STUDY OF MORITA THEORY RELATED TO SEMIRINGS, SEMIMODULES, MONOIDS AND ACTS

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled **"STUDY OF MORITA THEORY RELATED TO SEMIRINGS, SEMIMODULES, MONOIDS AND ACTS"** submitted by **Smt. Monali Das** who got her name registered on 7th February, 2018 (Index No.: 26/18/Maths./25) for the award of Ph. D. (Science) degree of Jadavpur University, is absolutely based upon her own work under the supervision of **Dr. Sujit Kumar Sardar** and that neither this thesis nor any part of it has been submitted for either any degree / diploma or any other academic award anywhere before.

01.04.2022

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Dedicated to
my respected parents
Shri Subhas Das and Smt. Mangala Das

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2022 Monali Das

Abstract

The thesis is a study of some problems of Morita theory related to semirings, semimodules, monoids and acts. First, the theory of Morita equivalence for semirings with identity is extended to cover a wider range of semirings, namely the semirings with local units. Various concepts such as prime subsemimodule, (right) strongly prime subsemimodule, uniformly strongly prime subsemimodule, locally nilpotent subsemimodule of a bisemimodule related to a Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ for semirings have been studied in order to prove that structures like prime radical, (right) strongly prime radical, uniformly strongly prime radical, Levitzki radical are preserved under Morita equivalence of semirings with identity. Then we study some topological properties of the prime spectrum of a semimodule P related to a Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ for semirings.

Concepts like (right) strongly prime sub-biacts, uniformly strongly prime sub-biacts, nil sub-biacts, nilpotent sub-biacts of a biact related to a Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ for monoids have been introduced using the idea of Morita equivalence of monoids and we obtain one-to-one inclusion preserving correspondence between the set of all (right) strongly prime (uniformly strongly prime, nil, nilpotent) ideals and the set of all (right) strongly prime (resp. uniformly strongly prime, nil, nilpotent) sub-biacts of the pairs (i) S, P (ii) S, Q (iii) T, P (iv) T, Q. Lastly, for a topological monoid S, we consider the category S-Top of topological S-acts and investigate some of its categorical aspects, which might help initiate the study of Morita theory for topological monoids.

Contents

List of Publications

A list of publications resulting from the work of this thesis is appended below.

(1) M. Das, S. Gupta and S. K. Sardar, Morita equivalence of semirings with local units, *Algebra and Discrete Mathematics*, Vol. 31, No. 1, pp. 37–60 (2021), DOI: 10.12958/adm1288.

(2) M. Das and S. K. Sardar, On some Morita invariant radicals of semirings, *Discussiones Mathematicae - General Algebra and Applications*, Vol. 43 (to be published).

(3) M. Das, S. K. Sardar and S. Gupta, On Categorical Properties of Topological S-Acts, *Southeast Asian Bulletin of Mathematics*, Vol. 46, No. 1, pp. 1-14 (2022).

(4) M. Das and S. K. Sardar, Topology on the prime spectrum of a semimodule related to a Morita context, (Communicated).

(5) M. Das and S. K. Sardar, On some Morita invariants of monoids, (Communicated).

Introduction

The study of monoid, that is a semigroup with identity, trailed behind that of other algebraic structures with more complex axioms such as groups or rings. The initial studies were carried out in the early twentieth century. Several sources [43, 53] attribute the first use of the term (in French) to Monsieur l'Abbé J. A. de Séguier in his book "Eléments de la théorie des groups abstraites", Paris 1904. In 1905, L. E. Dickson published an article "On semigroups and the general isomorphism between infinite groups", where he cites de Séguier. In 1916, O. J. Schmidt introduced the term semigroup in his book "Abstract Group Theory" (in Russian). However, these early definitions of 'semigroups' differed slightly from the modern notion. The first 'proper' semigroup theory began to emerge in the 1920s with the work of the Russian mathematician A. K. Suschkewitsch [89]. During the 1930s, the study of semigroups began to take off. Although the early studies on semigroup theory were highly motivated by existing works on both groups and rings, as the decade progressed, the theory gradually gained momentum, culminating in the publication of some highly influential papers: D. Rees [79], Clifford [13, 14] and P. Dubreil [17]. In more recent years the subject has developed its own characteristic problems, methods and results. Representation of a semigroup (monoid) by transformations of a set defines an act, which plays an essential role in the study of semigroup theory.

Historically, semirings first appear implicitly in [15] and later in [63], [57], [72] and [62] in connection with the study of ideals of a ring. They also appear in [42] and [45] in connection with the axiomatization of the natural numbers and nonnegative rational numbers. However, it was H. S. Vandiver who used the term "semi-ring" in his 1934 paper [93] to introduce an algebraic structure with two operations of addition and multiplication such that multiplication distributes over addition, while cancellation law of addition does not hold. Over the years, semirings have been studied by various researchers either in an attempt to broaden techniques coming from semigroup theory or ring theory, or in connection with applications. Subsequently, the theory of semirings has created a sustained research interest which is evident from various monographs such as [30, 31, 32, 33, 39]. Representation of a semiring R by transformations of a commutative semigroup defines an R-semimodule, which plays an essential role in the study of semiring theory.

We discuss some relevant history of Morita theory before sketching out our main thesis. In 1958, Morita established the Morita equivalence theory for rings with identity in his paper [68]. The classical Morita theory for rings has since been regarded as one of the most important and fundamental tools for studying the structures of rings. Morita theory has subsequently been generalized and studied from different angles. In 1974, Fuller [24] initiated the generalization of the theory of Morita equivalence to rings without identity. His results were further enriched by Sato [86] and Azumaya [8]. In 1983, Abrams [1] studied the Morita theory for rings with local units, where a ring is said to have local units if there is a set of commuting idempotents such that every element of the ring admits one of these idempotents as a two-sided identity. He considered the categories of all left modules over these rings which are unitary in a natural sense. He proved that two such module categories over the rings R and S, say, are equivalent if and only if there exists a unitary left R-module P which is a generator, the direct limit of a given kind of system of finitely generated projective modules, and such that S is isomorphic to the ring of certain endomorphisms of P. Ánh and Márki [5] further generalized Abrams' result to cover a wider range of rings by weakening the condition of commutativity of idempotents in question. In 1991, Garcia and Simon [27] studied the Morita theory for idempotent rings using a completely new technique of non-commutative localizations. Xu, Shum and Turner-Smith [97] introduced the concept of Morita-like equivalence which is an extension of the usual Morita equivalence from the class of rings with identity to a wider class of rings, using the matrix approach and replacement techniques. Later, Ouyang et al. [77, 78] and Garcia et al. [25] characterized and further studied Morita-like equivalence. Garcia and Marin studied the Morita theory for associative rings in [26]. Studies are still being conducted in this area by various researchers.

In 1972, U. Knauer [54] and B. Banaschewski [9] independently transferred the theory of Morita equivalence from rings to monoids. For a monoid A, Knauer considered the non-additive category A-Act of A-acts and described all monoids B such that the category B-Act is equivalent to the category A-Act. In particular, he found that the equivalence of these categories yields an isomorphism between the monoids A and B if A is a group or finite or commutative. In [9], Banaschewski observed that if A and B are Morita equivalent semigroups, in the sense that the categories A-Act and B-Act are equivalent (without any requirement of the acts being unitary in any sense), then A and B are isomorphic semigroups. Clearly one must define Morita equivalence in terms of some subcategories in order to obtain a notion differing from isomorphism. In 1995, based on the development in [5], Talwar [90] gave a generalization of Morita equivalence of monoids to that of semigroups with local units, where a semigroup S is said to have local units if for each $s \in S$ there exist idempotents e and f in S such that $es = s = sf$. For such a semigroup S, he considered the full subcategory FS-Act consisting of the unitary S-acts, which are fixed by the functor $S \otimes Hom_S(S, -)$ and called two such semigroups S and T to be Morita equivalent if FS-Act is equivalent to FT-Act. By analogy with ring theory [5], he then defined Morita context for semigroups and showed that the categories FS-Act and FT-Act are equivalent if and only if there exists a unitary Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ with θ, ϕ surjective. Over the years, several generalizations of Morita theory for various classes of semigroups have been obtained by many researchers (see Talwar [91], Chen and Shum [12], Laan and Márki [58], Lawson [61], Afara and Lawson [3], Steinberg [87]). Study of Morita invariants is also an important aspect of studying Morita theory. In [61] Lawson proved that several important subclasses of regular semigroups are Morita invariant, under the assumption that these semigroups have local units. In [59] Laan and Márki discussed some Morita invariant properties of semigroups. Along with several other results, they established isomorphism between the lattices of ideals of strongly Morita equivalent semigroups with weak local units. In [84] Sardar et al. studied Morita equivalence for monoids in connection with Γ-semigroups with unities and obtained some Morita invariants of monoids. Later Sardar and Gupta [83] further studied some Morita invariants of semigroups and showed that there is a lattice isomorphism between the set of all ideals and the set of all sub-biacts corresponding to a Morita context of semigroups.

