,\partial_j)\\
=&\mathrm{Ric}^{F_2}(\partial_i,\partial_j)\\
&-\overline{g}_{ij}\left[\varepsilon\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)+(n_2-1)\varepsilon\frac{{A_2}^2}{4}+n_1\varepsilon\frac{A_1A_2}{4}\right], && (4.3.13)\\
\overline{\mathrm{Ric}}(\partial_t,\partial_a)=&0, && (4.3.14)\\
\overline{\mathrm{Ric}}(\partial_a,\partial_i)=&0. && (4.3.15)
\end{aligned}
$$

This completes the proof. □

Proposition 4.3.3. *Let $(\overline{M}=B\times_{h_1}F_1\times_{h_2}F_2,\overline{g})$ be a generalized Friedmann-Robertson-Walker spacetime. Then the scalar curvature $\overline{S}$ of $(\overline{M},\overline{g})$ have the following expression*

$$
\begin{aligned}
\overline{S}=&-2\left[n_1\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right)+n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)\right]+\frac{(n_1-1)}{{h_1}^2}\left[2\varphi\Delta_g\varphi-n_1|\nabla_g\varphi|^2\right]\\
&+\frac{S^{F_2}}{{h_2}^2}-\left[n_1(n_1-1)\varepsilon\frac{{A_1}^2}{4}+n_2(n_2-1)\varepsilon\frac{{A_2}^2}{4}\right]-n_1n_2\varepsilon\frac{A_1A_2}{4}. \qquad (4.3.16)
\end{aligned}
$$

Proof. To prove this, we use Proposition 4.2.2 and it follows that

$$
\begin{aligned}
\overline{S}=&S^B-2\sum_{i=1}^{2}n_i\left(\frac{\Delta_B h_i}{h_i}\right)+\sum_{i=1}^{2}\frac{S^{F_i}}{{h_i}^2}-\sum_{i=1}^{2}n_i(n_i-1)\frac{|\mathrm{grad}_B h_i|_B^2}{{h_i}^2}\\
&-\sum_{i=1}^{2}\sum_{k=1,k\neq i}^{2}n_in_k\frac{g_B(\mathrm{grad}_B h_i,\mathrm{grad}_B h_k)}{h_ih_k},
\end{aligned}
$$

where S^B and S^{F_i} denote the scalar curvatures of the metrics g_B and g_i respectively.

This implies that

$$
\begin{aligned}
\overline{S}=&-2\left[n_1\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right)+n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)\right]+\frac{S^{F_1}}{h_1^2}+\frac{S^{F_2}}{h_2^2}\\
&-\left[n_1(n_1-1)\varepsilon\frac{{A_1}^2}{4}+n_2(n_2-1)\varepsilon\frac{{A_2}^2}{4}\right]-n_1n_2\varepsilon\frac{A_1A_2}{4}.
\end{aligned}
$$

Now we know that from [9],

$$
\begin{aligned}
\mathrm{Ric}^{F_1} =& \frac{1}{\varphi}[(n_1-2)H_g^{\varphi}(X_i,X_j)];\;\; i\neq j,\; i,j\in\{1,2,...,n_1\},\\
\mathrm{Ric}^{F_1} =& \frac{1}{\varphi^2}[(n_1-2)\varphi H_g^{\varphi}(X_i,X_i)+\{\varphi\Delta_g\varphi-(n_1-1)|\nabla_g\varphi|^2\}]\varepsilon_i\;;\;\; i=j.
\end{aligned}
$$

Taking trace on both sides of the above equation, we obtain

$$
\begin{aligned}
S^{F_1} =& \sum_{i=1}^{n_1} g_1^{ii}\mathrm{Ric}_{g_{1ii}}\\
=& \sum_{i=1}^{n_1} g_1^{ii}\mathrm{Ric}_{g_1}(\varphi X_i,\varphi X_i)\\
=& \sum_{i=1}^{n_1} \varepsilon_i\varphi^2\mathrm{Ric}_{g_1}(X_i,X_i)\\
=& \sum_{i=1}^{n_1} \varepsilon_i\left[(n_1-2)\varphi H_g^{\varphi}(X_i,X_i)+\{\varphi\Delta_g\varphi-(n_1-1)|\nabla_g\varphi|^2\}g(X_i,X_i)\right]\\
=& (n_1-2)\varphi\sum_{i=1}^{n_1}\varepsilon_i H_g^{\varphi}(X_i,X_i)+\{\varphi\Delta_g\varphi-(n_1-1)|\nabla_g\varphi|^2\}\sum_{i=1}^{n_1}\varepsilon_i^2\delta_{ii}\\
=& (n_1-2)\varphi\sum_{i=1}^{n_1} g^{ii}H^{\varphi}_{g,ii}+\{\varphi\Delta_g\varphi-(n_1-1)|\nabla_g\varphi|^2\}\sum_{i=1}^{n_1}\varepsilon_i^2\\
=& (n_1-2)\varphi\, tr(H_g^{\varphi})+n_1\{\varphi\Delta_g\varphi-(n_1-1)|\nabla_g\varphi|^2\}\\
=& (n_1-2)\varphi\Delta_g\varphi+n_1\{\varphi\Delta_g\varphi-(n_1-1)|\nabla_g\varphi|^2\}\\
=& 2(n_1-1)\varphi\Delta_g\varphi-n_1(n_1-1)|\nabla_g\varphi|^2.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
\overline{S} =& -2\left[n_1\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right)+n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)\right]+\frac{(n_1-1)}{{h_1}^2}\left[2\varphi\Delta_g\varphi-n_1|\nabla_g\varphi|^2\right]\\
&+\frac{S^{F_2}}{{h_2}^2}-\left[n_1(n_1-1)\varepsilon\frac{{A_1}^2}{4}+n_2(n_2-1)\varepsilon\frac{{A_2}^2}{4}\right]-n_1n_2\varepsilon\frac{A_1A_2}{4}.
\end{aligned}
$$

This completes the proof. □

Proposition 4.3.4. *Let $(\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2, \overline{g})$ be a generalized Friedmann-Robertson-Walker spacetime and $\overline{G}$ be its Einstein gravitational tensor field. Then we have the following equations*

$$\begin{aligned}
\overline{G}_{00} = &-\frac{(n_1-1)\varepsilon}{2{h_1}^2}[2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2] - \frac{\varepsilon S^{F_2}}{2{h_2}^2} - \frac{n_1}{2}(3-2\varepsilon-n_1)\frac{{A_1}^2}{4} \\
&-\frac{n_2}{2}(3-2\varepsilon-n_2)\frac{{A_2}^2}{4} - n_1(1-\varepsilon)\frac{A_1'}{2} - n_2(1-\varepsilon)\frac{A_2'}{2} \\
&+\frac{n_1 n_2}{2}\frac{A_1 A_2}{4}, \qquad (4.3.17)
\end{aligned}$$

$$\overline{G}_{a0} = 0, \;\; \overline{G}_{i0} = 0, \;\; \overline{G}_{ia} = 0, \qquad (4.3.18)$$

$$\begin{aligned}
\overline{G}_{ab} = &\frac{1}{\varphi}(n_1-2)H_g^{\varphi}(\partial_a,\partial_b) + \overline{g}_{ab}\Bigg[-\frac{(n_1-1)}{2{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\} - \frac{S^{F_2}}{2{h_2}^2} \\
&+(n_1-\varepsilon)\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) + n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right) + \frac{\varepsilon(n_1-1)(n_1-2)}{2}\frac{{A_1}^2}{4} \\
&+\frac{\varepsilon n_2(n_2-1)}{2}\frac{{A_2}^2}{4} + \frac{\varepsilon n_2(n_1-2)}{2}\frac{A_1A_2}{4}\Bigg]; \;\; a \neq b, \qquad (4.3.19)
\end{aligned}$$

$$\begin{aligned}
\overline{G}_{ab} = &\frac{1}{\varphi}(n_1-2)H_g^{\varphi}(\partial_a,\partial_a) + \frac{1}{\varphi}\varepsilon_a\Delta_g\varphi - \frac{(n_1-1)\varepsilon_a}{\varphi^2}|\nabla_g\varphi|^2 \\
&+\overline{g}_{aa}\Bigg[-\frac{(n_1-1)}{2{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\} - \frac{S^{F_2}}{2{h_2}^2} \\
&+(n_1-\varepsilon)\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) + n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right) + \frac{\varepsilon(n_1-1)(n_1-2)}{2}\frac{{A_1}^2}{4} \\
&+\frac{\varepsilon n_2(n_2-1)}{2}\frac{{A_2}^2}{4} + \frac{\varepsilon n_2(n_1-2)}{2}\frac{A_1A_2}{4}\Bigg]; \;\; a = b, \qquad (4.3.20)
\end{aligned}$$

$$\begin{aligned}
\overline{G}_{ij} = &G_{ij} + \overline{g}_{ij}\Bigg[-\frac{(n_1-1)}{2{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\} + n_1\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) \\
&+(n_2-\varepsilon)\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right) + \frac{\varepsilon n_1(n_1-1)}{2}\frac{{A_1}^2}{4} + \frac{\varepsilon(n_2-1)(n_2-2)}{2}\frac{{A_2}^2}{4} \\
&+\frac{\varepsilon n_1(n_2-2)}{2}\frac{A_1A_2}{4}\Bigg], \qquad (4.3.21)
\end{aligned}$$

where G_{ab} and G_{ij} are the local components of Einstein gravitational tensor field G of (F_1, g_1) and (F_2, g_2) respectively.

Proof. We know that the Einstein gravitational tensor field $\overline{G}$ of $(\overline{M}, \overline{g})$ is given by

$$\overline{G} = \overline{\mathrm{Ric}} - \frac{1}{2}\overline{S}\overline{g}.$$

Using this equation, we get

$$
\begin{aligned}
\overline{G}_{00} =& \overline{\mathrm{Ric}}(\partial_t,\partial_t) - \frac{1}{2}\overline{S}\overline{g}_{00}\\
=& -\left[n_1\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right)+n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)\right]\\
&-\frac{1}{2}\left[-2n_1\varepsilon\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right)-2n_2\varepsilon\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)\right.\\
&+\frac{(n_1-1)\varepsilon}{h_1^2}\{2\varphi\Delta_g\varphi-n_1|\nabla_g\varphi|^2\}+\frac{\varepsilon S^{F_2}}{h_2^2}\\
&\left.-n_1(n_1-1)\frac{{A_1}^2}{4}-n_2(n_2-1)\frac{{A_2}^2}{4}-n_1n_2\frac{A_1A_2}{4}\right]\\
=& -\frac{(n_1-1)\varepsilon}{2{h_1}^2}[2\varphi\Delta_g\varphi-n_1|\nabla_g\varphi|^2]-\frac{\varepsilon S^{F_2}}{2{h_2}^2}-\frac{n_1}{2}(3-2\varepsilon-n_1)\frac{{A_1}^2}{4}\\
&-\frac{n_2}{2}(3-2\varepsilon-n_2)\frac{{A_2}^2}{4}-n_1(1-\varepsilon)\frac{A_1'}{2}-n_2(1-\varepsilon)\frac{A_2'}{2}\\
&+\frac{n_1n_2}{2}\frac{A_1A_2}{4},
\end{aligned}
\tag{4.3.22}
$$

$$
\overline{G}_{a0} = 0,\ \ \overline{G}_{i0}=0,\ \ \overline{G}_{ia}=0,
\tag{4.3.23}
$$

$$
\begin{aligned}
\overline{G}_{ab} =& \overline{\mathrm{Ric}}(\partial_a,\partial_b) - \frac{1}{2}\overline{S}\overline{g}_{ab};\ \ a\neq b\\
=& \frac{1}{\varphi}(n_1-2)H_g^{\varphi}(\partial_a,\partial_b)-\overline{g}_{ab}\left[\varepsilon\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right)+(n_1-1)\varepsilon\frac{{A_1}^2}{4}\right.\\
&\left.+n_2\varepsilon\frac{A_1A_2}{4}\right]-\frac{1}{2}\overline{g}_{ab}\left[-2n_1\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right)-2n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)\right.\\
&+\frac{(n_1-1)}{{h_1}^2}\{2\varphi\Delta_g\varphi-n_1|\nabla_g\varphi|^2\}+\frac{S^{F_2}}{h_2^2}\\
&\left.-n_1(n_1-1)\varepsilon\frac{{A_1}^2}{4}-n_2(n_2-1)\varepsilon\frac{{A_2}^2}{4}-n_1n_2\varepsilon\frac{A_1A_2}{4}\right];\ a\neq b\\
=& \frac{1}{\varphi}(n_1-2)H_g^{\varphi}(\partial_a,\partial_b)+\overline{g}_{ab}\left[-\frac{(n_1-1)}{2{h_1}^2}\{2\varphi\Delta_g\varphi-n_1|\nabla_g\varphi|^2\}-\frac{S^{F_2}}{2{h_2}^2}\right.\\
&+(n_1-\varepsilon)\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right)+n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)+\frac{\varepsilon(n_1-1)(n_1-2)}{2}\frac{{A_1}^2}{4}\\
&\left.+\frac{\varepsilon n_2(n_2-1)}{2}\frac{{A_2}^2}{4}+\frac{\varepsilon n_2(n_1-2)}{2}\frac{A_1A_2}{4}\right];\ \ a\neq b,
\end{aligned}
\tag{4.3.24}
$$

$$
\begin{aligned}
\overline{G}_{ab} =& \overline{\mathrm{Ric}}(\partial_a,\partial_a) - \frac{1}{2}\overline{S}\overline{g}_{aa}; \;\; a=b \\
=& \frac{1}{\varphi}(n_1-2)H_g^{\varphi}(\partial_a,\partial_a) + \frac{1}{\varphi}\varepsilon_a\Delta_g\varphi - \frac{(n_1-1)\varepsilon_a}{\varphi^2}|\nabla_g\varphi|^2 \\
& - \overline{g}_{aa}\left[\varepsilon\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) + (n_1-1)\varepsilon\frac{{A_1}^2}{4} + n_2\varepsilon\frac{A_1A_2}{4}\right] \\
& - \frac{1}{2}\overline{g}_{aa}\left[-2n_1\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) - 2n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)\right. \\
& + \frac{(n_1-1)}{{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\} + \frac{S^{F_2}}{h_2^2} \\
& \left. - n_1(n_1-1)\varepsilon\frac{{A_1}^2}{4} - n_2(n_2-1)\varepsilon\frac{{A_2}^2}{4} - n_1n_2\varepsilon\frac{A_1A_2}{4}\right] \; ; \; a=b \\
=& \frac{1}{\varphi}(n_1-2)H_g^{\varphi}(\partial_a,\partial_a) + \frac{1}{\varphi}\varepsilon_a\Delta_g\varphi - \frac{(n_1-1)\varepsilon_a}{\varphi^2}|\nabla_g\varphi|^2 \\
& + \overline{g}_{aa}\left[-\frac{(n_1-1)}{2{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\} - \frac{S^{F_2}}{2{h_2}^2}\right. \\
& + (n_1-\varepsilon)\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) + n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right) + \frac{\varepsilon(n_1-1)(n_1-2)}{2}\frac{{A_1}^2}{4} \\
& \left. + \frac{\varepsilon n_2(n_2-1)}{2}\frac{{A_2}^2}{4} + \frac{\varepsilon n_2(n_1-2)}{2}\frac{A_1A_2}{4}\right]; \;\; a=b,
\end{aligned} \tag{4.3.25}
$$

$$
\begin{aligned}
\overline{G}_{ij} =& \overline{\mathrm{Ric}}(\partial_i,\partial_j) - \frac{1}{2}\overline{S}\overline{g}_{ij} \\
=& \mathrm{Ric}^{F_2}(\partial_i,\partial_j) - \overline{g}_{ij}\left[\varepsilon\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right) + (n_2-1)\varepsilon\frac{{A_2}^2}{4} + n_1\varepsilon\frac{A_1A_2}{4}\right] \\
& - \frac{1}{2}\overline{g}_{ij}\left[-2n_1\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) - 2n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)\right. \\
& + \frac{(n_1-1)}{{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\} + \frac{S^{F_2}}{h_2^2} - n_1n_2\varepsilon\frac{A_1A_2}{4} \\
& \left. - n_1(n_1-1)\varepsilon\frac{{A_1}^2}{4} - n_2(n_2-1)\varepsilon\frac{{A_2}^2}{4}\right] \\
=& \mathrm{Ric}^{F_2}(\partial_i,\partial_j) - \frac{1}{2}S^{F_2}g_{2ij} + \overline{g}_{ij}\left[-\frac{(n_1-1)}{2{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\}\right. \\
& + n_1\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) + (n_2-\varepsilon)\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right) + \frac{\varepsilon n_1(n_1-1)}{2}\frac{{A_1}^2}{4} \\
& \left. + \frac{\varepsilon(n_2-1)(n_2-2)}{2}\frac{{A_2}^2}{4} + \frac{\varepsilon n_1(n_2-2)}{2}\frac{A_1A_2}{4}\right]
\end{aligned}
$$

$$
\begin{aligned}
=&G_{ij}+\overline{g}_{ij}\Bigg[-\frac{(n_1-1)}{2{h_1}^2}\{2\varphi\Delta_g\varphi-n_1|\nabla_g\varphi|^2\}+n_1\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right)\\
&+(n_2-\varepsilon)\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)+\frac{\varepsilon n_1(n_1-1)}{2}\frac{{A_1}^2}{4}+\frac{\varepsilon(n_2-1)(n_2-2)}{2}\frac{{A_2}^2}{4}\\
&+\frac{\varepsilon n_1(n_2-2)}{2}\frac{A_1A_2}{4}\Bigg].
\end{aligned} \tag{4.3.26}
$$

This completes the proof. □

Proposition 4.3.5. *The Einstein equations in generalized Friedmann-Robertson-Walker spacetime* $(\overline{M},\overline{g})$ *with the cosmological constant* $\overline{\kappa}$ *are equivalent to the following reduced Einstein equations*

$$
\begin{aligned}
\overline{\kappa}=&\frac{(n_1-1)}{2{h_1}^2}\left[2\varphi\Delta_g\varphi-n_1|\nabla_g\varphi|^2\right]-\frac{\varepsilon n_1(n_1+n_2+2\varepsilon-3)}{2}\frac{{A_1}^2}{4}\\
&-\frac{\varepsilon n_2(n_2+2\varepsilon-3)}{2}\frac{{A_2}^2}{4}+\frac{\varepsilon n_1(2-2\varepsilon-n_2)}{2}\frac{A_1'}{2}\\
&+\frac{\varepsilon n_2(3-2\varepsilon-n_2)}{2}\frac{A_2'}{2},
\end{aligned} \tag{4.3.27}
$$

$$
G_{ij}=\varepsilon\overline{g}_{ij}\left(\frac{n_2}{2}-1\right)\left[n_1\frac{{A_1}^2}{4}+n_1\frac{A_1'}{2}+(n_2-1)\frac{A_2'}{2}-n_1\frac{A_1A_2}{4}\right]. \tag{4.3.28}
$$

Proof. Using (4.3.17) and $\overline{G}=-\overline{\kappa}\,\overline{g}$, we obtain

$$
\begin{aligned}
\overline{\kappa}=&\frac{(n_1-1)}{2{h_1}^2}\left[2\varphi\Delta_g\varphi-n_1|\nabla_g\varphi|^2\right]+\frac{S^{F_2}}{2{h_2}^2}-\frac{\varepsilon n_1(2\varepsilon+n_1-3)}{2}\frac{{A_1}^2}{4}\\
&-\frac{\varepsilon n_2(2\varepsilon+n_2-3)}{2}\frac{{A_2}^2}{4}+n_1\varepsilon(1-\varepsilon)\frac{A_1'}{2}\\
&+n_2\varepsilon(1-\varepsilon)\frac{A_2'}{2}-\frac{\varepsilon n_1n_2}{2}\frac{A_1A_2}{4}.
\end{aligned} \tag{4.3.29}
$$

Again by using (4.3.21), the Einstein equation $\overline{G}=-\overline{\kappa}\,\overline{g}$ and (4.3.29), we get

$$
G_{ij}=-\overline{g}_{ij}\left[\frac{S^{F_2}}{2h_2^2}+n_1\varepsilon\frac{{A_1}^2}{4}+n_1\varepsilon\frac{A_1'}{2}+\varepsilon(n_2-1)\frac{A_2'}{2}-n_1\varepsilon\frac{A_1A_2}{4}\right]. \tag{4.3.30}
$$

Now contracting (4.3.30) with g^{ij}, we have

$$
\frac{S^{F_2}}{h_2^2}=n_1n_2\varepsilon\frac{A_1A_2}{4}-\varepsilon n_1n_2\frac{{A_1}^2}{4}-\varepsilon n_1n_2\frac{A_1'}{2}-\varepsilon n_2(n_2-1)\frac{A_2'}{2}. \tag{4.3.31}
$$

Hence from (4.3.30) and (4.3.31), we obtain

$$
G_{ij}=\varepsilon\overline{g}_{ij}\left(\frac{n_2}{2}-1\right)\left[n_1\frac{{A_1}^2}{4}+n_1\frac{A_1'}{2}+(n_2-1)\frac{A_2'}{2}-n_1\frac{A_1A_2}{4}\right]. \tag{4.3.32}
$$

Using (4.3.31) in (4.3.29), we get

$$\begin{aligned}\overline{\kappa} =& \frac{(n_1-1)}{2{h_1}^2}\left[2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\right] - \frac{\varepsilon n_1(n_1+n_2+2\varepsilon-3)}{2}\frac{{A_1}^2}{4} \\ & - \frac{\varepsilon n_2(n_2+2\varepsilon-3)}{2}\frac{{A_2}^2}{4} + \frac{\varepsilon n_1(2-2\varepsilon-n_2)}{2}\frac{A_1'}{2} \\ & + \frac{\varepsilon n_2(3-2\varepsilon-n_2)}{2}\frac{A_2'}{2}. \end{aligned} \tag{4.3.33}$$

This completes the proof. □

Proposition 4.3.6. *The Einstein equations $\overline{G} = -\overline{\kappa}\,\overline{g}$ on $(\overline{M},\overline{g})$ with the cosmological constant $\overline{\kappa}$ induce the Einstein equations $G_{ij} = -\kappa_2 g_{2ij}$ on (F_2,g_2), where κ_2 is given by*

$$\kappa_2 = -\varepsilon {h_2}^2\left(\frac{n_2}{2}-1\right)\left[n_1\frac{{A_1}^2}{4} + n_1\frac{A_1'}{2} + (n_2-1)\frac{A_2'}{2} - n_1\frac{A_1A_2}{4}\right].$$

Proof. By using (4.3.3) and (4.3.28), we get $G_{ij} = -\kappa_2 g_{2ij}$ on (F_2,g_2), where

$$\kappa_2 = -\varepsilon {h_2}^2\left(\frac{n_2}{2}-1\right)\left[n_1\frac{{A_1}^2}{4} + n_1\frac{A_1'}{2} + (n_2-1)\frac{A_2'}{2} - n_1\frac{A_1A_2}{4}\right], \tag{4.3.34}$$

is the cosmological constant. □

Note 4.3.7. *One can also study the generalized Friedmann-Robertson-Walker space-time $(\overline{M},\overline{g})$ of type $\overline{M} = B\times_{h_1}F_1\times_{h_2}F_2$ equipped with the metric $\overline{g} = g_B\oplus {h_1}^2 g_1\oplus {h_2}^2 g_2$, where $g_2 = \frac{g}{\varphi^2}$, g being the pseudo-Euclidean metric on $\mathbb{R}^{n_2}$ and can compute the Ricci tensor of (F_i,g_i) and Einstein gravitational field tensor of $(\overline{M},\overline{g})$. After similar calculations we find out the following results for the cosmological constants of Einstein equations.*

Proposition 4.3.8. *The Einstein equations $\overline{G}_{AB} = -\overline{\kappa}\,\overline{g}_{AB}$ on $(\overline{M},\overline{g})$ with the cosmological constant $\overline{\kappa}$ induce the Einstein equations $G_{ab} = -\kappa_1 g_{1ab}$ on (F_1,g_1), where $\overline{\kappa}$ and κ_1 are given by*

$$\begin{aligned}\overline{\kappa} =& \frac{(n_2-1)}{2{h_2}^2}\left[2\varphi\Delta_g\varphi - n_2|\nabla_g\varphi|^2\right] - \frac{\varepsilon n_2(n_1+n_2+2\varepsilon-3)}{2}\frac{{A_2}^2}{4} \\ & - \frac{\varepsilon n_1(n_1+2\varepsilon-3)}{2}\frac{{A_1}^2}{4} + \frac{\varepsilon n_2(2-2\varepsilon-n_1)}{2}\frac{A_2'}{2} \\ & + \frac{\varepsilon n_1(3-2\varepsilon-n_1)}{2}\frac{A_1'}{2}, \end{aligned} \tag{4.3.35}$$

$$\kappa_1 = -\varepsilon {h_1}^2\left(\frac{n_1}{2}-1\right)\left[n_2\frac{{A_2}^2}{4} + n_2\frac{A_2'}{2} + (n_1-1)\frac{A_1'}{2} - n_2\frac{A_1A_2}{4}\right]. \tag{4.3.36}$$

Proof. Similar as Proposition 4.3.6. □

4.4 Example of generalized black holes

Using the above mentioned Proposition 4.3.7, we wish to show some examples of the generalized black hole solutions whose metrics can be written as a multiply warped product metric of the generalized Friedmann-Robertson-Walker spacetime $(\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2, \overline{g})$, where F_2 is conformal to the pseudo-Euclidean space $\mathbb{R}^{n_2}$. Then we reduce the Einstein equations $\overline{G}_{AB} = -\overline{\kappa}\ \overline{g}_{AB}$ into $G_{ab} = -\kappa_1 g_{1ab}$ by considering an n-dimensional Schwarzschild black hole and an n-dimensional Reissner-Nördstrom black hole.

4.4.1. *n*-dimensional Schwarzschild black hole

The metric of a Schwarzschild black hole [76] of dimension n is given by

$$\mathrm{ds}^2 = -p(r)\mathrm{dt}^2 + p(r)^{-1}\mathrm{dr}^2 + r^2\mathrm{d}\Omega^2_{n-2}, \tag{4.4.1}$$

where $p(r) = \left(1 - \frac{m}{r^{n-3}}\right), \mathrm{d}\Omega^2_{n-2} = \frac{(2\pi)^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}, \Gamma(z+1) = z\Gamma(z)$ and the geometric mass m indicates for the radius of horizon. Then this may be expressed [54] as a multiply warped product $\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2$ of dimension n equipped with the metric

$$\mathrm{ds}^2 = -\mathrm{d}\mu^2 + {h_1}^2(\mu)\mathrm{dt}^2 + {h_2}^2(\mu)\mathrm{d}\Omega^2_{n-2}, \tag{4.4.2}$$

where

$$\begin{aligned} h_1(\mu) =& \sqrt{\frac{m}{(F^{-1}(\mu))^{n-3}} - 1}, \\ h_2(\mu) =& F^{-1}(\mu). \end{aligned}$$

We consider F_2 is conformal to an $(n-2)$-dimensional pseudo-Euclidean space $(\mathbb{R}^{n-2}, g)$. Then $\mathrm{d}\Omega^2_{n-2} = \frac{1}{\varphi^2} d\Phi^2_{n-2}$, where $\mathrm{d}\Phi^2_{n-2}$ is the pseudo-Euclidean metric and $\varphi : \mathbb{R}^{n-2} \to \mathbb{R}$ is a smooth function.

The existence of the above functions $h_1(\mu)$ and $h_2(\mu)$ guarantees the reduction of Einstein equations $\overline{G}_{AB} = -\overline{\kappa}\ \overline{g}_{AB}$ into $G_{ab} = -\kappa_1 g_{1ab}$, where $\overline{\kappa}$ and κ_1 are the cosmological constants subject to the set of coupled differential equations (4.3.35) and (4.3.36) by the substitution of t by μ.

4.4.2. n-dimensional Reissner-Nördstrom black hole

The metric of a Reissner-Nördstrom black hole of dimension n (≥ 4) is given by

$$\mathrm{ds}^2 = -p(r)\mathrm{dt}^2 + p(r)^{-1}\mathrm{dr}^2 + r^2\mathrm{d}\Omega^2_{n-2}, \tag{4.4.3}$$

where $p(r) = \left(1 - \frac{m}{r^{n-3}} + \frac{q}{r^{2(n-3)}}\right)$; m and q are the geometric mass and charge of the black hole respectively and $\mathrm{d}\Omega_{n-2} = \frac{2\pi}{\Gamma\left(\frac{n-1}{2}\right)}$.

Then (4.4.3) can be written as an n-dimensional multiply warped product metric of the generalized Friedmann-Robertson-Walker spacetime ($\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2, \overline{g}$) furnished with the metric [54]

$$\mathrm{ds}^2 = -\mathrm{d}\mu^2 + {h_1}^2(\mu)\mathrm{dt}^2 + {h_2}^2(\mu)\mathrm{d}\Omega^2_{n-2}, \tag{4.4.4}$$

where

$$\begin{aligned} h_1(\mu) =& \sqrt{\frac{m}{(F^{-1}(\mu))^{n-3}} - \frac{q}{(F^{-1}(\mu))^{2n-6}} - 1}, \\ h_2(\mu) =& F^{-1}(\mu) \end{aligned}$$

with

$$\begin{aligned} \mu =& \int_{r_-}^{r} \sqrt{-p(r)^{-1}}\, dr = F(r), \;\; (say) \\ i.e., \, r =& F^{-1}(\mu). \end{aligned} \tag{4.4.5}$$

We consider F_2 is conformal to an $(n-2)$-dimensional pseudo-Euclidean space $(\mathbb{R}^{n-2}, g)$. Then $\mathrm{d}\Omega^2_{n-2} = \frac{1}{\varphi^2}\mathrm{d}\Phi^2_{n-2}$, where $\mathrm{d}\Phi^2_{n-2}$ is the pseudo-Euclidean metric and $\varphi : \mathbb{R}^{n-2} \to \mathbb{R}$ is a smooth function.

The existence of the above functions $h_1(\mu)$ and $h_2(\mu)$ guarantees the reduction of Einstein equations $\overline{G}_{AB} = -\overline{\kappa}\ \overline{g}_{AB}$ into $G_{ab} = -\kappa_1 g_{1ab}$, where $\overline{\kappa}$ and κ_1 are the cosmological constants subject to the set of coupled differential equations (4.3.35) and (4.3.36) by the substitution of t by μ.

Note 4.4.1. *One can also investigate the above singular metrics of n-dimensional Schwarzschild black hole and Reissner-Nördstrom black hole in view of the lightlike warped product [44]. Let us consider the n-dimensional Schwarzschild black hole metric given in (4.4.1) with respect to the coordinate system* $(t, r, x^1, x^2, ..., x^{n-2})$ *on* $(\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2, \overline{g})$. *Let u and v be two null coordinates such that* $u = t + r$ *and* $v = t - r$. *Then the metric given in (4.4.1) transforms into the metric*

$$\mathrm{ds}^2 = \frac{1}{4p(r)}[1 - p(r)^2][\mathrm{du}^2 + \mathrm{dv}^2] - 2[1 + p(r)^2]\mathrm{dudv} + \frac{1}{4}(u - v)^2 \mathrm{d}\Omega^2_{n-2}. \quad (4.4.6)$$

Clearly if we consider the condition $p(r) = 1$ *then the metric given in (4.4.6) becomes*

$$\mathrm{ds}^2 = -4\mathrm{dudv} + \frac{1}{4}(u - v)^2 \mathrm{d}\Omega^2_{n-2}. \quad (4.4.7)$$

Hence the absence of the terms du^2 *and* dv^2 *in (4.4.7) implies that u and v are all constants. Hence u and v are lightlike hypersurfaces of* $\overline{M}$. *Therefore, according to [44], it is possible to construct a lightlike warped product manifold. Then one can also do the further calculations in a similar way. We obtain the same result for the n-dimensional Reissner-Nördstrom black hole.*

4.5 Hyper-generalized quasi-Einstein $(HGQE)_n$ warped product spaces with non positive scalar curvature

In view of Proposition 1.2.4 and (1.1.19), we obtain the following result.

Result 4.5.1. *When* U, V *and* W *are mutually orthogonal and tangent to the base B, the warped product* $M = B \times_f F$ *is a hyper-generalized quasi-Einstein manifold with*

$$\begin{aligned} S_M(X,Y) = &\alpha g_M(X,Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] \\ &+ \delta[A(X)D(Y) + A(Y)D(X)] \end{aligned}$$

if and only if

$$
\begin{aligned}
(2.a)\ S_B(X,Y) =&\alpha g_B(X,Y) + \beta g_B(X,U) g_B(Y,U) + \gamma[g_B(X,U) g_B(Y,V) \\
&+ g_B(Y,U) g_B(X,V)] + \delta[g_B(X,U) g_B(Y,W) \\
&+ g_B(Y,U) g_B(X,W)] + \frac{k}{f} H^f(X,Y),
\end{aligned}
$$

$$(2.b)\ S_F(X,Y) = \mu g_F(X,Y),$$

$$(2.c)\ \mu = [\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2].$$

Lemma 4.5.2. *[75] Suppose f is a smooth function on a Riemannian manifold B, then for any vector X,*

$$\mathrm{div}(H^f)(X) = S(\nabla f, X) - \Delta(\mathrm{df})(X), \tag{4.5.1}$$

where $\Delta = d\delta + \delta d$ is the Laplacian on B which is acting on differential forms.

Now we prove the following proposition.

Proposition 4.5.3. *Suppose (B^m, g_B) is an $m(\geq 2)$ dimensional compact Riemannian manifold. Also, suppose that f is a nonconstant smooth function on B satisfying (2.a) for $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ and if the condition*

$$
\begin{aligned}
&\beta g_B(X,U) g_B(\nabla f, U) + \gamma[g_B(X,U) g_B(\nabla f, V) + g_B(\nabla f, U) g_B(X,V)] \\
&+\delta[g_B(X,U) g_B(\nabla f, W) + g_B(\nabla f, U) g_B(X,W)] = 0
\end{aligned}
$$

holds, then f satisfies (2.c) for $\mu \in \mathbb{R}$. Hence, for a compact Riemannian manifold F with $S_F(X,Y) = \mu g_F(X,Y)$, we can construct a compact hyper-generalized quasi Einstein warped product space $M = B \times_f F$ with

$$
\begin{aligned}
S_M(X,Y) =&\alpha g_M(X,Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] \\
&+ \delta[A(X)D(Y) + A(Y)D(X)],
\end{aligned}
$$

where U, V and W are mutually orthogonal and tangent to the base B.

Proof. By considering the trace of both sides of (2.a), we obtain

$$r = \alpha m - k\frac{\Delta f}{f} + \beta, \tag{4.5.2}$$

where r is the scalar curvature of B. From the second Bianchi identity, it follows that

$$\mathrm{dr} = 2\mathrm{div}(\mathrm{S}). \tag{4.5.3}$$

In view of (4.5.2) and (4.5.3), we get

$$\mathrm{div}S(X) = \frac{k}{2f^2}\{\Delta f \mathrm{df} - f\mathrm{d}(\Delta \mathrm{f})\}(X). \tag{4.5.4}$$

Also, we obtain

$$\mathrm{div}\left(\frac{1}{f}H^f\right)(X) = \sum_i (D_{E_i}\left(\frac{1}{f}H^f\right))(E_i, X) = -\frac{1}{f^2}H^f(\nabla f, X) + \frac{1}{f}\mathrm{div}H^f(X),$$

where X is a vector field and $\{E_1, E_2, ..., E_m\}$ is an orthonormal frame of B. Since $H^f(\nabla f, X) = (D_X \mathrm{df})(\nabla f) = \frac{1}{2}\mathrm{d}(|\nabla \mathrm{f}|^2)(X)$, the last equation becomes

$$\mathrm{div}\left(\frac{1}{f}H^f\right)(X) = -\frac{1}{2f^2}\mathrm{d}(|\nabla \mathrm{f}|^2)(X) + \frac{1}{f}\mathrm{div}H^f(X),$$

X is a vector field of B. Therefore, from (2.a) and (4.5.1), we get

$$\begin{aligned}
\mathrm{div}\left(\frac{1}{f}H^f\right)(X) =& \frac{1}{2f^2}\{(k-1)d(|\nabla f|^2) - 2fd(\Delta f) + 2\alpha f df\} \\
&+ \frac{1}{f}\beta g_B(X,U) g_B(\nabla f, U) \\
&+ \frac{1}{f}\gamma[g_B(X,U)g_B(\nabla f, V) + g_B(\nabla f, U)g_B(X,V)] \\
&+ \frac{1}{f}\delta[g_B(X,W)g_B(\nabla f, U) + g_B(\nabla f, W)g_B(X,U)].
\end{aligned} \tag{4.5.5}$$

But, (2.a) implies $\mathrm{divS_B} = \mathrm{div}\left(\frac{k}{f}H^f\right)$. So, from (4.5.4) and (4.5.5) it follows that $\mathrm{d}(-\mathrm{f}\Delta \mathrm{f} + (\mathrm{k}-1)|\nabla \mathrm{f}|^2 + \alpha \mathrm{f}^2) = 0$, i.e., $-f\Delta f + (k-1)|\nabla f|^2 + \alpha f^2 = \mu$, where μ is some constant. This completes the proof of the first part of the Proposition. Now if (F, g_F) is a k-dimensional compact Riemannian manifold with $S_F = \mu g_F$, then we can make a compact hyper-generalized quasi-Einstein warped product $M = B \times_f F$ with respect to the sufficient condition of the Result 4.5.1. □

Similarly, we obtain the following Result and Proposition where U, V and W are mutually orthogonal and tangent to the fibre F.

Result 4.5.4. *When U, V and W are mutually orthogonal and tangent to the fiber F, the warped product $M = B \times_f F$ is a hyper-generalized quasi Einstein manifold with*

$$\begin{aligned} S_M(X,Y) =& \alpha g_M(X,Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] \\ &+ \delta[A(X)D(Y) + A(Y)D(X)] \end{aligned}$$

if and only if

$$\begin{aligned} (2.d)\ \ S_B(X,Y) =& \alpha g_B(X,Y) + \frac{k}{f} H^f(X,Y), \\ (2.e)\ \ S_F(X,Y) =& g_F(X,Y)[\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2] + \beta f^4 g_F(X,U) g_F(Y,U) \\ &+ \gamma f^4[g_F(X,U)g_F(Y,V) + g_F(Y,U)g_F(X,V)] \\ &+ \delta f^4[g_F(X,U)g_F(Y,W) + g_F(Y,U)g_F(X,W)], \\ (2.f)\ \ \mu =& [\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2]. \end{aligned}$$

Proposition 4.5.5. *Suppose (B^m, g_B) is an $m(\geq 2)$ dimensional compact Riemannian manifold. Also, suppose that f is a nonconstant smooth function on B satisfying (2.d) for $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$. Hence, for a compact hyper-generalized quasi Einstein manifold F with*

$$\begin{aligned} S_F(X,Y) =& g_F(X,Y)[\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2 + \beta f^4 g_F(X,U) g_F(Y,U) \\ &+ \gamma f^4[g_F(X,U)g_F(Y,V) + g_F(Y,U)g_F(X,V)] \\ &+ \delta f^4[g_F(X,U)g_F(Y,W) + g_F(Y,U)g_F(X,W)], \end{aligned}$$

we can construct a compact hyper-generalized quasi Einstein warped product space $M = B \times_f F$ with

$$\begin{aligned} S_M(X,Y) =& \alpha g_M(X,Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] \\ &+ \delta[A(X)D(Y) + A(Y)D(X)], \end{aligned}$$

where U, V and W are mutually orthogonal and tangent to the fiber F.

Proof. By considering the trace of both sides of (2.d), we get

$$r = \alpha m - k\frac{\Delta f}{f}, \tag{4.5.6}$$

where r is the scalar curvature of B.

In view of (4.5.6) and (4.5.3), we get

$$\mathrm{div}S(X) = \frac{k}{2f^2}\{\Delta f \mathrm{df} - f\mathrm{d}(\Delta \mathrm{f})(X)\}. \tag{4.5.7}$$

So, from (2.d) and (4.5.1), we obtain

$$\mathrm{div}\left(\frac{1}{f}H^f\right)(X) = \frac{1}{2f^2}\{(k-1)\mathrm{d}(|\nabla \mathrm{f}|^2) - 2\mathrm{fd}(\Delta \mathrm{f}) + 2\alpha \mathrm{fdf}\}. \tag{4.5.8}$$

But, (2.d) implies $\mathrm{div}S_B = \mathrm{div}\left(\frac{k}{f}H^f\right)$. So, from (4.5.7) and (4.5.8) it follows that

$$\mathrm{d}(-\mathrm{f}\Delta \mathrm{f} + (\mathrm{k}-1)|\nabla \mathrm{f}|^2 + \alpha \mathrm{f}^2) = 0,$$
$$i.e., \ -f\Delta f + (k-1)|\nabla f|^2 + \alpha f^2 = \mu,$$

where μ is some constant. This completes the proof of the first part of the Proposition 4.5.5. Now if (F, g_F) is a k dimensional compact Riemannian manifold with

$$\begin{aligned} S_F(X,Y) =& g_F(X,Y)[\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2] + \beta f^4 g_F(X,U)g_F(Y,U) \\ &+ \gamma f^4[g_F(X,U)g_F(Y,V) + g_F(Y,U)g_F(X,V)] \\ &+ \delta f^4[g_F(X,U)g_F(Y,W) + g_F(Y,U)g_F(X,W)], \end{aligned}$$

then we can make a compact hyper-generalized quasi-Einstein warped product $M = B \times_f F$ with respect to the sufficient condition of the Result 4.5.4. □

Now we state the following theorem.

Theorem 4.5.6. *If $M = B \times_f F$ is a compact hyper-generalized quasi-Einstein warped product space of non positive scalar curvature, then the warped product will be a Riemannian product.*

Proof. See [92] for proof. □

4.6 The generators U,V and W as concurrent vector fields

Definition 4.6.1 (Concurrent vector field)**.** *[108] A vector field η is concurrent if it satisfies the following condition*

$$\nabla_X \eta = \lambda X, \tag{4.6.1}$$

where λ $(\neq 0)$ is a constant.

If $\lambda = 0$, then the vector field turns into a parallel vector field.

Here we take the concurrent vector fields U,V and W with respect to the associated 1-forms A,B and D respectively.

Then we get,

$$(\nabla_X A)(Y) = ag(X,Y), \tag{4.6.2}$$

$$(\nabla_X B)(Y) = bg(X,Y), \tag{4.6.3}$$

$$(\nabla_X D)(Y) = cg(X,Y), \tag{4.6.4}$$

where a,b and c are the non zero constants.

We suppose that α,β,γ and δ are constants and then considering covariant derivative of (1.1.19) with respect to Z , we get

$$\begin{aligned}(\nabla_Z S)(X,Y) =& \beta[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)] \\ &+ \gamma[(\nabla_Z A)(X)B(Y) + A(X)(\nabla_Z B)(Y) \\ &+ (\nabla_Z A)(Y)B(X) + A(Y)(\nabla_Z B)(X)] \\ &+ \delta[(\nabla_Z A)(X)D(Y) + A(X)(\nabla_Z D)(Y) \\ &+ (\nabla_Z A)(Y)D(X) + A(Y)(\nabla_Z D)(X)]. \end{aligned} \tag{4.6.5}$$

Now by using (4.6.2), (4.6.3) and (4.6.4) in (4.6.5), we get

$$\begin{aligned}(\nabla_Z S)(X,Y) =& \beta[ag(Z,X)A(Y) + ag(Z,Y)A(X)] \\ &+ \gamma[ag(Z,X)B(Y) + bg(Z,Y)A(X) + ag(Z,Y)B(X) \\ &+ bg(Z,X)A(Y)] + \delta[ag(Z,X)D(Y) + cg(Z,Y)A(X) \\ &+ ag(Z,Y)D(X) + cg(Z,X)A(Y)]. \end{aligned} \tag{4.6.6}$$

Taking contraction on (4.6.6) over X and Y, we get

$$\mathrm{dr}(Z) = 2a\beta A(Z) + 2\gamma[aB(Z) + bA(Z)] + 2\delta[aD(Z) + cA(Z)], \tag{4.6.7}$$

where r being the scalar curvature of this manifold.

From (1.1.22), we have

$$r = n\alpha + \beta. \tag{4.6.8}$$

Since $\alpha, \beta \in \mathbb{R}$, therefore

$$dr(X) = 0, \ for\ all\ X. \tag{4.6.9}$$

From (4.6.7) and (4.6.9), it follows that

$$\begin{aligned} &a\beta A(Z) + \gamma[aB(Z) + bA(Z)] + \delta[aD(Z) + cA(Z)] = 0, \\ &i.e., \ (a\beta + b\gamma + c\delta)A(Z) + a\gamma B(Z) + a\delta D(Z) = 0, \\ &i.e., \ D(Z) = -\left(\frac{a\beta + b\gamma + c\delta}{a\delta}\right)A(Z) - \frac{\gamma}{\delta}B(Z). \end{aligned} \tag{4.6.10}$$

Since a, b and c are the non zero constants, then with the help of (4.6.10) in (1.1.19), we get

$$S(X,Y) = \alpha g(X,Y) - \left(\frac{a\beta + 2b\gamma + 2c\delta}{a}\right)A(X)A(Y). \tag{4.6.11}$$

Therefore, the manifold turns into a quasi Einstein manifold. Hence, we get the following theorem.

Theorem 4.6.2. *If the associated scalars are constants and the associated vector fields of a $(HGQE)_n$ are concurrent, then the manifold turns into a quasi Einstein manifold.*

4.7 Ricci recurrent $(HGQE)_n$

Definition 4.7.1 (Ricci recurrent)**.** *[95] A $(HGQE)_n$ is Ricci recurrent if its Ricci tensor S of type $(0,2)$ obeys the following condition*

$$(\nabla_X S)(Y,Z) = E(X)S(Y,Z), \tag{4.7.1}$$

where $E(X)$ being a non zero 1-form.