Katsov and Nam [49] generalized the Morita theory for rings to semirings with identity heavily using the notion of tensor product [48] of semimodules. Later Katsov et al. [50] proved that being ideal-simple and congruence-simple are Morita invariant properties of semirings. Dutta and Das [20] introduced the notion of Morita context for semirings. In [81] Sardar et al. redefined Morita context for semirings using the notion of tensor product and connected Morita equivalence with Morita context for semirings. In [82] Sardar and Gupta studied some Morita invariants of semirings and along with several other results, they established lattice isomorphism between the sets of ideals of Morita equivalent semirings. Later in [36] they further showed that if R and S are Morita equivalent semirings via the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$, then there is a lattice isomorphism between the set of ideals of R and the set of subsemimodules of P.

Nowadays, Morita theory has crossed the threshold of algebra and has scattered in several branches of mathematics (see [23], [70], [80], [98]). But there remains much more to investigate on this topic. Now as discussed above, there have been several generalizations of Morita theory to settings other than rings with identity, one such case being the study of Morita equivalence for rings with local units [5]. At the same time semiring theory being a generalization of ring theory, one aspect of the study of semirings involves the investigation of the validity of the ring theoretic analogues in the semirings. Motivated by this consideration, we attempt to generalize the Morita theory to semirings with local units analogous to the one for rings with local units [5]. Also the more recent developments in the field of Morita equivalence of semirings [49, 82, 36] motivate us to further study some problems of Morita theory associated with semirings and semimodules. The topics we consider in this regard are the study of some Morita invariant radicals of semirings and the study of topology on the prime spectrum of a semimodule related to a Morita context. Again as noted earlier, generalizing in another direction, there have been several studies on Morita equivalence of monoids as well as semigroups with various kinds of local units. In some of these works, we observe a nice interplay among the various components of a Morita context for monoids, which motivates us to study certain Morita invariants of monoids. Lastly, we consider a topological monoid S, i.e., a monoid equipped with a topology in such a way that the monoid multiplication is continuous, and study some categorical aspects of the category S-Top of topological S-acts that might help initiate the study of Morita equivalence of topological monoids.

Now we present below a short summary of the thesis. The thesis consists of six chapters.

✠ In **Chapter 1**, we recall some necessary basic notions and results concerning category theory, monoids, acts, semirings, semimodules and topology in order to develop the thesis.

✠ In **Chapter 2**, we extend the theory of Morita equivalence for semirings with identity to cover a wider range of semirings, namely the semirings with local units in the sense that any two elements of the semiring have a common two-sided identity. For such a semiring R, we consider the category R-Sem of unitary R-semimodules and call two such semirings R and S to be Morita equivalent if the categories R-Sem and S-Sem are equivalent. First, we define locally projective unitary semimodule analogous to the notion of locally projective module [5] and observe some characterizations of locally projective generators. Then we proceed to develop certain tools to obtain some necessary and sufficient conditions for the Morita equivalence of two semirings with local units and study the relation between such equivalence and Morita context. Then we observe one characterization of the semirings with local units that are Morita equivalent to semirings with identity. We conclude the chapter by discussing some properties which remain invariant under Morita equivalence.

✠ In **Chapter 3**, we consider a Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ for semirings with identity and introduce notions like (right) strongly prime subsemimodule, uniformly strongly prime subsemimodule, locally nilpotent subsemimodule of a semimodule, using the idea of Morita equivalence of semirings. Then we obtain one-to-one inclusion preserving correspondence between the set of all prime ((right) strongly prime, uniformly strongly prime, locally nilpotent) ideals and the set of all prime (resp. (right) strongly prime, uniformly strongly prime, locally nilpotent) subsemimodules of the pairs (i) $R,\ P$ (ii) $R,\ Q$ (iii) $S,\ P$ (iv) $S,\ Q$. Finally with the help of these correspondences we prove that structures like prime radical, strongly prime radical, uniformly strongly prime radical and Levitzki radical of semirings are preserved under Morita equivalence.

✠ In **Chapter 4**, we topologize the prime spectrum $Spec(P)$ of a bisemimodule P related to a Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ for semirings with identity and investigate the interrelation between the properties of this space and the algebraic properties of the bisemimodule P. We further obtain homeomorphisms between the topological spaces of $Spec(R)$ and $Spec(P)$, $Spec(S)$ and $Spec(P)$, which in turn result in the homeomorphism between the spaces $Spec(R)$ and $Spec(S)$.

✠ In **Chapter 5**, we introduce notions like (right) strongly prime sub-biacts, uniformly strongly prime sub-biacts, nil sub-biacts, nilpotent sub-biacts of a monoid-act, using the idea of Morita equivalence of monoids and obtain one-to-one inclusion preserving correspondence between the set of all (right) strongly prime (uniformly strongly prime, nil, nilpotent) ideals and the set of all (right) strongly prime (resp. uniformly strongly prime, nil, nilpotent) sub-biacts of the pairs (i) $S,\ P$ (ii) $S,\ Q$ (iii) $T,\ P$ (iv) $T,\ Q$, where S, T, P, Q are connected in a way such that $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ is a Morita context for monoids. In addition, we observe that these correspondences in turn establish one-to-one inclusion preserving correspondence between the set of all (right) strongly prime (uniformly strongly prime, nil, nilpotent) ideals of S and T.

✠ In **Chapter 6**, we consider the category S-Top of topological S-acts for a topological monoid S and study some of its categorical aspects. First, we identify the product, coproduct, free object in S-Top. We define indecomposable topological S-act and observe the unique decomposition of a topological S-act into indecomposable topological subacts. Then we obtain a characterization of a projective topological S-act. Finally, we define generator in S-Top and obtain some of its characterizations.

The thesis is also appended with a list of some remarks and further scope of study that transpired from the present work.

List of Abbreviations and Notations

The notations and abbreviations used throughout the thesis are explained as and when they are introduced. Despite this, for the convenience of the readers, a list of notations and abbreviations used frequently in the thesis has been provided below.

$\varnothing$	The empty set
$\mathbb{N}$	The set of all non-negative integers
$\mathbb{Z}$	The set of all integers
$\mathbb{Z}^+$	The set of all positive integers
ρ^{tr}	Transitive closure of a relation ρ
$Ob(\mathcal{C})$	Class of objects of a category $\mathcal{C}$
$\|X\|$	Underlying set of $X \in Ob(\mathcal{C})$, where $\mathcal{C}$ is a concrete category
$Hom_{\mathcal{C}}(A, B)$	Set of all morphisms from object A to object B in category $\mathcal{C}$
$End(A)$	Set of all morphisms from object A to itself
$\prod\limits_{i \in I} C_i$	Product of a family of objects $(C_i)_{i\in I}$
$\coprod\limits_{i \in I} C_i$	Coproduct of a family of objects $(C_i)_{i\in I}$
$I_{\mathcal{C}}$	Identity functor on a category $\mathcal{C}$
id_A	Identity morphism on object A
$Id(S)$	Lattice of all ideals of a semiring (semigroup, monoid) S
$Sub(P)$	Lattice of all subsemimodules (sub-biacts) of a semimodule (biact) P
$E(R)$	Set of local units (slu) of a semiring R
${}_R\mathcal{M}$	Category of all left R-semimodules
$\mathcal{M}_S$	Category of all right S-semimodules
${}_R\mathcal{M}_S$	Category of all R-S-bisemimodules
R-Sem	Category of unitary left R-semimodules
$\bigoplus_{i\in I} M_i$	Direct sum of a family of semimodules $(M_i)_{i\in I}$
$\underrightarrow{lim}_I M_i$	Direct limit of a family of semimodules $(M_i)_{i\in I}$
$\dot{\cup}_I A_i$	Disjoint union of a collection of sets $(A_i)_{i\in I}$

$Hom_R(A, B)$	Set of all left(/right) R-semimodule homomorphisms from A to B
$End_R(A)$	Set of all left(/right) R-semimodule morphisms from A to itself
$tr(P)$	Trace ideal of a semimodule ${}_RP$
$\overline{X}$	Topological closure of a set X
$\langle a \rangle$	Ideal (subsemimodule, sub-biact) generated by a
S-Act	Category of all left S-acts
Act-S	Category of all right S-acts
S-Top	Category of left topological S-acts of a topological monoid (S, τ_S)
S-CReg	Category of Hausdorff completely regular topological S-acts
$\prod_{\alpha\in\Lambda} A_\alpha$	Product of $(A_\alpha)_{\alpha\in\Lambda}$ in S-Act
$\coprod_{\alpha\in\Lambda} A_\alpha$	Coproduct of $(A_\alpha)_{\alpha\in\Lambda}$ in S-Act
$F(X)$	Free S-act over a set X
$C(X, Y)$	Set of all continuous S-maps from topological S-act (X, τ_X) to (Y, τ_Y).

Chapter 1

Preliminaries

In this chapter, we recall some basic notions and results of category theory, monoids, acts, semirings, semimodules and topology in order to use them in the sequel.

1.1 Category

Here we recall some necessary notions of category theory from [64, 67, 11, 53, 2].

Definition 1.1.1. [2] A category is a quadruple $\mathcal{C} = (Ob, Hom, id, \circ)$ consisting of

(1) a class Ob, whose members are called objects,

(2) for each pair (A, B) of objects, a set $Hom_{\mathcal{C}}(A, B)$, whose members are called morphisms from A to B,

(3) for each object A, a morphism $id_A : A \to A$, called the identity on A,

(4) a composition law associating each pair of morphisms $f : A \to B$ and $g : B \to C$ with a morphism $g \circ f : A \to C$, called the composite of f and g,

subject to the following conditions:

(a) composition is associative; i.e., for morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$ equation $h \circ (g \circ f) = (h \circ g) \circ f$ holds,

(b) for any morphism $f : A \to B$, we have $id_B \circ f = f$ and $f \circ id_A = f$,

(c) the sets $Hom_{\mathcal{C}}(A, B)$ are pairwise disjoint.

Definition 1.1.2. [2] A category $\mathcal{C}$ is said to be a subcategory of a category $\mathcal{D}$ provided that the following conditions are satisfied:

(1) $Ob(\mathcal{C})$ is a subclass of $Ob(\mathcal{D})$,

(2) for each $A, B \in Ob(\mathcal{C})$, $Hom_{\mathcal{C}}(A, B) \subseteq Hom_{\mathcal{D}}(A, B)$,

(3) for each object A of $\mathcal{C}$, id_A is the same in $\mathcal{D}$ as in $\mathcal{C}$,

(4) the composition law in $\mathcal{C}$ is the restriction of the composition law in $\mathcal{D}$ to the morphisms of $\mathcal{C}$.

Definition 1.1.3. [2] A subcategory $\mathcal{C}$ of a category $\mathcal{D}$ is said to be a full subcategory if for each $A, B \in Ob(\mathcal{C})$, $Hom_{\mathcal{C}}(A, B) = Hom_{\mathcal{D}}(A, B)$.

Definition 1.1.4. [2] If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a (covariant) functor F from $\mathcal{C}$ to $\mathcal{D}$ is a function that assigns each object A of $\mathcal{C}$ to an object $F(A)$ of $\mathcal{D}$ and each morphism $f : A \to B$ of $\mathcal{C}$ to a morphism $F(f) : F(A) \to F(B)$ of $\mathcal{D}$ in such a way that

(1) F preserves composition; i.e., $F(f \circ g) = F(f) \circ F(g)$ whenever $f \circ g$ is defined, and

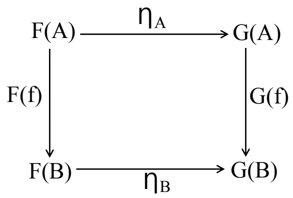
(2) F preserves identity morphisms; i.e., $F(id_A) = id_{F(A)}$ for each object A of $\mathcal{C}$.

Definition 1.1.5. [2] A functor $F : \mathcal{C} \to \mathcal{D}$ is called faithful provided that all the hom-set restrictions $F : Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{D}}(F(A), F(B))$ are injective for any $A, B \in Ob(\mathcal{C})$.

Remark 1.1.6. [2] For any category $\mathcal{C}$, there is the identity functor $I_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ which takes $A \in Ob(\mathcal{C})$ to itself and each morphism $f : A \to B$ in $\mathcal{C}$ to itself.

Remark 1.1.7. [2] If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are functors, then the composite $G \circ F : \mathcal{C} \to \mathcal{E}$ takes $A \in Ob(\mathcal{C})$ to $G(F(A))$ and each morphism $f : A \to B$ in $\mathcal{C}$ to $G(F(f)) : G(F(A)) \to G(F(B))$.

Definition 1.1.8. [11] Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\eta : F \to G$ is a collection of morphisms $\{\eta_A \mid \eta_A : F(A) \to G(A),\ A \in Ob(\mathcal{C})\}$ of $\mathcal{D}$ indexed by the objects of $\mathcal{C}$ and such that for every morphism $f : A \to B$ in $\mathcal{C}$, the following square commutes.



Definition 1.1.9. [2] Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\eta : F \to G$ is called a natural isomorphism[1] if for each $A \in Ob(\mathcal{C})$, η_A is an isomorphism.

[1] also known as natural equivalence in [53].

Definition 1.1.10. [2] Two functors $F, G : \mathcal{C} \to \mathcal{D}$ are said to be naturally isomorphic, denoted by $F \cong G$, provided that there exists a natural isomorphism from F to G.

Definition 1.1.11. [53] Two categories $\mathcal{C}$ and $\mathcal{D}$ are called equivalent categories if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G \cong I_{\mathcal{D}}$ and $G \circ F \cong I_{\mathcal{C}}$.

Definition 1.1.12. [11] Let I be a set and $(C_i)_{i \in I}$ a family of objects in a category $\mathcal{C}$. A product of that family is a pair $(P, (p_i)_{i \in I})$ where

(1) $P \in Ob(\mathcal{C})$,

(2) for every $i \in I$, $p_i : P \to C_i$ is a morphism of $\mathcal{C}$,

and this pair is such that for every other pair $(Q, (q_i)_{i \in I})$ where

(1) $Q \in Ob(\mathcal{C})$,

(2) for every $i \in I$, $q_i : Q \to C_i$ is a morphism of $\mathcal{C}$,

there exists a unique morphism $r : Q \to P$ such that for every index i, $q_i = p_i \circ r$.

We generally denote the product of a family of objects $(C_i)_{i \in I}$ by $\prod\limits_{i \in I} C_i$.