Also, it is known that

$$(\nabla_X S)(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{4.7.2}$$

Using (4.7.2) in (4.7.1), we get

$$E(X)S(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{4.7.3}$$

Using (1.1.19) in (4.7.3), we obtain

$$\begin{aligned}
&E(X)[\alpha g(Y,Z) + \beta A(Y)A(Z) + \gamma\{A(Y)B(Z) + A(Z)B(Y)\} \\
&\quad + \delta\{A(Y)D(Z) + A(Z)D(Y)\}] \\
&= X[\alpha g(Y,Z) + \beta A(Y)A(Z) + \gamma\{A(Y)B(Z) + A(Z)B(Y)\} \\
&\quad + \delta\{A(Y)D(Z) + A(Z)D(Y)\}] - [\alpha g(\nabla_X Y, Z) + \beta A(\nabla_X Y)A(Z) \\
&\quad + \gamma\{A(\nabla_X Y)B(Z) + A(Z)B(\nabla_X Y)\} + \delta\{A(\nabla_X Y)D(Z) \\
&\quad + A(Z)D(\nabla_X Y)\}] - [\alpha g(Y, \nabla_X Z) + \beta A(Y)A(\nabla_X Z) \\
&\quad + \gamma\{A(Y)B(\nabla_X Z) + A(\nabla_X Z)B(Y)\} + \delta\{A(Y)D(\nabla_X Z) \\
&\quad + A(\nabla_X Z)D(Y)\}].
\end{aligned} \tag{4.7.4}$$

Setting $Y = Z = U$ in (4.7.4), we have

$$\begin{aligned}
X(\alpha+\beta) - (\alpha+\beta)E(X) =& 2(\alpha+\beta)A(\nabla_X U) + 2\gamma B(\nabla_X U) \\
& + 2\delta D(\nabla_X U).
\end{aligned} \tag{4.7.5}$$

Since $A(\nabla_X U) = 0$, therefore (4.7.5) becomes

$$\begin{aligned}
X(\alpha+\beta) - (\alpha+\beta)E(X) &= 2\gamma B(\nabla_X U) + 2\delta D(\nabla_X U), \\
\text{i.e., } X(\alpha+\beta) - (\alpha+\beta)E(X) &= 2\gamma g(\nabla_X U, V) + 2\delta g(\nabla_X U, W), \\
\text{i.e., } X(\alpha+\beta) - (\alpha+\beta)E(X) &= -2\gamma g(\nabla_X V, U) - 2\delta g(\nabla_X W, U), \\
\text{i.e., } X(\alpha+\beta) - (\alpha+\beta)E(X) &= -2[g(\gamma\nabla_X V + \delta\nabla_X W, U)], \\
\text{i.e., } X(\alpha+\beta) - (\alpha+\beta)E(X) &= -2A(\nabla_X(\gamma V + \delta W)).
\end{aligned}$$

So, $A(\nabla_X(\gamma V + \delta W)) = 0$ if and only if $X(\alpha+\beta) - (\alpha+\beta)E(X) = 0$.
But $A(\nabla_X(\gamma V + \delta W)) = 0$ implies

$$\begin{aligned}
&\text{either, } \nabla_X(\gamma V + \delta W) \perp U, \\
&\qquad \text{or, } (\gamma V + \delta W) \text{ is a parallel vector field.}
\end{aligned} \tag{4.7.6}$$

Setting $Y = Z = V$ in (4.7.4), we obtain

$$X\alpha - \alpha E(X) = 2\alpha B(\nabla_X V) + 2\gamma A(\nabla_X V). \tag{4.7.7}$$

Since $B(\nabla_X V) = 0$, therefore (4.7.7) becomes

$$X\alpha - \alpha E(X) = 2\gamma A(\nabla_X V).$$

So, $A(\nabla_X V) = 0$ if and only if $X\alpha - \alpha E(X) = 0$. But $A(\nabla_X V) = 0$ implies

$$\text{either, } \nabla_X V \perp U,$$
$$\text{or, V is a parallel vector field.} \tag{4.7.8}$$

Setting $Y = Z = W$ in (4.7.4), we get

$$X\alpha - \alpha E(X) = 2\alpha D(\nabla_X W) + 2\delta A(\nabla_X W). \tag{4.7.9}$$

Since $D(\nabla_X W) = 0$, therefore (4.7.9) becomes

$$X\alpha - \alpha E(X) = 2\delta A(\nabla_X W).$$

So, $A(\nabla_X W) = 0$ if and only if $X\alpha - \alpha E(X) = 0$. But $A(\nabla_X W) = 0$ implies

$$\text{either, } \nabla_X W \perp U,$$
$$\text{or, W is a parallel vector field.} \tag{4.7.10}$$

Thus from (4.7.6), (4.7.8) and (4.7.10), we get the following theorem.

Theorem 4.7.2. *If $(HGQE)_n$ is Ricci recurrent, then*

(i) *Either* $\nabla_X(\gamma V + \delta W) \perp U$

or $(\gamma V + \delta W)$ *is a parallel vector field iff* $X(\alpha+\beta) - (\alpha+\beta)E(X) = 0$.

(ii) *Either* $\nabla_X V \perp U$

or V is a parallel vector field iff $X\alpha - \alpha E(X) = 0$.

(iii) *Either* $\nabla_X W \perp U$

or W is a parallel vector field iff $X\alpha - \alpha E(X) = 0$.

4.8 Einstein's field equation in $(HGQE)_n$

The Einstein's field equation is

$$S(X,Y) - \frac{r}{2} g(X,Y) + \lambda g(X,Y) = kT(X,Y), \tag{4.8.1}$$

where S is the $(0,2)$-type Ricci tensor, r being the scalar curvature, k and λ are the gravitational constant and cosmological constant respectively.

Considering without cosmological constant $(i.e., \lambda = 0)$, then (4.8.1) becomes

$$S(X,Y) - \frac{r}{2} g(X,Y) = kT(X,Y). \tag{4.8.2}$$

With the help of (1.1.19) in (4.8.2), we get

$$\begin{aligned} &\left(\alpha - \frac{r}{2}\right) g(X,Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] \\ &+\delta[A(X)D(Y) + A(Y)D(X)] = kT(X,Y). \end{aligned} \tag{4.8.3}$$

After covariant differentiation on (4.8.3) with respect to Z, we get

$$\begin{aligned} &\beta[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)] + \gamma[(\nabla_Z A)(X)B(Y) \\ &+A(X)(\nabla_Z B)(Y) + (\nabla_Z A)(Y)B(X) + A(Y)(\nabla_Z B)(X)] \\ &+\delta[(\nabla_Z A)(X)D(Y) + A(X)(\nabla_Z D)(Y) + (\nabla_Z A)(Y)D(X) \\ &+A(Y)(\nabla_Z D)(X)] = k(\nabla_Z T)(X,Y). \end{aligned} \tag{4.8.4}$$

Thus by virtue of (4.8.4), we have the following theorem.

Theorem 4.8.1. *If the associated* 1*-forms* A, B *and* D *in a* $(HGQE)_n$ *satisfying Einstein's field equation without cosmological constant are covariant constant, then the energy momentum is also covariant constant.*

4.9 $(HGQE)_4$ spacetime admitting space-matter tensor

Space-matter tensor $\tilde{P}$ of type $(0,4)$ has been introduced by Petrov [98]. He defined the space-matter tensor as follows

$$\tilde{P} = \tilde{R} + \frac{k}{2} g \wedge T - \sigma G, \tag{4.9.1}$$

$\tilde{R}$ being the curvature tensor of type $(0,4)$, T being the energy-momentum tensor of type $(0,2)$, k being the gravitational constant, σ being the energy density and $\wedge$ is the Kulkarni-Nomizu product defined in (1.3.6). Also, G is a tensor of type $(0,4)$ such that

$$G(X,Y,Z,N) = g(Y,Z)g(X,N) - g(X,Z)g(Y,N), \tag{4.9.2}$$

for all $X,Y,Z,N \in \chi(M)$. $\tilde{P}$ is called the space-matter tensor of type $(0,4)$ of M.

Here we study $(HGQE)_4$ spacetime when space-matter tensor is zero. From (4.9.1), we obtain

$$\begin{aligned}\tilde{P}(X,Y,Z,N) =& \tilde{R}(X,Y,Z,N) + \frac{k}{2}[g(Y,Z)T(X,N) + g(X,N)T(Y,Z)\\ &- g(X,Z)T(Y,N) - g(Y,N)T(X,Z)]\\ &- \sigma[g(Y,Z)g(X,N) - g(X,Z)g(Y,N)].\end{aligned} \tag{4.9.3}$$

If $\tilde{P} = 0$ in (4.9.3), we get

$$\begin{aligned}\tilde{R}(X,Y,Z,N) =& -\frac{k}{2}[g(Y,Z)T(X,N) + g(X,N)T(Y,Z)\\ &- g(X,Z)T(Y,N) - g(Y,N)T(X,Z)]\\ &+ \sigma[g(Y,Z)g(X,N) - g(X,Z)g(Y,N)].\end{aligned} \tag{4.9.4}$$

Using (1.1.19) and (4.8.2) in (4.9.4), we derive

$$\begin{aligned}\tilde{R}(X,Y,Z,N) =& \left(\sigma - \alpha + \frac{r}{2}\right)[g(Y,Z)g(X,N) - g(X,Z)g(Y,N)]\\ &- \frac{\beta}{2}[g(Y,Z)A(X)A(N) + g(X,N)A(Y)A(Z)\\ &- g(X,Z)A(Y)A(N) - g(Y,N)A(X)A(Z)]\\ &- \frac{\gamma}{2}[g(Y,Z)\{A(X)B(N) + A(N)B(X)\}\\ &+ g(X,N)\{A(Y)B(Z) + A(Z)B(Y)\}\\ &- g(X,Z)\{A(Y)B(N) + A(N)B(Y)\}\\ &- g(Y,N)\{A(X)B(Z) + A(Z)B(X)\}]\\ &- \frac{\delta}{2}[g(Y,Z)\{A(X)D(N) + A(N)D(X)\}\end{aligned}$$

$$
\begin{aligned}
&+g(X,N)\{A(Y)D(Z)+A(Z)D(Y)\}\\
&-g(X,Z)\{A(Y)D(N)+A(N)D(Y)\}\\
&-g(Y,N)\{A(X)D(Z)+A(Z)D(X)\}].
\end{aligned} \tag{4.9.5}
$$

In view of (1.1.24), (4.9.5) follows that the manifold is a manifold of hyper-generalized quasi constant curvature. Thus we get the following theorem.

Theorem 4.9.1. *A $(HGQE)_4$ spacetime satisfying Einstein's field equation without cosmological constant with zero space-matter tensor will be a spacetime of hyper-generalized quasi constant curvature.*

Finally, we study to get sufficient condition for which $(HGQE)_4$ may be a divergence free space-matter tensor. From (1.1.22), we get

$$r = n\alpha + \beta$$

$$i.e.,\ r = \text{constant}.$$

This implies $\mathrm{dr}(X) = 0$, for all X.

With the help of (4.8.2) and (4.9.3) we get

$$
\begin{aligned}
(\mathrm{div}P)(X,Y,Z) =&(\mathrm{div}R)(X,Y,Z) + \frac{1}{2}[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)]\\
&- g(Y,Z)\left[\frac{1}{4}\mathrm{dr}(X) + \mathrm{d}\sigma(X)\right]\\
&+ g(X,Z)\left[\frac{1}{4}\mathrm{dr}(Y) + \mathrm{d}\sigma(Y)\right].
\end{aligned} \tag{4.9.6}
$$

For a semi-Riemannian manifold,

$$(\mathrm{div}R)(X,Y,Z) = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \tag{4.9.7}$$

From (4.9.6) and (4.9.7), we deduce

$$
\begin{aligned}
(\mathrm{div}P)(X,Y,Z) =&\frac{3}{2}[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)]\\
&- g(Y,Z)\left[\frac{1}{4}\mathrm{dr}(X) + \mathrm{d}\sigma(X)\right]\\
&+ g(X,Z)\left[\frac{1}{4}\mathrm{dr}(Y) + \mathrm{d}\sigma(Y)\right].
\end{aligned} \tag{4.9.8}
$$

Let us assume that $(\mathrm{div}P)(X,Y,Z) = 0$ and taking contraction on (4.9.8) over Y and Z, we get $\mathrm{d}\sigma(X) = 0$. Thus we obtain the following theorem.

Theorem 4.9.2. *In a $(HGQE)_4$ spacetime satisfying Einstein's field equation without cosmological constant with divergence free space-matter tensor, the energy density is constant.*

Now using (1.1.19) in (4.9.8), we have

$$\begin{aligned}
&(\mathrm{div}P)(X,Y,Z)\\
=&\frac{3}{2}[\mathrm{d}\alpha(X)g(Y,Z)-\mathrm{d}\alpha(Y)g(X,Z)]+\frac{3}{2}[\mathrm{d}\beta(X)A(Y)A(Z)\\
&-\mathrm{d}\beta(Y)A(X)A(Z)]+\frac{3\beta}{2}[(\nabla_X A)(Y)A(Z)+A(Y)(\nabla_X A)(Z)\\
&-(\nabla_Y A)(X)A(Z)-A(X)(\nabla_Y A)(Z)]+\frac{3}{2}[\mathrm{d}\gamma(X)\{A(Y)B(Z)\\
&+B(Y)A(Z)\}-\mathrm{d}\gamma(Y)\{A(X)B(Z)+B(X)A(Z)\}]+\frac{3\gamma}{2}[(\nabla_X A)(Y)B(Z)\\
&+A(Y)(\nabla_X B)(Z)+(\nabla_X A)(Z)B(Y)+A(Z)(\nabla_X B)(Y)-(\nabla_Y A)(X)B(Z)\\
&-A(X)(\nabla_Y B)(Z)-(\nabla_Y A)(Z)B(X)-A(Z)(\nabla_Y B)(X)]\\
&+\frac{3}{2}[\mathrm{d}\delta(X)\{A(Y)D(Z)+D(Y)A(Z)\}-\mathrm{d}\delta(Y)\{A(X)D(Z)+D(X)A(Z)\}]\\
&+\frac{3\delta}{2}[(\nabla_X A)(Y)D(Z)+A(Y)(\nabla_X D)(Z)+(\nabla_X A)(Z)D(Y)\\
&+A(Z)(\nabla_X D)(Y)-(\nabla_Y A)(X)D(Z)-A(X)(\nabla_Y D)(Z)-(\nabla_Y A)(Z)D(X)\\
&-A(Z)(\nabla_Y D)(X)]-g(Y,Z)\left[\frac{1}{4}\mathrm{dr}(X)+\mathrm{d}\sigma(X)\right]\\
&+g(X,Z)\left[\frac{1}{4}\mathrm{dr}(Y)+\mathrm{d}\sigma(Y)\right].
\end{aligned}\tag{4.9.9}$$

Considering the conditions that $\sigma,\alpha,\beta,\gamma$ and δ are constants and the generator U is a parallel vector field (i.e., $\nabla_X U=0$), we get

$$\mathrm{dr}(X)=0,\ \mathrm{d}\sigma(X)=0,\ \forall X\ \text{ and }\ g(\nabla_X U,Y)=0\ \text{ i.e., }\ (\nabla_X A)(Y)=0. \tag{4.9.10}$$

In view of [56], we derive

$$\alpha+\beta=0,\gamma=0,\delta=0. \tag{4.9.11}$$

Using (4.9.10) and (4.9.11) in (4.9.9), we get $(\mathrm{div}P)(X,Y,Z)=0$.
Hence we get the following theorem.

Theorem 4.9.3. *If in a $(HGQE)_4$ spacetime with parallel vector field U satisfying Einstein's field equation without cosmological constant, the energy density and the associated scalars are constants, then the divergence of the space-matter tensor vanishes.*

4.10 General relativistic viscous fluid $(HGQE)_4$ spacetime

Let us consider (M^4, g) be a connected semi-Riemannian viscous fluid spacetime admitting heat flux obeying Einstein's field equation.

For the fluid matter distribution, the energy momentum tensor has been given by Ellis [48] as

$$\begin{aligned} T(X,Y) =& (\sigma + p)A(X)A(Y) + pg(X,Y) + A(X)B(Y) \\ &+ A(Y)B(X) + A(X)D(Y) + A(Y)D(X), \end{aligned} \tag{4.10.1}$$

with

$$\begin{aligned} &g(X,U) = A(X),\ g(X,V) = B(X),\ g(X,W) = D(X), \\ &A(U) = -1,\ B(V) = 1,\ D(W) = 1, \\ &g(U,V) = 0,\ g(V,W) = 0,\ g(U,W) = 0, \end{aligned}$$

where σ is the matter density, p is the isotropic pressure, U is the timelike velocity vector field, V is the heat conduction vector field and W is the stress vector field.

Using (4.10.1) in (4.8.1), we get

$$\begin{aligned} S(X,Y) =& (kp + \frac{r}{2} - \lambda)g(X,Y) + k(\sigma + p)A(X)A(Y) \\ &+ k[A(X)B(Y) + A(Y)B(X)] \\ &+ k[A(X)D(Y) + A(Y)D(X)]. \end{aligned} \tag{4.10.2}$$

Clearly, it follows that this spacetime is a $(HGQE)_4$ spacetime whose associated scalars are $(kp + \frac{r}{2} - \lambda)$, $k(\sigma + p)$, k and k. A, B and D are associated 1-forms and generators are U, V and W. Hence, we get the following theorem.

Theorem 4.10.1. *A viscous fluid space time admitting heat flux and obeying Einstein's field equation with cosmological constant is a connected semi-Riemannian hyper-generalized quasi Einstein manifold of dimension four.*

From (1.1.22), we get for (M^4, g)

$$r = 4\alpha + \beta. \tag{4.10.3}$$

Now using (1.1.19) and (4.10.3) in (4.10.2), we gain

$$\begin{aligned}\left(\frac{2kp+2\alpha+\beta-2\lambda}{2}\right) g(X,Y) =& [\beta - k(\sigma+p)]A(X)A(Y) \\ &+ (\gamma-k)[A(X)B(Y)+B(X)A(Y)] \\ &+ (\delta-k)[A(X)D(Y)+A(Y)D(X)]. \end{aligned}\tag{4.10.4}$$

Putting $X = Y = U$ in (4.10.4), we find

$$\sigma = \frac{2\alpha+3\beta-2\lambda}{2k}. \tag{4.10.5}$$

Taking contraction on (4.10.2) over X and Y, we deduce

$$r = 4(kp + \frac{r}{2} - \lambda) - k(\sigma + p). \tag{4.10.6}$$

In view of (4.10.3) and (4.10.5), (4.10.6) implies that

$$p = \frac{6\lambda - 6\alpha + \beta}{6k}. \tag{4.10.7}$$

By putting $X = Y = V$ and $X = Y = W$ in (4.10.4), we obtain the same value of p in each case given by

$$p = \frac{2\lambda - 2\alpha - \beta}{2k}. \tag{4.10.8}$$

As α, β are not constants, then in view of (4.10.5), (4.10.6) and (4.10.8) it follows that σ and p are not constants. Hence, we get the following theorem.

Theorem 4.10.2. *If a viscous fluid $(HGQE)_4$ spacetime admitting heat flux satisfies Einstein's field equation with cosmological constant, then isotropic pressure and energy density of the fluid can not be a constant.*

If α, β are constants, then from (4.10.5) and (4.10.7), it implies that σ and p are constants. As $\sigma > 0$, $p > 0$, so we obtain from (4.10.5) and (4.10.7) that $\lambda < \frac{2\alpha+3\beta}{2}$ and $\lambda > \frac{6\alpha-\beta}{6}$, which implies

$$\frac{6\alpha-\beta}{6} < \lambda < \frac{2\alpha+3\beta}{2}.$$

Also, (4.10.8) gives $\frac{2\alpha+\beta}{2} < \lambda$.

Hence, we get the following theorem.

Theorem 4.10.3. *If a viscous fluid $(HGQE)_4$ spacetime admitting heat flux satisfies Einstein's field equation with cosmological constant, then cosmological constant λ obeys the following condition either $\frac{6\alpha-\beta}{6} < \lambda < \frac{2\alpha+3\beta}{2}$ or, $\frac{2\alpha+\beta}{2} < \lambda$.*

Now we consider a hyper-generalized quasi Einstein spacetime satisfying Einstein's field equation without cosmological constant (i.e., $\lambda = 0$) whose matter content is viscous fluid. Putting $\lambda = 0$ in (4.10.2), then (4.10.2) becomes

$$\begin{aligned} S(X,Y) =&(kp+\frac{r}{2})g(X,Y)+k(\sigma+p)A(X)A(Y) \\ &+k[A(X)B(Y)+A(Y)B(X)] \\ &+k[A(X)D(Y)+A(Y)D(X)]. \end{aligned} \tag{4.10.9}$$

By comparing (1.1.19) and (4.10.9), we obtain

$$\alpha = kp+\frac{r}{2}, \beta = k(\sigma+p), \gamma = k, \delta = k. \tag{4.10.10}$$

Taking contraction on (4.10.9) over X and Y, we get

$$r = k(\sigma - 3p). \tag{4.10.11}$$

Using (4.10.11) in (4.10.9), it follows that

$$\begin{aligned} S(X,Y) =&\frac{k(\sigma-p)}{2}g(X,Y)+k(\sigma+p)A(X)A(Y) \\ &+k[A(X)B(Y)+A(Y)B(X)] \\ &+k[A(X)D(Y)+A(Y)D(X)]. \end{aligned} \tag{4.10.12}$$

Suppose Q is the Ricci operator given by $g(QX,Y) = S(X,Y)$ and $S(QX,Y) = S^2(X,Y)$. Therefore, we get $A(QX) = g(QX,U) = S(X,U)$,

$B(QX) = g(QX,V) = S(X,V)$ and $D(QX) = g(QX,W) = S(X,W)$.
Hence from (4.10.12) we have the following equation

$$\begin{aligned} S(QX,Y) =& \frac{k(\sigma - p)}{2} S(X,Y) + k(\sigma + p)S(X,U)A(Y) \\ & + k[S(X,U)B(Y) + A(Y)S(X,V)] \\ & + k[S(X,U)D(Y) + A(Y)S(X,W)]. \end{aligned} \tag{4.10.13}$$

Contracting (4.10.13) over X and Y, we get

$$\begin{aligned} S^2(X,X) = \|Q\|^2 =& \frac{k(\sigma - p)r}{2} + k(\sigma + p)S(U,U) \\ & + 2kS(U,V) + 2kS(U,W). \end{aligned} \tag{4.10.14}$$

From (1.1.19), (4.10.10) and (4.10.11), we obtain

$$S(U,U) = \beta - \alpha = \frac{k(\sigma + 3p)}{2}. \tag{4.10.15}$$

$$S(U,V) = -\gamma = -k. \tag{4.10.16}$$

$$S(U,W) = -\delta = -k. \tag{4.10.17}$$

Using (4.10.15), (4.10.16) and (4.10.17) in (4.10.14), we derive

$$\|Q\|^2 = k^2(\sigma^2 + 3p^2 - 4). \tag{4.10.18}$$

Hence, we can state the following theorem.

Theorem 4.10.4. *If a viscous fluid $(HGQE)_4$ spacetime satisfying Einstein's field equation without cosmological constant, then the square of the length of Ricci operator is $k^2(\sigma^2 + 3p^2 - 4)$.*

Now, if we consider

$$\sigma > 3p. \tag{4.10.19}$$

From (4.10.18) it follows that

$$k^2(\sigma^2 + 3p^2 - 4) > 0,$$

$$i.e., \quad \sigma^2 + 3p^2 > 4. \tag{4.10.20}$$

In view of (4.10.19) and (4.10.20), we obtain

$$\sigma^2 + \frac{\sigma^2}{3} > \sigma^2 + 3p^2 > 4,$$

which gives $\sigma > \sqrt{3}$. Hence, we get the following corollary.

Corollary 4.10.5. *In a viscous fluid $(HGQE)_4$ spacetime satisfying Einstein's field equation without cosmological constant with $\sigma > 3p$ and $p > 0$, the energy density is greater than $\sqrt{3}$.*

4.11 Example of $(HGQE)_4$ Spacetime

In this section, we give a non trivial example of $(HGQE)_4$ spacetime to ensure its existence. We take a Lorentzian metric g on M^4 by

$$ds^2 = g_{ij}dx^i dx^j = -\frac{k}{r}(dt)^2 + \frac{1}{\frac{c}{r} - 4}(dr)^2 + r^2(d\theta)^2 + (r\sin\theta)^2(d\phi)^2,$$

where $i, j = 1, 2, 3, 4$ and k, c are constants. Then non zero components of Christofell symbols, curvature tensors and Ricci tensors are given below.

$$\left.\begin{array}{r}\Gamma^2_{33} = 4r - c, \Gamma^1_{12} = -\dfrac{1}{2r}, \Gamma^2_{22} = \dfrac{c}{2r(c-4r)}, \Gamma^3_{32} = \Gamma^4_{42} = \dfrac{1}{r}, \\ \Gamma^4_{43} = \cot\theta, \Gamma^2_{44} = (4r-c)(\sin\theta)^2, \Gamma^3_{44} = -\dfrac{\sin(2\theta)}{2}\end{array}\right\} \quad (4.11.1)$$

$$\left.\begin{array}{r}R_{1221} = -\dfrac{k(c-3r)}{r^3(c-4r)}, R_{1331} = \dfrac{k(c-4r)}{2r^2}, R_{1441} = \dfrac{k(c-4r)(\sin\theta)^2}{2r^2} \\ R_{2332} = \dfrac{c}{2(4r-c)}, R_{2442} = \dfrac{c(\sin\theta)^2}{2(4r-c)}, R_{3443} = r(c-5r)(\sin\theta)^2 \\ R_{11} = -\dfrac{k}{r^3}, R_{22} = -\dfrac{3}{r(c-4r)}, R_{33} = -3, R_{44} = -3(\sin\theta)^2\end{array}\right\} \quad (4.11.2)$$

From (4.11.1) and (4.11.2) it follows that M^4 is a Lorentzian manifold of non zero scalar curvature ($= -\frac{8}{r^2}$). Now our aim is to show that this manifold is $(HGQE)_4$. Suppose α, β, γ and δ are the associated scalars and we consider these scalars by the following way

$$\alpha = -\frac{3}{r^2}, \beta = -\frac{4}{r^2}, \gamma = \frac{2}{r^2}, \delta = \frac{3}{r^2} \quad (4.11.3)$$

and the associated 1-forms are as follows

$$A_i(x) = \begin{cases} \sqrt{\frac{k}{r}} & \text{for} \quad i = 1 \\ 0 & \text{for} \quad i = 2,3,4 \end{cases} \quad ; \quad B_i(x) = \begin{cases} \frac{1}{2r^2} & \text{for} \quad i = 4 \\ 0 & \text{for} \quad i = 1,2,3 \end{cases}$$

and $D_i(x) = \begin{cases} -\frac{1}{3r^2} & \text{for} \quad i = 4 \\ 0 & \text{for} \quad i = 1,2,3 \end{cases}$

Thus we get,

(i) $R_{11} = \alpha g_{11} + \beta A_1 A_1 + \gamma[A_1 B_1 + B_1 A_1] + \delta[A_1 D_1 + D_1 A_1]$

(ii) $R_{22} = \alpha g_{22} + \beta A_2 A_2 + \gamma[A_2 B_2 + B_2 A_2] + \delta[A_2 D_2 + D_2 A_2]$

(iii) $R_{33} = \alpha g_{33} + \beta A_3 A_3 + \gamma[A_3 B_3 + B_3 A_3] + \delta[A_3 D_3 + D_3 A_3]$

(iv) $R_{44} = \alpha g_{44} + \beta A_4 A_4 + \gamma[A_4 B_4 + B_4 A_4] + \delta[A_4 D_4 + D_4 A_4]$

Since the other Ricci tensors except R_{11}, R_{22}, R_{33} and R_{44} are zero, so we have $R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma[A_i B_j + B_i A_j] + \delta[A_i D_j + D_i A_j], i, j = 1,2,3,4$. It is clearly seen that its scalar curvature $= 4\alpha - \beta = -\frac{8}{r^2}$. Therefore, (M^4, g) is a hyper-generalized quasi Einstein manifold. So we have the following example.

Example 4.11.1. *Suppose (M^4, g) is a Lorentzian manifold equipped with the Lorentzian metric g given by*

$$ds^2 = g_{ij} dx^i dx^j = -\frac{k}{r}(dt)^2 + \frac{1}{\frac{c}{r} - 4}(dr)^2 + r^2(d\theta)^2 + (r\sin\theta)^2(d\phi)^2,$$

where $i, j = 1,2,3,4$ and k, c are constants. Then (M^4, g) is a $(HGQE)_4$ space time with non constant and non zero scalar curvature.

4.12 A spacetime admitting vanishing $\mathscr{T}$-curvature tensor

In this unit we consider V_4 as a spacetime of dimension four in general relativity for entire study. The following results have been obtained from (4.2.6).

Theorem 4.12.1. *If $(c_0 + 4c_1 + c_2 + c_3 + c_5 + c_6) \neq 0$ where $c_0, c_1, c_2, c_3, c_5, c_6$ are smooth functions on an n dimensional pseudo-Riemannian manifold (M, g), then a $\mathscr{T}$-flat spacetime is an Einstein spacetime.*

Proof. For a $\mathscr{T}$-flat spacetime $\tilde{\mathscr{T}}(X,Y,Z,W)=0$. Then from (4.2.7), we obtain

$$\begin{aligned}0 =& c_0R(X,Y,Z,W)\\ &+c_1S(Y,Z)g(X,W)+c_2S(X,Z)g(Y,W)\\ &+c_3S(X,Y)g(Z,W)+c_4g(Y,Z)S(X,W)\\ &+c_5g(X,Z)S(Y,W)+c_6g(X,Y)S(Z,W)\\ &+rc_7\left[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\right].\end{aligned} \tag{4.12.1}$$

Taking contraction on both sides over X and W, we derive

$$S(Y,Z)=-\left[\frac{r(c_4+3c_7)}{(c_0+4c_1+c_2+c_3+c_5+c_6)}\right]g(Y,Z). \tag{4.12.2}$$

Let $\alpha=-\left[\frac{r(c_4+3c_7)}{c_0+4c_1+c_2+c_3+c_5+c_6}\right]$. Then (4.12.2) becomes

$$S(Y,Z)=\alpha g(Y,Z). \tag{4.12.3}$$

Clearly, if $(c_0+4c_1+c_2+c_3+c_5+c_6)\neq 0$ then this is an Einstein spacetime. □

Theorem 4.12.2. *If $c_0\neq 0$, $c_3+c_6=0$, $(c_1+c_2+c_4+c_5)=0$ and $(c_0+4c_1+c_2+c_3+c_5+c_6)\neq 0$ where $c_0,c_1,c_2,c_3,c_4,c_5,c_6$ are smooth functions on an n dimensional pseudo-Riemannian manifold (M,g), then a $\mathscr{T}$-flat spacetime is a spacetime with constant curvature.*

Proof. In view of (4.12.3), (4.12.1) implies that

$$\begin{aligned}R(X,Y,Z,W)=&-\left[\frac{(c_1+c_4)\alpha+rc_7}{c_0}\right][g(Y,Z)g(X,W)\\ &+\left[\frac{rc_7-(c_2+c_5)\alpha}{c_0}\right]g(X,Z)g(Y,W)]\\ &-\frac{\alpha(c_3+c_6)}{c_0}g(X,Y)g(Z,W).\end{aligned} \tag{4.12.4}$$

It clearly follows that if $c_0\neq 0$, $c_3+c_6=0, (c_1+c_2+c_4+c_5)=0$ and $(c_0+4c_1+c_2+c_3+c_5+c_6)\neq 0$ then

$$R(X,Y,Z,W)=\left[\frac{(c_1+c_4)\alpha+rc_7}{c_0}\right][g(X,Z)g(Y,W)-g(Y,Z)g(X,W)].$$

That is, a $\mathscr{T}$-flat spacetime is a spacetime with constant curvature with respect to the above conditions. □

Theorem 4.12.3. *The energy momentum tensor is covariant constant in $\mathscr{T}$-flat spacetime satisfying the Einstein's field equation with the cosmological constant.*

Proof. We consider a spacetime satisfying the Einstein's field equation with the cosmological constant (4.8.1).

In view of (4.12.3) and (4.8.1), we derive

$$T(X,Y) = \frac{1}{k}\left(\alpha - \frac{r}{2} + \lambda\right) g(X,Y). \tag{4.12.5}$$

By taking the covariant derivative with respect to Z on both sides, we gain

$$(\nabla_Z T)(X,Y) = -\frac{1}{k}\left[\frac{(c_4+3c_7)}{(c_0+4c_1+c_2+c_3+c_5+c_6)} + \frac{1}{2}\right] \mathrm{d}r(Z) g(X,Y). \tag{4.12.6}$$

As a $\mathscr{T}$-flat spacetime is an Einstein spacetime with the condition $(c_0+4c_1+c_2+c_3+c_5+c_6) \neq 0$, hence the scalar curvature r is a constant. Therefore,

$$\mathrm{d}r(Z) = 0, \;\; \forall Z. \tag{4.12.7}$$

(4.12.6) and (4.12.7) jointly imply that

$$(\nabla_Z T)(X,Y) = 0.$$

Thus the energy momentum tensor $T(X,Y)$ is covariant constant. □

Theorem 4.12.4. *If a spacetime M with $\mathscr{T}$-curvature tensor with respect to a Killing vector field ξ is curvature collineation then the Lie derivative of $\mathscr{T}$-curvature tensor vanishes along ξ.*

Proof. The geometrical symmetries of a spacetime can be written as

$$£_\xi A - 2\Omega A = 0, \tag{4.12.8}$$

where A is the physical or geometrical quantity, Ω is a scalar and $£_\xi$ represents the Lie derivative with respect to ξ.

For the metric inheritance symmetry we put $A = g$ in (4.12.8). Thus

$$\left(£_\xi g\right)(X,Y) - 2\Omega g(X,Y) = 0. \tag{4.12.9}$$

Clearly, in this case if $\Omega = 0$ then ξ becomes a Killing vector field. Let a spacetime M with $\mathscr{T}$-curvature tensor with respect to a Killing vector field ξ be curvature collineation. Thus we gain

$$\left(\pounds_{\xi} g\right)(X,Y) = 0. \tag{4.12.10}$$

As M is admitting a curvature collineation, hence we derive from (4.2.8) that

$$\left(\pounds_{\xi} S\right)(X,Y) = 0, \tag{4.12.11}$$

where S denotes the Ricci tensor.

We take the Lie derivative of (4.2.6) and then with the help of (4.2.8), (4.12.10) and (4.12.11), we derive $\left(\pounds_{\xi}\mathscr{T}\right)(X,Y)Z = 0$. □

Theorem 4.12.5. *Let a spacetime satisfying the Einstein's field equation with cosmological constant be $\mathscr{T}$-flat. The spacetime admits the matter collineation with respect to ξ if and only if ξ is a Killing vector field.*

Proof. The symmetry of energy momentum tensor T is called matter collineation and it is defined by

$$\left(\pounds_{\xi} T\right)(X,Y) = 0,$$

where ξ is the symmetry generating vector field and $\pounds_{\xi}$ is the operator of Lie derivative along ξ.

Let ξ be a Killing vector field of vanishing $\mathscr{T}$-curvature tensor. Therefore

$$\left(\pounds_{\xi} g\right)(X,Y) = 0. \tag{4.12.12}$$

Taking the Lie derivative on both the sides of (4.12.5) with respect to ξ, we have

$$\frac{1}{k}\left(\alpha - \frac{r}{2} + \lambda\right)\left(\pounds_{\xi} g\right)(X,Y) = \left(\pounds_{\xi} T\right)(X,Y). \tag{4.12.13}$$

Using (4.12.12) in (4.12.13), we have

$$\left(\pounds_{\xi} T\right)(X,Y) = 0. \tag{4.12.14}$$

This proves that the spacetime admits the matter collineation.

For the converse part, let $\left(\pounds_{\xi} T\right)(X,Y) = 0$. Therefore from (4.12.13), we find

$$\left(\pounds_{\xi} g\right)(X,Y) = 0.$$

This shows that ξ is a Killing vector field. □

Theorem 4.12.6. *Let a spacetime satisfying the Einstein's field equation be of vanishing $\mathscr{T}$-curvature tensor. The vector field ξ is a conformal Killing vector field if and only if the energy momentum tensor has the Lie inheritance property with respect to ξ.*

Proof. Let ξ be a conformal Killing vector field. Therefore,

$$\left(\pounds_{\xi} g\right)(X,Y) = 2\phi g(X,Y), \tag{4.12.15}$$

where ϕ is being a scalar.

Now, from (4.12.13), it follows that

$$\left(\alpha - \frac{r}{2} + \lambda\right) 2\phi g(X,Y) = k\left(\pounds_{\xi} T\right)(X,Y). \tag{4.12.16}$$

With the help of (4.12.5) in (4.12.16), we have

$$\left(\pounds_{\xi} T\right)(X,Y) = 2\phi T(X,Y). \tag{4.12.17}$$

This shows that the energy momentum tensor has the Lie inheritance property with respect to ξ.

For the converse part, let the energy momentum tensor have the Lie inheritance property with respect to ξ. Therefore,

$$\left(\pounds_{\xi} T\right)(X,Y) = 2\phi T(X,Y).$$

Clearly, (4.12.15) holds good. This proves that ξ is a conformal Killing vector field. □

4.13 General relativistic viscous fluid spacetime admitting vanishing $\mathscr{T}$-curvature tensor

In this unit we consider the general relativistic viscous fluid spacetime admitting vanishing $\mathscr{T}$-curvature tensor satisfying the Einstein's field equation without cosmological constant with the condition $\sigma + p = 0$ where p, σ are respectively the isotropic pressure and the energy density. Furthermore, $\sigma + p = 0$ implies that the fluid behaves like a cosmological constant [116] and it is also called the phantom barrier [27]. The choice $\sigma = -p$ leads to the rapid expansion of this spacetime in cosmology and it is called inflation [4]. We obtain the following theorems.

Theorem 4.13.1. *If a $\mathscr{T}$-flat general relativistic viscous fluid spacetime with the condition $\sigma + p = 0$ where p, σ are respectively the isotropic pressure and the energy density satisfies the Einstein's field equation without cosmological constant, then*

$$\|Q\|^2 = \frac{4k^2p^2(c_4+3c_7)^2}{(c_0+4c_1+c_2+c_3+2c_4+c_5+c_6+6c_7)^2},$$

where Q is the Ricci operator.

Proof. In a general relativistic viscous fluid spacetime with the condition $\sigma + p = 0$, the energy momentum tensor T takes the form [84]

$$T(X,Y) = pg(X,Y), \tag{4.13.1}$$

where p is the isotropic pressure, σ denotes the energy density and $g(U,U) = -1$, U is the velocity vector field of this flow.

The field equation of Einstein without cosmological constant takes the form

$$S(X,Y) - \frac{r}{2}g(X,Y) = kT(X,Y), \tag{4.13.2}$$

where r denotes the scalar curvature and $k \neq 0$.

Using (4.12.3) and (4.13.1) in (4.13.2), we have

$$\left(\alpha - \frac{r}{2} - kp\right) g(X,Y) = 0. \tag{4.13.3}$$

Taking contraction on both sides over X and Y, we derive

$$r=-\frac{2pk(c_0+4c_1+c_2+c_3+c_5+c_6)}{(c_0+4c_1+c_2+c_3+2c_4+c_5+c_6+6c_7)}. \tag{4.13.4}$$

From (4.12.3) and (4.13.4), it implies that

$$S(X,Y)=\frac{2pk(c_4+3c_7)}{(c_0+4c_1+c_2+c_3+2c_4+c_5+c_6+6c_7)}g(X,Y). \tag{4.13.5}$$

If Q is the Ricci operator then $g(QX,Y)=S(X,Y)$ and $S(QX,Y)=S^2(X,Y)$. From (4.13.5), we have

$$S(QX,Y)=\frac{4p^2k^2(c_4+3c_7)^2}{(c_0+4c_1+c_2+c_3+2c_4+c_5+c_6+6c_7)^2}g(X,Y). \tag{4.13.6}$$

Taking contraction on both sides over X and Y, we get

$$\|Q\|^2=\frac{4p^2k^2(c_4+3c_7)^2}{(c_0+4c_1+c_2+c_3+2c_4+c_5+c_6+6c_7)^2}. \tag{4.13.7}$$

□

Theorem 4.13.2. *If a $\mathscr{T}$-flat general relativistic viscous fluid spacetime with the condition $\sigma+p=0$ where p,σ are respectively the isotropic pressure and the energy density obeying the Einstein's field equation without cosmological constant satisfies the condition of timelike convergence then this spacetime also satisfies the relation*

$$\frac{p(c_4+3c_7)}{(c_0+4c_1+c_2+c_3+2c_4+c_5+c_6+6c_7)}<0.$$

Proof. The condition of timelike convergence [104] is given by

$$S(X,X)>0, \tag{4.13.8}$$

for any timelike vector field X.

From (4.13.1) and (4.13.2), it follows that

$$S(X,Y)-\frac{r}{2}g(X,Y)=kpg(X,Y). \tag{4.13.9}$$

Setting $X=Y=U$ in (4.13.9) and with the help of (4.13.4), we have

$$S(U,U)=-\frac{2pk(c_4+3c_7)}{(c_0+4c_1+c_2+c_3+2c_4+c_5+c_6+6c_7)}. \tag{4.13.10}$$

Since $k > 0$ and $S(U,U) > 0$, so we obtain

$$\frac{p(c_4 + 3c_7)}{(c_0 + 4c_1 + c_2 + c_3 + 2c_4 + c_5 + c_6 + 6c_7)} < 0. \tag{4.13.11}$$

□

Theorem 4.13.3. *For a purely electromagnetic distribution the scalar curvature of a $\mathscr{T}$-flat spacetime with the condition $\sigma + p = 0$ where p, σ are respectively the isotropic pressure and the energy density satisfying the Einstein's field equation without cosmological constant is zero.*

Proof. Taking contraction on both sides of (4.13.2) over X and Y, we gain

$$r = -kt, \tag{4.13.12}$$

where t is the trace of T.
Using (4.13.12) in (4.13.2), we derive

$$S(X,Y) = kT(X,Y) - \frac{kt}{2} g(X,Y). \tag{4.13.13}$$

For a purely electromagnetic distribution the Einstein's field equation without cosmological constant is given by

$$S(X,Y) = kT(X,Y). \tag{4.13.14}$$

From (4.13.13) and (4.13.14), it implies that $t = 0$. Hence, we obtain $r = 0$ from (4.13.12). □

4.14 General relativistic viscous fluid spacetime admitting divergence-free $\mathscr{T}$-curvature tensor

This part is devoted to study the general relativistic viscous fluid spacetime admitting the divergence-free $\mathscr{T}$-curvature tensor. We have the following theorems in this regard.

Theorem 4.14.1. *In a general relativistic viscous fluid spacetime admitting divergence-free $\mathscr{T}$-curvature tensor, if $c_1 + c_2 = 0, c_0 \neq 0$ and $c_3 = 0$ then the energy momentum tensor is of Codazzi type.*

Proof. From (4.2.6), we have

$$\begin{aligned}(\text{div } \mathscr{T})(X,Y,Z) =& (c_0 + c_1)(\nabla_X S)(Y,Z) + (c_2 - c_0)(\nabla_Y S)(X,Z) \\ &+ c_3(\nabla_Z S)(X,Y) + \left(\frac{c_4}{2} + c_7\right) g(Y,Z)dr(X) \\ &+ \left(\frac{c_5}{2} - c_7\right) g(X,Z)\text{dr}(Y) + \frac{c_6}{2} g(X,Y)\text{dr}(Z). \end{aligned} \tag{4.14.1}$$

Putting $(\text{div}\mathscr{T})(X,Y,Z) = 0$ and $\text{dr}(X) = 0$ in (4.14.1), we have

$$\begin{aligned}0 =& (c_0 + c_1)(\nabla_X S)(Y,Z) + (c_2 - c_0)(\nabla_Y S)(X,Z) \\ &+ c_3(\nabla_Z S)(X,Y). \end{aligned} \tag{4.14.2}$$

Clearly, if $c_1 + c_2 = 0, c_0 \neq 0$ and $c_3 = 0$, then we derive from (4.14.2) that

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z). \tag{4.14.3}$$

From (4.13.2) and (4.14.3), it implies that

$$(\nabla_X T)(Y,Z) = (\nabla_Y T)(X,Z).$$

Therefore, the energy momentum tensor is of Codazzi type. □

Theorem 4.14.2. *In a general relativistic viscous fluid spacetime admitting divergence-free $\mathscr{T}$-curvature tensor, if $c_1 + c_2 = 0$ and $c_3 = 0$ then the velocity vector field of the fluid is proportional to the gradient vector field of the energy density.*

Proof. It is already proved that the energy momentum tensor in the general relativistic viscous fluid spacetime is of Codazzi type. This implies that both the vorticity and shear of the fluid vanish and the velocity vector field is hyper-surface orthogonal. That is, the velocity vector field of the fluid is proportional to the gradient vector field of the energy density [52, 102]. □

Theorem 4.14.3. *For a general relativistic viscous fluid spacetime admitting divergence-free $\mathscr{T}$-curvature tensor, if $c_1 + c_2 = 0$ and $c_3 = 0$ then the possible local cosmological structure of this spacetime is of Petrov type I, D or O.*

Proof. Barnes [6] proved that if the shear and vorticity of a perfect fluid spacetime vanish then the velocity vector field U is hyper-surface orthogonal and the energy density is constant over the hyper-surface which is orthogonal to U. Hence, the local cosmological structure of this spacetime is of Petrov type I, D or O. $\square$

CHAPTER 5

Some solitons on warped product space

5.1 Introduction

Nowadays Ricci solitons and Riemann solitons with their generalizations are enjoying rapid growth by providing new techniques in understanding the geometry and topology of arbitrary Riemannian manifolds. Riemann soliton and Ricci soliton are self similar solution to Riemann flow and Ricci flow respectively. They are also important geometric partial differential equations highlighted in many fields of theoritical research and practical applications.

At the beginning of 90's, it is known that a Ricci soliton which is a compact gradient expanding or steady, is an Einstein manifold [59, 70]. Petersen and Wylie [97] gave a theorem in reference to Brinkmann [15] that warped product is nothing but a surface gradient Ricci soliton. Robert Bryant [19, 36] also made a Ricci soliton which is steady as a warped product $(0,+\infty)\times_f S^m$, where $m>1$ and in this case warping function denoted by f is radial. As the function f is not limited, hence we face two very simple questions which are given below.

(1) When a warped product having a limited warping function would be an h-almost η-Ricci soliton ?
(2) Are there any condition ? if yes, what are these conditions ?

In this chapter Theorem 5.6.4 partly provides an answer to these above questions. Motivated by the work of Kim et al. [75] we have Theorem 5.6.5. Our first theorem is the natural generalization from Einstein case to Ricci soliton case except the condition of compactness on the product which has been considered in [75]. By the way, one significant fact comes out during the study of h-almost η-Ricci soliton which are felt like a warped product. Actually, bases of them satisfy

$$\mathrm{Ric} + \nabla^2 \phi = \lambda g_B + \frac{m}{f} \nabla^2 f, \tag{5.1.1}$$

It is the generalization of Einstein metrics containing quasi-Einstein metrics. Theorem 5.6.5 sets up a criterion of compactness for shrinking gradient h-almost η-Ricci soliton warped product with respect to a condition that the base is compact.

In this chapter, we introduce a new notion of gradient h-almost η-Ricci soliton and study Riemann soliton in the frame of warped product Kenmotsu manifold. This chapter is divided into six units. The first one is introductory unit. Some basic definitions, ideas and results related to it belong to the preliminaries unit. Then Riemann soliton has been studied on warped product Kenmotsu manifold to deduce some conditions for its existence admitting W_2-curvature tensor, projective curvature tensor and Weyl-conformal curvature tensor. The fourth unit is added to ensure the existence of Riemann soliton on 5-dimensional warped product Kenmotsu manifold by constructing an example. In the fifth unit, Ricci soliton and gradient Ricci soliton have been discussed with pointwise bi-slant submanifolds of trans-Sasakian manifold to establish that the pointwise bi-slant submanifolds of trans-Sasakian manifold are Einstein manifold under certain conditions. The last unit is dealt with the existence of the gradient h-almost η-Ricci soliton warped product spaces. The nature of h-almost η-Ricci soliton and gradient h-almost η-Ricci soliton have been investigated admitting a concurrent vector field.