Definition 1.1.13. [11] Let I be a set and $(C_i)_{i \in I}$ a family of objects in a category $\mathcal{C}$. A coproduct of that family is a pair $(P, (s_i)_{i \in I})$ where

(1) $P \in Ob(\mathcal{C})$,

(2) for every $i \in I$, $s_i : C_i \to P$ is a morphism of $\mathcal{C}$,

and this pair is such that for every other pair $(Q, (t_i)_{i \in I})$ where

(1) $Q \in Ob(\mathcal{C})$,

(2) for every $i \in I$, $t_i : C_i \to Q$ is a morphism of $\mathcal{C}$,

there exists a unique morphism $r : P \to Q$ such that for every index i, $t_i = r \circ s_i$.

We generally denote the coproduct of a family of objects $(C_i)_{i \in I}$ by $\coprod\limits_{i \in I} C_i$.

Definition 1.1.14. [2] Let $A \underset{g}{\overset{f}{\rightrightarrows}} B$ be a pair of morphisms. A morphism $B \xrightarrow{\gamma} C$ is called a coequalizer of f and g, usually denoted by $\gamma = coeq(f, g)$, provided that the following conditions hold:

(1) $\gamma \circ f = \gamma \circ g$,

(2) for any morphism $\gamma' : B \to C'$ with $\gamma' \circ f = \gamma' \circ g$, there exists a unique morphism $\overline{\gamma} : C \to C'$ such that $\gamma' = \overline{\gamma} \circ \gamma$.

Definition 1.1.15. [11] Given a functor $F : \mathcal{D} \to \mathcal{C}$, a cone on F consists of

(1) an object $C \in Ob(\mathcal{C})$,

(2) for every object $D \in Ob(\mathcal{D})$, a morphism $p_D : C \to FD$ in $\mathcal{C}$, in such a way that for every morphism $d : D \to D'$ in $\mathcal{D}$, $p_{D'} = Fd \circ p_D$.

Definition 1.1.16. [11] Given a functor $F : \mathcal{D} \to \mathcal{C}$, a limit of F is a cone $(L, (p_D)_{D\in Ob(\mathcal{D})})$ on F such that, for every cone $(M, (q_D)_{D\in Ob(\mathcal{D})})$ on F, there exists a unique morphism $m : M \to L$ such that for every object $D \in Ob(\mathcal{D})$, $q_D = p_D \circ m$.

Remark 1.1.17. [11] When a functor $F : \mathcal{D} \to \mathcal{C}$ admits limit, it is unique up to isomorphism.

Definition 1.1.18. [11] Given a functor $F : \mathcal{D} \to \mathcal{C}$, a cocone on F consists of
(1) an object $C \in Ob(\mathcal{C})$,
(2) for every object $D \in Ob(\mathcal{D})$, a morphism $s_D : FD \to C$ in $\mathcal{C}$, in such a way that for every morphism $d : D' \to D$ in $\mathcal{D}$, $s_{D'} = s_D \circ Fd$.

Definition 1.1.19. [11] Given a functor $F : \mathcal{D} \to \mathcal{C}$, a colimit[2] of F is a cocone $(L, (s_D)_{D\in Ob(\mathcal{D})})$ on F such that, for every cocone $(M, (t_D)_{D\in Ob(\mathcal{D})})$ on F, there exists a unique morphism $m : L \to M$ such that for every object $D \in Ob(\mathcal{D})$, $t_D = m \circ s_D$.

Remark 1.1.20. [11] When a functor $F : \mathcal{D} \to \mathcal{C}$ admits colimit, it is unique up to isomorphism.

Remark 1.1.21. [11] Coproducts, coequalizers are special cases of the general notion of colimit (direct limit).

Definition 1.1.22. [67] Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D},\ G : \mathcal{D} \to \mathcal{C}$ be covariant functors. We call F to be left adjoint to G and G right adjoint to F and write $F \dashv G$ if there exists a natural equivalence of set-valued bifunctors

$$\eta : Hom_{\mathcal{D}}(F(-), -) \to Hom_{\mathcal{C}}(-, G(-)).$$

Theorem 1.1.23. *[53] If a pair of functors* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *constitutes an equivalence of categories, then* $F \dashv G$ *and* $G \dashv F$.

Theorem 1.1.24. *[53] Let* $F,\ F' : \mathcal{C} \to \mathcal{D}$ *and* $G,\ G' : \mathcal{D} \to \mathcal{C}$ *be functors with* $F \dashv G$. *Then* $F' \dashv G$ *if and only if* $F \cong F'$ *are naturally equivalent functors and* $F \dashv G'$ *if and only if* $G \cong G'$ *are naturally equivalent functors.*

Theorem 1.1.25. *[11] If the functor* $F : \mathcal{C} \to \mathcal{D}$ *has a left adjoint,* F *preserves all limits which turn out to exist in* $\mathcal{C}$.

Definition 1.1.26. [53] A category $\mathcal{C}$ is called a concrete category if all objects are (structured) sets, morphisms from A to B are (structure preserving) mappings from A to B, composition of morphisms is the composition of mappings and the identities are the identity mappings.

[2] also known as direct limit in [64]

Definition 1.1.27. [2] A morphism $f : A \to B$ is said to be an epimorphism provided that for all pairs $B \overset{h}{\underset{k}{\rightrightarrows}} C$ of morphisms such that $h \circ f = k \circ f$, it follows that $h = k$.

Definition 1.1.28. [2] A morphism $f : A \to B$ is said to be a monomorphism provided that for all pairs $C \overset{h}{\underset{k}{\rightrightarrows}} A$ of morphisms such that $f \circ h = f \circ k$, it follows that $h = k$.

Definition 1.1.29. [53] A morphism $f : A \to B$ is called a retraction if f is right invertible, i.e., there exists $g \in Hom(B, A)$ with $f \circ g = id_B$. B is called a retract of A.

Definition 1.1.30. [53] A morphism $f : A \to B$ is called a coretraction if f is left invertible, i.e., there exists $g \in Hom(B, A)$ with $g \circ f = id_A$. A is called a coretract of B.

Remark 1.1.31. [53] If $\mathcal{C}$ is a concrete category, then the following implications hold for $f : A \to B$,

$$retraction \Rightarrow surjective \Rightarrow epimorphism$$

$$coretraction \Rightarrow injective \Rightarrow monomorphism.$$

Definition 1.1.32. [2] A morphism $f : A \to B$ in a category $\mathcal{C}$ is called an isomorphism provided that there exists a morphism $g : B \to A$ with $g \circ f = id_A$ and $f \circ g = id_B$. Such a morphism g is called the inverse of f.

Definition 1.1.33. [53] Let $\mathcal{C}$ be a concrete category. $F \in Ob(\mathcal{C})$ is called a free object in $\mathcal{C}$, if there exist a set I and a mapping $\sigma : I \to |F|$ such that for every $X \in Ob(\mathcal{C})$ and every mapping $\xi : I \to |X|$, there exists exactly one $\xi^* \in Hom_{\mathcal{C}}(F, X)$ such that $\xi^* \circ \sigma = \xi$.

Definition 1.1.34. [53] $P \in Ob(\mathcal{C})$ is called projective in $\mathcal{C}$ if for every $f \in Hom_{\mathcal{C}}(P, Y)$ and every epimorphism $\pi \in Hom_{\mathcal{C}}(X, Y)$, there exists $\bar{f} \in Hom_{\mathcal{C}}(P, X)$ such that $\pi \circ \bar{f} = f$, whenever $X, Y \in Ob(\mathcal{C})$.

Remark 1.1.35. [53] Let $\mathcal{C}$ be a concrete category with surjective epimorphisms. Then every free object is projective in $\mathcal{C}$.

Definition 1.1.36. [67] A family of objects $\{U_i\}_{i \in I}$ is called a *family of generators* for a category $\mathcal{C}$ if for every pair of distinct morphisms $\alpha, \beta : A \to B$ in $\mathcal{C}$ there is a morphism $u : U_i \to A$ for some i such that $\alpha \circ u \neq \beta \circ u$. An object U in $\mathcal{C}$ is called a generator for $\mathcal{C}$ if $\{U\}$ is a family of generators for $\mathcal{C}$.

Remark 1.1.37. [67] If $\mathcal{C}$ has coproducts, then U is a generator for $\mathcal{C}$ if and only if for each $A \in Ob(\mathcal{C})$ there is an epimorphism $\gamma : \coprod_I U \to A$ for some set I.

1.2 Monoids and S-acts

We now recall the following preliminary notions of monoid (semigroup) theory.

Definition 1.2.1. [53] Let S be a non-empty set with a binary operation $*$ on S. Then the pair $(S, *)$ is called a semigroup if the operation $*$ is associative, i.e., $x * (y * z) = (x * y) * z$ for any $x, y, z \in S$. It is customary to write simply xy instead of $x * y$ when there is no confusion about the binary operation.

Definition 1.2.2. [53] A semigroup S is called a monoid if there exists an element $1_S \in S$, known as the identity element, such that $1_S x = x = x 1_S$ for all $x \in S$.

Definition 1.2.3. [53] Let S and T be two semigroups. Then a map $f : S \to T$ is said to be a semigroup morphism if for any $s, s' \in S$, $f(ss') = f(s)f(s')$.

Definition 1.2.4. [53] Let S and T be two monoids. Then a semigroup morphism $f : S \to T$ is said to be a monoid morphism if $f(1_S) = 1_T$.

Definition 1.2.5. [90] A semigroup S is said to have local units if for every $s \in S$ there exist idempotents $u_s, v_s \in S$ such that $u_s s = s = s v_s$.

Definition 1.2.6. [91] A semigroup S is said to have weak local units if for every $s \in S$ there exist $u_s, v_s \in S$ such that $u_s s = s = s v_s$.

Definition 1.2.7. [53] Let S be a semigroup (monoid). A non-empty subset I of S is called a left (right) ideal of S if $SI \subseteq I$ (resp. $IS \subseteq I$). A both-sided ideal (or simply an ideal) of S is a subset of S which is both a left and a right ideal of S.

Definition 1.2.8. [75] The intersection, K_S, of all ideals of a semigroup S, if non-empty, is called the kernel of S.

Remark 1.2.9. The set of all ideals of a semigroup (monoid) S having kernel forms a lattice, namely $Id(S)$, with inclusion as the partial order. The join and meet of this lattice are, respectively, the union[3] and the intersection of the ideals. If S does not have a kernel, then we adjoin the empty set in the collection to make $Id(S)$ a lattice.

Definition 1.2.10. [75] An ideal P of a semigroup (monoid) S is said to be a prime ideal if for any ideals I, J of S, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

[3]For lattice of ring ideals this is the sum.

Definition 1.2.11. [13] An element x of a semigroup (monoid) S is said to be nilpotent if $x^n \in K_S$ for some $n \in \mathbb{Z}^+$. An ideal I of S is said to be a nil ideal of S provided every element of I is nilpotent.

Definition 1.2.12. [88] An ideal I of a semigroup (monoid) S is called a nilpotent ideal of S if $I^n \subseteq K_S$ for some $n \in \mathbb{Z}^+$.

Definition 1.2.13. [53] Let S be a monoid. Then a set M together with a function $S \times M \to M$, denoted by $(s, m) \mapsto sm$, satisfying
(1) $1_S m = m$ and
(2) $(st)m = s(tm)$ for all $s, t \in S$ and $m \in M$
is called a left S-act and is denoted by ${}_S M$.

If S is a semigroup without identity, then a left S-act has only the property (2) above.

Definition 1.2.14. [53] Let S be a monoid. Then a set M together with a function $M \times S \to M$, denoted by $(m, s) \mapsto ms$, satisfying
(1) $m1_S = m$ and
(2) $m(st) = (ms)t$ for all $s, t \in S$ and $m \in M$
is called a right S-act and is denoted by M_S.

If S is a semigroup without identity then a right S-act has only the property (2) above.

Definition 1.2.15. [53] If M is simultaneously a left S-act and a right T-act such that $(sm)t = s(mt)$ for all $s \in S, m \in M$ and $t \in T$, then M is said to be an S-T-biact and is denoted by ${}_S M_T$.

Definition 1.2.16. [53] Let M be a left S-act (right S-act, S-T-biact). Then a non-empty subset N of M is said to be a subact (resp. subact, sub-biact) of M if $SN \subseteq N$ (resp. $NS \subseteq N$, $SNT \subseteq N$).

Definition 1.2.17. [35] Let M be a left S-act (right S-act, S-T-biact). Then the kernel of M, denoted as K_M, is defined to be the intersection of all subacts (resp. subacts, sub-biacts) of M, if non-empty.

Remark 1.2.18. The set of all subacts (subacts, sub-biacts) of a left S-act (resp. right S-act, S-T-biact) M (adjoined with the empty set, in case, the intersection of all subacts is empty) forms a lattice, namely $Sub(M)$, with inclusion as the partial order. The join and meet of this lattice are, respectively, the union and intersection of the subacts (subacts, sub-biacts).

Definition 1.2.19. [53] A subact B of ${}_SA$ is said to be generated by $X(\subseteq A)$ if $B = SX$. It is also denoted by $B = \langle X \rangle$. If X is finite then B is called finitely generated subact. In this case $B = \bigcup_{i=1}^{n} Sa_i$ for $X = \{a_1, a_2, ...a_n\}$.

We call ${}_SA$ a cyclic S-act if $A = \langle \{a\} \rangle$, where $a \in {}_SA$, and write $A = \langle a \rangle$. Then, clearly, ${}_SA = Sa$.

Definition 1.2.20. [53] An S-act ${}_SA$ is said to be decomposable if there exist two subacts ${}_SB, {}_SC \subseteq {}_SA$ such that ${}_SA = {}_SB \cup {}_SC$ and ${}_SB \cap {}_SC = \varnothing$. In this case ${}_SA = {}_SB \cup {}_SC$ is called a decomposition of ${}_SA$. Otherwise ${}_SA$ is called indecomposable.

Definition 1.2.21. [53] Let M and N be two left S-acts. Then a mapping $f : M \to N$ is called a left S-morphism if for all $s \in S$ and $m \in M$, $f(sm) = sf(m)$.

Definition 1.2.22. [53] Let M and N be two right S-acts. Then a mapping $f : M \to N$ is called a right S-morphism if for all $s \in S$ and $m \in M$, $f(ms) = f(m)s$.

Definition 1.2.23. [53] Let M and N be two S-T-biacts. Then a mapping $f : M \to N$ is called an S-T-bimorphism if it is both left S-morphism and right T-morphism.

Remark 1.2.24. The category formed by left S-acts together with the left S-morphisms is denoted by S-Act. Analogously, the right S-acts and the right S-morphisms form a category, denoted by Act-S. The category of all S-T-biacts together with S-T-bimorphisms is denoted by S-Act-T.

Definition 1.2.25. [53] Let $\times_{i\in I} X_i$ be the cartesian product of a family $(X_i)_{i\in I}$ of left S-acts. Define the projections $p_j : \times_{i\in I} X_i \to X_j,\ j \in I$, by $p_j((x_i)_{i\in I}) = x_j,\ j \in I,\ (x_i)_{i\in I} \in \times_{i\in I} X_i$. This cartesian product endowed with the S-action defined on it as componentwise multiplication by elements of S from the left is the product of $(X_i)_{i\in I}$ in S-Act and is denoted by $\prod_{i\in I} X_i$.

Definition 1.2.26. [53] Let $I \neq \varnothing$ be a set. Let $\dot{\bigcup}_{i\in I} X_i$ be the disjoint union of a family $(X_i)_{i\in I}$ of left S-acts with injections $u_j : X_j \to \dot{\bigcup}_{i\in I} X_i$ defined by $u_j(x) := (x, j),\ j \in I,\ x \in X_j$. Then the disjoint union together with the S-action defined on it as

$$S \times \dot{\bigcup_{i\in I}} X_i \to \dot{\bigcup_{i\in I}} X_i$$
$$(s, (x, j)) \mapsto (sx, j)$$

is the coproduct of $(X_i)_{i\in I}$ in S-Act and is denoted by $\coprod_{i\in I} X_i$.