5.2 Preliminaries

This unit briefly states some basic ideas and results.

Differentiating (1.1.33) with respect to a vector field X and using (1.1.30), we derive

$$(\nabla_X Q)\xi = -QX - 2nX. \tag{5.2.1}$$

From the symmetry of $\pounds_V \nabla$ in commutation formula [127]

$$(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) = -g\left(\left(\pounds_V \nabla\right)(X,Y),Z\right) - g\left(\left(\pounds_V \nabla\right)(X,Z),Y\right),$$

We obtain

$$\begin{aligned} 2g\left(\left(\pounds_V \nabla\right)(X,Y),Z\right) =& (\nabla_X \pounds_V g)(Y,Z) + (\nabla_Y \pounds_V g)(Z,X) \\ & - (\nabla_Z \pounds_V g)(X,Y). \end{aligned} \tag{5.2.2}$$

The following equations are known as commutation equations.

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z), \tag{5.2.3}$$

$$\pounds_V \nabla_X Y - \nabla_X \pounds_V Y - \nabla_{[V,X]} Y = (\pounds_V \nabla)(X,Y). \tag{5.2.4}$$

The following two identities will help us to prove Proposition 5.6.3.

$$\mathrm{div}(\nabla^2 \phi) = \mathrm{Ric}(\nabla\phi, .) + \mathrm{d}(\Delta\phi), \tag{5.2.5}$$

$$\frac{1}{2}\mathrm{d}(\mid \nabla\phi \mid^2) = (\nabla^2\phi)(\nabla\phi, .). \tag{5.2.6}$$

Now, by taking trace of (1.3.4), we gain

$$\mathrm{R} + h\Delta\psi = k\lambda + \mu.$$

The following result has been proved by Hamilton [59]

$$2\lambda\psi - \mid \nabla\psi \mid^2 + \Delta\psi = c, \tag{5.2.7}$$

where c is some constant. In this way, we have derived similar equation to (5.2.7) for gradient h-almost η- Ricci soliton warped product's base, cf. equation (5.6.1).

5.3 Riemann soliton on warped product Kenmotsu manifold

The purpose of this unit is to study the Riemann soliton in the frame of warped product Kenmotsu manifold. Let the warped product $M = M_1 \times_f M_2$ be a Kenmotsu manifold of dimension $(4n+1)$ where $\dim(M_1) = 2n+1$ and $\dim(M_2) = 2n$. We obtain some significant conditions for its existence by considering different cases. We also deduce the conditions when it admits W_2-curvature tensor, projective curvature tensor and Weyl-conformal curvature tensor.

From (1.3.6) and (1.3.7), it follows that

$$\begin{aligned}&2R(X_1,X_2,X_3,X_4)+2\alpha[g(X_1,X_3)g(X_2,X_4)-g(X_1,X_4)g(X_2,X_3)]\\&+[g(X_1,X_3)(\pounds_V g)(X_2,X_4)+g(X_2,X_4)(\pounds_V g)(X_1,X_3)\\&-g(X_1,X_4)(\pounds_V g)(X_2,X_3)-g(X_2,X_3)(\pounds_V g)(X_1,X_4)]=0. \qquad (5.3.1)\end{aligned}$$

The following two cases are considered to obtain the main results.

<u>Case 1.</u> Let $X_1, X_4, V \in \mathfrak{X}(M_1)$ and $X_2, X_3 \in \mathfrak{X}(M_2)$. Then we have

$$(\pounds_V g)(X_2,X_4)=g(\nabla_{X_2}V,X_4)+g(\nabla_{X_4}V,X_2)=g\left(\nabla^{M_1}_{X_4}V,X_2\right), \qquad (5.3.2)$$

$$(\pounds_V g)(X_1,X_3)=g(\nabla_{X_1}V,X_3)+g(\nabla_{X_3}V,X_1)=g\left(\nabla^{M_1}_{X_1}V,X_3\right), \qquad (5.3.3)$$

$$(\pounds_V g)(X_2,X_3)=g(\nabla_{X_2}V,X_3)+g(\nabla_{X_3}V,X_2)=2\left(\frac{Vf}{f}\right)g(X_2,X_3), \qquad (5.3.4)$$

$$\begin{aligned}(\pounds_V g)(X_1,X_4)=&g(\nabla_{X_1}V,X_4)+g(\nabla_{X_4}V,X_1)\\=&g\left(\nabla^{M_1}_{X_1}V,X_4\right)+g\left(\nabla^{M_1}_{X_4}V,X_1\right). \qquad (5.3.5)\end{aligned}$$

Using (5.3.2)-(5.3.5) in (5.3.1), we obtain

$$\begin{aligned}&2R(X_1,X_2,X_3,X_4)-2\alpha g(X_1,X_4)g(X_2,X_3)-2\left(\frac{Vf}{f}\right)g(X_1,X_4)g(X_2,X_3)\\&-g(X_2,X_3)\left[g\left(\nabla^{M_1}_{X_1}V,X_4\right)+g\left(\nabla^{M_1}_{X_4}V,X_1\right)\right]=0. \qquad (5.3.6)\end{aligned}$$

Taking contraction on both sides of the above relation over X_1 and X_4, we derive

$$S(X_2,X_3)=\left[(2n+1)\alpha+(2n+1)\left(\frac{Vf}{f}\right)+\mathrm{div(V)}\right]g(X_2,X_3). \qquad (5.3.7)$$

The Ricci tensor S satisfies the following condition

$$S(X_1,R(X_2,X_3)X_4)\xi - S(\xi,R(X_2,X_3)X_4)X_1 + S(X_1,X_2)R(\xi,X_3)X_4$$
$$-S(\xi,X_2)R(X_1,X_3)X_4 + S(X_1,X_3)R(X_2,\xi)X_4 - S(\xi,X_3)R(X_2,X_1)X_4$$
$$+S(X_1,X_4)R(X_2,X_3)\xi - S(\xi,X_4)R(X_2,X_3)X_1 = 0,$$

for any $X_1,X_2,X_3,X_4 \in \mathfrak{X}(M)$.

Taking inner product with ξ, we have

$$S(X_1,R(X_2,X_3)X_4) - S(\xi,R(X_2,X_3)X_4)\eta(X_1) + S(X_1,X_2)\eta(R(\xi,X_3)X_4)$$
$$-S(\xi,X_2)\eta(R(X_1,X_3)X_4) + S(X_1,X_3)\eta(R(X_2,\xi)X_4) - S(\xi,X_3)\eta(R(X_2,X_1)X_4)$$
$$+S(X_1,X_4)\eta(R(X_2,X_3)\xi) - S(\xi,X_4)\eta(R(X_2,X_3)X_1) = 0. \tag{5.3.8}$$

Using (5.3.7) and putting $X_4 = \xi$ in (5.3.8), we derive

$$\left[(2n+1)\alpha + (2n+1)\left(\frac{Vf}{f}\right) + \mathrm{div(V)}\right]\eta(R(X_2,X_3)X_1) = 0.$$

This implies for existence of Riemann soliton that

$$\left[(2n+1)\alpha + (2n+1)\left(\frac{Vf}{f}\right) + \mathrm{div(V)}\right] \neq 0.$$

Thus we obtain the following theorem.

Theorem 5.3.1. *Let the warped product $M = M_1 \times_f M_2$ be a $(4n+1)$-dimensional Kenmotsu manifold where $\dim(M_1) = 2n+1$ and $\dim(M_2) = 2n$. Let (g,V) be a Riemann soliton with soliton vector V. Then Riemann soliton exists in M provided*

$$\left[(2n+1)\alpha + (2n+1)\left(\frac{Vf}{f}\right) + \mathrm{div(V)}\right] \neq 0.$$

Remark 5.3.2. *From Theorem 5.3.1 and (1.3.7), it follows that the Riemann soliton on warped product Kenmotsu manifold is expanding, steady and shrinking if $\alpha > 0$, $\alpha = 0$ and $\alpha < 0$ respectively.*

Pokhariyal and Mishra introduced the notion of W_2-curvature tensor [99] in 1970. It is defined by

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{4n}[g(X,Z)QY - g(Y,Z)QX] \tag{5.3.9}$$

on (M^{4n+1}, g) where $X, Y, Z \in \mathfrak{X}(M)$.

The Ricci tensor S satisfies the following condition

$$\begin{aligned}&S(X_1, W_2(X_2,X_3)X_4)\xi - S(\xi, W_2(X_2,X_3)X_4)X_1 + S(X_1,X_2)W_2(\xi,X_3)X_4\\&-S(\xi,X_2)W_2(X_1,X_3)X_4 + S(X_1,X_3)W_2(X_2,\xi)X_4 - S(\xi,X_3)W_2(X_2,X_1)X_4\\&+S(X_1,X_4)W_2(X_2,X_3)\xi - S(\xi,X_4)W_2(X_2,X_3)X_1 = 0,\end{aligned}$$

for any $X_1, X_2, X_3, X_4 \in \mathfrak{X}(M)$.

Taking the inner product with respect to ξ, then the above equation becomes

$$\begin{aligned}&S(X_1, W_2(X_2,X_3)X_4) - S(\xi, W_2(X_2,X_3)X_4)\eta(X_1) + S(X_1,X_2)\eta(W_2(\xi,X_3)X_4)\\&-S(\xi,X_2)\eta(W_2(X_1,X_3)X_4)S(X_1,X_3)\eta(W_2(X_2,\xi)X_4) - S(\xi,X_3)\eta(W_2(X_2,X_1)X_4)\\&+S(X_1,X_4)\eta(W_2(X_2,X_3)\xi) - S(\xi,X_4)\eta(W_2(X_2,X_3)X_1) = 0, \qquad (5.3.10)\end{aligned}$$

Using (5.3.9) in (5.3.10), we derive

$$\left[(2n+1)\alpha + (2n+1)\left(\frac{Vf}{f}\right) + \mathrm{div(V)}\right][\eta(R(X_2,X_3)X_4, X_1)] = 0.$$

$$i.e., \left[(2n+1)\alpha + (2n+1)\left(\frac{Vf}{f}\right) + \mathrm{div(V)}\right] \neq 0,$$

Theorem 5.3.3. *Let the warped product $M = M_1 \times_f M_2$ be a $(4n+1)$-dimensional Kenmotsu manifold where $\dim(M_1) = 2n+1$ and $\dim(M_2) = 2n$ admitting W_2-curvature tensor. Let (g,V) be a Riemann soliton with soliton vector V. Then the Riemann soliton exists in M provided*

$$\left[(2n+1)\alpha + (2n+1)\left(\frac{Vf}{f}\right) + \mathrm{div(V)}\right] \neq 0.$$

Similarly, we state the following two theorems when the Riemann soliton on warped product Kenmotsu manifold admits the projective curvature tensor [80] and Weyl-conformal curvature tensor [128].

Theorem 5.3.4. *Let the warped product $M = M_1 \times_f M_2$ be a $(4n+1)$-dimensional Kenmotsu manifold where $\dim(M_1) = 2n+1$ and $\dim(M_2) = 2n$ admitting projective curvature tensor. Let (g,V) be a Riemann soliton with soliton vector V. Then the Riemann soliton exists in M provided*

$$\left[(2n+1)\alpha + (2n+1)\left(\frac{Vf}{f}\right) + \mathrm{div(V)}\right] \neq 0.$$

Theorem 5.3.5. *Let the warped product $M = M_1 \times_f M_2$ be a $(4n+1)$-dimensional Kenmotsu manifold where $\dim(M_1) = 2n+1$ and $\dim(M_2) = 2n$ admitting Weyl-conformal curvature tensor. Let (g,V) be a Riemann soliton with soliton vector V. Then the Riemann soliton exists in M provided*

$$\left[(2n+1)\alpha + (2n+1)\left(\frac{Vf}{f}\right) + \mathrm{div(V)}\right] \neq 0.$$

Remark 5.3.6. *From Theorem 5.3.3, Theorem 5.3.4, Theorem 5.3.5 and (1.3.7), it follows that the Riemann soliton on warped product Kenmotsu manifold admitting W_2-curvature tensor, projective curvature tensor and Weyl-conformal curvature tensor is expanding, steady and shrinking if $\alpha > 0$, $\alpha = 0$ and $\alpha < 0$ respectively.*

<u>Case 2.</u> Let $X_1, X_4 \in M_1$ and $X_2, X_3, V \in M_2$. Then from (5.3.1), it follows that

$$\begin{aligned}
&2R(X_1,X_2,X_3,X_4) - 2\alpha g(X_1,X_4)g(X_2,X_3) - g(X_1,X_4)(\pounds_V g)(X_2,X_3)\\
&- g(X_2,X_3)(\pounds_V g)(X_1,X_4) = 0.\\
\textit{i.e.,}\ &2R(X_1,X_2,X_3,X_4) - 2\alpha g(X_1,X_4)g(X_2,X_3) - g(X_1,X_4)[g(\nabla_{X_2}V,X_3)\\
&+ g(\nabla_{X_3}V,X_2)] - g(X_2,X_3)[g(\nabla_{X_1}V,X_4) + g(\nabla_{X_4}V,X_1)] = 0.\\
\textit{i.e.,}\ &2R(X_1,X_2,X_3,X_4) - 2\alpha g(X_1,X_4)g(X_2,X_3) - g(X_1,X_4)[g(\nabla^{M_2}_{X_2}V,X_3)\\
&+ g(\nabla^{M_2}_{X_3}V,X_2)] + g(X_1,X_4)\frac{g(V,X_2)}{f}g(\nabla^{M_1}f,X_3)\\
&+ g(X_1,X_4)\frac{g(V,X_3)}{f}g(\nabla^{M_1}f,X_2) = 0.
\end{aligned}$$

Taking contraction over X_1 and X_4, we obtain

$$\begin{aligned}
&(\pounds_V g)(X_2,X_3) - \frac{2}{2n+1}S^{M_2}(X_2,X_3)\\
&+\frac{2}{2n+1}[f^{\#} + (2n+1)\alpha]g(X_2,X_3) = 0,
\end{aligned} \tag{5.3.11}$$

where $f^{\#} = \frac{\Delta f}{f} + \frac{2n-1}{f^2}\|\nabla f\|^2$.

After covariant differentiation with respect to X_1, we obtain

$$(\nabla_{X_1}\pounds_V g)(X_2,X_3) - \frac{2}{2n+1}(\nabla_{X_1}S^{M_2})(X_2,X_3) = 0. \tag{5.3.12}$$

In view of (5.2.2), we get

$$g((\pounds_V\nabla)(X_1,X_2),X_3) = \frac{1}{2n+1}[(\nabla_{X_1}S^{M_2})(X_2,X_3)+(\nabla_{X_2}S^{M_2})(X_3,X_1) - (\nabla_{X_3}S^{M_2})(X_1,X_2)]. \tag{5.3.13}$$

The following relation is satisfied for a Kenmotsu manifold of dimension $(2n+1)$.

$$(\nabla_\xi Q)X_1 = -2QX_1 - 4nX_1 \tag{5.3.14}$$

Setting $X_2=\xi$ in (5.3.13) and using (5.2.1) and (5.3.14), we derive

$$(\pounds_V\nabla)(X_1,\xi) = -\frac{2}{2n+1}QX_1 - \frac{4n}{2n+1}X_1. \tag{5.3.15}$$

After covariant differentiation with respect to X_2 and using (1.1.29), we have

$$(\nabla_{X_2}\pounds_V\nabla)(X_1,\xi)+(\pounds_V\nabla)(X_1,X_2)+\frac{2}{2n+1}\eta(X_2)[QX_1+2nX_1] = -\frac{2}{2n+1}(\nabla_{X_2}Q)X_1$$

In view of the above result we derive from (5.2.3)

$$(\pounds_V R)(X_1,X_2)\xi = -\frac{2}{2n+1}[\eta(X_1)QX_2-\eta(X_2)QX_1+(\nabla_{X_1}Q)X_2-(\nabla_{X_2}Q)X_1] - \frac{4n}{2n+1}[\eta(X_1)X_2-\eta(X_2)X_1]. \tag{5.3.16}$$

Putting $X_2=\xi$ and using (5.2.1) and (5.3.14), we achieve $(\pounds_V R)(X_1,\xi)\xi=0$. Besides, from (1.1.31), we get

$$R(X_1,\xi)\xi = -X_1+\eta(X_1)\xi,$$

which gives

$$(\pounds_V R)(X_1,\xi)\xi + g(X_1,\pounds_V\xi)\xi - 2\eta(\pounds_V\xi)X_1 = [(\pounds_V\eta)X_1]\xi.$$

Since $(\pounds_V R)(X_1,\xi)\xi=0$, hence

$$g(X_1,\pounds_V\xi)\xi - 2\eta(\pounds_V\xi)X_1 = \{(\pounds_V\eta)X_1\}\xi. \tag{5.3.17}$$

With the help of (1.1.33), (5.3.11) becomes

$$(\pounds_V g)(X_1,\xi) = -\frac{2}{2n+1}[2n+(2n+1)\alpha+f^{\#}]\eta(X_1) \tag{5.3.18}$$

Taking Lie-differentiation with respect to V, we have

$$(\pounds_V\eta)X_1 - g(X_1,\pounds_V\xi) + \frac{2}{2n+1}[2n+(2n+1)\alpha+f^{\#}]\eta(X_1) = 0,$$
$$\eta(\pounds_V\xi) = \frac{1}{2n+1}[2n+(2n+1)\alpha+f^{\#}]. \tag{5.3.19}$$

Using these two equations in (5.3.17), we derive

$$[2n+(2n+1)\alpha+f^{\#}]\times[X_1-\eta(X_1)\xi] = 0. \tag{5.3.20}$$

Taking trace we obtain

$$[2n+(2n+1)\alpha+f^{\#}] = 0.$$

After contraction (5.3.16) becomes

$$(\pounds_V S^{M_2})(X_2,\xi) = \frac{1}{2n+1}\left[(8n+16n^2+2r)\eta(X_2)+X_2r\right]$$

where we use div(Q) $= \frac{1}{2}$grad r and tr(∇Q) = grad r.

Taking trace of (5.3.14) provides

$$\xi r = -2r-8n^2$$

Using the above equation, we derive

$$(\pounds_V S^{M_2})(X_2,\xi) = \frac{1}{2n+1}[\{8n(n+1)-\xi r\}\eta(X_2)+X_2r]$$

Hence we have the following theorem.

Theorem 5.3.7. *Let the warped product $M = M_1\times_f M_2$ be a $(4n+1)$-dimensional Kenmotsu manifold where* $\dim(M_1) = 2n+1$ *and* $\dim(M_2) = 2n$*. If (g,V) is a Riemann soliton with soliton vector V, then the soliton vector V and the Ricci tensors satisfy the relation*

$$(i)\ \left[2n+(2n+1)\alpha+\frac{\Delta f}{f}+\frac{2n-1}{f^2}\|\nabla f\|^2\right] = 0,$$
$$(ii)\ (\pounds_V S)(X_2,\xi) = \frac{1}{2n+1}[\{8n(n+1)-\xi r\}\eta(X_2)+X_2r],$$

where r is the scalar curvature and ξ is the potential vector field of M.

5.4 Example of Riemann soliton on warped product Kenmotsu manifold

In this unit an example of Riemann soliton on 5-dimensional warped product Kenmotsu manifold has been constructed. Moreover, the results obtained from the previous section have been verified at the end of the example.

We consider a manifold $M \subset \mathbb{R}^5$ of dimension five defined by

$$M = \{(x,y,z,u,v) \in \mathbb{R}^5 : z \neq 0\},$$

where (x,y,z,u,v) are the canonical co-ordinates of $\mathbb{R}^5$.

Let e_1, e_2, e_3, e_4, e_5 be five linearly independent vector fields. They are defined by

$$e_1 = e^{-z}\frac{\partial}{\partial y},\ e_2 = e^{-z}\frac{\partial}{\partial x},\ e_3 = \frac{\partial}{\partial z},\ e_4 = e^{-z}\frac{\partial}{\partial u}, e_5 = e^{-z}\frac{\partial}{\partial v}.$$

We can easily check that

$$[e_1,e_2] = e_1,\ [e_2,e_3] = e_2,\ [e_3,e_4] = -e_4,\ [e_2,e_5] = -e_5.$$

A tensor field φ of type (1,1) is defined on M by

$$\varphi(e_1) = e_2,\ \varphi(e_2) = -e_1,\ \varphi(e_3) = 0,\ \varphi(e_4) = e_5,\ \varphi(e_5) = e_4.$$

The Riemannian metric tensor g is defined by

$$g(e_i,e_j) = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j, \end{cases}$$

where $1 \leq i,j \leq 5$. Then g is given by

$$\begin{aligned} g =& e^{2z}(dx^2+dy^2+du^2+dv^2)+dz^2 \\ =&(dz^2+e^{2z}dx^2+e^{2z}dy^2)+e^{2z}(du^2+dv^2) \end{aligned}$$

It clearly follows that $M = M_1 \times_f M_2$ be a warped product manifold of dimension five where $\dim(M_1) = 3,\ \dim(M_2) = 2$ and $f : M_1 \to (0,\infty)$ is the warping function defined by $f(x,y,z) = e^z$.

Applying Koszul formula, we obtain

$$\begin{aligned}
&\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_4} e_4 = \nabla_{e_5} e_5 = -e_3,\ \nabla_{e_1} e_3 = e_1,\\
&\nabla_{e_2} e_3 = e_2,\ \nabla_{e_4} e_3 = e_4,\ \nabla_{e_5} e_3 = e_5, \qquad (5.4.1)
\end{aligned}$$

where ∇ is the Levi-Civita connection of g. It is easy to check that the manifold M is a Kenmotsu manifold. After some elementary steps, we have

$$\begin{aligned}
&R(e_1,e_2)e_1 = e_2,\ R(e_2,e_1)e_1 = -e_2,\ R(e_1,e_4)e_1 = e_4,\\
&R(e_4,e_1)e_1 = -e_4,\ R(e_1,e_5)e_1 = e_5,\ R(e_5,e_1)e_1 = -e_5,\\
&R(e_3,e_1)e_3 = e_1,\ R(e_1,e_3)e_3 = -e_1,\ R(e_3,e_2)e_3 = e_2,\\
&R(e_2,e_3)e_3 = -e_2,\ R(e_3,e_4)e_3 = e_4,\ R(e_4,e_3)e_3 = -e_4,\\
&R(e_3,e_5)e_3 = e_5,\ R(e_5,e_3)e_3 = -e_5. \qquad (5.4.2)
\end{aligned}$$

Let us consider a vector field V defined by

$$V = a\left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}\right], \qquad (5.4.3)$$

where a is a non-zero constant.

It is clearly seen that V has a constant divergence. As a consequence of (5.4.1), we have

$$(\pounds_V g)(e_i,e_j) = 0, \quad 1 \le i,j \le 5. \qquad (5.4.4)$$

Using (5.4.4) we see that (1.3.7) holds good with respect to V defined in (5.4.3) and $\alpha = -1$. Hence, g is a Riemann soliton.

Verification : In the above example, $n = 1$, $\alpha = -1$, and $\mathrm{div(V)} = 0$. Therefore,

$$\left[(2n+1)\alpha + (2n+1)\left(\frac{Vf}{f}\right) + \mathrm{div(V)}\right] \neq 0.$$

Hence Theorem 5.3.1, Theorem 5.3.3, Theorem 5.3.4 and Theorem 5.3.5 are verified by this example.

5.5 Ricci soliton and gradient Ricci soliton on pointwise bi-slant submanifolds of 3-dimensional trans-Sasakian manifold

This unit is dealt with Ricci soliton and gradient Ricci soliton on pointwise bi-slant submanifolds of trans-Sasakian manifold. The following theorems show that the pointwise bi-slant submanifolds of trans-Sasakian manifold are Einstein manifolds admitting Ricci soliton and gradient Ricci soliton under some certain conditions.

Theorem 5.5.1. *Let M be a pointwise bi-slant submanifold of a trans-Sasakian manifold $\tilde{M}$ with pointwise slant distributions $\mathscr{D}_1$ and $\mathscr{D}_2 \oplus \langle \xi \rangle$ with distinct slant angles $\theta_1[\neq n\pi + (-1)^n \theta_2]$ and θ_2 respectively admitting Ricci soliton. If M is a mixed totally geodesic submanifold and $F\xi = FP_2\xi$ then M is an Einstein manifold.*

Proof. Let M be a pointwise bi-slant submanifold of a trans-Sasakian manifold $\tilde{M}$ admitting Ricci soliton. Then for any $X, Y \in \Gamma(\mathscr{D}_1)$ and $Z \in \Gamma(\mathscr{D}_2 \oplus \langle \xi \rangle)$, we have

$$S(X,Y) + \frac{1}{2}\pounds_Z g(X,Y) + \lambda g(X,Y) = 0, \tag{5.5.1}$$

for some constant λ and the Lie derivative $\pounds_Z g$.

If we put $Z = \xi$ in (5.5.1), it can be written as

$$2S(X,Y) + 2\lambda g(X,Y) = -g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X). \tag{5.5.2}$$

On the other hand, for any $X, Y \in \Gamma(\mathscr{D}_1)$ and $Z \in \Gamma(\mathscr{D}_2 \oplus \langle \xi \rangle)$, we have

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X Y, Z) = g(\phi \tilde{\nabla}_X Y, \phi Z) - \eta(\tilde{\nabla}_X Y)\eta(Z)$$

From (1.1.34), we can write

$$\begin{aligned} g(\nabla_X Y, Z) =& g(\tilde{\nabla}_X \phi Y, \phi Z) - g((\tilde{\nabla}_X \phi)Y, \phi Z) \\ &+ \eta(Z)[-\alpha g(\phi X, Y) + \beta g(X,Y)]. \end{aligned}$$

Using (1.1.42), we obtain

$$\begin{aligned} g(\nabla_X Y, Z) =& g(\tilde{\nabla}_X P_1 Y, \phi Z) + g(\tilde{\nabla}_X FY, P_2 Z) + g(\tilde{\nabla}_X FY, FZ) \\ &+ \eta(Z)[-\alpha g(\phi X, Y) + \beta g(X,Y)]. \end{aligned}$$

Taking (1.1.25), (1.1.26) and (1.1.40), we have

$$\begin{aligned}g(\nabla_X Y,Z) =&g((\tilde{\nabla}_X\phi)(P_1Y),Z)-g(\tilde{\nabla}_X P_1^2Y,Z)-g(\tilde{\nabla}_X FP_1Y,Z)\\&-g(A_{FY}X,P_2Z)+g(\tilde{\nabla}_X FZ,P_1Y)-g((\tilde{\nabla}_X\phi)FZ,Y)\\&+g(\tilde{\nabla}_X\phi FZ,Y)+\eta(Z)[-\alpha g(\phi X,Y)+\beta g(X,Y)].\end{aligned}$$

Then from (1.1.25)-(1.1.27), (1.1.34)-(1.1.36), (1.1.42)-(1.1.45) and (1.1.47), it follows that

$$\begin{aligned}(\sin^2\theta_1-\sin^2\theta_2)g(\nabla_X Y,Z) =&g(A_{FP_1Y}Z-A_{FY}P_2Z,X)\\&+g(A_{FP_2Z}Y-A_{FZ}P_1Y,X)+\alpha\eta(Z)[g(X,P_1Y)\\&-g(\phi X,Y)]+\beta\eta(Z)(1+\cos^2\theta_2)g(X,Y)\\&-\alpha\eta(Z)\sin^2\theta_2 g(\phi X,Y)+\beta\eta(Z)\sin^2\theta_2 g(X,Y).\end{aligned}$$

If M is a mixed totally geodesic submanifold and using the condition $F\xi = FP_2\xi$ the above equation reduces to

$$\begin{aligned}(\sin^2\theta_2-\sin^2\theta_1)g(Y,\nabla_X Z) =&\alpha\eta(Z)[g(X,P_1Y)-g(\phi X,Y)]\\&+\beta\eta(Z)(1+\cos^2\theta_2)g(X,Y)\\&-\alpha\eta(Z)\sin^2\theta_2 g(\phi X,Y)\\&+\beta\eta(Z)\sin^2\theta_2 g(X,Y).\end{aligned}\tag{5.5.3}$$

Now interchanging X and Y and then adding with (5.5.3), we obtain

$$\begin{aligned}&(\sin^2\theta_2-\sin^2\theta_1)[g(Y,\nabla_X Z)+g(X,\nabla_Y Z)]\\=&\beta\eta(Z)(1+\cos^2\theta_2+\sin^2\theta_2)g(X,Y).\end{aligned}\tag{5.5.4}$$

Putting $Z=\xi$ and then using (5.5.2) and (5.5.4), we derive

$$S(X,Y)=\frac{-2\lambda+\beta\eta(Z)(1+\cos^2\theta_2+\sin^2\theta_2)}{(\sin^2\theta_2-\sin^2\theta_1)}g(X,Y),$$

where $\theta_1 \neq n\pi+(-1)^n\theta_2$. Therefore, M is an Einstein manifold. This completes the proof. $\square$

Theorem 5.5.2. *Let M be a pointwise bi-slant submanifold of 3-dimensional trans-Sasakian manifold $\tilde{M}$ of type (α, β) satisfying α, β $(\alpha^2 \neq \beta^2)$ being constants with pointwise slant distributions $\mathscr{D}_1$ and $\mathscr{D}_2 \oplus \langle \xi \rangle$ with distinct slant angles θ_1 and θ_2, respectively, admitting gradient Ricci soliton. If $A_{FX}F\xi = A_{FP_2\xi}X + A_{F\xi}P_1X$ for any $X \in \Gamma(\mathscr{D}_1)$ then M is an Einstein manifold.*

Proof. Let M be a pointwise bi-slant submanifold of 3-dimensional trans-Sasakian manifold $\tilde{M}$ with pointwise slant distributions $\mathscr{D}_1$ and $\mathscr{D}_2 \oplus \langle \xi \rangle$ with distinct slant angles θ_1 and θ_2 respectively satisfying gradient Ricci soliton. Let R, Q and r be the curvature tensor, Ricci operator and scalar curvature of pointwise bi-slant submanifold M respectively. Then for a potential function f, (1.3.7) reduces to

$$R(Z,W)Df = (\nabla_Z Q)W - (\nabla_W Q)Z,$$

where $Z, W \in \Gamma(\mathscr{D}_2 \oplus \langle \xi \rangle)$ and D denotes the gradient operator of g. Also from (1.1.38), it can be written as

$$QW = \left[\frac{r}{2} - (\alpha^2 - \beta^2)\right] W - \left[\frac{r}{2} - 3(\alpha^2 - \beta^2)\right] \eta(W)\xi. \tag{5.5.5}$$

Now differentiating (5.5.5) with respect to $V \in \Gamma(\mathscr{D}_2 \oplus \langle \xi \rangle)$ and then putting $V = \xi$ we can write

$$(\nabla_\xi Q)W = \frac{\mathrm{dr}(\xi)}{2}[W - \eta(W)\xi]. \tag{5.5.6}$$

Also we can write

$$g((\nabla_\xi Q)W - (\nabla_W Q)\xi, \xi) = 0.$$

From (5.5.5), we derive

$$g(R(\xi, W)Df, \xi) = 0. \tag{5.5.7}$$

Also, we have

$$R(Z,W)\xi = (\alpha^2 - \beta^2)(\eta(W)Z - \eta(Z)W). \tag{5.5.8}$$

Hence from (5.5.7) and (5.5.8), it follows that

$$Df = (\xi f)\xi \ \text{ with } \alpha^2 \neq \beta^2 \tag{5.5.9}$$

Again (1.3.7) gives

$$S(X,Y)+\lambda g(X,Y)=g(\nabla_Y(Df),X)=g(\tilde{\nabla}_Y(Df),X), \tag{5.5.10}$$

for any $X,Y\in\Gamma(\mathscr{D}_1)$.

Now we can write

$$\begin{aligned}S(X,Y)+\lambda g(X,Y)=&g(\phi(\tilde{\nabla}_Y(Df)),\phi X)\\ =&g(\tilde{\nabla}_Y\phi(Df),\phi X)-g((\tilde{\nabla}_Y\phi)Df,\phi X).\end{aligned}$$

Using (1.1.34) and (1.1.42), we obtain

$$\begin{aligned}S(X,Y)+\lambda g(X,Y)=&g(\tilde{\nabla}_Y P_2 Df,\phi X)+g(\tilde{\nabla}_Y F(Df),P_1X)\\ &+g(\tilde{\nabla}_Y F(Df),FX)+\alpha\eta(Df)g(Y,\phi X)\\ &+\beta\eta(Df)g(X,Y).\end{aligned}$$

Taking (1.1.26)-(1.1.27), (1.1.34) and (5.5.9), we derive

$$\begin{aligned}S(X,Y)+\lambda g(X,Y)=&-g(\phi(\tilde{\nabla}_Y P_2((\xi f)\xi)),X)+g(\tilde{\nabla}_Y F((\xi f)\xi)),P_1X)\\ &+g(\tilde{\nabla}_Y F((\xi f)\xi)),FX)+\alpha(\xi f)g(Y,\phi X)\\ &+\beta(\xi f)g(X,Y).\end{aligned}$$

Using (1.1.34), (1.1.40) and (1.1.42) the above relation gives

$$\begin{aligned}S(X,Y)+\lambda g(X,Y)=&-\alpha\eta(P_2\xi)(\xi f)g(X,Y)-\beta\eta(P_2\xi)(\xi f)g(X,\phi Y)\\ &-(\xi f)g(A_{F\xi}Y,P_1X)+(\xi f)g(A_{FX}Y,F\xi)\\ &-(\xi f)[g(\tilde{\nabla}_Y P_2^2\xi,X)+g(\tilde{\nabla}_Y FP_2\xi,X)]\\ &+\alpha(\xi f)g(Y,\phi X)+\beta(\xi f)g(X,Y).\end{aligned} \tag{5.5.11}$$

Taking the condition $A_{FX}F\xi=A_{FP_2\xi}X+A_{F\xi}P_1X$ and then using (1.1.38), (1.1.42) and (5.5.11), we get

$$\begin{aligned}S(X,Y)+\lambda g(X,Y)=&-\alpha\eta(P_2\xi)(\xi f)g(X,Y)-\beta\eta(P_2\xi)(\xi f)g(X,\phi Y)\\ &+\alpha(\xi f)g(Y,\phi X)+\beta(\xi f)g(X,Y).\end{aligned} \tag{5.5.12}$$

Interchanging X and Y and then adding with (5.5.12), it follows that

$$\begin{aligned}S(X,Y) =& -\alpha\eta(P_2\xi)(\xi f)g(X,Y) - \lambda g(X,Y) + \beta(\xi f)g(X,Y)\\ =& [\beta(\xi f) - \lambda - \alpha\eta(P_2\xi)]g(X,Y).\end{aligned}$$

Hence M is an Einstein manifold. This completes the proof. □

5.6 The conditions for existence of h-almost η-Ricci soliton warped product spaces

Now a Riemannian manifold (B^n, g_B) has been constructed as a base of a gradient h-almost η-Ricci soliton warped product $(M = B^n \times_f F^m, g, \nabla\psi, h, \eta, \lambda)$. We consider that ψ is the potential function and ψ being the lift of ϕ, which is a smooth function defined on B^n, that is, the crucial information of M will be carried base. Keeping in mind with these considerations, we set up some conditions on the functions which parametrize a gradient h-almost η-Ricci soliton by the almost η-Ricci soliton warped product. Hamilton's equation (5.2.7) for B^n is the first condition.

Proposition 5.6.1. *Let $M = B^n \times_f F^m$ be a warped product and ϕ defined on B is a smooth function such that $(M, g, \nabla\tilde{\phi}, h, \eta, \lambda)$ is a gradient h-almost η-Ricci soliton. Then we obtain*

$$2\lambda\phi - |\nabla\phi|^2 + \Delta\phi + \frac{m}{f}\nabla\phi(f) = c, \tag{5.6.1}$$

where c is a constant.

Proof. Hamilton [59] had proved that

$$2\lambda\tilde{\phi} - |\nabla\tilde{\phi}|^2 + \Delta\tilde{\phi} = c, \tag{5.6.2}$$

where c is some constant. Besides this,

$$\nabla\tilde{\phi} = \tilde{\nabla}\phi, \tag{5.6.3}$$

$$\Delta\tilde{\phi} = \Delta\phi + \frac{m}{f}\nabla\phi(f). \tag{5.6.4}$$

Using (5.6.3) and (5.6.4) in (5.6.2), we gain

$$2\lambda\phi - \mid \nabla\phi \mid^2 + \Delta\phi + \frac{m}{f}\nabla\phi(f) = c. \tag{5.6.5}$$

This completes the proof. □

Proposition 5.6.2. *Let $M = B^n \times_f F^m$ be a warped product and ϕ defined on B is a smooth function such that $(M, g, \nabla\tilde{\phi}, h, \eta, \lambda)$ is a gradient h-almost η-Ricci soliton, where $m > 1$. Then*

$$\mathrm{Ric}_B + hH^\phi = \lambda g_B + \frac{m}{f}H^f + \mu(\eta \otimes \eta), \tag{5.6.6}$$

$$\mathrm{Ric}_F = [\lambda f^2 + f\Delta f + (m-1) \mid \nabla f \mid^2 - hf\nabla\phi(f)]g_F + \mu(\eta \otimes \eta). \tag{5.6.7}$$

Proof. Clearly, it is seen that

$$\mathrm{Ric}(Y,Z) = \mathrm{Ric}_B(Y,Z) - \frac{m}{f}H^f(Y,Z), \forall\, Y,Z \in \Gamma(B) \tag{5.6.8}$$

The gradient h-almost η-Ricci soliton is

$$\begin{aligned} \mathrm{Ric} + h\nabla^2\tilde{\phi} &= \lambda g + \mu(\eta \otimes \eta). \\ \textit{i.e.}, \mathrm{Ric}(Y,Z) &= \lambda g_B(Y,Z) + \mu(\eta \otimes \eta)(Y,Z) - hH^\phi(Y,Z). \end{aligned} \tag{5.6.9}$$

From (5.6.8) and (5.6.9), it follows that

$$\mathrm{Ric}_B + hH^\phi = \lambda g_B + \frac{m}{f}H^f + \mu(\eta \otimes \eta). \tag{5.6.10}$$

Hence, this completes the proof of the first assertion of Proposition 5.6.2.

It is also observed from Proposition 1.2.4 that

$$\begin{aligned} \mathrm{Ric}(V,W) = &\mathrm{Ric}_F(V,W) \\ &- \left[\frac{\Delta f}{f} + (m-1)\frac{\mid \nabla f \mid^2}{f^2}\right] g(V,W), \forall\, V,W \in \Gamma(F). \end{aligned} \tag{5.6.11}$$

Also, from (1.3.4), we obtain

$$\mathrm{Ric}(V,W) = \lambda f^2 g_F(V,W) - h\nabla^2\tilde{\phi}(V,W) + \mu(\eta \otimes \eta)(V,W). \tag{5.6.12}$$

In view of (5.6.11) and (5.6.12), we have

$$\begin{aligned}\mathrm{Ric}_F(V,W) =& \lambda f^2 g_F(V,W) - h\nabla^2\tilde{\phi}(V,W) + \mu(\eta\otimes\eta)(V,W)\\ &+ f\left[\Delta f + \frac{(m-1)\mid\nabla f\mid^2}{f}\right] g_F(V,W).\end{aligned} \tag{5.6.13}$$

Since $\nabla\tilde{\phi}\in\Gamma(B)$ and using Proposition 1.2.2, we obtain

$$\nabla^2\tilde{\phi}(V,W) = g(D_V\nabla\tilde{\phi},W) = g\left(\frac{\nabla\tilde{\phi}(f)}{f}V,W\right) = f\nabla\phi(f)g_F(V,W). \tag{5.6.14}$$

In view of (5.6.14), (5.6.13) implies that

$$\begin{aligned}\mathrm{Ric}_F(V,W) =& [\lambda f^2 + f\Delta f + (m-1)\mid\nabla f\mid^2\\ &- hf\nabla\phi(f)]g_F(V,W) + \mu(\eta\otimes\eta)(V,W).\end{aligned} \tag{5.6.15}$$

Hence, this completes the proof of the second assertion of Proposition 5.6.2. □

Proposition 5.6.3. *Let (B^n,g) be a Riemannian manifold having two smooth functions ϕ and $f(>0)$ which are satisfying the following equations*

$$\mathrm{Ric} + h\nabla^2\phi = \lambda g + \frac{m}{f}\nabla^2 f + \mu(\eta\otimes\eta), \tag{5.6.16}$$

$$2\lambda\phi - \mid\nabla\phi\mid^2 + \Delta\phi + \frac{m}{f}\nabla\phi(f) = c, \tag{5.6.17}$$

for some constants m, c, λ and $\mu\in\mathbb{R}$ and $m\neq 0$. Then f and ϕ will satisfy the following equation

$$\lambda f^2 + f\Delta f + (m-1)\mid\nabla f\mid^2 - hf\nabla\phi(f) = \beta, \tag{5.6.18}$$

where $\beta\in\mathbb{R}$ is a constant, if it satisfies the condition

$$\begin{aligned}0 =& -hf\mathrm{d}(\nabla\phi(f)) + \frac{hf^2}{m}\mathrm{d}(h\mid\nabla\phi\mid^2) - \frac{hf^2}{m}\mathrm{d}(\mid\nabla\phi\mid^2)\\ &+ 2f\mu(\eta\otimes\eta)(\nabla f,.) + \frac{f^2}{m}\Delta\phi\mathrm{d}h - \frac{2h\mu f^2}{m}(\eta\otimes\eta)(\nabla\phi,.)\\ &- \frac{2f^2}{m}(\nabla^2\phi)(\nabla h,.) + \mathrm{d}hf(\nabla\phi(f)).\end{aligned} \tag{5.6.19}$$

Proof. By taking trace on both sides of (5.6.16), we have

$$\mathrm{S} = n\lambda + \frac{m}{f}\Delta f + \mu - h\Delta\phi, \tag{5.6.20}$$

where scalar curvature of B is S. Hence,

$$\mathrm{dS} = -\frac{m}{f^2}\Delta f \mathrm{d}f + \frac{m}{f}\mathrm{d}(\Delta f) - \Delta\phi \mathrm{d}h - h\mathrm{d}(\Delta\phi). \tag{5.6.21}$$

Now, we use the second contracted Bianchi identity, which is

$$-\frac{1}{2}\mathrm{dS} + \mathrm{div}(\mathrm{Ric}) = 0. \tag{5.6.22}$$

We obtain by computation from (5.6.16),

$$\begin{aligned}\mathrm{div}(\mathrm{Ric}) =& \frac{m}{f}\mathrm{Ric}(\nabla f,.) + \frac{m}{f}\mathrm{d}(\Delta f) - \frac{m}{2f^2}\mathrm{d}(\mid \nabla f \mid^2) \\ & - h\mathrm{Ric}(\nabla\phi,.) - h\mathrm{d}(\Delta\phi) - (\nabla^2\phi)(\nabla h,.)\end{aligned} \tag{5.6.23}$$

From (5.6.16), it follows that

$$\mathrm{Ric}(\nabla f,.) + h(\nabla^2\phi)(\nabla f,.) =\lambda \mathrm{d}f + \frac{m}{2f}\mathrm{d}(\mid \nabla f \mid^2) + \mu(\eta\otimes\eta)(\nabla f,.) \tag{5.6.24}$$

Replacing ∇f by $\nabla\phi$ in (5.6.24), we obtain

$$\begin{aligned}\mathrm{Ric}(\nabla\phi,.) =& \lambda \mathrm{d}\phi + \frac{m}{f}(\nabla^2 f)(\nabla\phi,.) \\ & + \mu(\eta\otimes\eta)(\nabla\phi,.) - \frac{h}{2}\mathrm{d}(\mid \nabla\phi \mid^2).\end{aligned} \tag{5.6.25}$$

Using (5.6.24) and (5.6.25) in (5.6.23), we gain

$$\begin{aligned}\mathrm{div}(\mathrm{Ric}) =& \frac{m\lambda}{f}\mathrm{d}f + \frac{m(m-1)}{2f^2}\mathrm{d}(\mid \nabla f \mid^2) + \frac{m\mu}{f}(\eta\otimes\eta)(\nabla f,.) \\ & - \frac{mh}{f}\mathrm{d}(\nabla\phi(f)) + \frac{m}{f}\mathrm{d}(\Delta f) - h\lambda \mathrm{d}\phi - h\mu(\eta\otimes\eta)(\nabla\phi,.) \\ & + \frac{h^2}{2}\mathrm{d}(\mid \nabla\phi \mid^2) - h\mathrm{d}(\Delta\phi) - (\nabla^2\phi)(\nabla h,.).\end{aligned} \tag{5.6.26}$$

Using (5.6.21) and (5.6.26) in (5.6.22), we obtain

$$\begin{aligned}0 =& \frac{m}{2f^2}\Delta f \mathrm{d}f + \frac{m}{2f}\mathrm{d}(\Delta f) + \frac{1}{2}\Delta\phi \mathrm{d}h \\ & - \frac{h}{2}\mathrm{d}(\Delta\phi) + \frac{m\lambda}{f}\mathrm{d}f + \frac{m(m-1)}{2f^2}\mathrm{d}(\mid \nabla f \mid^2) \\ & + \frac{m\mu}{f}(\eta\otimes\eta)(\nabla f,.) - \frac{mh}{f}\mathrm{d}(\nabla\phi(f)) - h\lambda \mathrm{d}\phi \\ & - h\mu(\eta\otimes\eta)(\nabla\phi,.) + \frac{h^2}{2}\mathrm{d}(\mid \nabla\phi \mid^2) - (\nabla^2\phi)(\nabla h,.).\end{aligned} \tag{5.6.27}$$

Multiplying the previous (5.6.27) by $\frac{2f^2}{m}$, we get

$$\begin{aligned}0 =&\mathrm{d}[f\Delta f+\lambda f^2+(m-1)\mid \nabla f\mid^2]-\frac{hf^2}{m}\mathrm{d}[\Delta\phi+2\lambda\phi-h\mid\nabla\phi\mid^2]\\&+\frac{f^2}{m}\Delta\phi\mathrm{d}h+2\mu f(\eta\otimes\eta)(\nabla f,.)-2hf\mathrm{d}(\nabla\phi(f))\\&-\frac{2h\mu f^2}{m}(\eta\otimes\eta)(\nabla\phi,.)-\frac{2f^2}{m}(\nabla^2\phi)(\nabla h,.).\end{aligned}$$

Using the hypothesis

$$2\lambda\phi-\mid\nabla\phi\mid^2+\Delta\phi+\frac{m}{f}\nabla\phi(f)=c,$$

we derive after some steps

$$\begin{aligned}0 =&\mathrm{d}(f\Delta f+\lambda f^2+(m-1)\mid\nabla f\mid^2)+hf\mathrm{d}f(\nabla\phi(f))-h\mathrm{d}f(\nabla\phi(f))\\&+\frac{hf^2}{m}\mathrm{d}(h\mid\nabla\phi\mid^2)-\frac{hf^2}{m}\mathrm{d}(\mid\nabla\phi\mid^2)+2f\mu(\eta\otimes\eta)(\nabla f,.)\\&+\frac{f^2}{m}\Delta\phi\mathrm{d}h-2hf\mathrm{d}(\nabla\phi(f))-\frac{2h\mu f^2}{m}(\eta\otimes\eta)(\nabla\phi,.)\\&-\frac{2f^2}{m}(\nabla^2\phi)(\nabla h,.)+\mathrm{d}hf(\nabla\phi(f)).\end{aligned}\tag{5.6.28}$$

If we consider that

$$\begin{aligned}0 =&-hf\mathrm{d}(\nabla\phi(f))+\frac{hf^2}{m}\mathrm{d}(h\mid\nabla\phi\mid^2)-\frac{hf^2}{m}\mathrm{d}(\mid\nabla\phi\mid^2)\\&+2f\mu(\eta\otimes\eta)(\nabla f,.)+\frac{f^2}{m}\Delta\phi\mathrm{d}h-\frac{2h\mu f^2}{m}(\eta\otimes\eta)(\nabla\phi,.)\\&-\frac{2f^2}{m}(\nabla^2\phi)(\nabla h,.)+\mathrm{d}hf(\nabla\phi(f)),\end{aligned}\tag{5.6.29}$$

then (5.6.29) becomes

$$\mathrm{d}\left(f\Delta f+\lambda f^2+(m-1)\mid\nabla f\mid^2-hf(\nabla\phi(f))\right)=0,\tag{5.6.30}$$

which is sufficient to complete the proof. □

Theorem 5.6.4. *Let $M=B^n\times_f F^m$ be a warped product and ϕ is a smooth function on B such that $(M,g,\nabla\tilde{\phi},h,\eta,\lambda)$ is a steady or expanding gradient h-almost η-Ricci soliton. Also, suppose that fiber F^m of this warped product with dimension greater than or equal to two and warping function f of it attains minimum as well as maximum with the condition (5.6.30). Then M will definitely be a Riemannian product if $(h-1)\nabla\phi(f)\geq\frac{(1-m)}{f}\mid\nabla f\mid^2$.*

Proof. Let $M = B^n \times_f F^m$, $m > 1$, be a gradient h-almost η-Ricci soliton satisfying (1.3.4). Then Proposition 5.6.2 indicates

$$\mathrm{Ric}_F = \beta g_F + \mu(\eta \otimes \eta), \tag{5.6.31}$$

where

$$\beta = \lambda f^2 + f\Delta f + (m-1) \mid \nabla f \mid^2 - hf(\nabla\phi(f)). \tag{5.6.32}$$

From Proposition 5.6.3, it is clear that β is a constant. (5.6.16) and (5.6.17) are guaranteed from (5.6.1) and (5.6.6) of Proposition 5.6.1 and Proposition 5.6.2 respectively, satisfying the condition (5.6.30). Suppose that $p, q \in B^n$ are the points where the warping function f reaches its minimum as well as maximum in B^n. Hence

$$\nabla f(p) = 0 = \nabla f(q), \tag{5.6.33}$$

$$\nabla f(p) \leq 0 \leq \nabla f(q). \tag{5.6.34}$$

As, $\lambda \leq 0$ and $f > 0$, we obtain

$$-\lambda(f(p))^2 \geq -\lambda(f(q))^2 \tag{5.6.35}$$

and plugging this with (5.6.33), we get

$$0 \geq f(p)\Delta f(p) = \beta - \lambda(f(p))^2 \geq \beta - \lambda(f(q))^2 = f(q)\Delta f(q) \geq 0. \tag{5.6.36}$$

(5.6.36) now implies

$$\beta - \lambda(f(p))^2 = \beta - \lambda(f(q))^2 = 0. \tag{5.6.37}$$

Hence, $\lambda < 0$ implies that $f(p) = f(q)$. That is, the warping function f is a constant function. When $\lambda = 0$, we obtain that $\beta = 0$ and equation (5.6.33) becomes

$$\begin{aligned} Lf =& (\Delta - \nabla\phi)f, [\text{where } L = \Delta - \nabla\phi] \\ =& \frac{(1-m)}{f} \mid \nabla f \mid^2 + (h-1)\nabla\phi(f) \end{aligned} \tag{5.6.38}$$

Clearly, $\frac{(1-m)}{f} \mid \nabla f \mid^2 \leq 0$. It is also seen that $Lf \leq 0$, if

$$(h-1)\nabla\phi(f) \geq \frac{(1-m)}{f} \mid \nabla f \mid^2 . \tag{5.6.39}$$

So, if $(h-1)\nabla\phi(f) \geq \frac{(1-m)}{f} \mid \nabla f \mid^2$, then by using strong maximum principle, it is obvious that f is constant. Therefore, in both cases M is a Riemannian product. □

Theorem 5.6.5. *Let $M = B^n \times_f F^m$ be a warped product and ϕ is a smooth function on B such that $(M, g, \nabla\tilde{\phi}, h, \eta, \lambda)$ is a shrinking gradient h-almost η-Ricci soliton having compact base and fiber of dimension greater than or equal to two. Then M will definitely be a compact manifold if $\int_{B^n}(1-h)f(\nabla\phi(f))dB > 0$.*

Proof. Let $M = B^n \times_f F^m$, $m > 1$, be a gradient h-almost η-Ricci soliton satisfying (1.3.4). From Theorem 5.6.4, it follows that $\mathrm{Ric}_F = \beta g_F + \mu(\eta \otimes \eta)$, where β is a constant which is given by (5.6.33) or equivalently

$$\begin{aligned}
\beta =& \lambda f^2 + f\Delta f + (m-1) \mid \nabla f \mid^2 - hf(\nabla\phi(f)) \\
=& \lambda f^2 + f(\Delta f - \nabla\phi(f)) + (m-1) \mid \nabla f \mid^2 + (1-h)f\nabla\phi(f) \\
=& \lambda f^2 + fLf + (m-1) \mid \nabla f \mid^2 + (1-h)f\nabla\phi(f).
\end{aligned} \tag{5.6.40}$$

Integrating on both sides, we have

$$\begin{aligned}
\beta \mathrm{vol}_\phi(B^n) =& \lambda \int_{B^n} f^2 e^{-\phi} \mathrm{d}B + (m-2) \int_{B^n} \mid \nabla f \mid^2 e^{-\phi} \mathrm{d}B \\
&+ \int_{B^n} (1-h)f(\nabla\phi(f))\mathrm{d}B.
\end{aligned} \tag{5.6.41}$$

As $m > 1$ and $\lambda > 0$, hence we conclude that $\beta > 0$ if $\int_{B^n}(1-h)f(\nabla\phi(f))dB > 0$. Therefore, by using Bonnet-Myers Theorem, it is obvious that F^m is compact and consequently $B^n \times_f F^m$ becomes a compact manifold. □

Theorem 5.6.6. *Let $\bar{M} = I \times_f M$ be a generalized Robertson-walker space time furnished by a metric $\bar{g} = -\mathrm{d}t^2 \oplus f^2 g$, where (M, g) is a Riemannian manifold and I is an open connected interval with the usual flat metric $-\mathrm{d}t^2$. If $(\bar{\bar{M}}, \bar{g}, u, h, \eta, \lambda)$ be a gradient h-almost η-Ricci soliton, for $u = \int_a^t f(r)\mathrm{d}r$, where $a \in I$ is a constant, then* $\mathrm{Ric} = (\lambda - h\dot{f})\bar{g} + \mu(\eta \otimes \eta)$.