Theorem 1.2.27. *[53] For ${}_SG \in Ob(S\text{-}Act)$ the following conditions are equivalent:*

(i) $_SG$ *is a generator in S-Act.*

(ii) *The functor* $Hom(_SG, -) : S - Act \to Set$ *is faithful.*

(iii) *Every* $_SX \in Ob(S\text{-}Act)$ *is an epimorphic image of* $\coprod_{Hom(_SG, _SX)} {}_SG$.

(iv) *For every* $_SX \in Ob(S\text{-}Act)$ *there exists a set* I *such that* $_SX$ *is an epimorphic image of* $\coprod_I {}_SG$.

(v) *There exists an epimorphism* $\pi : {}_SG \to {}_SS$.

(vi) $_SS$ *is a retract of* $_SG$.

(vii) $tr_S(G) = {}_SS$.

(viii) *There exists* $\varepsilon^2 = \varepsilon \in End(_SG)$ *such that* $\varepsilon(_SG) = {}_SSu \cong {}_SS$ *for some* $u \in G$.

Definition 1.2.28. [90] A left S-act (right S-act, S-T-biact) M is said to be unitary if $SM = M$ (resp. $MS = M$, $SMT = M$). We denote such an act by left US-act (resp. right US-act, US-UT-biact).

Definition 1.2.29. [90] A left (right) US-act is said to be a fixed act, denoted by left (resp. right) FS-act, if $S \otimes_S Hom_S(S, M) \cong M$ (resp. $Hom_S(S, M) \otimes_S S \cong M$).

Remark 1.2.30. The unitary left (right) S-acts together with left (resp. right) S-morphisms form a full subcategory of S-Act (resp. Act-S) which we denote by US-Act (resp. Act-US). The full subcategory of US-Act (Act-US) containing the fixed acts is denoted by FS-Act (resp. Act-FS).

Theorem 1.2.31. *[61] A left (right) US-act $_SM$ (resp. M_S) is a left (resp. right) FS-act, if and only if $S \otimes M \cong M$ (resp. $M \otimes S \cong M$).*

Definition 1.2.32. [53] For a right S-act M_S and a left S-act $_SN$ the tensor product of these two acts, denoted by $M \otimes_S N$, is the solution of the usual universal problem: that is, $M \otimes_S N = (M \times N)/\sigma$, where σ is the equivalence relation on $M \times N$ generated by $\Sigma = \{((xs, y), (x, sy)) : x \in M, y \in N, s \in S\}$. We denote the class of (x, y) by $x \otimes y$. When there is no ambiguity about the semigroup (monoid) S we write the tensor product as $M \otimes N$.

Remark 1.2.33. [53] For the biacts $_SM_R$ and $_RN_T$, the tensor product $M \otimes N$ can be made into an S-T-biact by defining $s(m \otimes n) = (sm \otimes n)$ and $(m \otimes n)t = (m \otimes nt)$ for $s \in S, t \in T, m \in M$ and $n \in N$.

Definition 1.2.34. [90, 84] A six-tuple $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ is known as a Morita context of monoids (semigroups), where S, T are monoids (resp. semigroups), ${}_SP_T$ and ${}_TQ_S$ are biacts, and $\theta : {}_S(P \otimes_T Q)_S \to {}_SS_S$ and $\phi : {}_T(Q \otimes_S P)_T \to {}_TT_T$ are biact homomorphisms such that for every $p, p' \in P$ and $q, q' \in Q$, $\theta(p \otimes q)p' = p\phi(q \otimes p')$ and $\phi(q \otimes p)q' = q\theta(p \otimes q')$.

Moreover, a Morita context of semigroups is called unitary if ${}_SP_T$ and ${}_TQ_S$ are unitary biacts. In the sense of this definition, every Morita context of monoids is unitary.

Definition 1.2.35. [53] Two monoids S and T are said to be Morita equivalent if the categories S-Act and T-Act (or equivalently, Act-S and Act-T) are two equivalent categories.

Definition 1.2.36. [90] Two semigroups S and T with local units are Morita equivalent if the categories FS-Act and FT-Act (or equivalently, Act-FS and Act-FT) are equivalent categories.

Definition 1.2.37. [91] Semigroups S and T are said to be strongly Morita equivalent if there exists a unitary Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ with θ and ϕ surjective.

Remark 1.2.38. [91] The notions of Morita equivalence and strong Morita equivalence coincide in the case of semigroups with local units.

Definition 1.2.39. [59] By a Morita invariant of a monoid (semigroup), we mean a property of monoid (resp. semigroup) which remains unchanged under (resp. strong) Morita equivalence.

Theorem 1.2.40. *[53] Let T and S be two Morita equivalent monoids via inverse equivalences $F : T$-Act $\to S$-Act and $G : S$-Act $\to T$-Act. Set $P = F(T)$ and $Q = G(S)$. Then P and Q are unitary biacts ${}_SP_T$ and ${}_TQ_S$ such that,*
(1) ${}_SP, {}_TQ, P_T$ and Q_S are, respectively, generators for S-Act, T-Act, Act-T and Act-S;
(2) $T \cong End_S(P) \cong End_S(Q)$ and $S \cong End_T(Q) \cong End_T(P)$;
(3) $F \cong Hom_T(Q, -) \cong Hom_S(Q, -)$ and $G \cong Hom_S(P, -) \cong Hom_T(P, -)$;
(4) ${}_SP_T \cong Hom_T(Q, T) \cong Hom_S(Q, S)$ and ${}_TQ_S \cong Hom_S(P, S) \cong Hom_T(P, T)$.

Theorem 1.2.41. *[59] Let S and T be semigroups with weak local units. If S and T are strongly Morita equivalent via the Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$, then the following maps*

$$\Theta : Id(T) \to Id(S), \quad \theta(J) := \{\theta(pj \otimes q) \mid p \in P, q \in Q, j \in J\},$$
$$\Phi : Id(S) \to Id(T), \quad \Phi(I) := \{\phi(qi \otimes p) \mid p \in P, q \in Q, i \in I\},$$

are mutually inverse isomorphisms between their corresponding lattices of ideals. These isomorphisms also preserve finitely generated ideals and principal ideals.

The following theorem generalizes Theorem 1.2.41 to sub-biacts.

Theorem 1.2.42. *[83] Let S and T be semigroups with weak local units. If S and T are strongly Morita equivalent via the Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$, then the following lattices are isomorphic:*
(1) the lattice of ideals of S,
(2) the lattice of ideals of T,
(3) the lattice of sub-biacts of ${}_SP_T$,
(4) the lattice of sub-biacts of ${}_TQ_S$.

Remark 1.2.43. [35] The corresponding pair of mappings that give respectively the isomorphisms between (1) and (3); (1) and (4) in the above theorem are explicitly written below.

$$f_1 : Id(S) \to Sub(P),\ f_1(I) := \{ip \mid i \in I, p \in P\} = IP \text{ and}$$
$$g_1 : Sub(P) \to Id(S),\ g_1(M) := \{\theta(m \otimes q) \mid m \in M, q \in Q\} = \theta(M \otimes Q)$$
$$f_2 : Id(S) \to Sub(Q),\ f_2(I) := \{qi \mid i \in I, q \in Q\} = QI \text{ and}$$
$$g_2 : Sub(Q) \to Id(S),\ g_2(N) := \{\theta(p \otimes n) \mid p \in P, n \in N\} = \theta(P \otimes N).$$

The mappings $f_3 : Id(T) \to Sub(P)$, $g_3 : Sub(P) \to Id(T)$, $f_4 : Id(T) \to Sub(Q)$, $g_4 : Sub(Q) \to Id(T)$ are defined in an analogous manner.

Remark 1.2.44. [35] Let S and T be two strongly Morita equivalent semigroups and K_P and K_S exist. Then $K_S = g_1(K_P)$ and $K_P = f_1(K_S)$.

1.3 Semirings and semimodules

Now we recall below some definitions and results of semiring theory.