Proof. Assume that $\zeta = \text{grad u}$, hence $\zeta = f(t)\partial_t$. Clearly, the vector field is orthogonal to M. Let ∂_t, ∂_1, ∂_2, ..., ∂_m are orthogonal bases of $\chi(\bar{M})$, then the Hessian tensor of u is given as follows.

$$H^u(\partial_t, \partial_t) = \bar{g}(\nabla_X \text{grad u}, Y).$$

Now, the following cases may arise. The first case when $X = Y = \partial_t$. For this, we get

$$\begin{aligned} H^u(\partial_t, \partial_t) =& \bar{g}(\nabla_{\partial_t} \text{grad u}, \partial_t) \\ =& \dot{f}\bar{g}(\partial_t, \partial_t). \end{aligned} \tag{5.6.42}$$

The second case when $X = \partial_t$ and $Y = \partial_i$, $i = 1, 2, 3, ..., m$. For this, we get

$$\begin{aligned} H^u(\partial_t, \partial_i) =& \bar{g}(\nabla_{\partial_t} \text{grad } u, \partial_i) \\ =& \dot{f}\bar{g}(\partial_t, \partial_i). \end{aligned} \tag{5.6.43}$$

At last, when $X = \partial_t$ and $Y = \partial_i$, $i = 1, 2, 3, ..., m$. For this, we obtain

$$\begin{aligned} H^u(\partial_i, \partial_j) =& \bar{g}(\nabla_{\partial_i} \text{grad } u, \partial_j) \\ =& f\bar{g}(\nabla_{\partial_i} \partial_t, \partial_j) \\ =& f\bar{g}\left(\frac{\dot{f}}{f}\partial_i, \partial_j\right) \\ =& \dot{f}\bar{g}(\partial_i, \partial_j). \end{aligned} \tag{5.6.44}$$

Hence, $H^u(X, Y) = \dot{f}\bar{g}(X, Y)$ and consequently

$$\begin{aligned} (\pounds_\xi \bar{g})(X, Y) =& \bar{g}(\nabla_X \text{grad } u, Y) + \bar{g}(\nabla_Y \text{grad } u, X) \\ =& 2H^u(X, Y) \\ =& 2\dot{f}\bar{g}(X, Y). \end{aligned} \tag{5.6.45}$$

Let $(\bar{\bar{M}}, \bar{g}, u, h, \eta, \lambda)$ be a gradient h-almost η-Ricci soliton, then

$$\begin{aligned} \text{Ric} + \frac{h}{2}\pounds_\xi \bar{g} = \lambda \bar{g} + \mu(\eta \otimes \eta) \\ i.e., \text{Ric} = (\lambda - h\dot{f})\bar{g} + \mu(\eta \otimes \eta) \end{aligned} \tag{5.6.46}$$

This completes the proof of Theorem 5.6.6. □

Theorem 5.6.7. *Let $(M,g,h,\varsigma,\lambda,\mu)$ be an h-almost η-Ricci soliton and ς be a concurrent vector field on M where $M = B^n \times_f F^m$ and $\varsigma_2 \neq 0$. Then F becomes an Einstein manifold for $U_1, U_2 \in \mathfrak{X}(B)$.*

Proof. We consider that $(M,g,h,\varsigma,\lambda,\mu)$ is a h-almost η-Ricci soliton. Then we have

$$\mathrm{Ric}(X,Y) + \frac{h}{2}\pounds_X g(X,Y) = \lambda g(X,Y) + \mu\eta(X)\eta(Y),$$

where $\eta(X) = g(X,U)$.

Since ς is a concurrent vector field, we obtain

$$\mathrm{Ric}(X,Y) + \frac{h}{2}(g(D_X\varsigma,Y) + g(D_Y\varsigma,X)) = \lambda g(X,Y) + \mu\eta(X)\eta(Y).$$

Hence we get

$$\mathrm{Ric}(X,Y) = (\lambda - h)g(X,Y) + \mu\eta(X)\eta(Y), \tag{5.6.47}$$

Putting $X = V \in \mathfrak{X}(F), Y = W \in \mathfrak{X}(F)$, and $U_1, U_2 \in \mathfrak{X}(B)$ then by using Proposition 1.2.4, it follows that

$$\mathrm{Ric}_F(V,W) = (\lambda - h)f^2 g_F(V,W) + \left[\frac{\Delta f}{f} + \frac{|\nabla f|^2}{f^2}(m-1)\right] f^2 g_F(V,W). \tag{5.6.48}$$

Since ς is concurrent and $\varsigma_2 \neq 0$, ς is concurrent and f is constant. Hence we have $\left[\frac{\Delta f}{f} + \frac{|\nabla f|^2}{f^2}(m-1)\right] = 0$ and also we obtain

$$\mathrm{Ric}_F(V,W) = (\lambda - h)f^2 g_F(V,W). \tag{5.6.49}$$

This implies that F is an Einstein manifold. □

Theorem 5.6.8. *Let $(M,g,h,u,\varsigma,\lambda,\mu)$ be a gradient h-almost η-Ricci soliton where $M = B^n \times_f F^m$. Then (B,g,u,λ) is a gradient Ricci soliton if h is a constant function and $U_1, U_2 \in \mathfrak{X}(F)$.*

Proof. Let $(M,g,h,u,\varsigma,\lambda,\mu)$ be a gradient h-almost η-Ricci soliton. Then we have

$$\mathrm{Ric}(X',X'') + hH^u(X',X'') = \lambda g(X',X'') + \mu\eta(X')\eta(X''). \tag{5.6.50}$$

Let $X' = Y \in \mathfrak{X}(B), X'' = Z \in \mathfrak{X}(B)$ and $U_1, U_2 \in \mathfrak{X}(F)$, then it follows that

$$\mathrm{Ric}(Y,Z) + hH_B^{u_1}(Y,Z) = \lambda g(Y,Z). \tag{5.6.51}$$

Using Proposition 1.2.4 we have

$$\mathrm{Ric}_B(Y,Z) - \frac{m}{f}H^f(Y,Z) + hH_B^{u_1}(Y,Z) = \lambda g(Y,Z). \tag{5.6.52}$$

Then we obtain

$$h(Y(Zu_1)) - h(\nabla_Y Z)u_1 - \frac{m}{f}(Y(Zf)) + \nabla_Y(Z(m\ln f)) - Z(Y(m\ln f))$$
$$+\mathrm{Ric}_B(Y,Z) = \lambda g_B(Y,Z).$$

Hence we get

$$Y(Z(hu_1 - m\ln f)) - (\nabla_Y Z)(hu_1 - m\ln f) + \mathrm{Ric}_B(Y,Z) = \lambda g_B(Y,Z).$$

It follows that

$$H_B^{\phi_1}(Y,Z) + \mathrm{Ric}_B(Y,Z) = \lambda g_B(Y,Z),$$

where $\phi_1 = hu_1 - m\ln f$, $h =$ constant and $u_1 = u$ at a fixed point on F. Hence we establish that (B,g,u,λ) is a gradient Ricci soliton. □

We end this chapter with these notable theorems.

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Pseudo-projective curvature tensor on warped product manifolds and its applications in space-times

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Abstract. In this paper we study the pseudo-projective curvature tensor on warped product manifolds. We obtain some significant results of the pseudo-projective curvature tensor on warped product manifolds in terms of its base and fiber manifolds. Moreover, we derive some interesting results which describe the geometry of base and fiber manifolds for a pseudo-projectively flat warped product manifold. Lastly, we study the pseudo-projective curvature tensor on generalized Robertson-Walker space-times and standard static space-times.

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Key words and phrases. Warped product, pseudo-projective curvature tensor, generalized Robertson-Walker space-times, standard static space-times.

§1. Introduction

Bishop and O'Neill [6] had given the idea of warped product in Riemannian manifolds. They introduced the notion of warped product for making a large class of complete manifolds having negative curvature. The main idea of this warped product actually appeared on account of a surface of revolution. Later, Nölker [13] also developed the concept of multiply warped product as a generalization of warped product. The warped product plays a very significant role in differential geometry, especially in mathematical physics and general relativity. Schwarzschild solution, Robertson-walker model, static model and Kruscal model etc. are the examples of warped products. There are so many exact solutions of Einstein field equations and modified field equations. These solutions can be written in terms of warped products.

The pseudo-projective curvature tensor had been defined by Prasad [15]. The pseudo-projective curvature tensor includes the projective curvature tensor. Many authors [8, 10, 11, 12] studied the pseudo-projective curvature tensor in different ways. The pseudo-projective curvature tensor has been studied in mathematics as well as physics as a research topic. Shenawy and Ünal [19] studied on the W_2-curvature tensor on warped product manifolds. In view of the above interesting works, we wish to study the pseudo-projective curvature tensor on warped product manifolds and space-times.

The aim of this paper is to study the geometry of pseudo-projective curvature tensor on warped product manifolds. Besides this we discuss its applications to Robertson-Walker space-times and standard static space-times. Hence this paper connects the pseudo-projective curvature tensor to warped product manifold, Robertson-Walker space-times and standard static space-times.

This paper has been arranged in the following way. In section 2, we state the concept of pseudo-projective curvature tensor and warped product manifolds. In section 3, we discuss some interesting results of pseudo-projective curvature tensor on warped product manifolds in terms of its base and fiber manifolds. In section 4, we study pseudo-projective curvature tensor on generalized Robertson-Walker space-times. The last section is devoted to the study of standard static space-times admitting the pseudo-projective curvature tensor.

§2. Preliminaries

In this part, we just recall some basic ideas on warped product and pseudo-projective curvature tensor.

Let (B, g_B) and (F, g_F) be two Riemannian manifolds with $\dim(B) > 0$ and $\dim(F) > 0$. Let $f : B \to (0, \infty)$ be a positive smooth function on B. Suppose the natural projections of the product manifold $B \times F$ are $\pi : B \times F \to B$ and $\eta : B \times F \to F$. The warped product $M = B \times_f F$ is the product manifold $B \times F$ furnished with the Riemannian structure such that

$$< X, X >=< \pi^*(X), \pi^*(X) > +f^2(\pi(X)) < \eta^*(X), \eta^*(X) >,$$

for each tangent vector $X \in \mathfrak{X}(M)$. Therefore, we obtain the metric relation $g_M = g_B \oplus f^2 g_F$. B and F are respectively the base and fiber of this warped product manifold. The function f is known as the warping function of this warped product.

Proposition 2.1 ([14]). *Let $M = B \times_f F$ be a warped product with Riemannian curvature tensor R. If $X, Y, Z \in \mathfrak{X}(B)$ and $U, V, W \in \mathfrak{X}(F)$, then*

$$(1)\quad R(X, Y)Z = R^B(X, Y)Z,$$

$$(2)\quad R(V, X)Y = \frac{H^f(X, Y)}{f} V,$$

$$(3)\quad R(X, Y)V = R(V, W)X = 0,$$

(4) $R(X,V)W = \dfrac{g(V,W)}{f} D^1_X(\nabla f)$,

(5) $R(V,W)U = R^F(V,W)U + \dfrac{\|\nabla f\|^2}{f^2}\left[g(W,U)V - g(V,U)W\right].$

Proposition 2.2 ([14])**.** *On the warped product $M = B \times_f F$ with $\dim(F) = d > 1$, let $X, Y \in \mathfrak{X}(B)$ and $V, W \in \mathfrak{X}(F)$. Then the Ricci tensor S_M of M are given by*

(1) $S_M(X,Y) = S_B(X,Y) - \dfrac{d}{f} H^f(X,Y)$,

(2) $S_M(X,V) = 0$,

(3) $S_M(V,W) = S_F(V,W) - g(V,W)f^{\#}, \quad f^{\#} = \dfrac{\Delta f}{f} + \dfrac{d-1}{f^2}\|\nabla f\|^2$,

where $\Delta f = tr\left(H^f\right)$ and H^f are respectively the Laplacian and the Hessian of f on B.

Proposition 2.3 ([7])**.** *Let $M = B \times_f F$ be a semi-Riemannian warped product furnished with the metric $g_M = g_B \oplus f^2 g_F$. Then the scalar curvature τ of M admits the following relation*

$$\tau = \tau_B + \frac{\tau_F}{f^2} - 2s\frac{\Delta_B(f)}{f} - s(s-1)\frac{\|\mathrm{grad}_B f\|_B^2}{f^2},$$

where $r = \dim(B)$ and $s = \dim(F)$.

The pseudo-projective curvature tensor $\bar{P}^*$ on a pseudo-Riemannian manifold is defined by

$$\begin{aligned}\bar{P}^*(X,Y,Z,W) &= a_1\bar{R}(X,Y,Z,W) + a_2[S(Y,Z)g(X,W) \\ &\quad - S(X,Z)g(Y,W)] - \frac{\tau}{n}\left(\frac{a_1}{n-1} + a_2\right) \\ &\quad \times [g(Y,Z)g(X,W) - g(X,Z)g(Y,W)],\end{aligned} \tag{2.1}$$

where a_1 and a_2 ($\neq 0$) are two constants, S is the Ricci tensor of $(0,2)$-type, the scalar curvature of the manifold is τ, $\bar{P}^*(X,Y,Z,W) = g(P^*(X,Y)Z,W)$, $\bar{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, where R is the Riemannian curvature tensor.

If $a_1 = 1$ and $a_2 = -\frac{1}{n-1}$, then Eq. (2.1) reduces to the projective curvature tensor. Moreover, if $P^* = 0$ for $n > 3$, then a pseudo-Riemannian manifold is called pseudo-projectively flat.

It clearly follows from Eq. (2.1) that

$$\begin{aligned}P^*(X,Y)Z &= a_1 R(X,Y)Z + a_2\left[S(Y,Z)X - S(X,Z)Y\right] \\ &\quad - \frac{\tau}{n}\left(\frac{a_1}{n-1} + a_2\right)\left[g(Y,Z)X - g(X,Z)Y\right].\end{aligned} \tag{2.2}$$

Remark. Suppose M is a semi-Riemannian manifold. Then

$$P^*(X,Y)Z + P^*(Y,Z)X + P^*(Z,X)Y = 0,$$

for $X, Y, Z \in \mathfrak{X}(M)$.

Proposition 2.4. *Suppose M is a semi-Riemannian manifold. Then the pseudo-projective curvature tensor vanishes if and only if the tensor P^* vanishes.*

A Riemannian metric g is said to be of Hessian type metric if $H^{f_1} = f_2 g$ for any two smooth functions f_1 and f_2, where H^{f_1} denotes the Hessian of the function f_1.

§3. Pseudo-projective curvature tensor on warped product manifolds

Here we study the pseudo-projective curvature tensor on warped product manifolds. We consider the warped product $M = M_1 \times_f M_2$ where $\dim(M) = n$, $\dim(M_1) = n_1$ and $\dim(M_2) = n_2$ such that $n = n_1 + n_2$, $n_i \neq 1$ for $i = 1, 2$. We denote R, R^i as the curvature tensor and S, S^i as the Ricci tensor on M, M_i respectively. On the other hand, ∇f, Δf and H^f are respectively the gradient, Laplacian and Hessian of f on M_1. D, D^i indicate the Levi-Civita connection with respect to the metric g, g_i for $i = 1,\ 2$ respectively. Throughout our entire study we use the relation $f^{\#} = \frac{\Delta f}{f} + \frac{n_2-1}{f^2}\|\nabla f\|^2$. Last of all, we denote the pseudo-projective curvature tensor and the tensor P^* on M and M_i by $\bar{P}^*$, P^* and $\bar{P}^*_i$, P^*_i respectively.

Now we obtain the following theorems for the pseudo-projective curvature tensor on warped product manifolds. These theorems describe the warped geometry in terms of its base and fiber manifolds.

Theorem 3.1. *Let $M = M_1 \times_f M_2$ be a warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. If $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$ for $i = 1, 2$, then*

$$\begin{aligned}
P^*(X_1,Y_1)Z_1 &= P_1^*(X_1,Y_1)Z_1 + \tau\left[\frac{n_2(n+n_1-1)}{nn_1(n-1)(n_1-1)}a_1 + \frac{n_2}{nn_1}a_2\right] \\
&\quad \times [g_1(Y_1,Z_1)X_1 - g_1(X_1,Z_1)Y_1] \\
&\quad + \frac{a_2 n_2}{f}\left[H^f(X_1,Z_1)Y_1 - H^f(Y_1,Z_1)X_1\right], \\
P^*(X_1,Y_1)Z_2 &= P^*(X_2,Y_2)Z_1 = 0, \\
P^*(X_1,Y_2)Z_1 &= \left(\frac{a_2 n_2 - a_1}{f}\right) H^f(X_1,Z_1)Y_2 - a_2 S^1(X_1,Z_1)Y_2 \\
&\qquad + \frac{\tau}{n}\left(\frac{a_1}{n-1} + a_2\right) g_1(X_1,Z_1)Y_2,
\end{aligned}$$

$$
\begin{aligned}
P^*(X_1,Y_2)Z_2 &= a_1 f g_2(Y_2,Z_2) D^1_{X_1}\nabla f + a_2 S^2(Y_2,Z_2)X_1 \\
&\quad - f^2\left[a_2 f^{\#} + \frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\right] g_2(Y_2,Z_2)X_1, \\
P^*(X_2,Y_2)Z_2 &= P^*_2(X_2,Y_2)Z_2 + \left[\left(\frac{n^2-n-n_2^2 f^2+n_2 f^2}{nn_2(n-1)(n_2-1)}\right)a_1\tau\right. \\
&\quad \left. + \left(\frac{n-n_2 f^2}{nn_2}\right)\tau a_2 - a_2 f^2 f^{\#} + a_1\|\nabla f\|^2\right] \\
&\quad \times \left[g_2(Y_2,Z_2)X_2 - g_2(X_2,Z_2)Y_2\right].
\end{aligned}
$$

Proof. Let $M = M_1 \times_f M_2$ be a warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. Let $\dim(M) = n$, $\dim(M_i) = n_i$ for $i = 1, 2$ and $n = n_1 + n_2$. If $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$ for $i = 1, 2$. Then, we obtain

$$
\begin{aligned}
P^*(X_1,Y_1)Z_1 &= a_1 R(X_1,Y_1)Z_1 + a_2\left[S(Y_1,Z_1)X_1 - S(X_1,Z_1)Y_1\right] \\
&\quad - \frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\left[g(Y_1,Z_1)X_1 - g(X_1,Z_1)Y_1\right] \\
&= a_1 R^1(X_1,Y_1)Z_1 + a_2\left[\left\{S^1(Y_1,Z_1) - \frac{n_2}{f}H^f(Y_1,Z_1)\right\}X_1\right. \\
&\quad \left. - \left\{S^1(X_1,Z_1) - \frac{n_2}{f}H^f(X_1,Z_1)\right\}Y_1\right] \\
&\quad - \frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\left[g_1(Y_1,Z_1)X_1 - g_1(X_1,Z_1)Y_1\right] \\
&= a_1 R^1(X_1,Y_1)Z_1 + a_2[S^1(Y_1,Z_1)X_1 - S^1(X_1,Z_1)Y_1] \\
&\quad - \frac{\tau}{n_1}\left(\frac{a_1}{n_1-1}+a_2\right)\left[g_1(Y_1,Z_1)X_1 - g_1(X_1,Z_1)Y_1\right] \\
&\quad + \left[\frac{\tau}{n_1}\left(\frac{a_1}{n_1-1}+a_2\right) - \frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\right] \\
&\quad \times \left[g_1(Y_1,Z_1)X_1 - g_1(X_1,Z_1)Y_1\right] \\
&\quad + \frac{a_2 n_2}{f}\left[H^f(X_1,Z_1)Y_1 - H^f(Y_1,Z_1)X_1\right] \\
&= P^*_1(X_1,Y_1)Z_1 + \tau\left[\frac{n_2(n+n_1-1)}{nn_1(n-1)(n_1-1)}a_1 + \frac{n_2}{nn_1}a_2\right] \\
&\quad \times \left[g_1(Y_1,Z_1)X_1 - g_1(X_1,Z_1)Y_1\right] \\
&\quad + \frac{a_2 n_2}{f}\left[H^f(X_1,Z_1)Y_1 - H^f(Y_1,Z_1)X_1\right], \\
P^*(X_1,Y_1)Z_2 &= a_1 R(X_1,Y_1)Z_2 + a_2\left[S(Y_1,Z_2)X_1 - S(X_1,Z_2)Y_1\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)[g(Y_1,Z_2)X_1-g(X_1,Z_2)Y_1]\\
&=0,\\
P^*(X_1,Y_2)Z_1 &= a_1R(X_1,Y_2)Z_1+a_2\left[S(Y_2,Z_1)X_1-S(X_1,Z_1)Y_2\right]\\
&-\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)[g(Y_2,Z_1)X_1-g(X_1,Z_1)Y_2]\\
&=-\left(\frac{a_1}{f}\right)H^f(X_1,Z_1)Y_2-a_2\bigg[S^1(X_1,Z_1)Y_2\\
&-\frac{n_2}{f}H^f(X_1,Z_1)Y_2\bigg]+\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)g_1(X_1,Z_1)Y_2\\
&=\left(\frac{a_2n_2-a_1}{f}\right)H^f(X_1,Z_1)Y_2-a_2S^1(X_1,Z_1)Y_2\\
&+\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)g_1(X_1,Z_1)Y_2,\\
P^*(X_1,Y_2)Z_2 &= a_1R(X_1,Y_2)Z_2+a_2\left[S(Y_2,Z_2)X_1-S(X_1,Z_2)Y_2\right]\\
&-\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)[g(Y_2,Z_2)X_1-g(X_1,Z_2)Y_2]\\
&=\left(\frac{a_1}{f}\right)g(Y_2,Z_2)D^1_{X_1}\nabla f+a_2\big[S^2(Y_2,Z_2)X_1\\
&-f^{\#}g(Y_2,Z_2)X_1\big]-\frac{\tau f^2}{n}\left(\frac{a_1}{n-1}+a_2\right)g_2(Y_2,Z_2)X_1\\
&=a_1fg_2(Y_2,Z_2)D^1_{X_1}\nabla f+a_2S^2(Y_2,Z_2)X_1\\
&-f^2\left[a_2f^{\#}+\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\right]g_2(Y_2,Z_2)X_1,\\
P^*(X_2,Y_2)Z_1 &= a_1R(X_2,Y_2)Z_1+a_2\left[S(Y_2,Z_1)X_2-S(X_2,Z_1)Y_2\right]\\
&-\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)[g(Y_2,Z_1)X_2-g(X_2,Z_1)Y_2]\\
&=0,\\
P^*(X_2,Y_2)Z_2 &= a_1R(X_2,Y_2)Z_2+a_2\left[S(Y_2,Z_2)X_2-S(X_2,Z_2)Y_2\right]\\
&-\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)[g(Y_2,Z_2)X_2-g(X_2,Z_2)Y_2]\\
&=a_1\left[R^2(X_2,Y_2)Z_2+\frac{\|\nabla f\|^2}{f^2}\{g(Y_2,Z_2)X_2-g(X_2,Z_2)Y_2\}\right]\\
&+a_2[\{S^2(Y_2,Z_2)X_2-f^{\#}g(Y_2,Z_2)X_2\}\\
&-\{S^2(X_2,Z_2)Y_2-f^{\#}g(X_2,Z_2)Y_2\}]\\
&-\frac{\tau f^2}{n}\left(\frac{a_1}{n-1}+a_2\right)[g_2(Y_2,Z_2)X_2-g_2(X_2,Z_2)Y_2]\\
&=a_1R^2(X_2,Y_2)Z_2+a_2\left[S^2(Y_2,Z_2)X_2-S^2(X_2,Z_2)Y_2\right]\\
&-\frac{\tau}{n_2}\left(\frac{a_1}{n_2-1}+a_2\right)[g_2(Y_2,Z_2)X_2-g_2(X_2,Z_2)Y_2]
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{\tau}{n_2}\left(\frac{a_1}{n_2-1}+a_2\right)-\frac{\tau f^2}{n}\left(\frac{a_1}{n-1}+a_2\right)\right.\\
&\left.-a_2 f^2 f^{\#}+a_1\|\nabla f\|^2\right][g_2(Y_2,Z_2)X_2-g_2(X_2,Z_2)Y_2]\\
&=P_2^*(X_2,Y_2)Z_2+\left[\left(\frac{n^2-n-n_2^2f^2+n_2f^2}{nn_2(n-1)(n_2-1)}\right)a_1\tau\right.\\
&\left.+\left(\frac{n-n_2f^2}{nn_2}\right)\tau a_2-a_2f^2f^{\#}+a_1\|\nabla f\|^2\right]\\
&\times[g_2(Y_2,Z_2)X_2-g_2(X_2,Z_2)Y_2].
\end{aligned}
$$

This completes the proof. □

Theorem 3.2. *Let $M=M_1\times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g=g_1\oplus f^2g_2$. Then*

$$
\begin{aligned}
\bar{P}_1^*(X_1,Y_1,Z_1,W_1)=\tau&\left[\frac{n_2(n+n_1-1)}{nn_1(n-1)(n_1-1)}a_1+\frac{n_2}{nn_1}a_2\right]\\
&\times[g_1(X_1,Z_1)g_1(Y_1,W_1)-g_1(Y_1,Z_1)g_1(X_1,W_1)]\\
&+\frac{a_2n_2}{f}\left[H^f(Y_1,Z_1)g_1(X_1,W_1)-H^f(X_1,Z_1)g_1(Y_1,W_1)\right],
\end{aligned}
$$

for $X_1,Y_1,Z_1,W_1\in\mathfrak{X}(M_1)$.

Proof. Let us assume that $M=M_1\times_f M_2$ be a pseudo-projectively flat warped product manifold. Therefore, in view of Theorem 3.1, we obtain

$$
\begin{aligned}
P_1^*(X_1,Y_1)Z_1=\tau&\left[\frac{n_2(n+n_1-1)}{nn_1(n-1)(n_1-1)}a_1+\frac{n_2}{nn_1}a_2\right]\\
&\times[g_1(X_1,Z_1)Y_1-g_1(Y_1,Z_1)X_1]\\
&+\frac{a_2n_2}{f}\left[H^f(Y_1,Z_1)X_1-H^f(X_1,Z_1)Y_1\right].
\end{aligned}
$$

Therefore, we derive

$$
\begin{aligned}
\bar{P}_1^*(X_1,Y_1,Z_1,W_1)&=g_1\left(P_1^*(X_1,Y_1)Z_1,W_1\right)\\
&=\tau\left[\frac{n_2(n+n_1-1)}{nn_1(n-1)(n_1-1)}a_1+\frac{n_2}{nn_1}a_2\right]\\
&\times[g_1(X_1,Z_1)g_1(Y_1,W_1)-g_1(Y_1,Z_1)g_1(X_1,W_1)]\\
&+\frac{a_2n_2}{f}\big[H^f(Y_1,Z_1)g_1(X_1,W_1)\\
&-H^f(X_1,Z_1)g_1(Y_1,W_1)\big].
\end{aligned}
$$

This completes the proof. □

Theorem 3.3. *Let $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. Then the base manifold M_1 is pseudo-projectively flat if and only if*

$$\begin{aligned}&\tau\left[\frac{n_2(n+n_1-1)}{nn_1(n-1)(n_1-1)}a_1 + \frac{n_2}{nn_1}a_2\right]\\ &\times[g_1(X_1,Z_1)g_1(Y_1,W_1) - g_1(Y_1,Z_1)g_1(X_1,W_1)]\\ &+\frac{a_2n_2}{f}\left[H^f(Y_1,Z_1)g_1(X_1,W_1) - H^f(X_1,Z_1)g_1(Y_1,W_1)\right] = 0,\end{aligned}$$

for $X_1, Y_1, Z_1, W_1 \in \mathfrak{X}(M_1)$.

Proof. Let the base manifold M_1 be pseudo-projectively flat. Then

$$\bar{P}_1^*(X_1, Y_1, Z_1, W_1) = 0.$$

Clearly, the proof follows from Theorem 3.2. □

Theorem 3.4. *Let $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. Then the scalar curvature τ_1 of M_1 is given by*

$$\tau_1 = \frac{1}{a_2}\left[\left(\frac{a_2n_2 - a_1}{f}\right)\Delta f + \frac{\tau n_1}{n}\left(\frac{a_1}{n-1} + a_2\right)\right].$$

Proof. Let us assume that $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold. Then Theorem 3.1 implies that

$$S^1(X_1, Z_1) = \frac{1}{a_2}\left[\left(\frac{a_2n_2 - a_1}{f}\right)H^f(X_1, Z_1) + \frac{\tau}{n}\left(\frac{a_1}{n-1} + a_2\right)g_1(X_1, Z_1)\right].$$

Taking contraction over X_1 and Z_1, we gain

$$\tau_1 = \frac{1}{a_2}\left[\left(\frac{a_2n_2 - a_1}{f}\right)\Delta f + \frac{\tau n_1}{n}\left(\frac{a_1}{n-1} + a_2\right)\right].$$

This completes the proof. □

*Remark***.** Proposition 2.3 [7] and Theorem 3.4 jointly imply that the scalar curvature τ_2 of (M_2, g_2) is a constant since the left hand side of the equation in Theorem 3.4 depends only on the base manifold (M_1, g_1).

Theorem 3.5. *Let $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. Then the pseudo-projective* *curvature tensor of* M_2 *is given by*

$$\begin{aligned}\bar{P}_2^*(X_2,Y_2,Z_2,W_2) = &\Bigg[\left(\frac{n^2-n-n_2^2f^2+n_2f^2}{nn_2(n-1)(n_2-1)}\right)a_1\tau+\left(\frac{n-n_2f^2}{nn_2}\right)\tau a_2 \\ &-a_2f^2f^\#+a_1\|\nabla f\|^2\Bigg][g_2(X_2,Z_2)g_2(Y_2,W_2) \\ &-g_2(Y_2,Z_2)g_2(X_2,W_2)],\end{aligned}$$

for $X_2,Y_2,Z_2,W_2\in\mathfrak{X}(M_2)$*.*

Proof. Let $M=M_1\times_f M_2$ be a pseudo-projectively flat warped product manifold. From Theorem 3.1, it follows that

$$\begin{aligned}0=P_2^*(X_2,Y_2)Z_2+&\Bigg[\left(\frac{n^2-n-n_2^2f^2+n_2f^2}{nn_2(n-1)(n_2-1)}\right)a_1\tau+\left(\frac{n-n_2f^2}{nn_2}\right)\tau a_2 \\ -a_2f^2f^\#+a_1\|\nabla f\|^2&\Bigg]\left[g_2(Y_2,Z_2)X_2-g_2(X_2,Z_2)Y_2\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{P}_2^*(X_2,Y_2,Z_2,W_2) &= g_2\left(P_2^*(X_2,Y_2)Z_2,W_2\right) \\ &=\Bigg[\left(\frac{n^2-n-n_2^2f^2+n_2f^2}{nn_2(n-1)(n_2-1)}\right)a_1\tau+\left(\frac{n-n_2f^2}{nn_2}\right)\tau a_2 \\ &\quad-a_2f^2f^\#+a_1\|\nabla f\|^2\Bigg][g_2(X_2,Z_2)g_2(Y_2,W_2) \\ &\quad-g_2(Y_2,Z_2)g_2(X_2,W_2)].\end{aligned}$$

This completes the proof. □

Theorem 3.6. *Let* $M=M_1\times_f M_2$ *be a pseudo-projectively flat warped product manifold furnished with the metric* $g=g_1\oplus f^2g_2$*. If the fiber manifold* M_2 *is Ricci flat, then the base manifold* M_1 *is of Hessian type.*

Proof. Let $M=M_1\times_f M_2$ be a pseudo-projectively flat warped product manifold. Then from Theorem 3.1, we derive

$$\begin{aligned}0=&\,a_1fg_2(Y_2,Z_2)D^1_{X_1}\nabla f+a_2S^2(Y_2,Z_2)X_1 \\ &-f^2\left[a_2f^\#+\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\right]g_2(Y_2,Z_2)X_1.\end{aligned}$$

Suppose that M_2 is Ricci flat. Then $S^2(X_2,Y_2)=0$ for any $X_2,Y_2\in\mathfrak{X}(M_2)$. Hence, we obtain from the above relation

$$D^1_{X_1}\nabla f=\frac{f}{a_1}\left[a_2f^\#+\frac{\tau}{n}\left(\frac{a_1}{n-1}+a_2\right)\right]X_1.$$

This implies that

$$H^f = \frac{f}{a_1}\left[a_2 f^{\#} + \frac{\tau}{n}\left(\frac{a_1}{n-1} + a_2\right)\right] g_1.$$

Hence, M_1 is of Hessian type. This completes the proof. □

Theorem 3.7. *Let $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. If the fiber manifold M_2 is Ricci flat, then the pointwise constant sectional curvature τ_2 of M_2 is given by*

$$\tau_2 = \frac{1}{a_1}\Bigg[-\left(\frac{n^2 - n - n_2^2 f^2 + n_2 f^2}{nn_2(n-1)(n_2-1)}\right) a_1\tau - \left(\frac{n - n_2 f^2}{nn_2}\right)\tau a_2 + a_2 f^2 f^{\#}$$
$$- a_1\|\nabla f\|^2 + \frac{\tau}{n}\left(\frac{a_1}{n-1} + a_2\right)\Bigg].$$

Proof. Let M_2 be Ricci flat. Therefore, from Eq. (2.1), we have

$$\bar{R}^2(X_2, Y_2, Z_2, W_2) = \frac{1}{a_1}\Bigg[\bar{P}_2^*(X_2, Y_2, Z_2, W_2) + \frac{\tau}{n}\left(\frac{a_1}{n-1} + a_2\right)$$
$$\times \{g_2(Y_2, Z_2)g_2(X_2, W_2) - g_2(X_2, Z_2)g_2(Y_2, W_2)\}\Bigg].$$

In view of Theorem 3.1, we derive from the above relation that

$$\bar{R}^2(X_2, Y_2, Z_2, W_2) = \frac{1}{a_1}\Bigg[-\left(\frac{n^2 - n - n_2^2 f^2 + n_2 f^2}{nn_2(n-1)(n_2-1)}\right) a_1\tau - \left(\frac{n - n_2 f^2}{nn_2}\right)\tau a_2$$
$$+ a_2 f^2 f^{\#} - a_1\|\nabla f\|^2 + \frac{\tau}{n}\left(\frac{a_1}{n-1} + a_2\right)\Bigg]$$
$$\times \{g_2(Y_2, Z_2)g_2(X_2, W_2) - g_2(X_2, Z_2)g_2(Y_2, W_2)\}.$$

This implies that M_2 has a pointwise constant sectional curvature and this curvature is given by

$$\tau_2 = \frac{1}{a_1}\Bigg[-\left(\frac{n^2 - n - n_2^2 f^2 + n_2 f^2}{nn_2(n-1)(n_2-1)}\right) a_1\tau - \left(\frac{n - n_2 f^2}{nn_2}\right)\tau a_2 + a_2 f^2 f^{\#}$$
$$- a_1\|\nabla f\|^2 + \frac{\tau}{n}\left(\frac{a_1}{n-1} + a_2\right)\Bigg].$$

This completes the proof. □

Theorem 3.8. *Let $M = M_1 \times_f M_2$ be a warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. If $H^f = 0$, $\Delta f = 0$ and M is pseudo-projectively flat, then M_2 is an Einstein manifold.*

Proof. Let M be pseudo-projectively flat. Therefore, M_1 is flat in view of Theorem 3.2. Furthermore, from Theorem 3.1, we obtain

$$\begin{aligned}0 &= a_1 f g_2(Y_2, Z_2) D^1_{X_1}\nabla f + a_2 S^2(Y_2, Z_2)X_1 \\ &\quad - f^2\left[a_2 f^{\#} + \frac{\tau}{n}\left(\frac{a_1}{n-1} + a_2\right)\right] g_2(Y_2, Z_2)X_1.\end{aligned} \tag{3.1}$$

Since $H^f(X_1, Y_1) = 0$ and $\Delta f = 0$. Therefore, we derive from Eq. (3.1) that

$$S^2(Y_2, Z_2) = \left[(n_2 - 1)\|\nabla f\|^2 + \frac{\tau f^2}{a_2 n}\left(\frac{a_1}{n-1} + a_2\right)\right] g_2(Y_2, Z_2).$$

This implies that M_2 is an Einstein manifold. This completes the proof. □

§4. Pseudo-projective curvature tensor on generalized Robertson-Walker space-times

Let (M, g) be a Riemannian manifold of dimension n. The function $f : I \to (0, \infty)$ is a smooth function where I is a connected and open subinterval of $\mathbb{R}$. Then the warped product manifold $\breve{M} = I \times_f M$ of dimension $(n + 1)$ equipped with the metric $\breve{g} = -dt^2 \oplus f^2 g$ is known as generalized Robertson-Walker space-time. Here dt^2 is the Euclidean metric on I. This structure is the generalization of Robertson-Walker space-times [9, 16, 17, 18]. We use ∂_t instead of $\frac{\partial}{\partial t} \in \mathfrak{X}(I)$ for simplicity in the following results.

With the help of Proposition 2.1, Proposition 2.2 and Eq. (2.2), we obtain the following theorem after some elementary calculations.

Theorem 4.1. *Let $\breve{M} = I \times_f M$ be a generalized Robertson-Walker space-time furnished with the metric $\breve{g} = -dt^2 \oplus f^2 g$. Then the curvature tensor $\breve{P}^*$ on $\breve{M}$ is given by*

$$\begin{aligned}
&\breve{P}^*(\partial_t, \partial_t)\partial_t = \breve{P}^*(\partial_t, \partial_t)X = \breve{P}^*(X, Y)\partial_t = 0,\\
&\breve{P}^*(\partial_t, X)\partial_t = \left[\left(\frac{na_2 - a_1}{f}\right)\ddot{f} - \frac{\tau}{n+1}\left(\frac{a_1}{n} + a_2\right)\right] X,\\
&\breve{P}^*(X, \partial_t)Y = \Bigg[\Bigg\{ - (a_1 + a_2) f\ddot{f} - (n-1)a_2\dot{f}^2\\
&\qquad + \frac{\tau f^2}{n+1}\left(\frac{a_1}{n} + a_2\right)\Bigg\} g(X, Y) - a_2 S(X, Y)\Bigg]\partial_t,\\
&\breve{P}^*(X, Y)Z = a_1 R(X, Y)Z + a_2\left[S(Y, Z)X - S(X, Z)Y\right]\\
&\qquad + \left[- a_1\dot{f}^2 + a_2 f\ddot{f} + a_2(n-1)\dot{f}^2 - \frac{\tau f^2}{n+1}\left(\frac{a_1}{n} + a_2\right)\right]\\
&\qquad \times \left[g(Y, Z)X - g(X, Z)Y\right],
\end{aligned}$$

for $X, Y, Z \in \mathfrak{X}(M)$ *and* $\partial_t \in \mathfrak{X}(I)$.

Theorem 4.2. *Let* $\breve{M} = I \times_f M$ *be a generalized Robertson-Walker space-time furnished with the metric* $\breve{g} = -dt^2 \oplus f^2 g$. *If* $\breve{M}$ *is pseudo-projectively flat, then the warping function* f *is given by*

$$f = \begin{cases} c_1 e^{\mu t} + c_2 e^{-\mu t}, & \textit{if } \mu^2 \textit{ is positive} \\ c_1 + c_2 t, & \textit{if } \mu^2 \textit{=0} \\ c_1 \cos \mu t + c_2 \sin \mu t, & \textit{if } \mu^2 \textit{ is negative} \end{cases}$$

where $\mu^2 = \frac{\tau(a_1 + n a_2)}{n(n+1)(n a_2 - a_1)}$ *and* c_1, c_2 *are two arbitrary constants.*

Proof. Let $\breve{M}$ be pseudo-projectively flat. Then from the second relation of Theorem 4.1, we have

$$\ddot{f} - \mu^2 f = 0.$$

Hence, by solving the above differential equation the warping function f is obtained and it is given by

$$f = \begin{cases} c_1 e^{\mu t} + c_2 e^{-\mu t}, & \text{if } \mu^2 \text{ is positive} \\ c_1 + c_2 t, & \text{if } \mu^2 \text{=0} \\ c_1 \cos \mu t + c_2 \sin \mu t, & \text{if } \mu^2 \text{ is negative} \end{cases}$$

where c_1, c_2 are two arbitrary constants. This completes the proof. □

Theorem 4.3. *Let* $\breve{M} = I \times_f M$ *be a generalized Robertson-Walker space-time furnished with the metric* $\breve{g} = -dt^2 \oplus f^2 g$. *If* $\breve{M}$ *is pseudo-projectively flat, then* M *is an Einstein manifold.*

Proof. Let $\breve{M}$ be pseudo-projectively flat. Then from the third relation of Theorem 4.1, we have

$$S(X, Y) = \frac{1}{a_2} \left[-(a_1 + a_2) f \ddot{f} - (n-1) a_2 \dot{f}^2 + \frac{\tau f^2}{n+1} \left(\frac{a_1}{n} + a_2 \right) \right] g(X, Y).$$

Hence, M is an Einstein manifold. This completes the proof. □

§5. Pseudo-projective curvature tensor on standard static space-times

Let (M, g) be a Riemannian manifold of dimension n. The function $f : M \to (0, \infty)$ is a smooth function. Then the warped product manifold $\breve{M} = I \times_f M$

of dimension $(n+1)$ equipped with the metric $\breve{g} = -f^2dt^2 \oplus g$ is known as standard static space-time. Here I is the connected, open subinterval of $\mathbb{R}$ and dt^2 is the Euclidean metric on I. This structure is the generalization of Einstein static universe [1, 2, 3, 4, 5]. We write ∂_t instead of $\frac{\partial}{\partial t} \in \mathfrak{X}(I)$ for expressing the following results in simpler way.

In view of Proposition 2.1, Proposition 2.2 and Eq. (2.2), we obtain the following theorem after some elementary calculations.

Theorem 5.1. *Let $\breve{M} = I \times_f M$ be a standard static space-time furnished with the metric $\breve{g} = -f^2dt^2 \oplus g$. Then the curvature tensor $\breve{P}^*$ on $\breve{M}$ is given by*

$$
\begin{aligned}
\breve{P}^*(\partial_t,\partial_t)\partial_t &= \breve{P}^*(\partial_t,\partial_t)X = \breve{P}^*(X,Y)\partial_t = 0,\\
\breve{P}^*(\partial_t,X)\partial_t &= f\left[a_1 D^1_X\nabla f - a_2\Delta f X - \frac{\tau f}{n+1}\left(\frac{a_1}{n}+a_2\right)X\right],\\
\breve{P}^*(\partial_t,X)Y &= \left[\left(\frac{a_1-a_2}{f}\right)H^f(X,Y) + a_2 S(X,Y)\right.\\
&\quad \left. - \frac{\tau}{n+1}\left(\frac{a_1}{n}+a_2\right)g(X,Y)\right]\partial_t,\\
\breve{P}^*(X,Y)Z &= a_1 R(X,Y)Z + a_2\left[S(Y,Z)X - S(X,Z)Y\right]\\
&\quad - \frac{a_2}{f}\left[H^f(Y,Z)X - H^f(X,Z)Y\right]\\
&\quad - \frac{\tau}{n+1}\left(\frac{a_1}{n}+a_2\right)\left[g(Y,Z)X - g(X,Z)Y\right],
\end{aligned}
$$

for $X, Y, Z \in \mathfrak{X}(M)$ and $\partial_t \in \mathfrak{X}(I)$.

Theorem 5.2. *Let $\breve{M} = I \times_f M$ be a standard static space-time furnished with the metric $\breve{g} = -f^2dt^2 \oplus g$. If $\breve{M}$ is pseudo-projectively flat, then $H^f = \frac{\Delta f}{n} g$.*

Proof. Let $\breve{M} = I \times_f M$ be pseudo-projectively flat. Then from the second relation of Theorem 5.1, we have

$$D^1_X\nabla f = \frac{1}{a_1}\left[a_2\Delta f + \frac{\tau f}{n+1}\left(\frac{a_1}{n}+a_2\right)\right]X$$

$$\text{i.e.,}\quad H^f = \frac{1}{a_1}\left[a_2\Delta f + \frac{\tau f}{n+1}\left(\frac{a_1}{n}+a_2\right)\right]g. \tag{5.1}$$

Taking trace on both sides, we obtain

$$\Delta f = \frac{nf\tau}{(n+1)(a_1-na_2)}\left(\frac{a_1}{n}+a_2\right). \tag{5.2}$$

Using Eq. (5.2) in Eq. (5.1), we derive $H^f = \frac{\Delta f}{n} g$. This completes the proof. □

Theorem 5.3. *Let $\breve{M} = I \times_f M$ be a standard static space-time furnished with the metric $\breve{g} = -f^2 dt^2 \oplus g$. If $\breve{M}$ is pseudo-projectively flat, then M is an Einstein manifold.*

Proof. Let $\breve{M} = I \times_f M$ be pseudo-projectively flat. We derive from the third relation of Theorem 5.1 by using Theorem 5.2 and Eq. (5.2) that

$$S(X,Y) = \frac{(1-n)\Delta f}{nf} g(X,Y).$$

This implies that M is an Einstein manifold. This completes the proof. □

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Biwarped product submanifolds in some structures of metallic Riemannian manifold

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Abstract
In this paper, we study the biwarped product submanifold in locally nearly metallic Riemannian manifold. We construct a non trivial example of a biwarped product submanifold in metallic Riemannian manifold. Moreover, we discuss a necessary and sufficient condition for such submanifolds to be locally trivial. Finally, we set up an inequality in locally nearly metallic Riemannian manifold for the second fundamental form with respect to some conditions. We also investigate the equality case.

Keywords Warped product · Biwarped product · Locally nearly metallic Riemannian manifold · Slant submanifold

Mathematics Subject Classification 53B25 · 53C15 · 53C40 · 53C42

1 Introduction

Firstly, the concept of the warped product in Riemannian manifolds had been developed by Bishop and O'Neill [1] to make a large class of complete manifolds with negative curvature. The concept of the warped product came due to a surface of revolution. Nölker [17] defined the notion of the multiply warped product from the concept of the warped product. Biwarped product is a special case of multiply warped product. The warped product has a great importance not only in differential geometry but also in mathematical physics, more specifically in general relativity. Robertson-walker model, Kruscal model, Schwarzschild solution and static

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model are warped products. Many exact solutions to Einstein field equations and modified field equations can be expressed in terms of the warped products.
Hretcanu et al. [9, 14] defined metallic Riemannian manifolds and their submanifolds from the concept of golden Riemannian manifolds which are studied in [5, 13]. Hretcanu et al. gave some properties of invariant, anti-invariant, slant [12], hemi slant [10] and semi slant submanifolds [3] of golden and metallic Riemannian manifolds. Besides, they discussed some integrability conditions of some distributions involved in such types of submanifolds. Moreover, they added some properties of golden and metallic Riemannian manifolds in [2, 11].
Two roots of the quadratic equation $x^2 - ax - b = 0$ are $\frac{a+\sqrt{a^2+4b}}{2}$ and $\frac{a-\sqrt{a^2+4b}}{2}$, where a and b are positive integers. It is clearly seen that out of these two roots one root is positive and the other root is negative. This positive root $\lambda_{a,b} = \frac{a+\sqrt{a^2+4b}}{2}$ is called the metallic number [7]. Metallic structure [6, 8] is a special case of the polynomial structure. We wish to study here on biwarped product submanifold in locally nearly metallic Riemannian manifold. The works [15, 16] by S. K. Hui et al. enlighten the present study.
In this note, we study the biwarped product submanifold in locally nearly metallic Riemannian manifold. In Sect. 2, we discuss some basic ideas. In Sect. 3, we construct a non trivial example of a biwarped product submanifold in metallic Riemannian manifold. In Sect. 4, we give a necessary and sufficient condition for such submanifolds to be locally trivial. In Sect. 5, we set up an inequality in locally nearly metallic Riemannian manifold for the second fundamental with respect to some conditions. We also investigate the equality case.

2 Preliminaries

In this section, we recall some basic definitions and formulas which are very important to our study. We discuss here about biwarped product manifolds, submanifolds of Riemannian and locally nearly metallic Riemannian manifolds respectively.

Biwarped product manifold:
Let M_0, M_1 and M_2 be three Riemannian manifolds and $M = M_0 \times M_1 \times M_2$ be their cartesian product. $\pi_i : M \to M_i$ is the canonical projection of M onto M_i, where $i \in \{0, 1, 2\}$. Let $\pi_{i*} : TM \to TM_i$ is the tangent map of $\pi_i : M \to M_i$, where $\Gamma(TM)$ is the Lie algebra of the vector fields of M.
If f_1 and f_2 are two positive real valued functions on M_0, then

$$g(X, Y) = g(\pi_{0*}X, \pi_{0*}Y) + (f_1 \circ \pi_1)^2 g(\pi_{1*}X, \pi_{1*}Y) + (f_2 \circ \pi_2)^2 g(\pi_{2*}X, \pi_{2*}Y),$$

$X, Y \in \Gamma(TM)$ defines a Riemannian metric on M. This is called the biwarped product metric.
The product manifold $M = M_0 \times M_1 \times M_2$ furnished by the metric g is called a biwarped product manifold and it is denoted by $M_0 \times_{f_1} M_1 \times_{f_2} M_2$. f_1 and f_2 are warping functions. M would be simply a Riemannian product if f_1 and f_2 are constant functions. If either f_1 or f_2 is a constant function, then M would be an ordinary warped product manifold. Moreover, if neither f_1 nor f_2 is a constant map, then M is called a proper biwarped product manifold.
Let $M = M_0 \times_{f_1} M_1 \times_{f_2} M_2$ be a biwarped product submanifold. Letting $\mathcal{D}^T = TM_T$, $\mathcal{D}^\perp = TM_\perp$, $\mathcal{D}^\subseteq = TM_\theta$ and $N =_{f_1} M_1 \times_{f_2} M_2$, we obtain [4, 18]

$$\nabla_X Z = \sum_{i=1}^{2}(X(\ln f_i))Z^i, \tag{2.1}$$

where $Z \in \Gamma(TN)$, $X \in \mathcal{D}^T$, ∇ is the Levi-Civita connection of M and M_i-component of Z is Z^i $(i = 1, 2)$.

Submanifolds of Riemannian manifolds:
Let M be a submanifold of a Riemannian manifold $\bar{M}$ with the induced metric g. Let ∇ and $\nabla^\perp$ be respectively the induced and the induced normal connections on M. Let $\Gamma(T^\perp M)$ be the set of all vector fields which are normal to M. Then the Gauss and Weingarten formulas are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.2}$$

$$\bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{2.3}$$

where $X, Y \in \Gamma(TM), \xi \in \Gamma(T^\perp M)$, h and A are respectively the second fundamental form and the shape operator of M. Now, h and A verify

$$g(h(X, Y), N) = g(A_N X, Y). \tag{2.4}$$

Let H be the mean curvature vector field of M. Then H can be calculated by $H = \frac{1}{\dim(M)}(\text{trace } h)$. If $h = 0$, then we say M is totally geodesic in $\bar{M}$. If $H = 0$, then we say M is minimal in $\bar{M}$. M is said to be totally umbilical if $h(X, Y) = g(X, Y)H$, for any $X, Y \in \Gamma(TM)$.
Let $\mathcal{D}^1$ and $\mathcal{D}^2$ be two distributions of M. If $h(X, Y) = 0$, for all $X, Y \in \mathcal{D}^1$, then M is called $\mathcal{D}^1$-geodesic. If $h(X, V) = 0$, for all $X \in \mathcal{D}^1$ and $V \in \mathcal{D}^2$, then M is called $(\mathcal{D}^1, \mathcal{D}^2)$-mixed geodesic.

Submanifolds of locally nearly metallic Riemannian manifolds:
A differentiable manifold N_k of even dimensional furnished by Riemannian metric g and metallic structure J is said to be a locally nearly metallic Riemannian manifold denoted by $(\bar{M}, J, g)$ if

$$\left.\begin{aligned} g(JX, JY) &= ag(JX, Y) + bg(X, Y), \\ g(JX, Y) &= g(X, JY), \\ (\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X &= 0, \end{aligned}\right\} \tag{2.5}$$

for all $X, Y \in \Gamma(TN_k)$ and a, b are positive integers.
If we consider $a = b = 1$ in (2.5), then the manifold N_k becomes a locally nearly golden Riemannian manifold.
Let M be a submanifold of dimension n of an almost Hermitian manifold $\bar{M}$ of dimension $2m$. We consider a local orthonormal frame field $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m}\}$ which is restricted to M, $e_1, \dots, e_n$ and $e_{n+1}, \dots, e_{2m}$ are respectively tangent and normal to M.
Let h^r_{ij}, $1 \le i, j \le n$, $n + 1 \le r \le 2m$ be the coefficients of the second fundamental form h in view of the local frame field. Hence, we obtain

$$\left.\begin{aligned} h^r_{ij} &= g(h(e_i, e_j), e_r) = g(A_{e_r} e_i, e_j), \\ \|h\|^2 &= \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \end{aligned}\right\} \tag{2.6}$$

For all $X \in \Gamma(TM)$ and $W \in \Gamma(T^\perp M)$, we can write

$$JX = TX + PX, \tag{2.7}$$

$$JW = tW + pW. \tag{2.8}$$

where TX, PX are respectively the tangential and normal components of JX and tW, pW are respectively the tangential and normal components of JW.
One can easily verify from (2.5) and (2.7) that

$$g(TX, Y) = g(X, TY), \text{ for } X, Y \in T_pM.$$

The angle $\theta(X)$ between JX and T_pM is known as Wirtinger angle of X, where $X \in T_pM$ is a non zero vector. If $\theta(X)$ is constant in M, then the submanifold M is said to be slant, where θ is the slant angle of M. Totally real and holomorphic submanifolds are two slant submanifolds having slant angles $\frac{\pi}{2}$ and 0 respectively. That is $J(T_pM) \subseteq T_p^{\perp}M$ and $J(T_pM) \subseteq T_pM$ are for the totally real and holomorohic submanifolds respectively. If a slant submanifold is neither totally real nor holomorphic, then it is called a proper slant submanifold.
Hence, M is a pointwise slant submanifold of $\bar{M}$ if and only if

$$T^2X = \cos^2\theta(aT + bI)X, \quad \text{for } X \in \Gamma(TM). \tag{2.9}$$

Using (2.7), (2.8) and the metallic structure, we derive

$$g(TX, TY) = \cos^2\theta[ag(TX, Y) + bg(X, Y)], \tag{2.10}$$

$$g(PX, PY) = \sin^2\theta[ag(TX, Y) + bg(X, Y)], \tag{2.11}$$

for $X, Y \in \Gamma(TM)$.

3 Example of a biwarped product submanifold in metallic Riemannian manifold

We construct a proper biwarped product submanifolds of type $M_T \times_f M_\perp \times_\sigma M_\theta$ in metallic Riemannian manifold.
We consider a metallic Riemannian manifold $\mathbb{R}^{14}$ furnished by the metallic structure $J : \mathbb{R}^{14} \to \mathbb{R}^{14}$ defined by

$$\begin{aligned} &J(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11}, X_{12}, X_{13}, X_{14}) \\ &\quad = (\lambda X_1, \bar{\lambda}X_2, \lambda X_3, \bar{\lambda}X_4, \lambda X_5, \bar{\lambda}X_6, \lambda X_7, \bar{\lambda}X_8, \lambda X_9, \bar{\lambda}X_{10}, \lambda X_{11}, \bar{\lambda}X_{12}, \lambda X_{13}, \bar{\lambda}X_{14}), \end{aligned}$$

where the metallic number is $\lambda = \lambda_{a,b} = \frac{a+\sqrt{a^2+4b}}{2}$; a, b are two positive integers and $\bar{\lambda} = a - \lambda$.
We consider a submanifold M in $\mathbb{R}^{14}$ where $(y_1, y_2, \ldots, y_{14})$ is the natural coordinates of $\mathbb{R}^{14}$ and they are given by

$$\begin{aligned} &y_1 = z_1\cos z_4, \quad y_2 = z_2\cos z_4, \quad y_3 = z_1\cos z_5, \quad y_4 = z_2\cos z_5, \quad y_5 = z_1\sin z_4, \\ &y_6 = z_2\sin z_4, \quad y_7 = z_1\sin z_5, \quad y_8 = z_2\sin z_5, \quad y_9 = z_1\cos z_3, \quad y_{10} = z_2\cos z_3, \\ &y_{11} = z_1\sin z_3, \quad y_{12} = z_2\sin z_3, \quad y_{13} = z_4 + z_5, \quad y_{14} = z_4 - z_5, \end{aligned}$$

where $z_1, z_2 \neq 0, 1$ and $z_3, z_4, z_5 \in (0, \frac{\pi}{2})$.
Therefore, the local frame of the tangent bundle $\Gamma(TM)$ of M are spanned by

$$\begin{aligned} Z_1 &= \cos z_4\frac{\partial}{\partial y_1} + \cos z_5\frac{\partial}{\partial y_3} + \sin z_4\frac{\partial}{\partial y_5} + \sin z_5\frac{\partial}{\partial y_7} + \cos z_3\frac{\partial}{\partial y_9} + \sin z_3\frac{\partial}{\partial y_{11}}, \\ Z_2 &= \cos z_4\frac{\partial}{\partial y_2} + \cos z_5\frac{\partial}{\partial y_4} + \sin z_4\frac{\partial}{\partial y_6} + \sin z_5\frac{\partial}{\partial y_8} + \cos z_3\frac{\partial}{\partial y_{10}} + \sin z_3\frac{\partial}{\partial y_{12}}, \end{aligned}$$

$$Z_3 = -z_1 \sin z_3 \frac{\partial}{\partial y_9} - z_2 \sin z_3 \frac{\partial}{\partial y_{10}} + z_1 \cos z_3 \frac{\partial}{\partial y_{11}} + z_2 \cos z_3 \frac{\partial}{\partial y_{12}},$$
$$Z_4 = -z_1 \sin z_4 \frac{\partial}{\partial y_1} - z_2 \sin z_4 \frac{\partial}{\partial y_2} + z_1 \cos z_4 \frac{\partial}{\partial y_5} + z_2 \cos z_4 \frac{\partial}{\partial y_6} + \frac{\partial}{\partial y_{13}} + \frac{\partial}{\partial y_{14}},$$
$$Z_5 = -z_1 \sin z_5 \frac{\partial}{\partial y_3} - z_2 \sin z_5 \frac{\partial}{\partial y_4} + z_1 \cos z_5 \frac{\partial}{\partial y_7} + z_2 \cos z_5 \frac{\partial}{\partial y_8} + \frac{\partial}{\partial y_{13}} - \frac{\partial}{\partial y_{14}}.$$

Clearly J satisfies $J^2 X = (aJ + bI)X$ and $g(JX, Y) = g(X, JY)$ for any $X, Y \in \mathbb{R}^{14}$. We obtain

$$\begin{aligned} JZ_4 &= -\lambda z_1 \sin z_4 \frac{\partial}{\partial y_1} - \bar{\lambda} z_2 \sin z_4 \frac{\partial}{\partial y_2} + \lambda z_1 \cos z_4 \frac{\partial}{\partial y_5} + \bar{\lambda} z_2 \cos z_4 \frac{\partial}{\partial y_6} \\ &\quad + \lambda \frac{\partial}{\partial y_{13}} + \bar{\lambda} \frac{\partial}{\partial y_{14}}, \\ JZ_5 &= -\lambda z_1 \sin z_5 \frac{\partial}{\partial y_3} - \bar{\lambda} z_2 \sin z_5 \frac{\partial}{\partial y_4} + \lambda z_1 \cos z_5 \frac{\partial}{\partial y_7} + \bar{\lambda} z_2 \cos z_5 \frac{\partial}{\partial y_8} \\ &\quad + \lambda \frac{\partial}{\partial y_{13}} - \bar{\lambda} \frac{\partial}{\partial y_{14}}, \end{aligned}$$
$$g(JZ_4, Z_4) = g(JZ_5, Z_5) = \lambda(z_1^2 + 1) + \bar{\lambda}(z_2^2 + 1),$$
$$\|Z_1\| = \|Z_2\| = \sqrt{3}, \quad \|Z_3\| = \sqrt{z_1^2 + z_2^2}, \quad \|Z_4\| = \|Z_5\| = \sqrt{z_1^2 + z_2^2 + 2}.$$
$$\|JZ_4\| = \|JZ_5\| = \sqrt{\lambda^2(z_1^2 + 1) + \bar{\lambda}^2(z_2^2 + 1)}$$

Therefore, $\mathcal{D}^T = \text{span}\{Z_1, Z_2\}$, $\mathcal{D}^\perp = \text{span}\{Z_3\}$ and $\mathcal{D}^\theta = \text{span}\{Z_4, Z_5\}$ are a holomorphic, totally real and proper pointwise slant distribution having slant function

$$\begin{aligned} \theta &= \cos^{-1} \frac{g(JZ_4, Z_4)}{\|Z_4\| \, \|JZ_4\|} = \cos^{-1} \frac{g(JZ_5, Z_5)}{\|Z_5\| \, \|JZ_5\|} \\ &= \cos^{-1} \frac{\lambda(z_1^2 + 1) + \bar{\lambda}(z_2^2 + 1)}{\sqrt{z_1^2 + z_2^2 + 2} \sqrt{\lambda^2(z_1^2 + 1) + \bar{\lambda}^2(z_2^2 + 1)}}. \end{aligned}$$

Thus, M is a biwarped product submanifold of the metallic Riemannian manifold $(\mathbb{R}^{14}, J, g)$. We see that $\mathcal{D}^T$ is totally geodesic, $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ are both integrable. Let the integral submanifolds $\mathcal{D}^T$, $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ be denoted by M_T, $M_\perp$ and M_θ respectively. Thus, the induced metric tensor of M is given by

$$\begin{aligned} ds^2 &= 3(dz_1^2 + dz_2^2) + (z_1^2 + z_2^2) dz_3^2 + (z_1^2 + z_2^2 + 2)(dz_4^2 + dz_5^2) \\ &= g_{M_T} + (z_1^2 + z_2^2) g_{M_\perp} + (z_1^2 + z_2^2 + 2) g_{M_\theta} \end{aligned}$$

Hence, $M = M_T \times_f M_\perp \times_\sigma M_\theta$ is a proper biwarped product submanifold in metallic Riemannian manifold $(\mathbb{R}^{14}, J, g)$ with warping functions $f = \sqrt{z_1^2 + z_2^2}$ and $\sigma = \sqrt{z_1^2 + z_2^2 + 2}$ respectively.

4 Biwarped product submanifold of locally nearly metallic Riemannian manifold

In this section, we study the biwarped product submanifolds of a locally nearly metallic Riemannian manifold $\bar{M}$ in the form $M_T \times_f M_\perp \times_\sigma M_\theta$, where M_T, $M_\perp$ and M_θ are respectively the holomorphic, totally real and proper slant submanifolds. If we consider $\mathcal{D}^T = TM_T$, $\mathcal{D}^\perp = TM_\perp$ and $\mathcal{D}^\theta = TM_\theta$, then the tangent and normal bundles of M can be respectively decomposed as

$$TM = \mathcal{D}^T \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta, \quad T^\perp M = J\mathcal{D}^T \oplus P\mathcal{D}^\perp \oplus \delta,$$

where δ is the J-invariant subbundle of $T^\perp M$.
The following two lemmas are very helpful for further study.

Lemma 4.1 *Let $M = M_T \times_f M_\perp \times_\sigma M_\theta$ be a biwarped product submanifold of a locally nearly metallic Riemannian manifold $\bar{M}$. Then we derive*

$$\text{(i) } g(h(U, V), JX) = 0,$$
$$\text{(ii) } g(h(U, V), PZ) = 0,$$
$$\text{(iii) } g(h(U, X), JY) = \frac{1}{3} JU(\ln f) g(X, Y),$$

where $U, V \in \Gamma(\mathcal{D}^T)$, $X, Y \in \Gamma(\mathcal{D}^\perp)$ and $Z \in \Gamma(\mathcal{D}^\theta)$.

Proof For all $U, V \in \Gamma(\mathcal{D}^T)$ and $X \in \Gamma(\mathcal{D}^\perp)$, we obtain

$$g(h(U, V), JX) = g(\bar{\nabla}_U V, JX) = g(\bar{\nabla}_U JV, X) - g((\bar{\nabla}_U J)V, X).$$

From (2.1), it follows that

$$g(h(U, V), JX) = g(\bar{\nabla}_U V, JX) = U(\ln f) g(JV, X) - g((\bar{\nabla}_U J)V, X).$$

Since $g(JV, X) = 0$, we find

$$g(h(U, V), JX) = -g((\bar{\nabla}_U J)V, X). \tag{4.1}$$

Replacing U and V by V and U respectively in (4.1), we derive

$$g(h(U, V), JX) = -g((\bar{\nabla}_V J)U, X). \tag{4.2}$$

By adding (4.1), (4.2) and using (2.5), we see

$$g(h(U, V), JX) = 0.$$

Hence, (i) follows.
By a similar manner, we can prove (ii).
Now, we wish to prove the third assertion of the lemma. For all $U \in \Gamma(\mathcal{D}^T)$ and $X, Y \in \Gamma(\mathcal{D}^\perp)$, we obtain

$$g(h(U, X), JY) = g(\bar{\nabla}_X U, JY) = g(\bar{\nabla}_X JU, Y) - g((\bar{\nabla}_X J)U, Y).$$

From (2.1) and (2.5), it implies that

$$\begin{aligned} g(h(U, X), JY) &= JU(\ln f) g(X, Y) + g((\bar{\nabla}_U J)X, Y). \\ &= JU(\ln f) g(X, Y) + g(\bar{\nabla}_U JX, Y) - g(\bar{\nabla}_U X, JY) \end{aligned}$$

From (2.2), (2.3), (2.4) and (2.5), we find

$$2g(h(U,X),JY) = JU(\ln f)g(X,Y) - g(h(U,Y),JX). \tag{4.3}$$

Putting $X = Y$ and $Y = X$, we obtain

$$2g(h(U,Y),JX) = JU(\ln f)g(X,Y) - g(h(U,X),JY). \tag{4.4}$$

From (4.3) and (4.4), it follows that

$$\begin{aligned} 2g(h(U,X),JY) &= JU(\ln f)g(X,Y) \\ &\quad - \frac{1}{2}[JU(\ln f)g(X,Y) - g(h(U,X),JY)] \\ \text{i.e., } g(h(U,X),JY) &= \frac{1}{3}JU(\ln f)g(X,Y). \end{aligned}$$

Hence, (iii) follows. This completes the proof. □

Lemma 4.2 *Let $M = M_T \times_f M_\perp \times_\sigma M_\theta$ be a biwarped product submanifold of a locally nearly metallic Riemannian manifold $\bar{M}$. Then we derive*

$$\text{(i) } g(h(U,X),PZ) = -\frac{1}{2}g(h(U,Z),JX) = 0,$$
$$\text{(ii) } g(h(U,Z),PW) = \frac{1}{3}[JU(\ln\sigma)g(Z,W) - U(\ln\sigma)g(TZ,W)],$$

where $U \in \Gamma(\mathcal{D}^T)$, $X \in \Gamma(\mathcal{D}^\perp)$ and $Z, W \in \Gamma(\mathcal{D}^\theta)$.

Proof For all $U \in \Gamma(\mathcal{D}^T)$, $X \in \Gamma(\mathcal{D}^\perp)$ and $Z \in \Gamma(\mathcal{D}^\theta)$, we get

$$\begin{aligned} g(h(U,X),PZ) &= g(\bar{\nabla}_X U, PZ) \\ &= g(\bar{\nabla}_X U, JZ) - g(\bar{\nabla}_X U, TZ) \\ &= g(\bar{\nabla}_X JU, Z) - g((\bar{\nabla}_X J)U, Z) - g(\bar{\nabla}_X U, TZ). \end{aligned}$$

In view of (2.5), (2.1) and the condition of orthogonality of two vector fields, we derive

$$\begin{aligned} g(h(U,X),PZ) &= -g((\bar{\nabla}_X J)U, Z) \\ &= g((\bar{\nabla}_U J)X, Z) \\ &= g(\bar{\nabla}_U JX, Z) - g(\bar{\nabla}_U X, JZ) \\ &= -g(\bar{\nabla}_U Z, JX) - g(\bar{\nabla}_U X, TZ) - g(\bar{\nabla}_U X, PZ) \\ &= -g(\bar{\nabla}_U Z, JX) - g(\bar{\nabla}_U X, PZ) \\ &= -g(h(U,Z),JX) - g(h(U,X),PZ). \end{aligned}$$

This implies that

$$g(h(U,X),PZ) = -\frac{1}{2}g(h(U,Z),JX), \tag{4.5}$$

which is the first equality of the first assertion of the lemma.
Also, we find

$$\begin{aligned} g(h(U,Z),JX) &= g(\bar{\nabla}_Z U, JX) \\ &= g(\bar{\nabla}_Z JU, X) - g((\bar{\nabla}_Z J)U, X). \end{aligned}$$

In view of (2.5), (2.1) and the condition of orthogonality of two vector fields, we derive

$$\begin{aligned} g(h(U,Z),JX) &= -g((\bar{\nabla}_Z J)U, X) \\ &= g((\bar{\nabla}_U J)Z, X) \\ &= g(\bar{\nabla}_U JZ, X) - g(\bar{\nabla}_U Z, JX) \\ &= g(\bar{\nabla}_U TZ, X) + g(\bar{\nabla}_U PZ, X) - g(\bar{\nabla}_U Z, JX). \end{aligned}$$

Since $g(\bar{\nabla}_U TZ, X) = 0$, thus by using (2.2), (2.3) and (2.4), we find

$$\begin{aligned} g(h(U,Z),JX) &= g(\bar{\nabla}_U PZ, X) - g(\bar{\nabla}_U Z, JX) \\ &= -g(h(U,X),PZ) - g(h(U,Z),JX). \end{aligned}$$

This implies that

$$g(h(U,Z),JX) = -\frac{1}{2}g(h(U,X),PZ). \tag{4.6}$$

From (4.5) and (4.6), we obtain

$$g(h(U,X),PZ) = 0.$$

Hence, the second equality of the first assertion of the lemma is proved.
Now, we wish to prove the second assertion of the lemma. For all $U \in \Gamma(\mathcal{D}^T)$ and $Z, W \in \Gamma(\mathcal{D}^\theta)$, we have

$$\begin{aligned} g(h(U,Z),PW) &= g(\bar{\nabla}_Z U, PW). \\ &= g(\bar{\nabla}_Z U, JW) - g(\bar{\nabla}_Z U, TW) \\ &= g(\bar{\nabla}_Z JU, W) - g((\bar{\nabla}_Z J)U, W) - g(\bar{\nabla}_Z U, TW) \\ &= JU(\ln\sigma)g(Z,W) + g((\bar{\nabla}_U J)Z, W) - U(\ln\sigma)g(Z,TW) \\ &= JU(\ln\sigma)g(Z,W) + g(\bar{\nabla}_U JZ, W) - g(\bar{\nabla}_U Z, JW) \\ &\quad - U(\ln\sigma)g(Z,TW) \\ &= JU(\ln\sigma)g(Z,W) + g(\bar{\nabla}_U TZ, W) + g(\bar{\nabla}_U PZ, W) \\ &\quad - g(\bar{\nabla}_U Z, TW) - g(\bar{\nabla}_U Z, PW) - U(\ln\sigma)g(Z,TW) \end{aligned}$$

From (2.1), (2.2), (2.3) and (2.4), we have

$$\begin{aligned} g(h(U,Z),PW) &= JU(\ln\sigma)g(Z,W) - U(\ln\sigma)g(Z,TW) \\ &\quad - g(\bar{\nabla}_U W, PZ) - g(\bar{\nabla}_U Z, PW). \\ &= JU(\ln\sigma)g(Z,W) - U(\ln\sigma)g(Z,TW) \\ &\quad - g(h(U,W),PZ) - g(h(U,Z),PW). \end{aligned}$$

This implies that

$$\begin{aligned} 2g(h(U,Z),PW) &= JU(\ln\sigma)g(Z,W) - U(\ln\sigma)g(Z,TW) \\ &\quad - g(h(U,W),PZ). \end{aligned} \tag{4.7}$$

Interchanging Z by W, we have

$$\begin{aligned} 2g(h(U,W),PZ) &= JU(\ln\sigma)g(Z,W) - U(\ln\sigma)g(Z,TW) \\ &\quad - g(h(U,Z),PW). \end{aligned} \tag{4.8}$$

Using (4.7) and (4.8), we derive

$$g(h(U, Z), PW) = \frac{1}{3}[JU(\ln \sigma)g(Z, W) - U(\ln \sigma)g(TZ, W)],$$

Hence, the second part is proved. This completes the proof. □

Putting $W = TW$ in the second part of the Lemma 4.2, we obtain

$$\begin{aligned} g(h(U, Z), PTW) &= \frac{1}{3}[JU(\ln \sigma)g(Z, TW) - U(\ln \sigma)g(TZ, TW)] \\ &= \frac{1}{3}[JU(\ln \sigma)g(Z, TW) \\ &\quad - U(\ln \sigma)\cos^2\theta\{ag(TZ, W) + bg(Z, W)\}] \\ &= \frac{1}{3}[JU(\ln \sigma)g(Z, TW) - a\cos^2\theta U(\ln \sigma)g(TZ, W) \\ &\quad - b\cos^2\theta U(\ln \sigma)g(Z, W)]. \end{aligned} \tag{4.9}$$

Now, we give a necessary and sufficient condition for such submanifolds to be locally trivial.

Theorem 4.3 *Let M be a biwarped product submanifold of type $M_T \times_f M_\perp \times_\sigma M_\theta$ of a locally nearly metallic Riemannian manifold $(\bar{M}, J, g)$ such that the invariant normal subbundle $\delta = \{0\}$. Then M is locally trivial if and only if M is $(\mathcal{D}^T, \mathcal{D}^\perp)$ and $(\mathcal{D}^T, \mathcal{D}^\theta)$-mixed geodesic.*

Proof Let M be a biwarped product submanifold of type $M_T \times_f M_\perp \times_\sigma M_\theta$ of a locally nearly metallic Riemannian manifold $(\bar{M}, J, g)$ such that the invariant normal subbundle $\delta = \{0\}$. Let M be locally trivial. Then both the warping functions f and σ are constants. Since f is constant, so $JU(\ln f) = 0$. Therefore, by Lemma 4.1, we see that $g(h(U, X), JY) = 0$ for any $U \in \mathcal{D}^T$ and $X, Y \in \mathcal{D}^\perp$. Also, from Lemma 4.2 and the decomposition of the normal bundles of M, we gain $h(U, X) = 0$. Consequently, it implies that M is $(\mathcal{D}^T, \mathcal{D}^\perp)$-mixed geodesic. On the other side, since the function σ is constant, so $JU(\ln \sigma) = 0$ and $U(\ln \sigma) = 0$. Therefore, from Lemma 4.2, we find $g(h(U, Z), PW) = 0$ for $U \in \mathcal{D}^T$ and $Z, W \in \mathcal{D}^\theta$. Also, from Lemma 4.2 and the decomposition of the normal bundles of M, we gain $h(U, Z) = 0$. Consequently, it implies that M is $(\mathcal{D}^T, \mathcal{D}^\theta)$-mixed geodesic.
For the converse part of the theorem, let M be $(\mathcal{D}^T, \mathcal{D}^\perp)$ and $(\mathcal{D}^T, \mathcal{D}^\theta)$-mixed geodesic. If M is $(\mathcal{D}^T, \mathcal{D}^\perp)$-mixed geodesic, then $h(U, X) = 0$ for any $U \in \mathcal{D}^T$ and $X \in \mathcal{D}^\perp$. Hence, from Lemma 4.1, we see $JU(\ln f) = 0$. Therefore, f is a constant function. On the other side, if M is $(\mathcal{D}^T, \mathcal{D}^\theta)$-mixed geodesic, then $h(U, Z) = 0$ for any $U \in \mathcal{D}^T$ and $Z \in \mathcal{D}^\theta$. Hence, from Lemma 4.2, we obtain

$$JU(\ln \sigma)g(Z, W) - U(\ln \sigma)g(TZ, W) = 0. \tag{4.10}$$

Putting $U = JU$ in (4.10), we get

$$\begin{aligned} &J^2U(\ln \sigma)g(Z, W) - JU(\ln \sigma)g(TZ, W) = 0 \\ &\quad \text{i.e., } (aJ + bI)U(\ln \sigma)g(Z, W) - JU(\ln \sigma)g(TZ, W) = 0 \\ &\quad \text{i.e., } aJU(\ln \sigma)g(Z, W) + bU(\ln \sigma)g(Z, W) \\ &\qquad - JU(\ln \sigma)g(TZ, W) = 0. \end{aligned} \tag{4.11}$$

Putting $Z = TZ$ in (4.11) and using (4.10), we have

$$aJU(\ln \sigma)g(TZ, W) + bU(\ln \sigma)g(TZ, W) - JU(\ln \sigma)g(T^2Z, W) = 0$$

$$\begin{aligned}
&\text{i.e., } aJU(\ln\sigma)g(TZ,W) + bU(\ln\sigma)g(TZ,W)\\
&\quad - JU(\ln\sigma)[a\cos^2\theta g(TZ,W) + b\cos^2\theta g(Z,W)] = 0\\
&\text{i.e., } a(1-\cos^2\theta)JU(\ln\sigma)g(TZ,W) + b(1-\cos^2\theta)JU(\ln\sigma)g(Z,W) = 0\\
&\text{i.e., } a\sin^2\theta JU(\ln\sigma)g(TZ,W) + b\sin^2\theta JU(\ln\sigma)g(Z,W) = 0.\\
&\text{i.e., } \sin^2\theta JU(\ln\sigma)[ag(TZ,W) + bg(Z,W)] = 0. \qquad (4.12)
\end{aligned}$$

Since M is a proper biwarped product submanifold of type $M_T \times_f M_\perp \times_\sigma M_\theta$ of a locally nearly metallic Riemannian manifold $(\bar{M}, J, g)$, $\sin\theta \neq 0$. Also, since a, b are positive integers, $g(TZ, W) \neq 0$ and $g(Z, W) \neq 0$ for $Z, W \in \mathcal{D}^\theta$, hence $ag(TZ, W)+bg(Z, W) \neq 0$. Therefore, from (4.12) we can conclude that $JU(\ln\sigma) = 0$. Consequently, σ is a constant function. Therefore, M is locally trivial. This completes the proof. □

Remark 4.4 From Theorem 4.3, it follows that a proper biwarped product submanifold $M = M_T \times_f M_\perp \times_\sigma M_\theta$ in a locally nearly metallic Riemannian manifold is neither $(\mathcal{D}^T, \mathcal{D}^\perp)$-mixed geodesic nor $(\mathcal{D}^T, \mathcal{D}^\theta)$-mixed geodesic.

5 Inequality for the second fundamental form

In this section, we give a sharp inequality for the second fundamental form with respect to some conditions. We also investigate its equality case.

Let $M = M_T \times_f M_\perp \times_\sigma M_\theta$ be a proper biwarped product submanifold of a locally nearly metallic Riemannian manifold $(\bar{M}, J, g)$ of dimension $2m$. We choose a local orthogonal basis $\{e_1, \ldots, e_n\}$ of the tangent bundle TM in such a manner that $g(Je_i, e_j) = g(Te_i, e_j) = 0$ for $i \neq j$ and

$$\begin{aligned}
\mathcal{D}^T &= \mathrm{span}\{e_1, \ldots, e_t, e_{t+1} = Je_1, \ldots, e_{2t} = Je_t\},\\
\mathcal{D}^\perp &= \mathrm{span}\{e_{2t+1} = \hat{e}_1, \ldots, e_{2t+p} = \hat{e}_p\},\\
\mathcal{D}^\theta &= \mathrm{span}\{e_{2t+p+1} = e_1^*, \ldots, e_{2t+p+q} = e_q^*, e_{2t+p+q+1} = \sec\theta e_1^*, \ldots, e_n = \sec\theta e_q^*\},
\end{aligned}$$

in which $\{e_1, \ldots, e_t\}$, $\{\hat{e}_1, \ldots, \hat{e}_p\}$ and $\{e_1^*, \ldots, e_q^*\}$ are three orthonormal set of vectors. Therefore, $\dim M_T = 2t$, $\dim M_\perp = p$ and $\dim M_\theta = 2q$. Furthermore, the orthonormal basis $\{E_1, \ldots, E_{2m-n-p-2q}\}$ of the normal bundle $T^\perp M$ are given by

$$\begin{aligned}
J\mathcal{D}^\perp &= \mathrm{span}\{E_1 = J\hat{e}_1, \ldots, E_p = J\hat{e}_p\},\\
P\mathcal{D}^\theta &= \mathrm{span}\{E_{p+1} = \csc\theta Pe_1^*, \ldots, E_{p+q} = \csc\theta Pe_q^*,\\
E_{p+q+1} &= \csc\theta\sec\theta PTe_1^*, \ldots, E_{p+2q} = \csc\theta\sec\theta PTe_q^*\},\\
\delta &= \mathrm{span}\{E_{p+2q+1}, \ldots, E_{2m-n-p-2q}\}.
\end{aligned}$$

Theorem 5.1 *Let M be a biwarped product submanifold of type $M_T \times_f M_\perp \times_\sigma M_\theta$ of a locally nearly metallic Riemannian manifold $(\bar{M}, J, g)$. Then the second fundamental form h satisfies*

$$\begin{aligned}
\|h\|^2 \geq \frac{2bp}{9}\|\nabla(\ln f)\|^2 + \frac{2}{9}\big[&bq\csc^2\theta + ax\cot^2\theta + bq\cot^2\theta + abx\csc^2\theta\\
&+ b^2q\csc^2\theta + a^3x\cot^2\theta\cos^2\theta + a^2bq\cot^2\theta\cos^2\theta + b^2q\cot^2\theta\\
&+ 2abx\cot^2\theta\big]\|\nabla(\ln\sigma)\|^2 + \frac{2}{9}\big[ap + aq\csc^2\theta - 2x\csc^2\theta
\end{aligned}$$

$$+ a^2 x \csc^2\theta + abq\csc^2\theta - 2a^2 x\cot^2\theta - 2abq\cot^2\theta$$
$$- 2bx\csc^2\theta\big]g(J\nabla(\ln\sigma), \nabla(\ln\sigma)), \tag{5.1}$$

where $\dim M_\perp = p$, $\dim M_\theta = 2q$ *and* $x = \sum_{r=1}^{q} g(Te_r^*, e_r^*)$.
The equality occurs in (5.1) *when* M_T *is totally geodesic in* $\bar{M}$ *and* $M_\perp$, M_θ *are totally umbilical in* $\bar{M}$. *Furthermore,* M *is neither* $(\mathcal{D}^T, \mathcal{D}^\perp)$*-mixed geodesic nor* $(\mathcal{D}^T, \mathcal{D}^\theta)$*-mixed geodesic in* $\bar{M}$.