Definition 1.3.1. [31] A semiring is an algebra $(R, +, \cdot, 0_R)$ such that
(1) $(R, +, 0_R)$ is a commutative monoid with identity element 0_R,
(2) $(R, \cdot)$ is a semigroup,
(3) multiplication distributes over addition from either side and
(4) $0_Rr = 0_R = r0_R$ for all $r \in R$.
If moreover there exists an element $1_R \in R$ such that $(R, \cdot, 1_R)$ is a monoid with identity 1_R, then R is called a semiring with identity.

Definition 1.3.2. [31] Let R and S be two semirings. Then a function $f : R \to S$ is called a semiring homomorphism if
(1) $f(0_R) = 0_S$,
(2) $f(r + r') = f(r) + f(r')$ and
(3) $f(rr') = f(r)f(r')$ for all $r, r' \in R$.
If R is a semiring with identity, then we must also have $f(1_R) = 1_S$.

Definition 1.3.3. [31] Let R and S be two semirings. Then a semiring homomorphism $f : R \to S$ is called a semiring isomorphism if it is bijective.

Definition 1.3.4. [31] A left R-semimodule over a semiring R is a commutative monoid $(M, +, 0_M)$ together with a scalar multiplication $R \times M \to M$ which satisfies the following identities for all $r, r' \in R$ and $m, m' \in M$:
(1) $(rr')m = r(r'm)$,
(2) $r(m + m') = rm + rm'$,
(3) $(r + r')m = rm + r'm$,
(4) $r0_M = 0_M = 0_R m$.
If R is a semiring with identity, then we must also have $1_R m = m$ for all $m \in M$.

Definition 1.3.5. [31] Let M and N be two left R-semimodules. Then a monoid homomorphism $f : M \to N$ is called a left R-homomorphism if $f(rm) = rf(m)$ for all $r \in R$ and $m \in M$.

Definition 1.3.6. [31] A right R-semimodule over a semiring R is a commutative monoid $(M, +, 0_M)$ together with a scalar multiplication $M \times R \to M$ which satisfies the following identities for all $r, r' \in R$ and $m, m' \in M$:
(1) $m(rr') = (mr)r'$,
(2) $(m + m')r = mr + m'r$,
(3) $m(r + r') = mr + mr'$,
(4) $0_M r = 0_M = m0_R$.
If R is a semiring with identity, then we must also have $m1_R = m$ for all $m \in M$.

Definition 1.3.7. [31] Let M and N be two right R-semimodules. Then a monoid homomorphism $f : M \to N$ is called a right R-homomorphism if $f(mr) = f(m)r$ for all $r \in R$ and $m \in M$.

Definition 1.3.8. [31] For given semirings R and S, an R-S-bisemimodule M, denoted by ${}_RM_S$, is a commutative monoid which is both a left R-semimodule and a right S-semimodule, with $(rm)s = r(ms)$ for all $r \in R$, $s \in S$ and $m \in M$.

Definition 1.3.9. [31] Let M and N be two R-S-bisemimodules. Then a monoid homomorphism $f : M \to N$ is called an R-S-bisemimodule homomorphism if $f(rms) = rf(m)s$ for all $r \in R$, $s \in S$ and $m \in M$.

Definition 1.3.10. [31] Let M and N be two left R-semimodules (right R-semimodules, R-S-bisemimodules). Then a left R-homomorphism (resp. right R-homomorphism, R-S-bisemimodule homomorphism) is called a left R-isomorphism (resp. right R-isomorphism, R-S-bisemimodule isomorphism) if it is bijective.

Remark 1.3.11. The category formed by left R-semimodules is denoted by ${}_R\mathcal{M}$. Its right analogue is denoted by $\mathcal{M}_R$. Also the category of R-S-bisemimodules is denoted by ${}_R\mathcal{M}_S$.

Definition 1.3.12. [31] Let R be a semiring and $\{M_i \mid i \in \Omega\}$ be a family of left R-semimodules. Then $\times_{i\in\Omega} M_i$ has the structure of a left semimodule under componentwise addition and scalar multiplication. This left semimodule is said to be the direct product of the R-semimodules M_i and is denoted by $\prod_{i\in\Omega} M_i$. Similarly,

$$\coprod_{i\in\Omega} = \{(m_i) \in \prod M_i \mid m_i = 0 \text{ for all but finitely-many indices } i\}$$

is a left R-semimodule and is said to be the coproduct of the R-semimodules M_i. The coproduct is also known as the direct sum of the family of the R-semimodules M_i and is denoted by $\bigoplus_{i\in\Omega} M_i$.

Remark 1.3.13. [31] For each $h \in \Omega$, there are canonical homomorphisms $\pi_h : \prod M_i \to M_h$ and $\iota_h : M_h \to \coprod M_i$ defined respectively by $\pi_h : (m_i) \mapsto m_h$ and $\iota_h : m_h \mapsto (u_i)$, where $u_i = \delta_{ih} m_h$.

Definition 1.3.14. [31] A semiring R is called additively cancellative if $a + x = a + y$ implies $x = y$ for all $a, x, y \in R$.

Definition 1.3.15. [31] A semimodule M is called additively cancellative if $a+x = a+y$ implies $x = y$ for all $a, x, y \in M$.

Definition 1.3.16. [31] A semiring R is called additively idempotent if $a + a = a$ for all $a \in R$.

Definition 1.3.17. [31] A semimodule M is called additively idempotent if $a + a = a$ for all $a \in M$.

Definition 1.3.18. [31] If for each element a of a semiring R there exists an element $b \in R$ such that $a + b + a = a$ then the semiring is said to be additively regular.

Definition 1.3.19. [31] If for each element a of a semimodule M there exists an element $b \in M$ such that $a+b+a=a$ then the semimodule is said to be additively regular.

Definition 1.3.20. [31] A semiring R is said to be zero-sum free if $a+b=0_R$ implies $a=b=0_R$ for all $a,b \in R$.

Definition 1.3.21. [31] A semimodule M is said to be zero-sum free if $a+b=0_M$ implies $a=b=0_M$ for all $a,b \in M$.

Definition 1.3.22. [31] A non-empty subset I of a semiring R is called an ideal of R if $i+j \in I$ and $ri, ir \in I$ for any $i,j \in I$ and $r \in R$.

Remark 1.3.23. [31] The set $Id(R)$ of all ideals of a semiring R forms a lattice with the intersection of two ideals as meet and the sum of two ideals as join.

Definition 1.3.24. [31] A semiring R is called ideal-simple if it does not contain any non-trivial ideal.

Definition 1.3.25. [31] A non-empty subset N of a semimodule ${}_RM$ is called a subsemimodule of M if $n+n' \in N$ and $rn \in N$ for any $n,n' \in N$ and $r \in R$. Subsemimodules of right semimodules and of bisemimodules are defined analogously.

Remark 1.3.26. [31] The set $Sub(M)$ of all subsemimodules of an R-S-bisemimodule M forms a lattice with the intersection of two subsemimodules as meet and the sum of two subsemimodules as join.

Definition 1.3.27. [31] A semimodule M is called subsemimodule-simple if it does not contain any non-trivial subsemimodule.

Definition 1.3.28. [31] An ideal I of a semiring R is called finitely generated if there exists a finite subset A of R such that $I=\langle A\rangle = RAR$.

Definition 1.3.29. [31] A subsemimodule N of a semimodule ${}_RM$ is called finitely generated if there exists a finite subset A of R such that $N=RA=\{\sum_{i=1}^{k} r_i m_i \mid r_i \in R, m_i \in A\}$.

Definition 1.3.30. [31] A semiring R is said to be Noetherian if every ascending chain of ideals terminates.

Definition 1.3.31. [31] A semimodule M is said to be Noetherian if every ascending chain of subsemimodules terminates.

Definition 1.3.32. [40] An ideal I of a semiring R is called a k-ideal of R if for $x \in R$, $y \in I$, $x + y \in I$ implies $x \in I$.

Definition 1.3.33. [31] A subsemimodule N of a semimodule M is said to be a k-subsemimodule[4] of M if for $x \in M$, $y \in N$, $x + y \in N$ implies $x \in N$.