Proof From the definition of the second fundamental form h, we have

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) = \sum_{r=1}^{2m-n-p-2q}\sum_{i,j=1}^{n} g^2(h(e_i, e_j), E_r). \tag{5.2}$$

Now, by decomposing (5.2) for the normal subbundles $T^\perp M$ of M as follows

$$\|h\|^2 = \sum_{r=1}^{p}\sum_{i,j=1}^{n} g^2(h(e_i, e_j), J\hat{e}_r) + \sum_{r=p+1}^{p+2q}\sum_{i,j=1}^{n} g^2(h(e_i, e_j), E_r)$$
$$+ \sum_{r=p+2q+1}^{2m-n-p-2q}\sum_{i,j=1}^{n} g^2(h(e_i, e_j), E_r). \tag{5.3}$$

We omit the last δ-components terms in (5.3) and by using the orthonormal bases of TM and $T^\perp M$, we have

$$\|h\|^2 \geq \sum_{r=1}^{p}\sum_{i,j=1}^{2t} g^2(h(e_i, e_j), J\hat{e}_r) + 2\sum_{r=1}^{p}\sum_{i=1}^{2t}\sum_{j=1}^{p} g^2(h(e_i, \hat{e}_j), J\hat{e}_r)$$
$$+ \sum_{r=1}^{p}\sum_{i,j=1}^{p} g^2(h(\hat{e}_i, \hat{e}_j), J\hat{e}_r) + 2\sum_{r=1}^{p}\sum_{i=1}^{2t}\sum_{j=1}^{2q} g^2(h(e_i, e_j^*), J\hat{e}_r)$$
$$+ \sum_{r=1}^{p}\sum_{i,j=1}^{2q} g^2(h(e_i^*, e_j^*), J\hat{e}_r) + 2\sum_{r=1}^{p}\sum_{i=1}^{2q}\sum_{j=1}^{p} g^2(h(e_i^*, \hat{e}_j), J\hat{e}_r)$$
$$+ \csc^2\theta\sum_{r=1}^{q}\sum_{i,j=1}^{2t}\big[g^2(h(e_i, e_j), Pe_r^*) + \sec^2\theta g^2(h(e_i, e_j), PTe_r^*)\big]$$
$$+ 2\csc^2\theta\sum_{r=1}^{q}\sum_{i=1}^{2t}\sum_{j=1}^{p}\big[g^2(h(e_i, \hat{e}_j), Pe_r^*) + \sec^2\theta g^2(h(e_i, \hat{e}_j), PTe_r^*)\big]$$
$$+ \csc^2\theta\sum_{r=1}^{q}\sum_{i,j=1}^{p}\big[g^2(h(\hat{e}_i, \hat{e}_j), Pe_r^*) + \sec^2\theta g^2(h(\hat{e}_i, \hat{e}_j), PTe_r^*)\big]$$
$$+ 2\csc^2\theta\sum_{r=1}^{q}\sum_{i=1}^{p}\sum_{j=1}^{2q}\big[g^2(h(\hat{e}_i, e_j^*), Pe_r^*) + \sec^2\theta g^2(h(\hat{e}_i, e_j^*), PTe_r^*)\big]$$
$$+ \csc^2\theta\sum_{r=1}^{q}\sum_{i,j=1}^{2q}\big[g^2(h(e_i^*, e_j^*), Pe_r^*) + \sec^2\theta g^2(h(e_i^*, e_j^*), PTe_r^*)\big]$$

$$+ 2\csc^2\theta \sum_{r=1}^{q}\sum_{i=1}^{2t}\sum_{j=1}^{2q}\big[g^2(h(e_i, e_j^*), Pe_r^*)$$
$$+ \sec^2\theta g^2(h(e_i, e_j^*), PTe_r^*)\big]. \tag{5.4}$$

Clearly, there is no connection for warped products for the third, fifth, sixth, ninth, tenth and eleventh terms in (5.4). Hence, we omit these positive terms. With the help of Lemmas 4.1, 4.2 and (4.9), we see that

$$\|h\|^2 \geq 2\sum_{r=1}^{p}\sum_{i=1}^{2t}\sum_{j=1}^{p}\left[\frac{1}{3}Je_i(\ln f)g(\hat{e}_j, \hat{e}_r)\right]^2$$
$$+ 2\csc^2\theta \sum_{r=1}^{q}\sum_{i=1}^{2t}\sum_{j=1}^{2q}\left[\frac{1}{3}\{Je_i(\ln\sigma)g(e_j^*, e_r^*) - e_i(\ln\sigma)g(Te_j^*, e_r^*)\}\right]^2$$
$$+ 2\csc^2\theta\sec^2\theta \sum_{r=1}^{q}\sum_{i=1}^{2t}\sum_{j=1}^{2q}\left[\frac{1}{3}\{Je_i(\ln\sigma)g(e_j^*, Te_r^*)\right.$$
$$\left. - a\cos^2\theta e_i(\ln\sigma)g(Te_j^*, e_r^*) - b\cos^2\theta e_i(\ln\sigma)g(e_j^*, e_r^*)\}\right]^2$$
$$= \frac{2p}{9}\sum_{i=1}^{2t}\big[Je_i(\ln f)\big]^2 + \frac{2q\csc^2\theta}{9}\sum_{i=1}^{2t}\big[Je_i(\ln\sigma)\big]^2$$
$$+ \frac{2\csc^2\theta}{9}\sum_{i=1}^{2t}\sum_{r=1}^{q}\big[e_i(\ln\sigma)\big]^2 g(Te_r^*, Te_r^*)$$
$$- \frac{4\csc^2\theta}{9}\sum_{i=1}^{2t}\sum_{r=1}^{q}\big[Je_i(\ln\sigma)e_i(\ln\sigma)\big]g(Te_r^*, e_r^*)$$
$$+ \frac{2\csc^2\theta\sec^2\theta}{9}\sum_{i=1}^{2t}\sum_{r=1}^{q}\big[Je_i(\ln\sigma)\big]^2 g(Te_r^*, Te_r^*)$$
$$+ \frac{2a^2\cot^2\theta}{9}\sum_{i=1}^{2t}\sum_{r=1}^{q}\big[e_i(\ln\sigma)\big]^2 g(Te_r^*, Te_r^*)$$
$$+ \frac{2b^2q\cot^2\theta}{9}\sum_{i=1}^{2t}[e_i(\ln\sigma)]^2$$
$$- \frac{4a\csc^2\theta}{9}\sum_{i=1}^{2t}\sum_{r=1}^{q}\big[Je_i(\ln\sigma)e_i(\ln\sigma)\big]g(Te_r^*, Te_r^*)$$
$$- \frac{4b\csc^2\theta}{9}\sum_{i=1}^{2t}\sum_{r=1}^{q}\big[Je_i(\ln\sigma)e_i(\ln\sigma)\big]g(Te_r^*, e_r^*)$$
$$+ \frac{4ab\cot^2\theta}{9}\sum_{i=1}^{2t}\sum_{r=1}^{q}\big[e_i(\ln\sigma)\big]^2 g(Te_r^*, e_r^*)$$
$$= \frac{2p}{9}\big[ag(J\nabla(\ln f), \nabla(\ln f)) + b\|\nabla(\ln f)\|^2\big]$$

$$
\begin{aligned}
&+\frac{2q\csc^2\theta}{9}\big[ag(J\nabla(\ln\sigma),\nabla(\ln\sigma))+b\|\nabla(\ln\sigma)\|^2\big]\\
&+\frac{2\csc^2\theta}{9}\|\nabla(\ln\sigma)\|^2\big[a\cos^2\theta\sum_{r=1}^{q}g(Te_r^*,e_r^*)+bq\cos^2\theta\big]\\
&-\frac{4\csc^2\theta}{9}g(J\nabla(\ln\sigma),\nabla(\ln\sigma))\sum_{r=1}^{q}g(Te_r^*,e_r^*)\\
&+\frac{2\csc^2\theta\sec^2\theta}{9}\big[ag(J\nabla(\ln\sigma),\nabla(\ln\sigma))+b\|\nabla(\ln\sigma)\|^2\big]\\
&\times\left[a\cos^2\theta\sum_{r=1}^{q}g(Te_r^*,Te_r^*)+bq\cos^2\theta\right]\\
&+\frac{2a^2\cot^2\theta}{9}\|\nabla(\ln\sigma)\|^2\big[a\cos^2\theta\sum_{r=1}^{q}g(Te_r^*,e_r^*)+bq\cos^2\theta\big]\\
&+\frac{2b^2q\cot^2\theta}{9}\|\nabla(\ln\sigma)\|^2\\
&-\frac{4a\csc^2\theta}{9}g(J\nabla(\ln\sigma),\nabla(\ln\sigma))\big[a\cos^2\theta\sum_{r=1}^{q}g(Te_r^*,e_r^*)+bq\cos^2\theta\big]\\
&-\frac{4b\csc^2\theta}{9}g(J\nabla(\ln\sigma),\nabla(\ln\sigma))\sum_{r=1}^{q}g(Te_r^*,e_r^*)\\
&+\frac{4ab\cot^2\theta}{9}\|\nabla(\ln\sigma)\|^2\sum_{r=1}^{q}g(Te_r^*,e_r^*)\\
=&\ \frac{2bp}{9}\|\nabla(\ln f)\|^2+\frac{2}{9}\big[bq\csc^2\theta+ax\cot^2\theta+bq\cot^2\theta+abx\csc^2\theta\\
&+b^2q\csc^2\theta+a^3x\cot^2\theta\cos^2\theta+a^2bq\cot^2\theta\cos^2\theta+b^2q\cot^2\theta\\
&+2abx\cot^2\theta\big]\|\nabla(\ln\sigma)\|^2+\frac{2}{9}\big[ap+aq\csc^2\theta-2x\csc^2\theta\\
&+a^2x\csc^2\theta+abq\csc^2\theta-2a^2x\cot^2\theta-2abq\cot^2\theta\\
&-2bx\csc^2\theta\big]g(J\nabla(\ln\sigma),\nabla(\ln\sigma)),
\end{aligned}
$$

where $x=\sum_{r=1}^{q}g(Te_r^*,e_r^*)$. Thus we obtain the inequality.
Now, we wish to consider the equality case. We obtain by omitting the third term in (5.3) that

$$h(TM,TM)\perp\delta. \tag{5.5}$$

By vanishing the first term and omitting the seventh term in (5.4), we see

$$h(\mathcal{D}^T,\mathcal{D}^T)\perp J\mathcal{D}^\perp \text{ and } h(\mathcal{D}^T,\mathcal{D}^T)\perp P\mathcal{D}^\theta. \tag{5.6}$$

From (5.5) and (5.6), it follows that

$$h(\mathcal{D}^T,\mathcal{D}^T)=0. \tag{5.7}$$

Also, by leaving the third and ninth terms in (5.4), we find

$$h(\mathcal{D}^\perp,\mathcal{D}^\perp)\perp J\mathcal{D}^\perp \text{ and } h(\mathcal{D}^\perp,\mathcal{D}^\perp)\perp P\mathcal{D}^\theta. \tag{5.8}$$

Hence, we can conclude from (5.5) and (5.8) that

$$h(\mathcal{D}^{\perp}, \mathcal{D}^{\perp}) = 0. \tag{5.9}$$

On the other side, by omitting the fifth and eleventh terms in (5.4), we derive

$$h(\mathcal{D}^{\theta}, \mathcal{D}^{\theta}) \perp J\mathcal{D}^{\perp} \text{ and } h(\mathcal{D}^{\theta}, \mathcal{D}^{\theta}) \perp P\mathcal{D}^{\theta}. \tag{5.10}$$

Therefore, we have from (5.5) and (5.10) that

$$h(\mathcal{D}^{\theta}, \mathcal{D}^{\theta}) = 0. \tag{5.11}$$

Furthermore, from leaving the sixth and tenth terms in (5.4), we have

$$h(\mathcal{D}^{\perp}, \mathcal{D}^{\theta}) \perp J\mathcal{D}^{\perp} \text{ and } h(\mathcal{D}^{\perp}, \mathcal{D}^{\theta}) \perp P\mathcal{D}^{\theta}. \tag{5.12}$$

Thus, from (5.5) and (5.12) that

$$h(\mathcal{D}^{\perp}, \mathcal{D}^{\theta}) = 0. \tag{5.13}$$

By vanishing the eighth term in (5.4) with (5.5), we derive

$$h(\mathcal{D}^{T}, \mathcal{D}^{\perp}) \subset J\mathcal{D}^{\perp}. \tag{5.14}$$

By a similar fashion, vanishing the forth term in (5.4) with (5.5), we find

$$h(\mathcal{D}^{T}, \mathcal{D}^{\theta}) \subset P\mathcal{D}^{\theta}. \tag{5.15}$$

Since M_T is totally geodesic in M, hence by using (5.7), (5.9) and (5.13), we conclude that M_T is totally geodesic in $\bar{M}$. On the other hand, since $M_{\perp}$ and M_{θ} are totally umbilical in M, hence by using (5.9), (5.11), (5.14) and (5.15), we can say that $M_{\perp}$ and M_{θ} are both totally umbilical in $\bar{M}$. Moreover, from Remark 4.4, Eqs. (5.14) and (5.15), it follows that M is neither $(\mathcal{D}^{T}, \mathcal{D}^{\perp})$-mixed geodesic nor $(\mathcal{D}^{T}, \mathcal{D}^{\theta})$-mixed geodesic in $\bar{M}$. This completes the proof of the theorem. □

Conclusion 5.2 Metallic structure is a polynomial structure. Here, we have discussed about the biwarped product submanifolds in nearly metallic Riemannian manifolds. We have obtained a necessary and sufficient condition for those submanifolds which are locally trivial. Also we have given an inequality in locally nearly metallic Riemannian manifold for the second fundamental with respect to some conditions. Metallic structure is a generalization of Golden structure, defined on Riemannian manifolds. If we consider $a = b = 1$ in this paper, metallic Riemannian manifolds becomes Golden Riemannian manifolds. Also, we can apply these results on in some structures of Golden Riemannian manifolds.

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RESEARCH ARTICLE

Application of Hyper-generalized Quasi-Einstein Spacetimes in General Relativity

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Abstract In this paper, we study the hyper-generalized quasi-Einstein ($HGQE$) warped product spaces with non-positive scalar curvature. This note deals with investigating of some geometric and physical properties of $(HGQE)_n$ manifolds. Next, we study the general relativistic viscous fluid $(HGQE)_4$ spacetimes with some physical applications. Lastly, we show the existence of $(HGQE)_4$ spacetimes by constructing a non-trivial example.

Keywords Einstein manifold · Hyper-generalized quasi-Einstein manifold · Warped product space · Einstein's field equation · Energy–momentum tensor · General relativistic viscous fluid

Mathematics Subject Classification 53C20 · 53C25 · 53C35 · 53C50

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1 Introduction

An $n(>2)$-dimensional semi-Riemannian manifold (M^n, g) is said to be an Einstein manifold if its Ricci tensor S of type (0, 2) satisfies the following condition

$$S = \frac{r}{n} g, \tag{1.1}$$

on M, where r is the scalar curvature of (M^n, g). Equation (1.1) is called the Einstein metric condition [1].

The notion of quasi-Einstein manifold has been developed by Chaki and Maity [2]. According to them, a Riemannian manifold (M^n, g), $(n > 2)$ is said to be a quasi-Einstein manifold if its nonzero Ricci tensor S of type (0, 2) satisfies the following condition

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y), \tag{1.2}$$

on M, where α and β are real-valued, nonzero scalar functions on (M^n, g). A is a nonzero 1-form such that

$$g(X, U) = A(X), g(U, U) = 1. \tag{1.3}$$

A is known as an associated 1-form and U is known as a generator of (M^n, g). This kind of manifold of dimension n is denoted by $(QE)_n$. If $\beta = 0$ in Eq. (1.2), then $(QE)_n$ turns into an Einstein manifold.

Then, the notion of generalized quasi-Einstein manifold has been introduced by Chaki [3]. According to him, a Riemannian manifold (M^n, g), $(n \geq 3)$ is said to be a generalized quasi-Einstein manifold denoted by $G(QE)_n$ if its nonzero Ricci tensor S of type (0, 2) satisfies the following condition

$$\begin{aligned} S(X, Y) =& \alpha g(X, Y) + \beta A(X)A(Y) \\ &+ \gamma[A(X)B(Y) + A(Y)B(X)], \end{aligned} \tag{1.4}$$

on M, where α, β and γ are real-valued, nonzero scalar functions on (M^n, g) in which $\beta \neq 0$, $\gamma \neq 0$. A and B are two nonzero 1-forms such that

$$g(X,U) = A(X), g(X,V) = B(X), \quad g(U,V) = 0, g(U,U) = 1, g(V,V) = 1. \tag{1.5}$$

Here, α, β and γ are known as associated scalars. A and B are called associated 1-forms. U and V are generators of this manifold.

Shaikh et al. [4] introduced the notion of hyper-generalized quasi-Einstein ($HGQE$) manifold. According to them, a Riemannian manifold (M^n, g), $(n > 2)$ is said to be a hyper-generalized quasi-Einstein manifold if its Ricci tensor S of type (0, 2) is nonzero and the following condition

$$\begin{aligned} S(X,Y) =& \alpha g(X,Y) + \beta A(X)A(Y) \\ &+ \gamma[A(X)B(Y) + A(Y)B(X)] \\ &+ \delta[A(X)D(Y) + A(Y)D(X)], \end{aligned} \tag{1.6}$$

for all $X, Y \in \chi(M)$, is satisfied. Here, α, β, γ and δ are real-valued, nonzero scalar functions on (M^n, g). A, B and D are nonzero 1-forms such that

$$g(X,U) = A(X), g(X,V) = B(X), g(X,W) = D(X), \tag{1.7}$$

U, V and W are the mutually orthogonal unit vector fields, i.e.,

$$g(U,V) = g(V,W) = g(U,W) = 0; \quad g(U,U) = g(V,V) = g(W,W) = 1. \tag{1.8}$$

α, β, γ and δ are called associated scalars. A, B and D are called associated 1-forms. U, V and W are called generators of this manifold. This manifold of dimension n is denoted by $(HGQE)_n$.

Shaikh et al. [4] studied on hyper-generalized quasi-Einstein manifolds with some geometric properties of it. Kim and Kim [5] studied on compact Einstein warped product spaces with non-positive scalar curvature. Güler and Demirbağ [6] dealt with some Ricci conditions on hyper-generalized quasi-Einstein manifolds. Pahan et al. [7] worked on multiply warped products quasi-Einstein manifolds with quarter-symmetric connection and they have discussed on compact super quasi-Einstein warped product with non-positive scalar curvature. Motivated by these works, presently we study about hyper-generalized quasi-Einstein warped product spaces with non-positive scalar curvature. Later, we apply our results on some physical properties of hyper-generalized quasi-Einstein manifold.

Let $\{e_i : i = 1, 2, 3, \ldots, n\}$ be an orthogonal frame field at any point of the manifold. Then, by putting $X = Y = e_i$ in Eq. (1.6) and taking summation over i $(1 \leq i \leq n)$, we get

$$r = n\alpha + \beta, \tag{1.9}$$

where r is the scalar curvature of the manifold.

We consider U as the timelike velocity vector field, V as the heat flux vector field and W as the stress vector field, i.e.,

$$g(U,U) = -1, g(V,V) = 1, g(W,W) = 1. \tag{1.10}$$

Many geometers worked with various types of curvature tensors in differential geometry. Tripathi [8] improved Chen–Ricci inequality for curvature like tensors and its applications. Chen and Yano [9] introduced the notion of quasi-constant curvature. According to them, a Riemannian manifold (M^n, g), $(n \geq 3)$ is said to be a quasi-constant curvature if it is conformally flat and its curvature tensor R of type (0, 4) satisfies the following condition

$$\begin{aligned} R(X,Y,Z,N) =& a_1[g(Y,Z)g(X,N) \\ &- g(X,Z)g(Y,N)] \\ &+ a_2[g(Y,Z)A(X)A(N) \\ &- g(X,Z)A(Y)A(N) \\ &+ g(X,N)A(Y)A(Z) \\ &- g(Y,N)A(X)A(Z)], \end{aligned}$$

where A is a 1-form and a_1, a_2 are both nonzero scalars.

Now, we define a Riemannian manifold (M^n, g), $(n \geq 3)$ to be hyper-generalized quasi-constant curvature if it is conformally flat and the curvature tensor of it has the following form

$$\begin{aligned} R(X,Y,Z,N) =& b_1[g(Y,Z)g(X,N) - g(X,Z)g(Y,N)] \\ &+ b_2[g(Y,Z)A(X)A(N) + g(X,N)A(Y)A(Z) \\ &- g(X,Z)A(Y)A(N) - g(Y,N)A(X)A(Z)] \\ &+ b_3[g(Y,Z)\{A(X)B(N) + A(N)B(X)\} \\ &+ g(X,N)\{A(Y)B(Z) + A(Z)B(Y)\} \\ &- g(X,Z)\{A(Y)B(N) + A(N)B(Y)\} \\ &- g(Y,N)\{A(X)B(Z) + A(Z)B(X)\}] \\ &+ b_4[g(Y,Z)\{A(X)D(N) + A(N)D(X)\} \\ &+ g(X,N)\{A(Y)D(Z) + A(Z)D(Y)\} \\ &- g(X,Z)\{A(Y)D(N) + A(N)D(Y)\} \\ &- g(Y,N)\{A(X)D(Z) + A(Z)D(X)\}], \end{aligned} \tag{1.11}$$

where A, B, D are 1-forms and b_1, b_2, b_3, b_4 are nonzero scalars.

The notions of cartesian (or direct) products have fruitful generalizations in the notion of warped products. The concept of warped product arose due to a surface of revolution. Two natural extensions of warped product manifolds are twisted products and convolution manifolds. Einstein's field equations and modified field equations have many exact solutions. These solutions are warped products.

For example, Robertson–Walker models and the Schwarzschild solution are warped products. It was initiated by Bishop and O' Neill [10] for studying manifolds with negative curvature.

Let (B, g_B) and (F, g_F) be two Riemannian manifolds of positive dimensions and $f : B \to (0, \infty)$ be a positive smooth function on B. Let $\pi : B \times F \to B$ and $\eta : B \times F \to F$ be the natural projection of the product manifold $B \times F$. The warped product $M = B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian structure such that

$$<X, X> \; = \; <\pi^*(X), \pi^*(X)> \\ + f^2(\pi(X)) <\eta^*(X), \eta^*(X)> ,$$

for any tangent vector $X \in \chi(M)$. Thus, we get $g_M = g_B + f^2 g_F$. Here, B and F are base and fiber, respectively. f is called the warping function of the warped product. So we obtain the following proposition [11].

Proposition 1.1 *The Ricci curvature S_M of the warped product $M = B \times_f F$ with $dimF = k$ satisfies*

(1) $S_M(X, Y) = S_B(X, Y) - \frac{k}{f} H^f(X, Y)$,
(2) $S_M(X, V) = 0$,
(3) $S_M(V, W) = S_F(V, W) - g(V, W) f^{\#}, f^{\#} = \frac{-\Delta f}{f} + \frac{k-1}{f^2} |\nabla f|^2$,

for any horizontal vectors X, Y (i.e., $X, Y \in \chi(B)$) and any vertical vectors V, W (i.e., $V, W \in \chi(F)$), where Δf and H^f denote the Laplacian of f (i.e., $\Delta f = -tr(H^f)$) and the Hessian of f, respectively.

In view of Proposition 1.1, we obtain the following theorem.

Theorem 1.1 *Suppose $M = B \times_f F$ is an warped product manifold as well as a hyper-generalized quasi-Einstein manifold. Then, its Ricci tensor satisfies the following conditions.*

(*i*) *When U, V and W are mutually orthogonal and tangent to the base B, then the Ricci tensors of B and F satisfy the following conditions*

$$\begin{aligned} (a) S_B(X, Y) =& \alpha g_B(X, Y) + \beta g_B(X, U) g_B(Y, U) \\ &+ \gamma [g_B(X, U) g_B(Y, V) \\ &+ g_B(Y, U) g_B(X, V)] \\ &+ \delta [g_B(X, U) g_B(Y, W) \\ &+ g_B(Y, U) g_B(X, W)] + \frac{k}{f} H^f(X, Y), \\ (b) S_F(X, Y) =& g_F(X, Y)[\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2]. \end{aligned}$$

(ii) When U, V and W are mutually orthogonal and tangent to the fiber F, then the Ricci tensors of B and F satisfy the following conditions

$$\begin{aligned} (a) S_B(X, Y) =& \alpha g_B(X, Y) + \frac{k}{f} H^f(X, Y), \\ (b) S_F(X, Y) =& g_F(X, Y)[\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2] \\ &+ \beta f^4 g_F(X, U) g_F(Y, U) \\ &+ \gamma f^4 [g_F(X, U) g_F(Y, V) \\ &+ g_F(Y, U) g_F(X, V)] \\ &+ \delta f^4 [g_F(X, U) g_F(Y, W) \\ &+ g_F(Y, U) g_F(X, W)]. \end{aligned}$$

The proof of Theorem 1.1 is similar to Theorem 2.1 of the paper [12].

2 Hyper-generalized Quasi-Einstein Warped Product Spaces with Non-Positive Scalar Curvature

In view of Proposition 1.1, we obtain the following result where Eq. (1.2) turns into

Result 2.1 *When U, V and W are mutually orthogonal and tangent to the base B, the warped product $M = B \times_f F$ is a hyper-generalized quasi-Einstein manifold with*

$$\begin{aligned} S_M(X, Y) =& \alpha g_M(X, Y) + \beta A(X) A(Y) \\ &+ \gamma [A(X) B(Y) + A(Y) B(X)] \\ &+ \delta [A(X) D(Y) + A(Y) D(X)] \end{aligned}$$

if and only if

$$\begin{aligned} (2.a) S_B(X, Y) =& \alpha g_B(X, Y) + \beta g_B(X, U) g_B(Y, U) \\ &+ \gamma [g_B(X, U) g_B(Y, V) \\ &+ g_B(Y, U) g_B(X, V)] \\ &+ \delta [g_B(X, U) g_B(Y, W) \\ &+ g_B(Y, U) g_B(X, W)] \\ &+ \frac{k}{f} H^f(X, Y), \\ (2.b) S_F(X, Y) =& \mu g_F(X, Y), \\ (2.c) \mu =& [\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2]. \end{aligned}$$

The complete proof of the below lemma is given in [5].

Lemma 2.1 *Suppose f is a smooth function on a Riemannian manifold B, then for any vector X,*

$$div(H^f)(X) = S(\nabla f, X) - \Delta(df)(X), \tag{2.1}$$

where $\Delta = d\delta + \delta d$ is the Laplacian on B which is acting on differential forms.

Now we give the following proposition.

Proposition 2.1 *Suppose (B^m, g_B) is an $m(\geq 2)$-dimensional compact Riemannian manifold. Also, suppose that f is a nonconstant smooth function on B satisfying (2.a) for $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ and if the condition*

$$\begin{aligned}&\beta g_B(X,U)g_B(\nabla f,U)+\gamma[g_B(X,U)g_B(\nabla f,V)\\&\quad+g_B(\nabla f,U)g_B(X,V)]\\&\quad+\delta[g_B(X,U)g_B(\nabla f,W)\\&\quad+g_B(\nabla f,U)g_B(X,W)]=0\end{aligned}$$

holds, then f satisfies (2.c) for $\mu \in \mathbb{R}$. Hence, for a compact Riemannian manifold F with $S_F(X,Y)=\mu g_F(X,Y)$, we can construct a compact hyper-generalized quasi-Einstein warped product space $M = B \times_f F$ with

$$\begin{aligned}S_M(X,Y)=&\alpha g_M(X,Y)+\beta A(X)A(Y)\\&+\gamma[A(X)B(Y)+A(Y)B(X)]\\&+\delta[A(X)D(Y)+A(Y)D(X)],\end{aligned}$$

where U, V and W are mutually orthogonal and tangent to the base B.

Proof By considering the trace of both sides of (2.a), we obtain

$$r=\alpha m-k\frac{\Delta f}{f}+\beta, \tag{2.2}$$

where r is the scalar curvature of B. From the second Bianchi identity, it follows that

$$dr=2div(S). \tag{2.3}$$

In view of Eqs. (2.2) and (2.3), we get

$$divS(X)=\frac{k}{2f^2}\{\Delta f df-fd(\Delta f)\}(X). \tag{2.4}$$

Also, we obtain

$$\begin{aligned}div(\frac{1}{f}H^f)(X)&=\sum_i(D_{E_i}(\frac{1}{f}H^f))(E_i,X)\\&=-\frac{1}{f^2}H^f(\nabla f,X)+\frac{1}{f}divH^f(X),\end{aligned}$$

where X is a vector field and $\{E_1,E_2,\ldots,E_m\}$ is an orthonormal frame of B. Since $H^f(\nabla f,X)=(D_X df)(\nabla f)=\frac{1}{2}d(|\nabla f|^2)(X)$, the last equation becomes

$$div(\frac{1}{f}H^f)(X)=-\frac{1}{2f^2}d(|\nabla f|^2)(X)+\frac{1}{f}divH^f(X),$$

X is a vector field of B. Therefore, from (2.a) and Eq. (2.1), we get

$$\begin{aligned}div(\frac{1}{f}H^f)(X)=&\frac{1}{2f^2}\{(k-1)d(|\nabla f|^2)\\&-2fd(\Delta f)+2\alpha f df\}\\&+\frac{1}{f}\beta g_B(X,U)g_B(\nabla f,U)\\&+\frac{1}{f}\gamma[g_B(X,U)g_B(\nabla f,V)\\&+g_B(\nabla f,U)g_B(X,V)]\\&+\frac{1}{f}\delta[g_B(X,W)g_B(\nabla f,U)\\&+g_B(\nabla f,W)g_B(X,U)].\end{aligned} \tag{2.5}$$

But, (2.a) implies $divS_B = div(\frac{k}{f}H^f)$. So, from Eqs. (2.4) and (2.5), it follows that $d(-f\Delta f+(k-1)|\nabla f|^2+\alpha f^2)=0$, i.e., $-f\Delta f+(k-1)|\nabla f|^2+\alpha f^2=\mu$, where μ is some constant. This completes the proof of the first part of the Proposition. Now if (F, g_F) is a k-dimensional compact Riemannian manifold with $S_F=\mu g_F$, then we can make a compact hyper-generalized quasi-Einstein warped product $M = B\times_f F$ with respect to the sufficient Result 2.1. □

Similarly, we obtain the following result and proposition where U, V and W are mutually orthogonal and tangent to the fiber F.

Result 2.2 *When U, V and W are mutually orthogonal and tangent to the fiber F, the warped product $M = B\times_f F$ is a hyper-generalized quasi-Einstein manifold with*

$$\begin{aligned}S_M(X,Y)=&\alpha g_M(X,Y)+\beta A(X)A(Y)\\&+\gamma[A(X)B(Y)+A(Y)B(X)]\\&+\delta[A(X)D(Y)+A(Y)D(X)]\end{aligned}$$

if and only if

$$(2.d)S_B(X,Y)=\alpha g_B(X,Y)+\frac{k}{f}H^f(X,Y),$$

$$\begin{aligned}(2.e)S_F(X,Y)=&g_F(X,Y)[\alpha f^2\\&-f\Delta f+(k-1)|\nabla f|^2]+\beta f^4 g_F(X,U)g_F(Y,U)\\&+\gamma f^4[g_F(X,U)g_F(Y,V)+g_F(Y,U)g_F(X,V)]\\&+\delta f^4[g_F(X,U)g_F(Y,W)+g_F(Y,U)g_F(X,W)],\end{aligned}$$

$$(2.f)\mu=[\alpha f^2-f\Delta f+(k-1)|\nabla f|^2].$$

Proposition 2.2 *Suppose (B^m, g_B) is an $m(\geq 2)$-dimensional compact Riemannian manifold. Also, suppose that f is a nonconstant smooth function on B satisfying (2.d) for $\alpha\in\mathbb{R}$ and $k\in\mathbb{N}$. Hence, for a compact hyper-generalized quasi-Einstein manifold F with*

$$\begin{aligned}S_F(X,Y)=&g_F(X,Y)[\alpha f^2-f\Delta f\\&+(k-1)|\nabla f|^2+\beta f^4 g_F(X,U)g_F(Y,U)\\&+\gamma f^4[g_F(X,U)g_F(Y,V)\\&+g_F(Y,U)g_F(X,V)]\\&+\delta f^4[g_F(X,U)g_F(Y,W)\\&+g_F(Y,U)g_F(X,W)],\end{aligned}$$

we can construct a compact hyper-generalized quasi-Einstein warped product space $M=B\times_f F$ with

$$\begin{aligned}S_M(X,Y)=&\alpha g_M(X,Y)+\beta A(X)A(Y)\\&+\gamma[A(X)B(Y)+A(Y)B(X)]\\&+\delta[A(X)D(Y)+A(Y)D(X)],\end{aligned}$$

where U, V and W are mutually orthogonal and tangent to the fiber F.

Proof By considering the trace of both sides of (2.d), we get

$$r=\alpha m-k\frac{\Delta f}{f}, \tag{2.6}$$

where r is the scalar curvature of B.

In view of Eqs. (2.6) and (2.3), we get

$$divS(X)=\frac{k}{2f^2}\{\Delta f df-fd(\Delta f)(X)\}. \tag{2.7}$$

So, from (2.d) and Eq. (2.1), we obtain

$$\begin{aligned}div(\frac{1}{f}H^f)(X)=&\frac{1}{2f^2}\{(k-1)d(|\nabla f|^2)\\&-2fd(\Delta f)+2\alpha f df\}.\end{aligned} \tag{2.8}$$

But, (2.d) implies $divS_B=div(\frac{k}{f}H^f)$. So, from Eqs. (2.7) and (2.8), it follows that

$$d(-f\Delta f+(k-1)|\nabla f|^2+\alpha f^2)=0,$$
$$i.e., -f\Delta f+(k-1)|\nabla f|^2+\alpha f^2=\mu,$$

where μ is some constant. This completes the proof of the first part of Proposition 2.2. Now if (F,g_F) is a k-dimensional compact Riemannian manifold with

$$\begin{aligned}S_F(X,Y)=&g_F(X,Y)[\alpha f^2-f\Delta f\\&+(k-1)|\nabla f|^2]+\beta f^4 g_F(X,U)g_F(Y,U)\\&+\gamma f^4[g_F(X,U)g_F(Y,V)\\&+g_F(Y,U)g_F(X,V)]\\&+\delta f^4[g_F(X,U)g_F(Y,W)\\&+g_F(Y,U)g_F(X,W)],\end{aligned}$$

then we can make a compact hyper-generalized quasi-Einstein warped product $M=B\times_f F$ with respect to the sufficient Result 2.2. □

Now we state the following theorem.

Theorem 2.1 *If $M=B\times_f F$ is a compact hyper-generalized quasi-Einstein warped product space of non-positive scalar curvature, then the warped product will be a Riemannian product.*

The proof of Theorem 2.1 is similar to Theorem 2.1 of the paper [13].

3 The Generators U, V and W as Concurrent Vector Fields

A vector field η is concurrent if it satisfies the following condition [14]

$$\nabla_X\eta=\lambda X, \tag{3.1}$$

where λ ($\neq 0$) is a constant. If $\lambda=0$, then the vector field turns into a parallel vector field.

Here, we take the concurrent vector fields U, V and W with respect to the associated 1-forms A, B and D, respectively.

Then, we get,

$$(\nabla_X A)(Y)=ag(X,Y), \tag{3.2}$$

$$(\nabla_X B)(Y)=bg(X,Y), \tag{3.3}$$

$$(\nabla_X D)(Y)=cg(X,Y), \tag{3.4}$$

where a, b and c are the nonzero constants.

We suppose that α,β,γ and δ are constants and then considering covariant derivative of Eq. (1.6) with respect to Z , we get

$$\begin{aligned}(\nabla_Z S)(X,Y)=&\beta[(\nabla_Z A)(X)A(Y)+A(X)(\nabla_Z A)(Y)]\\&+\gamma[(\nabla_Z A)(X)B(Y)+A(X)(\nabla_Z B)(Y)\\&+(\nabla_Z A)(Y)B(X)+A(Y)(\nabla_Z B)(X)]\\&+\delta[(\nabla_Z A)(X)D(Y)+A(X)(\nabla_Z D)(Y)\\&+(\nabla_Z A)(Y)D(X)+A(Y)(\nabla_Z D)(X)].\end{aligned} \tag{3.5}$$

Now by using Eqs. (3.2), (3.3) and (3.4) in Eq. (3.5), we get

$$\begin{aligned}(\nabla_Z S)(X,Y)=&\beta[ag(Z,X)A(Y)+ag(Z,Y)A(X)]\\&+\gamma[ag(Z,X)B(Y)+bg(Z,Y)A(X)\\&+ag(Z,Y)B(X)+bg(Z,X)A(Y)]\\&+\delta[ag(Z,X)D(Y)+cg(Z,Y)A(X)\\&+ag(Z,Y)D(X)+cg(Z,X)A(Y)].\end{aligned} \tag{3.6}$$

Taking contraction on Eq. (3.6) over X and Y, we get

$$\begin{aligned}dr(Z)=&2a\beta A(Z)+2\gamma[aB(Z)+bA(Z)]\\&+2\delta[aD(Z)+cA(Z)],\end{aligned} \tag{3.7}$$

where r being the scalar curvature of this manifold.

From Eq. (1.9), we have

$$r = n\alpha + \beta. \tag{3.8}$$

Since $\alpha, \beta \in \mathbb{R}$, therefore

$$dr(X) = 0, forall X. \tag{3.9}$$

From Eqs. (3.7) and (3.9), it follows that

$$\begin{aligned}
&a\beta A(Z) + \gamma[aB(Z) + bA(Z)] + \delta[aD(Z) + cA(Z)] = 0,\\
&i.e., (a\beta + b\gamma + c\delta)A(Z) + a\gamma B(Z) + a\delta D(Z) = 0,\\
&i.e., D(Z) = -\left(\frac{a\beta + b\gamma + c\delta}{a\delta}\right)A(Z) - \frac{\gamma}{\delta}B(Z).
\end{aligned} \tag{3.10}$$

Since a, b and c are the nonzero constants, then with the help of Eq. (3.10) in Eq. (1.6), we get

$$S(X,Y) = \alpha g(X,Y) - \left(\frac{a\beta + 2b\gamma + 2c\delta}{a}\right)A(X)A(Y). \tag{3.11}$$

Therefore, the manifold turns into a quasi-Einstein manifold. Hence, we get the following theorem.

Theorem 3.1 *If the associated scalars are constants and the associated vector fields of a $(HGQE)_n$ are concurrent, then the manifold turns into a quasi-Einstein manifold.*

4 Ricci Recurrent $(HGQE)_n$

A $(HGQE)_n$ is Ricci recurrent if its Ricci tensor S of type (0, 2) obeys the following condition [15]

$$(\nabla_X S)(Y,Z) = E(X)S(Y,Z),$$

where $E(X)$ being a nonzero 1-form.

By considering the manifold Ricci recurrent, we get

$$(\nabla_X S)(Y,Z) = E(X)S(Y,Z). \tag{4.1}$$

Also, it is known that

$$(\nabla_X S)(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{4.2}$$

Using Eq. (4.2) in Eq. (4.1), we get

$$E(X)S(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{4.3}$$

Using Eq. (1.6) in Eq. (4.3), we obtain

$$\begin{aligned}
&E(X)[\alpha g(Y,Z) + \beta A(Y)A(Z) + \gamma\{A(Y)B(Z) + A(Z)B(Y)\}\\
&\quad + \delta\{A(Y)D(Z) + A(Z)D(Y)\}]\\
&\quad = X[\alpha g(Y,Z) + \beta A(Y)A(Z) + \gamma\{A(Y)B(Z)\\
&\quad + A(Z)B(Y)\} + \delta\{A(Y)D(Z) + A(Z)D(Y)\}]\\
&\quad - [\alpha g(\nabla_X Y, Z) + \beta A(\nabla_X Y)A(Z)\\
&\quad + \gamma\{A(\nabla_X Y)B(Z) + A(Z)B(\nabla_X Y)\} + \delta\{A(\nabla_X Y)D(Z)\\
&\quad + A(Z)D(\nabla_X Y)\}] - [\alpha g(Y, \nabla_X Z) + \beta A(Y)A(\nabla_X Z)\\
&\quad + \gamma\{A(Y)B(\nabla_X Z) + A(\nabla_X Z)B(Y)\} + \delta\{A(Y)D(\nabla_X Z)\\
&\quad + A(\nabla_X Z)D(Y)\}].
\end{aligned} \tag{4.4}$$

Setting $Y = Z = U$ in Eq. (4.4), we have

$$\begin{aligned}
X(\alpha+\beta) - (\alpha+\beta)E(X) = &2(\alpha+\beta)A(\nabla_X U) + 2\gamma B(\nabla_X U)\\
&+ 2\delta D(\nabla_X U).
\end{aligned} \tag{4.5}$$

Since $A(\nabla_X U) = 0$, therefore Eq. (4.5) becomes

$$\begin{aligned}
&X(\alpha+\beta) - (\alpha+\beta)E(X) = 2\gamma B(\nabla_X U) + 2\delta D(\nabla_X U),\\
&i.e., X(\alpha+\beta) - (\alpha+\beta)E(X)\\
&\quad = 2\gamma g(\nabla_X U, V) + 2\delta g(\nabla_X U, W),\\
&i.e., X(\alpha+\beta) - (\alpha+\beta)E(X)\\
&\quad = -2\gamma g(\nabla_X V, U) - 2\delta g(\nabla_X W, U),\\
&i.e., X(\alpha+\beta) - (\alpha+\beta)E(X)\\
&\quad = -2[g(\gamma\nabla_X V + \delta\nabla_X W, U)],\\
&i.e., X(\alpha+\beta) - (\alpha+\beta)E(X)\\
&\quad = -2A(\nabla_X(\gamma V + \delta W)).
\end{aligned}$$

So, $A(\nabla_X(\gamma V + \delta W)) = 0$ if and only if $X(\alpha+\beta) - (\alpha+\beta)E(X) = 0$. But $A(\nabla_X(\gamma V + \delta W)) = 0$ implies

$$\begin{aligned}
&either, \nabla_X(\gamma V + \delta W) \perp U,\\
&\quad or, (\gamma V + \delta W) is\ a\ parallel\ vector\ field.
\end{aligned} \tag{4.6}$$

Setting $Y = Z = V$ in Eq. (4.4), we obtain

$$X\alpha - \alpha E(X) = 2\alpha B(\nabla_X V) + 2\gamma A(\nabla_X V). \tag{4.7}$$

Since $B(\nabla_X V) = 0$, therefore Eq. (4.7) becomes

$$X\alpha - \alpha E(X) = 2\gamma A(\nabla_X V).$$

So, $A(\nabla_X V) = 0$ if and only if $X\alpha - \alpha E(X) = 0$. But $A(\nabla_X V) = 0$ implies

$$\begin{aligned}
&either, \nabla_X V \perp U,\\
&\quad or, V\ is\ a\ parallel\ vector\ field.
\end{aligned} \tag{4.8}$$

Setting $Y = Z = W$ in Eq. (4.4), we get

$$X\alpha - \alpha E(X) = 2\alpha D(\nabla_X W) + 2\delta A(\nabla_X W). \tag{4.9}$$

Since $D(\nabla_X W) = 0$, therefore Eq. (4.9) becomes

$$X\alpha - \alpha E(X) = 2\delta A(\nabla_X W).$$

So, $A(\nabla_X W) = 0$ if and only if $X\alpha - \alpha E(X) = 0$. But $A(\nabla_X W) = 0$ implies

$$\begin{aligned} &\textit{either}, \nabla_X W \perp U, \\ &\quad \textit{or}, W \textit{ is a parallel vector field}. \end{aligned} \tag{4.10}$$

Thus, from Eqs. (4.6), (4.8) and (4.10), we get the following theorem.

Theorem 4.1 *If $(HGQE)_n$ is Ricci recurrent, then*

$$\begin{aligned} &(i) \textit{Either} \nabla_X(\gamma V + \delta W) \perp U \\ &\quad \textit{or} (\gamma V + \delta W) \textit{ is a parallel vector field iff} \\ &X(\alpha + \beta) - (\alpha + \beta)E(X) = 0. \\ &(ii) \textit{Either} \nabla_X V \perp U \\ &\quad \textit{or } V \textit{ is a parallel vector field iff } X\alpha - \alpha E(X) = 0. \\ &(iii) \textit{Either} \nabla_X W \perp U \\ &\quad \textit{or } W \textit{ is a parallel vector field iff } X\alpha - \alpha E(X) = 0. \end{aligned}$$

5 Einstein's Field Equation in a $(HGQE)_n$

The Einstein's field equation is

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y), \tag{5.1}$$

where r being the scalar curvature. k and λ are the gravitational constant and cosmological constant, respectively.

Considering without cosmological constant (*i.e.*, $\lambda = 0$), then Eq. (5.1) becomes

$$S(X,Y) - \frac{r}{2}g(X,Y) = kT(X,Y). \tag{5.2}$$

With the help of Eq. (1.6) in Eq. (5.2), we get

$$\begin{aligned} &(\alpha - \frac{r}{2})g(X,Y) + \beta A(X)A(Y) \\ &\quad + \gamma[A(X)B(Y) + A(Y)B(X)] \\ &\quad + \delta[A(X)D(Y) + A(Y)D(X)] = kT(X,Y). \end{aligned} \tag{5.3}$$

After covariant differentiation on Eq. (5.3) with respect to Z, we get

$$\begin{aligned} &\beta[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)] + \gamma[(\nabla_Z A)(X)B(Y) \\ &\quad + A(X)(\nabla_Z B)(Y) + (\nabla_Z A)(Y)B(X) + A(Y)(\nabla_Z B)(X)] \\ &\quad + \delta[(\nabla_Z A)(X)D(Y) + A(X)(\nabla_Z D)(Y) + (\nabla_Z A)(Y)D(X) \\ &\quad + A(Y)(\nabla_Z D)(X)] = k(\nabla_Z T)(X,Y). \end{aligned} \tag{5.4}$$

Thus, by virtue of Eq. (5.4), we get the following theorem.

Theorem 5.1 *If the associated 1-forms A, B and D in a $(HGQE)_n$ are covariant constant, then the energy–momentum is also covariant constant.*

6 $(HGQE)_4$ Spacetime Admitting Space-matter Tensor

Space-matter tensor $\tilde{P}$ of type (0, 4) has been introduced by Petrov [16]. He defined the space-matter tensor as follows

$$\tilde{P} = \tilde{R} + \frac{k}{2}g \wedge T - \sigma G, \tag{6.1}$$

$\tilde{R}$ being the curvature tensor of type (0, 4), T being the energy–momentum tensor of type (0, 2), k being the gravitational constant and σ being the energy density. Also, G is a tensor of type (0, 4) such that

$$G(X,Y,Z,N) = g(Y,Z)g(X,N) - g(X,Z)g(Y,N), \tag{6.2}$$

for all $X, Y, Z, N \in \chi(M)$. Kulkarni–Nomizu product $E \wedge F$ of two (0, 2)-type tensors E and F is as follows.

$$\begin{aligned} (E \wedge F)(X,Y,Z,N) &= E(Y,Z)F(X,N) + E(X,N)F(Y,Z) \\ &\quad - E(X,Z)F(Y,N) - E(Y,N)F(X,Z), \end{aligned} \tag{6.3}$$

for $X, Y, Z, N \in \chi(M)$. $\tilde{P}$ is called the space-matter tensor of type (0, 4) of M.

Here, we study $(HGQE)_4$ spacetime when space-matter tensor is zero. From Eq. (6.1), we obtain

$$\begin{aligned} \tilde{P}(X,Y,Z,N) =& \tilde{R}(X,Y,Z,N) \\ &+ \frac{k}{2}[g(Y,Z)T(X,N) + g(X,N)T(Y,Z) \\ &- g(X,Z)T(Y,N) - g(Y,N)T(X,Z)] \\ &- \sigma[g(Y,Z)g(X,N) - g(X,Z)g(Y,N)]. \end{aligned} \tag{6.4}$$

If $\tilde{P} = 0$ in Eq. (6.4), we get

$$\begin{aligned} \tilde{R}(X,Y,Z,N) =& -\frac{k}{2}[g(Y,Z)T(X,N) + g(X,N)T(Y,Z) \\ &- g(X,Z)T(Y,N) - g(Y,N)T(X,Z)] \\ &+ \sigma[g(Y,Z)g(X,N) - g(X,Z)g(Y,N)]. \end{aligned} \tag{6.5}$$

Using Eqs. (1.6) and (5.2) in Eq. (6.5), we derive

$$\begin{aligned}\tilde{R}(X,Y,Z,N) =&(\sigma-\alpha+\frac{r}{2})[g(Y,Z)g(X,N)-g(X,Z)g(Y,N)]\\&-\frac{\beta}{2}[g(Y,Z)A(X)A(N)+g(X,N)A(Y)A(Z)\\&-g(X,Z)A(Y)A(N)-g(Y,N)A(X)A(Z)]\\&-\frac{\gamma}{2}[g(Y,Z)\{A(X)B(N)+A(N)B(X)\}\\&+g(X,N)\{A(Y)B(Z)+A(Z)B(Y)\}\\&-g(X,Z)\{A(Y)B(N)+A(N)B(Y)\}\\&-g(Y,N)\{A(X)B(Z)+A(Z)B(X)\}]\\&-\frac{\delta}{2}[g(Y,Z)\{A(X)D(N)+A(N)D(X)\}\\&+g(X,N)\{A(Y)D(Z)+A(Z)D(Y)\}\\&-g(X,Z)\{A(Y)D(N)+A(N)D(Y)\}\\&-g(Y,N)\{A(X)D(Z)+A(Z)D(X)\}].\end{aligned} \tag{6.6}$$

In view of Eq. (1.11), (6.6) follows that the manifold is a manifold of hyper-generalized quasi-constant curvature. Thus, we get the following theorem.

Theorem 6.1 *A $(HGQE)_4$ spacetime satisfying Einstein's field equation with zero space-matter tensor will be a spacetime of hyper-generalized quasi-constant curvature.*

Finally, we study to get sufficient condition for which $(HGQE)_4$ may be a divergence free space-matter tensor. From Eq. (1.9), we get

$$r = n\alpha+\beta,$$
$$i.e., r = \text{ constant }.$$

This implies $dr(X)=0$, for all X.

With the help of Eqs. (5.2) and (6.4), we get

$$\begin{aligned}(divP)(X,Y,Z) =&(divR)(X,Y,Z)\\&+\frac{1}{2}[(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)]\\&-g(Y,Z)[\frac{1}{4}dr(X)+d\sigma(X)]\\&+g(X,Z)[\frac{1}{4}dr(Y)+d\sigma(Y)].\end{aligned} \tag{6.7}$$

For a semi-Riemannian manifold,

$$(divR)(X,Y,Z) = (\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z). \tag{6.8}$$

From Eqs. (6.7) and (6.8), we deduce

$$\begin{aligned}(divP)(X,Y,Z) =&\frac{3}{2}[(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)]\\&-g(Y,Z)[\frac{1}{4}dr(X)+d\sigma(X)]\\&+g(X,Z)[\frac{1}{4}dr(Y)+d\sigma(Y)].\end{aligned} \tag{6.9}$$

Let us assume that $(divP)(X,Y,Z)=0$ and taking contraction on Eq. (6.9) over Y and Z, we get $d\sigma(X)=0$.

Thus, we obtain the following theorem.

Theorem 6.2 *In a $(HGQE)_4$ spacetime satisfying Einstein's field equation with divergence free space-matter tensor, the energy density is constant.*

Now using Eq. (1.6) in Eq. (6.9), we have

$$\begin{aligned}(divP)(X,Y,Z) =& \frac{3}{2}[d\alpha(X)g(Y,Z)\\&-d\alpha(Y)g(X,Z)]+\frac{3}{2}[d\beta(X)A(Y)A(Z)\\&-d\beta(Y)A(X)A(Z)]+\frac{3\beta}{2}[(\nabla_X A)(Y)A(Z)+A(Y)(\nabla_X A)(Z)\\&-(\nabla_Y A)(X)A(Z)-A(X)(\nabla_Y A)(Z)]\\&+\frac{3}{2}[d\gamma(X)\{A(Y)B(Z)+B(Y)A(Z)\}\\&-d\gamma(Y)\{A(X)B(Z)+B(X)A(Z)\}]\\&+\frac{3\gamma}{2}[(\nabla_X A)(Y)B(Z)+A(Y)(\nabla_X B)(Z)+(\nabla_X A)(Z)B(Y)\\&+A(Z)(\nabla_X B)(Y)-(\nabla_Y A)(X)B(Z)\\&-A(X)(\nabla_Y B)(Z)-(\nabla_Y A)(Z)B(X)\\&-A(Z)(\nabla_Y B)(X)]+\frac{3}{2}[d\delta(X)\{A(Y)D(Z)+D(Y)A(Z)\}\\&-d\delta(Y)\{A(X)D(Z)+D(X)A(Z)\}]\\&+\frac{3\delta}{2}[(\nabla_X A)(Y)D(Z)+A(Y)(\nabla_X D)(Z)\\&+(\nabla_X A)(Z)D(Y)+A(Z)(\nabla_X D)(Y)\\&-(\nabla_Y A)(X)D(Z)-A(X)(\nabla_Y D)(Z)-(\nabla_Y A)(Z)D(X)\\&-A(Z)(\nabla_Y D)(X)]-g(Y,Z)[\frac{1}{4}dr(X)+d\sigma(X)]\\&+g(X,Z)[\frac{1}{4}dr(Y)+d\sigma(Y)].\end{aligned} \tag{6.10}$$

Considering the conditions that $\sigma,\alpha,\beta,\gamma$ and δ are constants and the generator U is a parallel vector field (i.e., $\nabla_X U=0$). Therefore, we get

$$\begin{aligned}&dr(X)=0, d\sigma(X)=0, \forall X\\&\quad and g(\nabla_X U,Y)=0, i.e., (\nabla_X A)(Y)=0.\end{aligned} \tag{6.11}$$

In view of [6], we derive

$$\alpha+\beta=0, \gamma=0, \delta=0. \tag{6.12}$$

Using Eqs. (6.11) and (6.12) in Eq. (6.10), we get $(divP)(X,Y,Z)=0$.

Hence, we get the following theorem.

Theorem 6.3 *If in a $(HGQE)_4$ spacetime with parallel vector field U satisfying Einstein's field equation, the energy density and the associated scalars are constants, then the divergence of the space-matter tensor is zero.*

7 General Relativistic Viscous Fluid $(HGQE)_4$ Spacetime

Let us consider (M^4, g) be a connected semi-Riemannian viscous fluid spacetime admitting heat flux obeying Einstein's field equation. The Einstein's field equation is given by

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y), \tag{7.1}$$

for all $X, Y \in \chi(M)$, where S is the (0, 2)-type Ricci tensor, r is the scalar curvature, λ is the cosmological constant and k is the gravitational constant.

For the fluid matter distribution, the energy–momentum tensor has been given by Ellis [17] as

$$\begin{aligned} T(X,Y) =& (\sigma + p)A(X)A(Y) + pg(X,Y) + A(X)B(Y) \\ &+ A(Y)B(X) + A(X)D(Y) + A(Y)D(X), \end{aligned} \tag{7.2}$$

with

$$\begin{aligned} &g(X,U) = A(X), g(X,V) = B(X), g(X,W) = D(X), \\ &A(U) = -1, B(V) = 1, D(W) = 1, \\ &g(U,V) = 0, g(V,W) = 0, g(U,W) = 0, \end{aligned}$$

where σ is the matter density, p is the isotropic pressure, U is the timelike velocity vector field, V is the heat conduction vector field and W is the stress vector field.

Using Eq. (7.2) in Eq. (7.1), we get

$$\begin{aligned} S(X,Y) =& (kp + \frac{r}{2} - \lambda)g(X,Y) + k(\sigma + p)A(X)A(Y) \\ &+ k[A(X)B(Y) + A(Y)B(X)] \\ &+ k[A(X)D(Y) + A(Y)D(X)]. \end{aligned} \tag{7.3}$$

Clearly, it follows that this spacetime is a $(HGQE)_4$ spacetime whose associated scalars are $(kp + \frac{r}{2} - \lambda)$, $k(\sigma + p)$, k and k. A, B and D are associated 1-forms and generators are U, V and W. Hence, we get the following theorem.

Theorem 7.1 *A viscous fluid spacetime admitting heat flux and obeying Einstein's field equation with cosmological constant is a connected semi-Riemannian hyper-generalized quasi-Einstein manifold of dimension four.*

From Eq. (1.9), we get for (M^4, g)

$$r = 4\alpha + \beta. \tag{7.4}$$

Now using Eqs. (1.6) and (7.4) in Eq. (7.3), we gain

$$\begin{aligned} &\left(\frac{2kp + 2\alpha + \beta - 2\lambda}{2}\right)g(X,Y) \\ &\quad = [\beta - k(\sigma + p)]A(X)A(Y) \\ &\quad + (\gamma - k)[A(X)B(Y) + B(X)A(Y)] \\ &\quad + (\delta - k)[A(X)D(Y) + A(Y)D(X)]. \end{aligned} \tag{7.5}$$

Putting $X = Y = U$ in Eq. (7.5), we find

$$\sigma = \frac{2\alpha + 3\beta - 2\lambda}{2k}. \tag{7.6}$$

Taking contraction on Eq. (7.3) over X and Y, we deduce

$$r = 4(kp + \frac{r}{2} - \lambda) - k(\sigma + p). \tag{7.7}$$

In view of Eqs. (7.4) and (7.6), (7.7) implies that

$$p = \frac{6\lambda - 6\alpha + \beta}{6k}. \tag{7.8}$$

By putting $X = Y = V$ and $X = Y = W$ in Eq. (7.5), we obtain the same value of p in each case given by

$$p = \frac{2\lambda - 2\alpha - \beta}{2k}. \tag{7.9}$$

As α, β are not constants, then in view of Eqs. (7.6), (7.7) and (7.9) it follows that σ and p are not constants. Hence, we get the following theorem.

Theorem 7.2 *If a viscous fluid $(HGQE)_4$ spacetime admitting heat flux satisfies Einstein's field equation with cosmological constant, then isotropic pressure and energy density of the fluid cannot be a constant.*

If α, β are constants, then from Eqs. (7.6) and (7.8), it implies that σ and p are constants. As $\sigma > 0$, $p > 0$, so we obtain from Eqs. (7.6) and (7.8) that $\lambda < \frac{2\alpha + 3\beta}{2}$ and $\lambda > \frac{6\alpha - \beta}{6}$, which implies

$$\frac{6\alpha - \beta}{6} < \lambda < \frac{2\alpha + 3\beta}{2}.$$

Also, Eq. (7.9) gives $\frac{2\alpha + \beta}{2} < \lambda$.

Hence, we get the following theorem.

Theorem 7.3 *If a viscous fluid $(HGQE)_4$ spacetime admitting heat flux satisfies Einstein's field equation with cosmological constant, then cosmological constant λ obeys the following condition either, $\frac{6\alpha - \beta}{6} < \lambda < \frac{2\alpha + 3\beta}{2}$ or, $\frac{2\alpha + \beta}{2} < \lambda$.*

Now we consider a hyper-generalized quasi-Einstein spacetime satisfying Einstein's field equation without cosmological constant (i.e., $\lambda = 0$) whose matter content is viscous fluid. Putting $\lambda = 0$ in Eq. (7.3), then Eq. (7.3) becomes

$$S(X,Y) = (kp + \frac{r}{2})g(X,Y) + k(\sigma + p)A(X)A(Y) + k[A(X)B(Y) + A(Y)B(X)] + k[A(X)D(Y) + A(Y)D(X)]. \tag{7.10}$$

By comparing Eqs. (1.6) and (7.10), we obtain

$$\alpha = kp + \frac{r}{2}, \beta = k(\sigma + p), \gamma = k, \delta = k. \tag{7.11}$$

Taking contraction on Eq. (7.10) over X and Y, we get

$$r = k(\sigma - 3p). \tag{7.12}$$

Using Eq. (7.12) in Eq. (7.10), it follows that

$$S(X,Y) = \frac{k(\sigma - p)}{2} g(X,Y) + k(\sigma + p)A(X)A(Y) + k[A(X)B(Y) + A(Y)B(X)] + k[A(X)D(Y) + A(Y)D(X)]. \tag{7.13}$$

Suppose Q is the Ricci operator given by $g(QX,Y) = S(X,Y)$ and $S(QX,Y) = S^2(X,Y)$. Therefore, we get $A(QX) = g(QX,U) = S(X,U)$, $B(QX) = g(QX,V) = S(X,V)$ and $D(QX) = g(QX,W) = S(X,W)$.

Hence, from Eq. (7.13), we have the following equation

$$S(QX,Y) = \frac{k(\sigma - p)}{2} S(X,Y) + k(\sigma + p)S(X,U)A(Y) + k[S(X,U)B(Y) + A(Y)S(X,V)] + k[S(X,U)D(Y) + A(Y)S(X,W)]. \tag{7.14}$$

Contracting Eq. (7.14) over X and Y, we get

$$S^2(X,X) = \|Q\|^2 = \frac{k(\sigma - p)r}{2} + k(\sigma + p)S(U,U) + 2kS(U,V) + 2kS(U,W). \tag{7.15}$$

From Eqs. (1.6), (7.11) and (7.12), we obtain

$$S(U,U) = \beta - \alpha = \frac{k(\sigma + 3p)}{2}. \tag{7.16}$$

$$S(U,V) = -\gamma = -k. \tag{7.17}$$

$$S(U,W) = -\delta = -k. \tag{7.18}$$

Using Eqs. (7.16), (7.17) and (7.18) in Eq. (7.15), we derive

$$\|Q\|^2 = k^2(\sigma^2 + 3p^2 - 4). \tag{7.19}$$

Hence, we can state the following theorem.

Theorem 7.4 *If a viscous fluid $(HGQE)_4$ spacetime satisfying Einstein's field equation without cosmological constant, then the square of the length of Ricci operator is $k^2(\sigma^2 + 3p^2 - 4)$.*

Now, if we consider

$$\sigma > 3p. \tag{7.20}$$

From Eq. (7.19), it follows that

$$k^2(\sigma^2 + 3p^2 - 4) > 0, \quad i.e., \sigma^2 + 3p^2 > 4. \tag{7.21}$$

In view of Eqs. (7.20) and (7.21), we obtain

$$\sigma^2 + \frac{\sigma^2}{3} > \sigma^2 + 3p^2 > 4,$$

which gives

$$\sigma > \sqrt{3}.$$

Hence, we get the following corollary.

Corollary 7.1 In a viscous fluid $(HGQE)_4$ spacetime satisfying Einstein's field equation without cosmological constant with $\sigma > 3p$ and $p > 0$, the energy density is greater than $\sqrt{3}$.

8 Example of $(HGQE)_4$ Spacetime

In this section, we give a non-trivial example of $(HGQE)_4$ spacetime to ensure its existence. We take a Lorentzian metric g on M^4 by

$$ds^2 = g_{ij}dx^i dx^j = -\frac{k}{r}(dt)^2 + \frac{1}{\frac{c}{r} - 4}(dr)^2 + r^2(d\theta)^2 + (r\sin\theta)^2(d\phi)^2,$$

where $i,j = 1,2,3,4$ and k, c are constants. Then, nonzero components of Christoffel symbols, curvature tensors and Ricci tensors are given below.

$$\left.\begin{aligned} &\Gamma^2_{33} = 4r - c, \Gamma^1_{12} = -\frac{1}{2r}, \Gamma^2_{22} = \frac{c}{2r(c-4r)}, \Gamma^3_{32} = \Gamma^4_{42} = \frac{1}{r}, \\ &\Gamma^4_{43} = \cot\theta, \Gamma^2_{44} = (4r-c)(\sin\theta)^2, \Gamma^3_{44} = -\frac{\sin(2\theta)}{2} \end{aligned}\right\} \tag{8.1}$$

$$\left.\begin{aligned} &R_{1221} = -\frac{k(c-3r)}{r^3(c-4r)}, R_{1331} = \frac{k(c-4r)}{2r^2}, R_{1441} = \frac{k(c-4r)(\sin\theta)^2}{2r^2} \\ &R_{2332} = \frac{c}{2(4r-c)}, R_{2442} = \frac{c(\sin\theta)^2}{2(4r-c)}, R_{3443} = r(c-5r)(\sin\theta)^2 \\ &R_{11} = -\frac{k}{r^3}, R_{22} = -\frac{3}{r(c-4r)}, R_{33} = -3, R_{44} = -3(\sin\theta)^2 \end{aligned}\right\}. \tag{8.2}$$

From Eqs. (8.1) and (8.2), it follows that M^4 is a Lorentzian manifold of nonzero scalar curvature $(= -\frac{8}{r^2})$. Now our aim is to show that this manifold is $(HGQE)_4$. Suppose α, β, γ and δ are the associated scalars and we consider these scalars by the following way

$$\alpha = -\frac{3}{r^2}, \beta = -\frac{4}{r^2}, \gamma = \frac{2}{r^2}, \delta = \frac{3}{r^2} \tag{8.3}$$

and the associated 1-forms are as follows

$$A_i(x) = \begin{cases} \sqrt{\frac{k}{r}} & \text{for} \quad i = 1 \\ 0 & \text{for} \quad i = 2, 3, 4 \end{cases};$$

$$B_i(x) = \begin{cases} \frac{1}{2r^2} & \text{for} \quad i = 4 \\ 0 & \text{for} \quad i = 1, 2, 3 \end{cases}$$

$$\text{and}\, D_i(x) = \begin{cases} -\frac{1}{3r^2} & \text{for} \quad i = 4 \\ 0 & \text{for} \quad i = 1, 2, 3 \end{cases}$$

Thus, we get,

$(i) R_{11} = \alpha g_{11} + \beta A_1 A_1 + \gamma[A_1 B_1 + B_1 A_1] + \delta[A_1 D_1 + D_1 A_1]$
$(ii) R_{22} = \alpha g_{22} + \beta A_2 A_2 + \gamma[A_2 B_2 + B_2 A_2] + \delta[A_2 D_2 + D_2 A_2]$
$(iii) R_{33} = \alpha g_{33} + \beta A_3 A_3 + \gamma[A_3 B_3 + B_3 A_3] + \delta[A_3 D_3 + D_3 A_3]$
$(iv) R_{44} = \alpha g_{44} + \beta A_4 A_4 + \gamma[A_4 B_4 + B_4 A_4] + \delta[A_4 D_4 + D_4 A_4]$.

Since the other Ricci tensors except R_{11}, R_{22}, R_{33} and R_{44} are zero, so we have
$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma[A_i B_j + B_i A_j] + \delta[A_i D_j + D_i A_j]$, $i, j = 1, 2, 3, 4$. It is clearly seen that its scalar curvature $= 4\alpha - \beta = -\frac{8}{r^2}$. Therefore, (M^4, g) is a hyper-generalized quasi-Einstein manifold.

Example 8.1 Suppose (M^4, g) is a Lorentzian manifold equipped with the Lorentzian metric g given by

$$ds^2 = g_{ij} dx^i dx^j = -\frac{k}{r}(dt)^2 + \frac{1}{\frac{c}{r} - 4}(dr)^2 + r^2 (d\theta)^2 + (r \sin\theta)^2 (d\phi)^2,$$

where $i, j = 1, 2, 3, 4$ and k, c are constants. Then, (M^4, g) is a $(HGQE)_4$ spacetime with nonconstant and nonzero scalar curvature.

Conclusion : Hyper-generalized quasi-Einstein manifolds play a very significant role in general relativity and cosmology. It has wide applications in general relativistic viscous fluid spacetime admitting heat flux and stress. General relativity describes a description of gravity as a geometric property of spacetime. The curvature of spacetime is directly related to the energy and momentum. Also we know the cosmological constant to be a homogeneous energy density which causes the expansion of the universe to accelerate. Here, we obtain geometric and physical properties of hyper-generalized quasi-Einstein spacetimes in general relativity and cosmology with some certain conditions.

Compliance with ethical standards

Relevance of the Work in Broad Context $(HGQE)_4$ is considered as base space of general relativistic viscous fluid spacetime. It plays significant role in general relativity. Warped product arose due to surface's revolution. Exact solutions of Einstein's field equations are warped products. So it is essential to study Einstein's field equation, space-matter tensor, warped product on $(HGQE)_n$.

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ORIGINAL PAPER

A new way to study on generalized Friedmann–Robertson–Walker spacetime

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Abstract: In this paper, we study the generalized Friedmann–Robertson–Walker spacetime in a new way. We know that the generalized Friedmann-Robertson-Walker metric and solutions of the Einstein field equations can be expressed in terms of Lorentzian warped products. We consider a multiply warped product metric of the generalized Friedmann-Robertson-Walker spacetime of type $\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2$ with the warping functions h_1, h_2 associated to the submanifolds F_1, F_2 with dimensions n_1, n_2, respectively and the submanifold F_1 is conformal to $(\mathbb{R}^{n_1}, g)$, a pseudo-Euclidean space. Then we show that the Einstein equations $\overline{G}_{AB} = -\overline{\kappa}\,\overline{g}_{AB}$ on $(\overline{M}, \overline{g})$ with a cosmological constant $\overline{\kappa}$ is reduced to the Einstein equations $G_{ij} = -\kappa_2 g_{2ij}$ on the submanifold (F_2, g_2) with the cosmological constant κ_2. Furthermore, we consider some black hole solutions as typical examples. Then we derive the corresponding Einstein equations and the reduced Einstein equations for each black hole solution.

Keywords: Generalized Friedmann–Robertson–Walker spacetime; Multiply warped product; Einstein equations; Black hole solution.

1. Introduction

Definition 1.1 Let (M^n, g) be a semi-Riemannian manifold of dimension n. Then G is said to be an Einstein gravitational tensor field of M if it satisfies the relation

$$G(X,Y) = \mathrm{Ric}(X,Y) - \frac{1}{2}Sg(X,Y)$$

for every $X, Y \in \mathfrak{X}(M)$, where S is the scalar curvature tensor on M.

Therefore the Einstein field equations can be written in the form

$$\mathrm{Ric}(X,Y) - \frac{1}{2}Sg(X,Y) + \kappa g(X,Y) = \lambda T(X,Y),$$

where T is the stress-energy tensor, κ is the cosmological constant and λ is the Einstein gravitational constant. The basic solutions of the Einstein field equations have been studied in Lorentzian geometry and general relativity and they can be expressed in terms of the warped products [1]. In Lorentzian geometry some well-known solutions of the Einstein field equations such as Schwarzschild and Friedmann-Robertson-Walker metrics can be expressed in terms of the warped products. The generalized Friedmann-Robertson-Walker metric and solutions of the Einstein field equations can be expressed in terms of the Lorentzian warped products. Different models like the general relativistic model of gravitation and cosmological model provided the importance to find the Einstein equations. The warped product geometry is used to solve the partial differential equations since we can easily use the method of separation of variables. In five dimensional warped product geometry [2], the world has been considered as a higher dimensional universe expressed in terms of warped product geometry. Albert Einstein provided a static solution of the field equations and introduced the cosmological constant [3]. Recently, the cosmological constants were studied by many authors on various spaces [4–7].

Definition 1.2 Let (M^n, g) be a semi-Riemannian manifold of dimension $n\,(\geq 4)$. Then M is said to be an Einstein manifold if its Ricci tensor Ric satisfies the condition $\mathrm{Ric}(X,Y) = \lambda g(X,Y)$ for every $X, Y \in \mathfrak{X}(M)$, where λ is a real constant on M.

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Note 1. $\mathrm{Ric}(X,Y)=0$ for $n=1$ and $\mathrm{Ric}(X,Y)=\frac{K}{2}\lambda g(X,Y)$ for $n=2$. Hence a 2-dimensional semi-Riemannian manifold is Einstein if and only if it has a constant sectional curvature and $(M,\ g)$ is Einstein for $n=3$ if and only if it has a constant sectional curvature.

Definition 1.3 Let (B,g_B) and (F,g_F) be two pseudo-Riemannian manifolds with $\dim(B)=n\,(>0)$, $\dim(F)=m\,(>0)$ and h be a positive and smooth function on B. Then the warped product $\overline{M}=B\times_h F$ is the product manifold $B\times F$ endowed with the metric tensor $g_{\overline{M}}=g_B+h^2g_F$ defined by

$$g_{\overline{M}}=\pi^*(g_B)+(h\circ\pi)^2\sigma^*(g_F),$$

where $\pi: B\times F\to B$ and $\sigma: B\times F\to F$ are the natural projections and * denotes the pull-back operator. Here B and F are called the base and fiber of $\overline{M}$, respectively. The function h is called the warping function of the warped product [8].

The concept of warped product was first introduced by Bishop and O'Neil [9] to construct the examples of Riemannian manifold with negative curvature. Now we can generalize the warped products to multiply warped products.

Definition 1.4 10] A multiply warped product is the product manifold $\overline{M}=B\times_{h_1}F_1\times_{h_2}F_2...\times_{h_m}F_m$ endowed with the metric tensor $\overline{g}=g_B\oplus h_1^2g_{F_1}\oplus h_2^2g_{F_2}\oplus h_3^2g_{F_3}\oplus ....\oplus h_m^2g_{F_m}$ defined by

$$\overline{g}=\pi^*(g_B)\oplus(h_1\circ\pi)^2\sigma_1^*(g_{F_1})\oplus...\oplus(h_m\circ\pi)^2\sigma_m^*(g_{F_m}),$$

where π and $\sigma_i\,(i=1,2,...,m)$ are the natural projections of $B\times F_1\times F_2.....\times F_m$ onto $B,F_1,F_2,...,F_{m-1}$ and F_m, respectively. For each $i\in\{1,2,...,m\}$ the function $h_i: B\to(0,\infty)$ is smooth and (F_i,g_{F_i}) is a pseudo-Riemannian manifold.

Note 2. In particular, when $B=(c,d)$ equipped with the negative definite metric $g_B=-dt^2$, where $c<d$ and (F_i,g_{F_i}) is a Riemannian manifold for each $i\in\{1,2,...,m\}$, then we call $(\overline{M},\overline{g})$ as the generalized Robertson–Walker spacetimes.

Many authors studied the warped product manifolds and locally conformally flat manifolds, see [11, 12]. There are several studies correlating the warped product Einstein manifolds under various conditions on the curvature and symmetry, see [13–16]. It is well-known that the Einstein condition on warped geometries requires that the fibers must be necessarily Einstein [17]. In 2000, B. Ünal [10] derived the covariant derivative formulas for multiply warped products and also studied the geodesic equations for such type of spaces. In 2000, J. Choi [18] investigated the curvature of a multiply warped product with C^0-warping functions and represented the interior Schwarzschild spacetime as a multiply warped product spacetime with warping functions. In 2005, F. Dobarro and B. Ünal [19] studied the Ricci-flat and Einstein-Lorentzian multiply warped products and provided some results on the generalized Kasner spacetimes. In 2005 [20], authors obtained the necessary and sufficient conditions for a static spacetime to be locally conformally flat. In 2016, D. Dumitru [21] calculated the warping functions for multiply generalized Robertson-Walker space-time to be an Einstein manifold when all fibers are Ricci flat. In 2017, F. Gholami, F. Darabi and A. Haji-Badali [22] studied the multiply warped product metrics and reduced the Einstein equations for generalized Friedmann-Robrtson-Walker spacetime. In 2017, Sousa and Pina [23] studied the warped product semi-Riemannian Einstein manifolds under consideration that the base is conformal to an n-dimensional pseudo-Euclidean space and invariant under the action of an $(n-1)$-dimensional group. More recently, in [24], the authors generalized the work of Sousa and Pina for multiply warped product semi-Riemannian Einstein manifolds.

So, there are several studies correlating the warped product manifolds, multiply warped product manifolds, Einstein-Lorentzian multiply warped product manifolds, generalized Kasner spacetimes, static spacetime with conformal condition and generalized Friedmann-Robrtson-Walker spacetime etc. It is well-known that the generalized Friedmann-Robertson-Walker metric and solutions of the Einstein field equations can be expressed in terms of the Lorentzian warped products. The multiply warped product $(\overline{M},\overline{g})$ is a Lorentzian multiply warped product when it satisfies **Note 2**. Then the Lorentzian multiply warped product $(\overline{M},\overline{g})$ is called a generalized Robertson-Walker spacetime. In this paper we consider a multiply warped product metric of the generalized Friedmann-Robertson-Walker spacetime of type $\overline{M}=B\times_{h_1}F_1\times_{h_2}F_2$ with $\dim(B)=1$, the warping functions h_1,h_2 associated to the submanifolds F_1,F_2 with dimensions n_1,n_2, respectively and the submanifold F_1 is conformal to $(\mathbb{R}^{n_1},g)$, a pseudo-Euclidean space. A new way to study on generalized Friedmann-Robertson-Walker spacetime means we discuss the Einstein gravitational field tensors and the cosmological constant in generalized Friedmann-Robertson-Walker spacetime $(\overline{M},\overline{g})$ of type $\overline{M}=B\times_{h_1}F_1\times_{h_2}F_2$ equipped with the metric $\overline{g}=g_B\oplus {h_1}^2g_1\oplus {h_2}^2g_2$, where $g_1=\frac{g}{\varphi^2}$, g being the pseudo-Euclidean metric on $\mathbb{R}^{n_1}$ with respect to the co-ordinates $x=(x_1,x_2,...,x_{n_1})$, $g_{ij}=\delta_{ij}\varepsilon_i$ and $\varphi:\mathbb{R}^{n_1}\to\mathbb{R}$ is a smooth function.

We organize the paper as follows: in section 2, we recall some elementary notions about multiply warped product manifolds. In section 3, we compute the Ricci tensor of

(F_i, g_i) and Einstein gravitational field tensor of $(\overline{M}, \overline{g})$. Then we show that the Einstein equations $\overline{G}_{AB} = -\overline{\kappa}\,\overline{g}_{AB}$ on $(\overline{M}, \overline{g})$ with the cosmological constant $\overline{\kappa}$ is reducible to the Einstein equations $G_{ij} = -\kappa_2 g_{ij}$ on F_2 with the cosmological constant κ_2 such that $\overline{\kappa}, \kappa_2$ are in terms of h_1, h_2, n_1 and n_2. In section 4, we consider some black hole solutions as typical examples [25, 26]. Then we derive the corresponding Einstein equations and the reduced Einstein equations for each black hole solution.

2. Preliminaries

In this section, we recall some basic results for multiply warped product manifolds [19] which will be needed throughout the current work. Let f be a smooth function on a semi-Riemannian manifold (M, g) of dimension n. Then the Hessian of f is defined by $H^f(X, Y) = X(Yf) - (\nabla_X Y)f$ and Laplacian of f is defined by $\Delta f = \mathrm{trace}_g(H^f)$, or $\Delta = \mathrm{div}(\mathrm{grad})$, where grad, div and ∇ are the gradient, divergence and covariant derivative operators, respectively.

Proposition 2.1 *Let $M = B \times_{f_1} M_1 \times \ldots \times_{f_m} M_m$ be a pseudo-Riemannian multiply warped product endowed with the metric tensor $g = g_B \oplus f_1^2 g_{M_1} \oplus f_2^2 g_{M_2} \oplus \ldots \oplus f_m^2 g_{M_m}$ and also let $X, Y, Z \in \mathcal{L}(B)$ and $V \in \mathcal{L}(M_i)$, $W \in \mathcal{L}(M_j)$.* Then

$$\mathrm{Ric}(X, Y) = \mathrm{Ric}^B(X, Y) - \sum_{i=1}^{m} \left(\frac{n_i}{f_i}\right) H_B^{f_i}(X, Y), \tag{1}$$

$$\mathrm{Ric}(V, X) = 0, \tag{2}$$

$$\mathrm{Ric}(V, W) = 0; \quad \mathit{for}\; i \neq j, \tag{3}$$

$$\begin{aligned} \mathrm{Ric}(V, W) &= \mathrm{Ric}^{M_i}(V, W) \\ &\quad - \left[\frac{\Delta_B f_i}{f_i} + (n_i - 1)\frac{|\mathrm{grad}_B f_i|_B^2}{f_i^2} \right. \\ &\quad \left. + \sum_{k=1, k\neq i}^{m} n_k \frac{g_B(\mathrm{grad}_B f_i, \mathrm{grad}_B f_k)}{f_i f_k}\right] g(V, W); \\ &\quad \mathit{for}\; i = j, \end{aligned} \tag{4}$$

where $\mathrm{Ric}, \mathrm{Ric}^{\mathrm{B}}$ and $\mathrm{Ric}^{\mathrm{M_i}}$ are the Ricci curvature tensors of the metrics g, g_B and g_{M_i}, respectively.

Proposition 2.2 *Let $M = B \times_{f_1} M_1 \times \ldots \times_{f_m} M_m$ be a pseudo-Riemannian multiply warped product with the metric tensor $g = g_B \oplus f_1^2 g_{M_1} \oplus f_2^2 g_{M_2} \oplus \ldots \oplus f_m^2 g_{M_m}$. Then the scalar curvature S of (M, g) admits the following expressions*

$$\begin{aligned} S = &S^B - 2\sum_{i=1}^{m} n_i \frac{\Delta_B f_i}{f_i} + \sum_{i=1}^{m} \frac{S^{M_i}}{f_i^2} - \sum_{i=1}^{m} n_i(n_i - 1)\frac{|\mathrm{grad}_B f_i|_B^2}{f_i^2} \\ &- \sum_{i=1}^{m} \sum_{k=1, k\neq i}^{m} n_i n_k \frac{g_B(\mathrm{grad}_B f_i, \mathrm{grad}_B f_k)}{f_i f_k}, \end{aligned} \tag{5}$$

where S^B and S^{M_i} are the scalar curvatures of the metrics g_B and g_{M_i}, respectively.

3. Generalized Friedmann–Robertson–Walker Spacetime

The Friedmann-Robertson-Walker metric is an exact solution of the Einstein's field equations in four dimensional spacetime. It describes an isotropic, homogeneous, contracting or expanding universe which may be simply or multiply connected. This metric can be written in the following general form

$$\overline{g}(x^\alpha) = \varepsilon dt^2 + f^2(t) g_{ab}(x) dx^a dx^b, \tag{6}$$

where $a, b \in \{1, 2, 3\}$.

Definition 3.1 Let (F_1, g_1) and (F_2, g_2) be two Riemannian manifolds and B be a manifold of dimension one. Also, let $h_i : B \to (0, \infty)$, $i \in \{1, 2\}$ be smooth functions. The Lorentzian multiply warped product is the product manifold $\overline{M} = B \times F_1 \times F_2$ equipped with the metric $\overline{g}$ on $\overline{M}$ given by

$$\overline{g}(x^\alpha) = \varepsilon dt^2 + {h_1}^2(t) g_{ab}(x^\mu) dx^a dx^b + {h_2}^2(t) g_{ij}(x^k) dx^i dx^j \tag{7}$$

with the local components

$$\begin{aligned} \overline{g}_{00} &= \overline{g}(\partial_t, \partial_t) = \varepsilon,\; \overline{g}_{ab} = {h_1}^2(t) g_{1ab}(x^\mu), \overline{g}_{ij} \\ &= {h_2}^2(t) g_{2ij}(x^k),\; \overline{g}_{ia} = 0,\; \overline{g}_{0i} = 0, \end{aligned} \tag{8}$$

where $\varepsilon^2 = 1$, $(x^\mu), (x^k)$ and t are the co-ordinate systems on F_1, F_2 and B, respectively. It is also noted that $a, b \in \{1, 2, \ldots, n_1\}$, $i, j \in \{n_1 + 1, \ldots, n_1 + n_2\}$ and $\alpha \in \{1, \ldots, n_1 + n_2\}$. We use $\partial_t = \frac{\partial}{\partial t}, \partial_i = \frac{\partial}{\partial x^i}, \partial_a = \frac{\partial}{\partial x^a}$. We consider $h_1' = \frac{dh_1}{dt}, h_2' = \frac{dh_2}{dt}, A_1 = \frac{2h_1'}{h_1}, A_2 = \frac{2h_2'}{h_2}$.

Now we obtain the following results in terms of the Ricci tensor and scalar curvature of generalized Friedmann-Robertson-Walker spacetime $(\overline{M}, \overline{g})$ of type $\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2$ equipped with the metric $\overline{g} = g_B \oplus {h_1}^2 g_1 \oplus {h_2}^2 g_2$, where $g_1 = \frac{g}{\varphi^2}$, g being the pseudo-Euclidean metric on $\mathbb{R}^{n_1}$.

Proposition 3.2 *Let $(\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2, \overline{g})$ be a generalized Friedmann-Robertson-Walker spacetime. Then we have*

$$\overline{\mathrm{Ric}}(\partial_t, \partial_t) = -n_1\left(\frac{{A_1}^2}{4} + \frac{A_1'}{2}\right) - n_2\left(\frac{{A_2}^2}{4} + \frac{A_2'}{2}\right), \tag{9}$$

$$\begin{aligned}\overline{\mathrm{Ric}}(\partial_a, \partial_b) =& \frac{1}{\varphi}(n_1 - 2)H_g^{\phi}(\partial_a, \partial_b) - \overline{g}_{ab}\left[\varepsilon\left(\frac{{A_1}^2}{4} + \frac{A_1'}{2}\right)\right.\\ &\left. + (n_1 - 1)\varepsilon\frac{{A_1}^2}{4} + n_2\varepsilon\frac{A_1A_2}{4}\right];\ a \neq b,\end{aligned} \tag{10}$$

$$\begin{aligned}\overline{\mathrm{Ric}}(\partial_a, \partial_b) =& \frac{1}{\varphi}(n_1 - 2)H_g^{\phi}(\partial_a, \partial_a) + \frac{1}{\varphi}\varepsilon_a\Delta_g\varphi\\ & - \frac{1}{\varphi^2}(n_1 - 1)\varepsilon_a|\nabla_g\varphi|^2 - \overline{g}_{ab}\left[\varepsilon\left(\frac{{A_1}^2}{4} + \frac{A_1'}{2}\right)\right.\\ &\left. + (n_1 - 1)\varepsilon\frac{{A_1}^2}{4} + n_2\varepsilon\frac{A_1A_2}{4}\right];\ a = b,\end{aligned} \tag{11}$$

$$\begin{aligned}&\overline{\mathrm{Ric}}(\partial_i, \partial_j) = \mathrm{Ric}^{F_2}(\partial_i, \partial_j)\\ &- \overline{g}_{ij}\left[\varepsilon\left(\frac{{A_2}^2}{4} + \frac{A_1'}{2}\right) + (n_2 - 1)\varepsilon\frac{{A_2}^2}{4} + n_1\varepsilon\frac{A_1A_2}{4}\right],\end{aligned} \tag{12}$$

$$\overline{\mathrm{Ric}}(\partial_t, \partial_a) = 0, \tag{13}$$

$$\overline{\mathrm{Ric}}(\partial_a, \partial_i) = 0, \tag{14}$$

where local components of the Ricci tensor on (F_2, g_2) is $\mathrm{Ric}^{F_2}(\partial_i, \partial_j)$.

Proof Here $(\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2, \overline{g})$ be a generalized Friedmann-Robertson-Walker spacetime equipped with the metric $\overline{g} = g_B \oplus {h_1}^2 g_1 \oplus {h_2}^2 g_2$, where $g_1 = \frac{g}{\varphi^2}$, g being the pseudo-Euclidean metric on $\mathbb{R}^{n_1}$. In view of Proposition 2.1, we obtain

$$\begin{aligned}\overline{\mathrm{Ric}}(\partial_t, \partial_t) &= \mathrm{Ric}^B(\partial_t, \partial_t) - \sum_{i=1}^{2}\left(\frac{n_i}{h_i}\right)H_B^{h_i}(\partial_t, \partial_t)\\ &= -\left[\left(\frac{n_1}{h_1}\right)H_B^{h_1}(\partial_t, \partial_t) + \left(\frac{n_2}{h_2}\right)H_B^{h_2}(\partial_t, \partial_t)\right]\\ &= -\left[\left(\frac{n_1}{h_1}\right)\ddot{h_1} + \left(\frac{n_2}{h_2}\right)\ddot{h_2}\right];\ since\ H_B^{h_i} = \ddot{h_i}\\ &= -n_1\left(\frac{{A_1}^2}{4} + \frac{A_1'}{2}\right) - n_2\left(\frac{{A_2}^2}{4} + \frac{A_2'}{2}\right),\end{aligned} \tag{15}$$

$$\begin{aligned}\overline{\mathrm{Ric}}(\partial_a, \partial_b) &= \mathrm{Ric}^{F_1}(\partial_a, \partial_b) - \left[\frac{\Delta_B h_1}{h_1} + (n_1 - 1)\frac{|\mathrm{grad}_B h_1|_B^2}{{h_1}^2}\right.\\ &\left. + n_2\frac{g_B(\mathrm{grad}_B h_1, \mathrm{grad}_B h_2)}{h_1h_2}\right]\overline{g}(\partial_a, \partial_b)\\ &= \frac{1}{\varphi}(n_1 - 2)H_g^{\phi}(\partial_a, \partial_b) - \overline{g}_{ab}\left[\varepsilon\left(\frac{{A_1}^2}{4} + \frac{A_1'}{2}\right)\right.\\ &\left. + (n_1 - 1)\varepsilon\frac{{A_1}^2}{4} + n_2\varepsilon\frac{A_1A_2}{4}\right];\ a \neq b,\end{aligned} \tag{16}$$

$$\begin{aligned}\overline{\mathrm{Ric}}(\partial_a, \partial_b) &= \mathrm{Ric}^{F_1}(\partial_a, \partial_a) - \left[\frac{\Delta_B h_1}{h_1} + (n_1 - 1)\frac{|\mathrm{grad}_B h_1|_B^2}{{h_1}^2}\right.\\ &\left. + n_2\frac{g_B(\mathrm{grad}_B h_1, \mathrm{grad}_B h_2)}{h_1h_2}\right]\overline{g}(\partial_a, \partial_a)\\ &= \frac{1}{\varphi}(n_1 - 2)H_g^{\phi}(\partial_a, \partial_a) + \frac{1}{\varphi}\varepsilon_a\Delta_g\varphi\\ & - \frac{1}{\varphi^2}(n_1 - 1)\varepsilon_a|\nabla_g\varphi|^2 - \overline{g}_{aa}\left[\varepsilon\left(\frac{{A_1}^2}{4} + \frac{A_1'}{2}\right)\right.\\ &\left. + (n_1 - 1)\varepsilon\frac{{A_1}^2}{4} + n_2\varepsilon\frac{A_1A_2}{4}\right];\ a = b,\end{aligned} \tag{17}$$

$$\begin{aligned}\overline{\mathrm{Ric}}(\partial_i, \partial_j) &= \mathrm{Ric}^{F_2}(\partial_i, \partial_j) - \left[\frac{\Delta_B h_2}{h_2} + (n_2 - 1)\frac{|\mathrm{grad}_B h_2|_B^2}{{h_2}^2}\right.\\ &\left. + n_1\frac{g_B(\mathrm{grad}_B h_1, \mathrm{grad}_B h_2)}{h_1h_2}\right]\overline{g}(\partial_i, \partial_j)\\ &= \mathrm{Ric}^{F_2}(\partial_i, \partial_j)\\ &- \overline{g}_{ij}\left[\varepsilon\left(\frac{{A_2}^2}{4} + \frac{A_2'}{2}\right) + (n_2 - 1)\varepsilon\frac{{A_2}^2}{4} + n_1\varepsilon\frac{A_1A_2}{4}\right],\end{aligned} \tag{18}$$

$$\overline{\mathrm{Ric}}(\partial_t, \partial_a) = 0, \tag{19}$$

$$\overline{\mathrm{Ric}}(\partial_a, \partial_i) = 0. \tag{20}$$

This completes the proof. □

Proposition 3.3 *Let $(\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2, \overline{g})$be a generalized Friedmann-Robertson-Walker spacetime. Then the scalar curvature $\overline{S}$of $(\overline{M}, \overline{g})$have the following expression*

$$\begin{aligned}\overline{S} =& -2\left[n_1\left(\frac{{A_1}^2}{4} + \frac{A_1'}{2}\right) + n_2\left(\frac{{A_2}^2}{4} + \frac{A_2'}{2}\right)\right]\\ & + \frac{(n_1 - 1)}{{h_1}^2}\left[2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\right] + \frac{S^{F_2}}{{h_2}^2}\\ & - \left[n_1(n_1 - 1)\varepsilon\frac{{A_1}^2}{4} + n_2(n_2 - 1)\varepsilon\frac{{A_2}^2}{4}\right]\\ & - n_1n_2\varepsilon\frac{A_1A_2}{4}.\end{aligned} \tag{21}$$

Proof To prove this Proposition 3.3, we use Proposition 2.2 and it follows that

$$\overline{S} = S^B - 2\sum_{i=1}^{2} n_i\left(\frac{\Delta_B h_i}{h_i}\right) + \sum_{i=1}^{2}\frac{S^{F_i}}{{h_i}^2} - \sum_{i=1}^{2} n_i(n_i-1)\frac{|\mathrm{grad}_B h_i|_B^2}{{h_i}^2} - \sum_{i=1}^{2}\sum_{k=1,k\neq i}^{2} n_i n_k \frac{g_B(\mathrm{grad}_B h_i, \mathrm{grad}_B h_k)}{h_i h_k},$$

where S^B and S^{F_i} denote the scalar curvatures of the metrics g_B and g_i, respectively.

This implies that

$$\overline{S} = -2\left[n_1\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) + n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)\right] + \frac{S^{F_1}}{h_1^2} + \frac{S^{F_2}}{h_2^2} - \left[n_1(n_1-1)\varepsilon\frac{{A_1}^2}{4} + n_2(n_2-1)\varepsilon\frac{{A_2}^2}{4}\right] - n_1 n_2 \varepsilon\frac{A_1A_2}{4}.$$

Now we know that from [17],

$$\mathrm{Ric}^{F_1} = \frac{1}{\varphi}[(n_1-2)H_g^{\varphi}(X_i,X_j)];\ i\neq j,\ i,j\in\{1,2,\ldots,n_1\},$$

$$\mathrm{Ric}^{F_1} = \frac{1}{\varphi^2}[(n_1-2)\varphi H_g^{\varphi}(X_i,X_i) + \{\varphi\Delta_g\varphi - (n_1-1)|\nabla_g\varphi|^2\}]\varepsilon_i;\ i=j.$$

Taking trace on both sides of the above equation, we obtain

$$\begin{aligned}
S^{F_1} &= \sum_{i=1}^{n_1} g_1^{ii}\mathrm{Ric}_{g_{1_{ii}}} \\
&= \sum_{i=1}^{n_1} g_1^{ii}\mathrm{Ric}_{g_1}(\varphi X_i, \varphi X_i) \\
&= \sum_{i=1}^{n_1} \varepsilon_i\varphi^2\mathrm{Ric}_{g_1}(X_i,X_i) \\
&= \sum_{i=1}^{n_1} \varepsilon_i\Big[(n_1-2)\varphi H_g^{\varphi}(X_i,X_i) \\
&\qquad + \{\varphi\Delta_g\varphi - (n_1-1)|\nabla_g\varphi|^2\}g(X_i,X_i)\Big] \\
&= (n_1-2)\varphi\sum_{i=1}^{n_1}\varepsilon_i H_g^{\varphi}(X_i,X_i) \\
&\qquad + \{\varphi\Delta_g\varphi - (n_1-1)|\nabla_g\varphi|^2\}\sum_{i=1}^{n_1}\varepsilon_i^2\delta_{ii} \\
&= (n_1-2)\varphi\sum_{i=1}^{n_1} g^{ii}H_{g,_{ii}}^{\varphi} + \{\varphi\Delta_g\varphi - (n_1-1)|\nabla_g\varphi|^2\}\sum_{i=1}^{n_1}\varepsilon_i^2 \\
&= (n_1-2)\varphi\, tr(H_g^{\varphi}) + n_1\{\varphi\Delta_g\varphi - (n_1-1)|\nabla_g\varphi|^2\} \\
&= (n_1-2)\varphi\Delta_g\varphi + n_1\{\varphi\Delta_g\varphi - (n_1-1)|\nabla_g\varphi|^2\} \\
&= 2(n_1-1)\varphi\Delta_g\varphi - n_1(n_1-1)|\nabla_g\varphi|^2.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\overline{S} &= -2\left[n_1\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) + n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)\right] \\
&\quad + \frac{(n_1-1)}{{h_1}^2}\left[2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\right] \\
&\quad + \frac{S^{F_2}}{{h_2}^2} - \left[n_1(n_1-1)\varepsilon\frac{{A_1}^2}{4} + n_2(n_2-1)\varepsilon\frac{{A_2}^2}{4}\right] \\
&\quad - n_1n_2\varepsilon\frac{A_1A_2}{4}.
\end{aligned}$$

This completes the proof. □

Proposition 3.4 *Let* $(\overline{M} = B\times_{h_1}F_1\times_{h_2}F_2, \overline{g})$*be a generalized Friedmann-Robertson-Walker spacetime and* $\overline{G}$*be its Einstein gravitational tensor field. Then we have the following equations*

$$\begin{aligned}
\overline{G}_{00} &= -\frac{(n_1-1)\varepsilon}{2{h_1}^2}[2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2] \\
&\quad - \frac{\varepsilon S^{F_2}}{2{h_2}^2} - \frac{n_1}{2}(3-2\varepsilon-n_1)\frac{{A_1}^2}{4} \\
&\quad - \frac{n_2}{2}(3-2\varepsilon-n_2)\frac{{A_2}^2}{4} - n_1(1-\varepsilon)\frac{A_1'}{2} \\
&\quad - n_2(1-\varepsilon)\frac{A_2'}{2} \\
&\quad + \frac{n_1n_2}{2}\frac{A_1A_2}{4},
\end{aligned} \tag{22}$$

$$\overline{G}_{a0} = 0,\ \overline{G}_{i0} = 0,\ \overline{G}_{ia} = 0, \tag{23}$$

$$\begin{aligned}
\overline{G}_{ab} &= \frac{1}{\varphi}(n_1-2)H_g^{\varphi}(\partial_a,\partial_b) \\
&\quad + \overline{g}_{ab}\Bigg[-\frac{(n_1-1)}{2{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\} - \frac{S^{F_2}}{2{h_2}^2} \\
&\quad + (n_1-\varepsilon)\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) + n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right) \\
&\quad + \frac{\varepsilon(n_1-1)(n_1-2)}{2}\frac{{A_1}^2}{4} \\
&\quad + \frac{\varepsilon n_2(n_2-1)}{2}\frac{{A_2}^2}{4} + \frac{\varepsilon n_2(n_1-2)}{2}\frac{A_1A_2}{4}\Bigg];\ a\neq b,
\end{aligned} \tag{24}$$

$$\begin{aligned}
\overline{G}_{ab} &= \frac{1}{\varphi}(n_1-2)H_g^{\varphi}(\partial_a,\partial_a) + \frac{1}{\varphi}\varepsilon_a\Delta_g\varphi - \frac{(n_1-1)\varepsilon_a}{\varphi^2}|\nabla_g\varphi|^2 \\
&\quad + \overline{g}_{aa}\Bigg[-\frac{(n_1-1)}{2{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\} - \frac{S^{F_2}}{2{h_2}^2} \\
&\quad + (n_1-\varepsilon)\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) + n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right) \\
&\quad + \frac{\varepsilon(n_1-1)(n_1-2)}{2}\frac{{A_1}^2}{4} \\
&\quad + \frac{\varepsilon n_2(n_2-1)}{2}\frac{{A_2}^2}{4} + \frac{\varepsilon n_2(n_1-2)}{2}\frac{A_1A_2}{4}\Bigg];\ a=b,
\end{aligned} \tag{25}$$

$$\overline{G}_{ij} = G_{ij} + \overline{g}_{ij}\Bigg[-\frac{(n_1-1)}{2{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\} + n_1\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) + (n_2-\varepsilon)\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right) + \frac{\varepsilon n_1(n_1-1)}{2}\frac{{A_1}^2}{4} + \frac{\varepsilon(n_2-1)(n_2-2)}{2}\frac{{A_2}^2}{4} + \frac{\varepsilon n_1(n_2-2)}{2}\frac{A_1A_2}{4}\Bigg], \tag{26}$$

where G_{ab} and G_{ij} are the local components of Einstein gravitational tensor field G of (F_1, g_1) and (F_2, g_2), respectively.

Proof We know that the Einstein gravitational tensor field $\overline{G}$ of $(\overline{M}, \overline{g})$ is given by

$$\overline{G} = \overline{\mathrm{Ric}} - \frac{1}{2}\overline{S}\overline{g}.$$

Using this equation, we get

$$\begin{aligned}\overline{G}_{00} =& \overline{\mathrm{Ric}}(\partial_t,\partial_t) - \frac{1}{2}\overline{S}\overline{g}_{00}\\ =& -\left[n_1\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) + n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)\right]\\ &-\frac{1}{2}\Bigg[-2n_1\varepsilon\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) - 2n_2\varepsilon\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)\\ &+\frac{(n_1-1)\varepsilon}{h_1^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\} + \frac{\varepsilon S^{F_2}}{h_2^2}\\ &-n_1(n_1-1)\frac{{A_1}^2}{4} - n_2(n_2-1)\frac{{A_2}^2}{4} - n_1n_2\frac{A_1A_2}{4}\Bigg]\\ =& -\frac{(n_1-1)\varepsilon}{2{h_1}^2}[2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2]\\ &-\frac{\varepsilon S^{F_2}}{2{h_2}^2} - \frac{n_1}{2}(3-2\varepsilon-n_1)\frac{{A_1}^2}{4}\\ &-\frac{n_2}{2}(3-2\varepsilon-n_2)\frac{{A_2}^2}{4} - n_1(1-\varepsilon)\frac{A_1'}{2}\\ &-n_2(1-\varepsilon)\frac{A_2'}{2}\\ &+\frac{n_1n_2}{2}\frac{A_1A_2}{4},\end{aligned} \tag{27}$$

$$\overline{G}_{a0} = 0,\ \overline{G}_{i0} = 0,\ \overline{G}_{ia} = 0, \tag{28}$$

$$\begin{aligned}\overline{G}_{ab} =& \overline{\mathrm{Ric}}(\partial_a,\partial_b) - \frac{1}{2}\overline{S}\overline{g}_{ab};\ a\neq b\\ =& \frac{1}{\varphi}(n_1-2)H_g^{\varphi}(\partial_a,\partial_b) - \overline{g}_{ab}\Bigg[\varepsilon\Bigg(\frac{{A_1}^2}{4}\\ &+\frac{A_1'}{2}\Bigg) + (n_1-1)\varepsilon\frac{{A_1}^2}{4}\\ &+n_2\varepsilon\frac{A_1A_2}{4}\Bigg] - \frac{1}{2}\overline{g}_{ab}\Bigg[-2n_1\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right)\\ &-2n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)\\ &+\frac{(n_1-1)}{{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\} + \frac{S^{F_2}}{h_2^2}\\ &-n_1(n_1-1)\varepsilon\frac{{A_1}^2}{4} - n_2(n_2-1)\varepsilon\frac{{A_2}^2}{4}\\ &-n_1n_2\varepsilon\frac{A_1A_2}{4}\Bigg];\ a\neq b\\ =& \frac{1}{\varphi}(n_1-2)H_g^{\varphi}(\partial_a,\partial_b)\\ &+\overline{g}_{ab}\Bigg[-\frac{(n_1-1)}{2{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\} - \frac{S^{F_2}}{2{h_2}^2}\\ &+(n_1-\varepsilon)\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) + n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)\\ &+\frac{\varepsilon(n_1-1)(n_1-2)}{2}\frac{{A_1}^2}{4}\\ &+\frac{\varepsilon n_2(n_2-1)}{2}\frac{{A_2}^2}{4} + \frac{\varepsilon n_2(n_1-2)}{2}\frac{A_1A_2}{4}\Bigg];\ a\neq b,\end{aligned} \tag{29}$$

$$\begin{aligned}\overline{G}_{ab} =& \overline{\mathrm{Ric}}(\partial_a,\partial_a) - \frac{1}{2}\overline{S}\overline{g}_{aa};\ a=b\\ =& \frac{1}{\varphi}(n_1-2)H_g^{\varphi}(\partial_a,\partial_a) + \frac{1}{\varphi}\varepsilon_a\Delta_g\varphi - \frac{(n_1-1)\varepsilon_a}{\varphi^2}|\nabla_g\varphi|^2\\ &-\overline{g}_{aa}\left[\varepsilon\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) + (n_1-1)\varepsilon\frac{{A_1}^2}{4} + n_2\varepsilon\frac{A_1A_2}{4}\right]\\ &-\frac{1}{2}\overline{g}_{aa}\Bigg[-2n_1\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) - 2n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)\\ &+\frac{(n_1-1)}{{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\} + \frac{S^{F_2}}{h_2^2}\\ &-n_1(n_1-1)\varepsilon\frac{{A_1}^2}{4} - n_2(n_2-1)\varepsilon\frac{{A_2}^2}{4} - n_1n_2\varepsilon\frac{A_1A_2}{4}\Bigg];\ a=b\\ =& \frac{1}{\varphi}(n_1-2)H_g^{\varphi}(\partial_a,\partial_a) + \frac{1}{\varphi}\varepsilon_a\Delta_g\varphi - \frac{(n_1-1)\varepsilon_a}{\varphi^2}|\nabla_g\varphi|^2\\ &+\overline{g}_{aa}\Bigg[-\frac{(n_1-1)}{2{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\} - \frac{S^{F_2}}{2{h_2}^2}\\ &+(n_1-\varepsilon)\left(\frac{{A_1}^2}{4}+\frac{A_1'}{2}\right) + n_2\left(\frac{{A_2}^2}{4}+\frac{A_2'}{2}\right)\\ &+\frac{\varepsilon(n_1-1)(n_1-2)}{2}\frac{{A_1}^2}{4}\\ &+\frac{\varepsilon n_2(n_2-1)}{2}\frac{{A_2}^2}{4} + \frac{\varepsilon n_2(n_1-2)}{2}\frac{A_1A_2}{4}\Bigg];\ a=b,\end{aligned} \tag{30}$$

$$\overline{G}_{ij} = \overline{\mathrm{Ric}}(\partial_i, \partial_j) - \frac{1}{2}\overline{S}\overline{g}_{ij}$$
$$= \mathrm{Ric}^{F_2}(\partial_i, \partial_j) - \overline{g}_{ij}\Big[\varepsilon\Big(\frac{{A_2}^2}{4} + \frac{A_2'}{2}\Big)$$
$$+ (n_2 - 1)\varepsilon\frac{{A_2}^2}{4} + n_1\varepsilon\frac{A_1A_2}{4}\Big]$$
$$-\frac{1}{2}\overline{g}_{ij}\Big[-2n_1\Big(\frac{{A_1}^2}{4} + \frac{A_1'}{2}\Big) - 2n_2\Big(\frac{{A_2}^2}{4} + \frac{A_2'}{2}\Big)$$
$$+ \frac{(n_1 - 1)}{{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\} + \frac{S^{F_2}}{h_2^2} - n_1n_2\varepsilon\frac{A_1A_2}{4}$$
$$-n_1(n_1 - 1)\varepsilon\frac{{A_1}^2}{4} - n_2(n_2 - 1)\varepsilon\frac{{A_2}^2}{4}\Big]$$
$$= \mathrm{Ric}^{F_2}(\partial_i, \partial_j) - \frac{1}{2}S^{F_2}g_{2ij}$$
$$+ \overline{g}_{ij}\Big[-\frac{(n_1 - 1)}{2{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\} + n_1\Big(\frac{{A_1}^2}{4} + \frac{A_1'}{2}\Big)$$
$$+ (n_2 - \varepsilon)\Big(\frac{{A_2}^2}{4} + \frac{A_2'}{2}\Big)$$
$$+ \frac{\varepsilon n_1(n_1 - 1)}{2}\frac{{A_1}^2}{4} + \frac{\varepsilon(n_2 - 1)(n_2 - 2)}{2}\frac{{A_2}^2}{4}$$
$$+ \frac{\varepsilon n_1(n_2 - 2)}{2}\frac{A_1A_2}{4}\Big]$$
$$= G_{ij} + \overline{g}_{ij}\Big[-\frac{(n_1 - 1)}{2{h_1}^2}\{2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\}$$
$$+ n_1\Big(\frac{{A_1}^2}{4} + \frac{A_1'}{2}\Big)$$
$$+ (n_2 - \varepsilon)\Big(\frac{{A_2}^2}{4} + \frac{A_2'}{2}\Big)$$
$$+ \frac{\varepsilon n_1(n_1 - 1)}{2}\frac{{A_1}^2}{4} + \frac{\varepsilon(n_2 - 1)(n_2 - 2)}{2}\frac{{A_2}^2}{4}$$
$$+ \frac{\varepsilon n_1(n_2 - 2)}{2}\frac{A_1A_2}{4}\Big]. \tag{31}$$

This completes the proof. □

Proposition 3.5 *The Einstein equations in generalized Friedmann-Robertson-Walker spacetime* $(\overline{M}, \overline{g})$*with the cosmological constant* $\overline{\kappa}$ *are equivalent to the following reduced Einstein equations*

$$\begin{aligned}\overline{\kappa} = {} & \frac{(n_1 - 1)}{2{h_1}^2}\Big[2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\Big] \\ & - \frac{\varepsilon n_1(n_1 + n_2 + 2\varepsilon - 3)}{2}\frac{{A_1}^2}{4} \\ & - \frac{\varepsilon n_2(n_2 + 2\varepsilon - 3)}{2}\frac{{A_2}^2}{4} + \frac{\varepsilon n_1(2 - 2\varepsilon - n_2)}{2}\frac{A_1'}{2} \\ & + \frac{\varepsilon n_2(3 - 2\varepsilon - n_2)}{2}\frac{A_2'}{2},\end{aligned} \tag{32}$$

$$G_{ij} = \varepsilon\overline{g}_{ij}\Big(\frac{n_2}{2} - 1\Big)\Big[n_1\frac{{A_1}^2}{4} + n_1\frac{A_1'}{2} + (n_2 - 1)\frac{A_2'}{2} - n_1\frac{A_1A_2}{4}\Big]. \tag{33}$$

Proof Using the equation (22) and $\overline{G} = -\overline{\kappa}\,\overline{g}$, we obtain

$$\begin{aligned}\overline{\kappa} = {} & \frac{(n_1 - 1)}{2{h_1}^2}\Big[2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\Big] + \frac{S^{F_2}}{2{h_2}^2} \\ & - \frac{\varepsilon n_1(2\varepsilon + n_1 - 3)}{2}\frac{{A_1}^2}{4} \\ & - \frac{\varepsilon n_2(2\varepsilon + n_2 - 3)}{2}\frac{{A_2}^2}{4} + n_1\varepsilon(1 - \varepsilon)\frac{A_1'}{2} \\ & + n_2\varepsilon(1 - \varepsilon)\frac{A_2'}{2} - \frac{\varepsilon n_1n_2}{2}\frac{A_1A_2}{4}.\end{aligned} \tag{34}$$

Again by using the equation (26), the Einstein equation $\overline{G} = -\overline{\kappa}\,\overline{g}$ and the equation (34), we get

$$\begin{aligned}G_{ij} = {} & -\overline{g}_{ij}\Big[\frac{S^{F_2}}{2h_2^2} + n_1\varepsilon\frac{{A_1}^2}{4} + n_1\varepsilon\frac{A_1'}{2} \\ & + \varepsilon(n_2 - 1)\frac{A_2'}{2} - n_1\varepsilon\frac{A_1A_2}{4}\Big].\end{aligned} \tag{35}$$

Now contracting the equation (35) with g^{ij}, we have

$$\begin{aligned}\frac{S^{F_2}}{h_2^2} = {} & n_1n_2\varepsilon\frac{A_1A_2}{4} - \varepsilon n_1n_2\frac{{A_1}^2}{4} - \varepsilon n_1n_2\frac{A_1'}{2} \\ & - \varepsilon n_2(n_2 - 1)\frac{A_2'}{2}.\end{aligned} \tag{36}$$

Hence from the equations (35) and (36), we obtain

$$G_{ij} = \varepsilon\overline{g}_{ij}\Big(\frac{n_2}{2} - 1\Big)\Big[n_1\frac{{A_1}^2}{4} + n_1\frac{A_1'}{2} + (n_2 - 1)\frac{A_2'}{2} - n_1\frac{A_1A_2}{4}\Big]. \tag{37}$$

Using the equation (36) in the equation (34), we get

$$\begin{aligned}\overline{\kappa} = {} & \frac{(n_1 - 1)}{2{h_1}^2}\Big[2\varphi\Delta_g\varphi - n_1|\nabla_g\varphi|^2\Big] \\ & - \frac{\varepsilon n_1(n_1 + n_2 + 2\varepsilon - 3)}{2}\frac{{A_1}^2}{4} \\ & - \frac{\varepsilon n_2(n_2 + 2\varepsilon - 3)}{2}\frac{{A_2}^2}{4} + \frac{\varepsilon n_1(2 - 2\varepsilon - n_2)}{2}\frac{A_1'}{2} \\ & + \frac{\varepsilon n_2(3 - 2\varepsilon - n_2)}{2}\frac{A_2'}{2}.\end{aligned} \tag{38}$$

This completes the proof. □

Proposition 3.6 *The Einstein equations* $\overline{G} = -\overline{\kappa}\,\overline{g}$ *on* $(\overline{M}, \overline{g})$ *with the cosmological constant* $\overline{\kappa}$ *induce the Einstein equations* $G_{ij} = -\kappa_2 g_{2ij}$ *on* (F_2, g_2), *where* κ_2*is given by*

$$\kappa_2 = -\varepsilon{h_2}^2\Big(\frac{n_2}{2} - 1\Big)\Big[n_1\frac{{A_1}^2}{4} + n_1\frac{A_1'}{2} + (n_2 - 1)\frac{A_2'}{2} - n_1\frac{A_1A_2}{4}\Big].$$

Proof By using the equations (8) and (33), we get $G_{ij} =$

$-\kappa_2 g_{2ij}$ on (F_2, g_2), where the cosmological constant κ_2 is given by

$$\kappa_2 = -\varepsilon {h_2}^2\left(\frac{n_2}{2}-1\right)\left[n_1\frac{{A_1}^2}{4}+n_1\frac{A_1'}{2}+(n_2-1)\frac{A_2'}{2}-n_1\frac{A_1A_2}{4}\right]. \tag{39}$$

□

Note 3. One can also study the generalized Friedmann-Robertson-Walker spacetime $(\overline{M}, \overline{g})$ of type $\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2$ equipped with the metric $\overline{g} = g_B \oplus {h_1}^2 g_1 \oplus {h_2}^2 g_2$, where $g_2 = \frac{g}{\varphi^2}$, g being the pseudo-Euclidean metric on $\mathbb{R}^{n_2}$ and can compute the Ricci tensor of (F_i, g_i) and Einstein gravitational field tensor of $(\overline{M}, \overline{g})$. After similar calculations we find out the following results for the cosmological constants of Einstein equations.

Proposition 3.7 *The Einstein equations $\overline{G}_{AB} = -\overline{\kappa}\,\overline{g}_{AB}$ on $(\overline{M}, \overline{g})$ with the cosmological constant $\overline{\kappa}$ induce the Einstein equations $G_{ab} = -\kappa_1 g_{1ab}$ on* (F_1, g_1), where $(F_1, g_1)\overline{\kappa}$ *and κ_1 are given by*

$$\begin{aligned}\overline{\kappa} = &\frac{(n_2-1)}{2{h_2}^2}\left[2\varphi\Delta_g\varphi - n_2|\nabla_g\varphi|^2\right] - \frac{\varepsilon n_2(n_1+n_2+2\varepsilon-3)}{2}\frac{{A_2}^2}{4}\\ &- \frac{\varepsilon n_1(n_1+2\varepsilon-3)}{2}\frac{{A_1}^2}{4} + \frac{\varepsilon n_2(2-2\varepsilon-n_1)}{2}\frac{A_2'}{2}\\ &+ \frac{\varepsilon n_1(3-2\varepsilon-n_1)}{2}\frac{A_1'}{2},\end{aligned} \tag{40}$$

$$\kappa_1 = -\varepsilon {h_1}^2\left(\frac{n_1}{2}-1\right)\left[n_2\frac{{A_2}^2}{4}+n_2\frac{A_2'}{2}+(n_1-1)\frac{A_1'}{2} - n_2\frac{A_1A_2}{4}\right]. \tag{41}$$

Proof Similar as Proposition 3.6.

4. Example of generalized black holes

Using the above mentioned Proposition 3.7, we wish to show some examples of the generalized black hole solutions whose metrics can be written as a multiply warped product metric of the generalized Friedmann-Robertson-Walker spacetime $(\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2, \overline{g})$, where F_2 is conformal to the pseudo-Euclidean space $\mathbb{R}^{n_2}$. Then we reduce the Einstein equations $\overline{G}_{AB} = -\overline{\kappa}\,\overline{g}_{AB}$ into $G_{ab} = -\kappa_1 g_{1ab}$ by considering an n-dimensional Schwarzschild black hole and an n-dimensional Reissner-Nördstrom black hole.

4.1. *n*-dimensional Schwarzschild black hole

The metric of a Schwarzschild black hole [25] of dimension n is given by

$$ds^2 = -p(r)dt^2 + p(r)^{-1}dr^2 + r^2 d\Omega^2_{n-2}, \tag{42}$$

where $p(r) = \left(1 - \frac{m}{r^{n-3}}\right)$, $d\Omega^2_{n-2} = \frac{(2\pi)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(z+1) = z\Gamma(z)$ and the geometric mass m indicates for the radius of horizon. Then this may be expressed [22] as a multiply warped product $\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2$ of dimension n equipped with the metric

$$ds^2 = -d\mu^2 + {h_1}^2(\mu)dt^2 + {h_2}^2(\mu)d\Omega^2_{n-2}, \tag{43}$$

where

$$h_1(\mu) = \sqrt{\frac{m}{(F^{-1}(\mu))^{n-3}} - 1},$$

$$h_2(\mu) = F^{-1}(\mu).$$

We consider F_2 is conformal to an $(n-2)$-dimensional pseudo-Euclidean space $(\mathbb{R}^{n-2}, g)$. Then $d\Omega^2_{n-2} = \frac{1}{\varphi^2}d\Phi^2_{n-2}$, where $d\Phi^2_{n-2}$ is the pseudo-Euclidean metric and $\varphi : \mathbb{R}^{n-2} \to \mathbb{R}$ is a smooth function.

The existence of the above functions $h_1(\mu)$ and $h_2(\mu)$ guarantees the reduction of Einstein equations $\overline{G}_{AB} = -\overline{\kappa}\,\overline{g}_{AB}$ into $G_{ab} = -\kappa_1 g_{1ab}$, where $\overline{\kappa}$ and κ_1 are the cosmological constants subject to the set of coupled differential equations (40) and (41) by the substitution of t by μ.

4.2. *n*-dimensional Reissner-Nördstrom black hole

The metric of a Reissner-Nördstrom black hole of dimension n (≥ 4) is given by

$$ds^2 = -p(r)dt^2 + p(r)^{-1}dr^2 + r^2 d\Omega^2_{n-2}, \tag{44}$$

where $p(r) = \left(1 - \frac{m}{r^{n-3}} + \frac{q}{r^{2(n-3)}}\right)$; m and q are the geometric mass and charge of the black hole, respectively, and $d\Omega_{n-2} = \frac{2\pi}{\Gamma\left(\frac{n-1}{2}\right)}$.

Then equation (44) can be written as an n-dimensional multiply warped product metric of the generalized Friedmann-Robertson-Walker spacetime $(\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2, \overline{g})$ furnished with the metric [22

$$ds^2 = -d\mu^2 + {h_1}^2(\mu)dt^2 + {h_2}^2(\mu)d\Omega^2_{n-2}, \tag{45}$$

where

$$h_1(\mu) = \sqrt{\frac{m}{(F^{-1}(\mu))^{n-3}} - \frac{q}{(F^{-1}(\mu))^{2n-6}} - 1},$$

$$h_2(\mu) = F^{-1}(\mu)$$

with

$$\mu = \int_{r_-}^{r} \sqrt{-p(r)^{-1}}\, dr = F(r), \ (say)$$
$$i.e., r = F^{-1}(\mu). \tag{46}$$

We consider F_2 is conformal to an $(n-2)$-dimensional pseudo-Euclidean space $(\mathbb{R}^{n-2}, g)$. Then $d\Omega_{n-2}^2 = \frac{1}{\varphi^2} d\Phi_{n-2}^2$, where $d\Phi_{n-2}^2$ is the pseudo-Euclidean metric and $\varphi : \mathbb{R}^{n-2} \to \mathbb{R}$ is a smooth function.

The existence of the above functions $h_1(\mu)$ and $h_2(\mu)$ guarantees the reduction of Einstein equations $\overline{G}_{AB} = -\overline{\kappa}\,\overline{g}_{AB}$ into $G_{ab} = -\kappa_1 g_{1ab}$, where $\overline{\kappa}$ and κ_1 are the cosmological constants subject to the set of coupled differential equations (40) and (41) by the substitution of t by μ.

5. Conclusions

One can also investigate the above singular metrics of n-dimensional Schwarzschild black hole and Reissner-Nördstrom black hole in view of the lightlike warped product [27]. Let us consider the n-dimensional Schwarzschild black hole metric given in (42) with respect to the coordinate system $(t, r, x^1, x^2, \ldots, x^{n-2})$ on $(\overline{M} = B \times_{h_1} F_1 \times_{h_2} F_2, \overline{g})$. Let u and v be two null coordinates such that $u = t + r$ and $v = t - r$. Then the metric given in (42) transforms into the metric

$$ds^2 = \frac{1}{4p(r)}[1 - p(r)^2][du^2 + dv^2] - 2[1 + p(r)^2]dudv + \frac{1}{4}(u - v)^2 d\Omega_{n-2}^2. \tag{47}$$

Clearly if we consider the condition $p(r) = 1$ then the metric given in (47) becomes

$$ds^2 = -4dudv + \frac{1}{4}(u - v)^2 d\Omega_{n-2}^2. \tag{48}$$

Hence the absence of the terms du^2 and dv^2 in (48) implies that u and v are all constants. Hence u and v are lightlike hypersurfaces of $\overline{M}$. Therefore, according to [27], it is possible to construct a lightlike warped product manifold. Then one can also do the further calculations in a similar way.

We obtain the same result for the n-dimensional Reissner-Nördstrom black hole.

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APPLICATION OF $\mathcal{T}$-CURVATURE TENSOR IN SPACETIMES

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Abstract. In this paper we show that $\mathcal{T}$-flat spacetime is Einstein with constant curvature and the energy momentum tensor of this spacetime satisfying the Einstein's field equation with the cosmological constant is covariant constant. Then we find the length of the Ricci operator and derive some geometric properties for a $\mathcal{T}$-flat general relativistic viscous fluid spacetime. We also see that for a purely electromagnetic distribution the scalar curvature of a $\mathcal{T}$-flat spacetime satisfying the Einstein's field equation without cosmological constant vanishes. Lastly we study the general relativistic viscous fluid spacetime with the divergence-free $\mathcal{T}$-curvature tensor with respect to some conditions and the possible local cosmological structure is of Petrov type I, D or O.

1. Introduction

This paper is dealt with some investigations in the theory of general relativity with respect to the coordinate vanishing method in differential geometry. In this type of study a spacetime of general relativity is considered like a connected pseudo-Riemannian manifold of dimension four equipped with the Lorentzian metric g having signature (-, +, +, +). The field equation of Einstein [3] follows that the energy momentum tensor is of divergence free. If the energy momentum tensor is covariant constant then this demand is fulfilled. Chaki and Roy [11] had proved that a general relativistic spacetime admitting the covariant constant energy momentum tensor is Ricci symmetric. Many authors [13, 16, 5, 18, 17] had studied spacetimes in different

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ways on different manifolds and different curvature tensors.
Let (M, g) be an n-dimensional pseudo-Riemannian manifold and $\mathfrak{X}(M)$ be the Lie algebra of vector fields in M. We consider $X, Y, Z, W \in \mathfrak{X}(M)$ throughout the entire study.

Definition 1.1. A pseudo-Riemannian manifold (M, g) is a differentiable manifold M equipped with an everywhere non-degenerate, smooth, symmetric metric tensor g.

Tripathi and Gupta [12] had developed the notion of $\mathcal{T}$- curvature tensor in pseudo-Riemannian manifolds. They defined $\mathcal{T}$- curvature tensor as follows.

Definition 1.2. In an n-dimensional pseudo-Riemannian manifold (M, g), a $\mathcal{T}$- curvature tensor is a tensor of type $(1, 3)$ defined by

$$
\begin{aligned}
\mathcal{T}(X,Y)Z &= c_0 R(X,Y)Z \\
&\quad + c_1 S(Y,Z)X + c_2 S(X,Z)Y + c_3 S(X,Y)Z \\
&\quad + c_4 g(Y,Z)QX + c_5 g(X,Z)QY + c_6 g(X,Y)QZ \\
&\quad + r c_7 \left[g(Y,Z)X - g(X,Z)Y \right],
\end{aligned} \tag{1.1}
$$

where $X, Y, Z \in \mathfrak{X}(M)$; $c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7$ are smooth functions on M; S, Q, R, r, g are respectively the Ricci tensor, Ricci operator, curvature tensor, scalar curvature and pseudo-Riemannian metric tensor.

Definition 1.3. The Riemannian curvature tensor R of type $(0, 4)$ on M is a quadrilinear mapping $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ defined by $R(X, Y, Z, W) = g(R(X, Y)Z, W)$ for any $X, Y, Z, W \in \mathfrak{X}(M)$.

$\mathcal{T}$-curvature tensor reduces to many other curvature tensors for different values of $c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7$.

Definition 1.4. A $\mathcal{T}$-curvature tensor of type $(0,4)$ is defined by

$$\begin{aligned}\mathcal{T}(X,Y,Z,W) &= c_0R(X,Y,Z,W)\\ &\quad+c_1S(Y,Z)g(X,W)+c_2S(X,Z)g(Y,W)\\ &\quad+c_3S(X,Y)g(Z,W)+c_4g(Y,Z)S(X,W)\\ &\quad+c_5g(X,Z)S(Y,W)+c_6g(X,Y)S(Z,W)\\ &\quad+rc_7\left[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\right],\end{aligned} \tag{1.2}$$

where $X,Y,Z,W\in\mathfrak{X}(M)$, R is the Riemannian curvature tensor, S is the Ricci tensor, g is the pseudo-Riemannian metric tensor and $\mathcal{T}(X,Y,Z,W)=g\left(\mathcal{T}(X,Y)Z,W\right)$.

Definition 1.5. A spacetime is called an Einstein spacetime if the Ricci tensor S of type $(0,2)$ satisfies the relation $S=\frac{r}{n}$, $n>2$ on M where r is the scalar curvature of (M^n,g).

Definition 1.6. A spacetime is called $\mathcal{T}$-flat if the $\mathcal{T}$-curvature tensor of type $(0,4)$ satisfies the relation $\mathcal{T}(X,Y,Z,W)=0$ on M for any $X,Y,Z,W\in\mathfrak{X}(M)$.

Definition 1.7. A spacetime is called a spacetime with constant curvature if the curvature tensor satisfies the relation $R(X,Y,Z,W)=g(X,Z)g(Y,W)-g(X,W)g(Y,Z)$ on M for any $X,Y,Z,W\in\mathfrak{X}(M)$.

Definition 1.8. If a spacetime M admits a symmetry then it is said to be a curvature collineation (CC) [8, 9, 6] if

$$\left(\pounds_\xi R\right)(X,Y)Z = 0, \tag{1.3}$$

where R is the Riemannian curvature tensor.

Definition 1.9. The vector field ξ is said to be a Killing vector field if it satisfies the relation $\left(\pounds_\xi g\right)(X,Y)=0$ where $X,Y\in\mathfrak{X}(M)$.

Definition 1.10. The vector field ξ is said to be a conformal Killing vector field if it satisfies the relation $\left(\pounds_\xi g\right)(X,Y)=2\phi g(X,Y)$ where $X,Y\in\mathfrak{X}(M)$ and ϕ is being a scalar.

Definition 1.11. A spacetime is called $\mathcal{T}$-conservative if $(div\ \mathcal{T})(X,Y,Z)=0$.

Definition 1.12. A $(0, 2)$-type symmetric tensor field F in a pseudo-Riemannian manifold (M^n, g) is called Codazzi type if $(\nabla_X F)(Y, Z) = (\nabla_Y F)(X, Z)$ for $X, Y, Z \in \mathfrak{X}(M)$.

This paper has been arranged in the following manner. In the first unit we give introduction. In Section 2 we study spacetime admitting vanishing $\mathcal{T}$-curvature tensor and some geometric properties have been derived. Section 3 is devoted to the general relativistic viscous fluid spacetime admitting vanishing $\mathcal{T}$-curvature tensor. In Section 4 we discuss the general relativistic viscous fluid spacetime admitting divergence-free $\mathcal{T}$-curvature tensor.

2. A spacetime admitting vanishing $\mathcal{T}$-curvature tensor

In this unit we consider V_4 as a spacetime of dimension 4 in general relativity for our entire study. We obtain the following results.

Theorem 2.1. *If* $(c_0 + 4c_1 + c_2 + c_3 + c_5 + c_6) \neq 0$ *then a* $\mathcal{T}$*-flat spacetime is an Einstein spacetime.*

Proof. For a $\mathcal{T}$-flat spacetime $\mathcal{T}(X, Y, Z, W) = 0$. Then from the equation (1.2), we obtain

$$
\begin{aligned}
0 &= c_0 R(X, Y, Z, W) \\
&\quad + c_1 S(Y, Z) g(X, W) + c_2 S(X, Z) g(Y, W) \\
&\quad + c_3 S(X, Y) g(Z, W) + c_4 g(Y, Z) S(X, W) \\
&\quad + c_5 g(X, Z) S(Y, W) + c_6 g(X, Y) S(Z, W) \\
&\quad + r c_7 \left[g(Y, Z) g(X, W) - g(X, Z) g(Y, W) \right].
\end{aligned} \tag{2.1}
$$

Taking contraction on both sides over X and W, we derive

$$S(Y, Z) = -\left[\frac{r(c_4 + 3c_7)}{(c_0 + 4c_1 + c_2 + c_3 + c_5 + c_6)} \right] g(Y, Z). \tag{2.2}$$

Let $\alpha = -\left[\frac{r(c_4+3c_7)}{c_0+4c_1+c_2+c_3+c_5+c_6} \right]$. Then the equation (2.2) becomes

$$S(Y, Z) = \alpha g(Y, Z). \tag{2.3}$$

Clearly, if $(c_0 + 4c_1 + c_2 + c_3 + c_5 + c_6) \neq 0$ then this is an Einstein spacetime. □

Theorem 2.2. *If $c_0 \neq 0$, $c_3 + c_6 = 0$, $(c_1 + c_2 + c_4 + c_5) = 0$ and $(c_0 + 4c_1 + c_2 + c_3 + c_5 + c_6) \neq 0$ then a $\mathcal{T}$-flat spacetime is a spacetime with constant curvature.*

Proof. In view of the equation (2.3), the equation (2.1) implies that

$$R(X,Y,Z,W) = -\left[\frac{(c_1+c_4)\,\alpha + rc_7}{c_0}\right][g(Y,Z)g(X,W) + \left[\frac{rc_7 - (c_2+c_5)\alpha}{c_0}\right] g(X,Z)g(Y,W)] - \frac{\alpha(c_3+c_6)}{c_0} g(X,Y)g(Z,W). \tag{2.4}$$

It clearly follows that if $c_0 \neq 0$, $c_3 + c_6 = 0, (c_1 + c_2 + c_4 + c_5) = 0$ and $(c_0 + 4c_1 + c_2 + c_3 + c_5 + c_6) \neq 0$ then

$$R(X,Y,Z,W) = \left[\frac{(c_1+c_4)\,\alpha + rc_7}{c_0}\right][g(X,Z)g(Y,W) - g(Y,Z)g(X,W)].$$

That is, a $\mathcal{T}$-flat spacetime is a spacetime with constant curvature with respect to the above conditions. □

Theorem 2.3. *The energy momentum tensor is covariant constant in $\mathcal{T}$-flat spacetime satisfying the Einstein's field equation with the cosmological constant.*

Proof. We consider a spacetime satisfying the Einstein's field equation with the cosmological constant

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y), \tag{2.5}$$

where S, λ, r, k and $T(X,Y)$ are being the Ricci tensor, cosmological constant, scalar curvature, gravitational constant and energy momentum tensor respectively.
In view of the equations (2.3) and (2.5), we derive

$$T(X,Y) = \frac{1}{k}\left(\alpha - \frac{r}{2} + \lambda\right) g(X,Y). \tag{2.6}$$

By taking the covariant derivative with respect to Z on both sides, we gain

$$(\nabla_Z T)(X,Y) = -\frac{1}{k}\left[\frac{(c_4+3c_7)}{(c_0+4c_1+c_2+c_3+c_5+c_6)} + \frac{1}{2}\right] dr(Z)g(X,Y). \tag{2.7}$$

As a $\mathcal{T}$-flat spacetime is an Einstein spacetime with the condition $(c_0 + 4c_1 + c_2 + c_3 + c_5 + c_6) \neq 0$, hence the scalar curvature r is a constant. Therefore,

$$dr(Z) = 0, \ \forall Z. \tag{2.8}$$

The equations (2.7) and (2.8) jointly imply that

$$(\nabla_Z T)(X,Y) \;=\; 0.$$

Thus the energy momentum tensor $T(X,Y)$ is covariant constant. □

Theorem 2.4. *If a spacetime M with $\mathcal{T}$-curvature tensor with respect to a Killing vector field ξ is curvature collineation then the Lie derivative of $\mathcal{T}$-curvature tensor vanishes along ξ.*

Proof. The geometrical symmetries of a spacetime can be written as

$$\pounds_\xi A - 2\Omega A \;=\; 0, \tag{2.9}$$

where A is the physical or geometrical quantity, Ω is a scalar and $\pounds_\xi$ represents the Lie derivative with respect to ξ.

For the metric inheritance symmetry we put $A = g$ in the equation (2.9). Thus

$$(\pounds_\xi g)(X,Y) - 2\Omega g(X,Y) \;=\; 0. \tag{2.10}$$

Clearly, in this case if $\Omega = 0$ then ξ becomes a Killing vector field. Let a spacetime M with $\mathcal{T}$-curvature tensor with respect to a Killing vector field ξ be curvature collineation. Thus we gain

$$(\pounds_\xi g)(X,Y) \;=\; 0. \tag{2.11}$$

As M is admitting a curvature collineation, hence we derive from the equation (1.3) that

$$(\pounds_\xi S)(X,Y) \;=\; 0, \tag{2.12}$$

where S denotes the Ricci tensor.

We take the Lie derivative of the equation (1.1) and then with the help of the equations (1.3), (2.11) and (2.12), we derive $(\pounds_\xi \mathcal{T})(X,Y)Z = 0$. □

Theorem 2.5. *Let a spacetime satisfying the Einstein's field equation be of zero $\mathcal{T}$-curvature tensor. The spacetime admits the matter collineation with respect to ξ if and only if ξ is a Killing vector field.*

Proof. The symmetry of energy momentum tensor T is called matter collineation and it is defined by

$$(\pounds_\xi T)(X,Y) = 0,$$

where ξ is the symmetry generating vector field and $\pounds_\xi$ is the operator of Lie derivative along ξ.
Let ξ be a Killing vector field of vanishing $\mathcal{T}$-curvature tensor. Therefore

$$(\pounds_\xi g)(X,Y) = 0. \tag{2.13}$$

Taking the Lie derivative on both the sides of the equation (2.6) with respect to ξ, we have

$$\frac{1}{k}\left(\alpha - \frac{r}{2} + \lambda\right)(\pounds_\xi g)(X,Y) = (\pounds_\xi T)(X,Y). \tag{2.14}$$

Using the equation (2.13) in the equation (2.14), we have

$$(\pounds_\xi T)(X,Y) = 0. \tag{2.15}$$

This proves that the spacetime admits the matter collineation.
For the converse part, let $(\pounds_\xi T)(X,Y) = 0$. Therefore from the equation (2.14), we find

$$(\pounds_\xi g)(X,Y) = 0.$$

This shows that ξ is a Killing vector field. □

Theorem 2.6. *Let a spacetime satisfying the Einstein's field equation be of vanishing $\mathcal{T}$-curvature tensor. The vector field ξ is a conformal Killing vector field if and only if the energy momentum tensor has the Lie inheritance property with respect to ξ.*

Proof. Let ξ be a conformal Killing vector field. Therefore,

$$(\pounds_\xi g)(X,Y) = 2\phi g(X,Y), \tag{2.16}$$

where ϕ is being a scalar.
Now, from the equation (2.14), it follows that

$$\left(\alpha - \frac{r}{2} + \lambda\right) 2\phi g(X,Y) = k(\pounds_\xi T)(X,Y). \tag{2.17}$$

With the help of the equation (2.6) in the equation (2.17), we have

$$(\pounds_\xi T)(X,Y) = 2\phi T(X,Y). \tag{2.18}$$

This shows that the energy momentum tensor has the Lie inheritance property with respect to ξ.

For the converse part, let the energy momentum tensor have the Lie inheritance property with respect to ξ. Therefore,

$$(\pounds_\xi T)(X,Y) = 2\phi T(X,Y).$$

Clearly, the equation (2.16) holds good. This proves that ξ is a conformal Killing vector field. □

3. General relativistic viscous fluid spacetime admitting vanishing $\mathcal{T}$-curvature tensor

In this unit we consider the general relativistic viscous fluid spacetime admitting vanishing $\mathcal{T}$-curvature tensor satisfying the Einstein's field equation without cosmological constant with the condition $\sigma + p = 0$ where p, σ are respectively the isotropic pressure and the energy density. Furthermore, $\sigma + p = 0$ implies that the fluid behaves like a cosmological constant [7] and it is also called the phantom barrier [15]. The choice $\sigma = -p$ leads to the rapid expansion of this spacetime in cosmology and it is called inflation [10]. We obtain the following theorems.

Theorem 3.1. *If a $\mathcal{T}$-flat general relativistic viscous fluid spacetime with the condition $\sigma + p = 0$ where p, σ are respectively the isotropic pressure and the energy density satisfies the Einstein's field equation without cosmological constant, then*

$$\|Q\|^2 = \frac{4k^2p^2(c_4+3c_7)^2}{(c_0+4c_1+c_2+c_3+2c_4+c_5+c_6+6c_7)^2},$$

where Q is the Ricci operator.

Proof. In a general relativistic viscous fluid spacetime with the condition $\sigma + p = 0$, the energy momentum tensor T takes the form [3]

$$T(X,Y) = pg(X,Y), \tag{3.1}$$

where p is the isotropic pressure, σ denotes the energy density and $g(U,U)=-1$, U is the velocity vector field of this flow.

The field equation of Einstein without cosmological constant takes the form

$$S(X,Y)-\frac{r}{2}g(X,Y) \;=\; kT(X,Y), \tag{3.2}$$

where r denotes the scalar curvature and $k\neq 0$.

Using the equations (2.3) and (3.1) in the equation (3.2), we have

$$\left(\alpha-\frac{r}{2}-kp\right)g(X,Y) \;=\; 0. \tag{3.3}$$

Taking contraction on both sides over X and Y, we derive

$$r \;=\; -\frac{2pk(c_0+4c_1+c_2+c_3+c_5+c_6)}{(c_0+4c_1+c_2+c_3+2c_4+c_5+c_6+6c_7)}. \tag{3.4}$$

From the equations (2.3) and (3.4), it implies that

$$S(X,Y) \;=\; \frac{2pk(c_4+3c_7)}{(c_0+4c_1+c_2+c_3+2c_4+c_5+c_6+6c_7)}g(X,Y). \tag{3.5}$$

If Q is the Ricci operator then $g(QX,Y)=S(X,Y)$ and $S(QX,Y)=S^2(X,Y)$. From the equation (3.5), we have

$$S(QX,Y) \;=\; \frac{4p^2k^2(c_4+3c_7)^2}{(c_0+4c_1+c_2+c_3+2c_4+c_5+c_6+6c_7)^2}g(X,Y). \tag{3.6}$$

Taking contraction on both sides over X and Y, we get

$$\|Q\|^2 \;=\; \frac{4p^2k^2(c_4+3c_7)^2}{(c_0+4c_1+c_2+c_3+2c_4+c_5+c_6+6c_7)^2}. \tag{3.7}$$

□

Theorem 3.2. *If a $\mathcal{T}$-flat general relativistic viscous fluid spacetime with the condition $\sigma+p=0$ where p,σ are respectively the isotropic pressure and the energy density obeying the Einstein's field equation without cosmological constant satisfies the condition of timelike convergence then this spacetime also satisfies the relation*

$$\frac{p(c_4+3c_7)}{(c_0+4c_1+c_2+c_3+2c_4+c_5+c_6+6c_7)} \;<\; 0.$$

Proof. The condition of timelike convergence [14] is given by

$$S(X,X) \;>\; 0, \tag{3.8}$$

for any timelike vector field X.

From the equations (3.1) and (3.2), it follows that

$$S(X,Y) - \frac{r}{2}g(X,Y) = kpg(X,Y). \tag{3.9}$$

Setting $X = Y = U$ in the equation (3.9) and with the help of the equation (3.4), we have

$$S(U,U) = -\frac{2pk(c_4 + 3c_7)}{(c_0 + 4c_1 + c_2 + c_3 + 2c_4 + c_5 + c_6 + 6c_7)}. \tag{3.10}$$

Since $k > 0$ and $S(U,U) > 0$, so we obtain

$$\frac{p(c_4 + 3c_7)}{(c_0 + 4c_1 + c_2 + c_3 + 2c_4 + c_5 + c_6 + 6c_7)} < 0. \tag{3.11}$$

□

Theorem 3.3. *For a purely electromagnetic distribution the scalar curvature of a $\mathcal{T}$-flat spacetime with the condition $\sigma + p = 0$ where p, σ are respectively the isotropic pressure and the energy density satisfying the Einstein's field equation without cosmological constant is zero.*

Proof. Taking contraction on both sides of the equation (3.2) over X and Y, we gain

$$r = -kt, \tag{3.12}$$

where t is the trace of T.

Using the equation (3.12) in the equation (3.2), we derive

$$S(X,Y) = kT(X,Y) - \frac{kt}{2}g(X,Y). \tag{3.13}$$

For a purely electromagnetic distribution the Einstein's field equation without cosmological constant is given by

$$S(X,Y) = kT(X,Y). \tag{3.14}$$

From the equations (3.13) and (3.14), it implies that $t = 0$. Hence, we obtain $r = 0$ from the equation (3.12). □

4. General relativistic viscous fluid spacetime admitting divergence-free $\mathcal{T}$-curvature tensor

This part is devoted to the study of the general relativistic viscous fluid spacetime admitting the divergence-free $\mathcal{T}$-curvature tensor. We have the following theorems in this regard.

Theorem 4.1. *In a general relativistic viscous fluid spacetime admitting divergence-free $\mathcal{T}$-curvature tensor, if $c_1 + c_2 = 0$ and $c_3 = 0$ then the energy momentum tensor is of Codazzi type.*

Proof. From the equation (1.1), we have

$$(div\ \mathcal{T})(X, Y, Z) = (c_0 + c_1)(\nabla_X S)(Y, Z) + (c_2 - c_0)(\nabla_Y S)(X, Z) + c_3(\nabla_Z S)(X, Y) + \left(\frac{c_4}{2} + c_7\right) g(Y, Z)dr(X) + \left(\frac{c_5}{2} - c_7\right) g(X, Z)dr(Y) + \frac{c_6}{2} g(X, Y)dr(Z). \tag{4.1}$$

Putting $(div\ \mathcal{T})(X, Y, Z) = 0$ and $dr(X) = 0$ in the equation (4.1), we have

$$0 = (c_0 + c_1)(\nabla_X S)(Y, Z) + (c_2 - c_0)(\nabla_Y S)(X, Z) + c_3(\nabla_Z S)(X, Y). \tag{4.2}$$

Clearly, if $c_1 + c_2 = 0$ and $c_3 = 0$, then we derive from the equation (4.2) that

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z). \tag{4.3}$$

From the equations (3.2) and (4.3), it implies that

$$(\nabla_X T)(Y, Z) = (\nabla_Y T)(X, Z).$$

Therefore, the energy momentum tensor is of Codazzi type. □

Theorem 4.2. *In a general relativistic viscous fluid spacetime admitting divergence-free $\mathcal{T}$-curvature tensor, if $c_1 + c_2 = 0$ and $c_3 = 0$ then the velocity vector field of the fluid is proportional to the gradient vector field of the energy density.*

Proof. It is already proved that the energy momentum tensor in the general relativistic viscous fluid spacetime is of Codazzi type. This implies that both the vorticity and shear of the fluid vanish and the velocity vector field is hyper-surface orthogonal. That is, the velocity vector field of the fluid is proportional to the gradient vector field of the energy density [4, 2]. □

Theorem 4.3. *For a general relativistic viscous fluid spacetime admitting divergence-free $\mathcal{T}$-curvature tensor, if $c_1 + c_2 = 0$ and $c_3 = 0$ then the possible local cosmological structure of this spacetime is of Petrov type I, D or O.*

Proof. Barnes [1] proved that if the shear and vorticity of a perfect fluid spacetime vanish then the velocity vector field U is hyper-surface orthogonal and the energy density is constant over the hyper-surface which is orthogonal to U. Hence, the local cosmological structure of this spacetime is of Petrov type I, D or O. □

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