Definition 1.3.34. [40] An ideal I of a semiring R is called an h-ideal of R if for $y_1, y_2 \in I$, $x, z \in R$, $x + y_1 + z = y_2 + z$ implies $x \in I$.

Definition 1.3.35. [69] A subsemimodule N of a semimodule M is said to be an h-subsemimodule of M if for $y_1, y_2 \in N$, $x, z \in M$, $x + y_1 + z = y_2 + z$ implies $x \in N$.

Definition 1.3.36. [40] The k-closure of an ideal I of a semiring R is denoted by $\widehat{I}$ and is defined by $\widehat{I} = \{x \in R \mid x + i \in I, \text{ for some } i \in I\}$. $\widehat{I}$ is the smallest k-ideal of R containing I. I is a k-deal if and only if $I = \widehat{I}$.

Definition 1.3.37. [31] The k-closure of a subsemimodule N of a semimodule M is denoted by $\widehat{N}$ and is defined by $\widehat{N} = \{x \in M \mid x + p \in N, \text{ for some } p \in N\}$.

Definition 1.3.38. [85] The h-closure of an ideal I of a semiring R is denoted by $\widetilde{I}$ and is defined by $\widetilde{I} = \{x \in R \mid x + y_1 + z = y_2 + z \text{ for some } y_1, y_2 \in I, \ z \in R\}$. $\widetilde{I}$ is the smallest h-ideal of R containing I. I is an h-ideal if and only if $I = \widetilde{I}$.

Definition 1.3.39. [95] The h-closure of a subsemimodule N of a semimodule M is denoted by $\widetilde{N}$ and is defined by $\widetilde{N} = \{x \in M \mid x + y_1 + z = y_2 + z \text{ for some } y_1, y_2 \in N, \ z \in M\}$.

Remark 1.3.40. The set of all k-ideals (h-ideals) of a semiring R forms a lattice with the intersection of two k-ideals (resp. h-ideals) as meet and k-closure (resp. h-closure) of the sum of two k-ideals (resp. h-ideals) as join.

Remark 1.3.41. For an R-S-bisemimodule M, the set of all k-subsemimodules (h-subsemimodules) of M forms a lattice with the intersection of two k-subsemimodules (resp. h-subsemimodules) as meet and k-closure (resp. h-closure) of the sum of two k-subsemimodules (resp. h-subsemimodules) as join.

Definition 1.3.42. [82, 34] A semiring R is said to be k-ideal-simple (h-ideal-simple) if it does not contain any non-trivial k-ideal (resp. h-ideal).

[4] known as subtractive subsemimodule in [31].

Definition 1.3.43. [36, 34] A semimodule is said to be k-subsemimodule-simple (h-subsemimodule-simple) if it does not contain any non-trivial k-subsemimodule (resp. h-subsemimodule).

Definition 1.3.44. [31] An equivalence relation ρ defined on a semiring R is called a congruence if $r\rho r'$ and $s\rho s'$ in R implies $(r+s)\rho(r'+s')$ and $(rs)\rho(r's')$.

Remark 1.3.45. [31] The set of all congruences on a semiring R, denoted by $Con(R)$, forms a lattice with meet and join of two congruences, say ρ_1, ρ_2, defined as follows:
(1) $r(\rho_1 \wedge \rho_2)r'$ if and only if $r\rho_1 r'$ and $r\rho_2 r'$.
(2) $r(\rho_1 \vee \rho_2)r'$ if and only if there exists a sequence

$$r = s_0 \rightarrow s_1 \rightarrow \cdots \rightarrow s_n = r'$$

such that $s_i\rho_1 s_{i+1}$ or $s_i\rho_2 s_{i+1}$ for all $i = 0, 1, \ldots, n-1$.

Definition 1.3.46. [31] An equivalence relation ρ defined on an R-S-bisemimodule M is called a congruence if $m\rho m'$ and $n\rho n'$ in M and $r \in R,\ s \in S$ implies $(m+n)\rho(m'+n')$, $(rm)\rho(rm')$ and $(ms)\rho(m's)$.

Remark 1.3.47. [31] The set of all congruences on an R-S-bisemimodule M, denoted by $Con(M)$, forms a lattice.

Definition 1.3.48. [31] A semiring R is called congruence-simple if it does not contain any non-trivial congruence.

Definition 1.3.49. [31] A semimodule M is called congruence-simple if it does not contain any non-trivial congruence.

Definition 1.3.50. [31] An ideal I of a semiring R defines a congruence $\mathcal{B}_I$ on R, called the Bourne congruence, given by $r\mathcal{B}_I r'$ if and only if there exist $a, a' \in I$ satisfying $r + a = r' + a'$.

Definition 1.3.51. [31] An ideal I of a semiring R defines a congruence $\mathcal{I}_I$ on R, called the Iizuka congruence, given by $r\mathcal{I}_I r'$ if and only if there exist $a, a' \in I$ and $r'' \in R$ satisfying $r + a + r'' = r' + a' + r''$.

Definition 1.3.52. [28] A congruence ρ on a semiring R is called a ring congruence if the factor semiring R/ρ is a ring.

Definition 1.3.53. [31] A subsemimodule N of a semimodule M defines a congruence $\mathcal{B}_N$ on M, called the Bourne congruence, given by $m\mathcal{B}_N m'$ if and only if there exist $a, a' \in N$ satisfying $m + a = m' + a'$.

Definition 1.3.54. [31] A subsemimodule N of a semimodule M defines a congruence $\mathcal{I}_N$ on M, called the Iizuka congruence, given by $m\mathcal{I}_N m'$ if and only if there exist $a, a' \in N$ and $m'' \in M$ satisfying $m + a + m'' = m' + a' + m''$.

Definition 1.3.55. [36] A congruence ρ on a semimodule M is called a module congruence if the factor semimodule M/ρ is a module.

Definition 1.3.56. [31] A proper ideal I of a semiring R is called prime ideal if for ideals A, B of R, $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$.

Definition 1.3.57. [31] For an ideal I of R, prime radical of I is denoted by $\sqrt{I}$ and defined to be the intersection of all prime ideals of R containing I.

Definition 1.3.58. [31] The prime radical of the zero ideal of a semiring R is said to be the prime radical of the semiring R.

Definition 1.3.59. [31] A semimodule ${}_RP$ is said to be projective if for any surjective homomorphism $f : M \to N$ between semimodules ${}_RM$, ${}_RN$ and any left R-homomorphism $g : P \to N$, there exists a left R-homomorphism $h : P \to M$ such that $fh = g$.

Theorem 1.3.60. *[31] Let R be a semiring with identity. Then a semimodule ${}_RP$ is projective if and only if it is a retract of a free semimodule F, i.e., there exist a free semimodule R^n for some positive integer n, a surjection $\tau : R^n \to P$ and an injection $\mu : P \to R^n$ such that $\tau\mu = id_P$.*

Theorem 1.3.61. *[31] If $\{P_i | \ i \in \Omega\}$ is a family of left R-semimodules then $P = \bigoplus_{i\in\Omega} P_i$ is projective if and only if each P_i is projective.*

Definition 1.3.62. [49] The trace ideal $tr(P)$ of a semimodule ${}_RP$ is defined as $tr(P) = \sum\limits_{f\in Hom_R(P,R)} f(P)$.

Definition 1.3.63. [49] Let R be a semiring with identity. Then a semimodule ${}_RP \in Ob({}_R\mathcal{M})$ is said to be a generator for the category ${}_R\mathcal{M}$ if the regular semimodule ${}_RR$ is a retract of a finite direct sum $\oplus_i P$ of the semimodule ${}_RP$.

Theorem 1.3.64. *[49] Let R be a semiring with identity. Then for any semimodule ${}_RP \in Ob({}_R\mathcal{M})$ the following are equivalent:*
(1) P is a generator for ${}_RM$.
(2) $tr(P) = R$.

(3) $_RR \in Ob(_R\mathcal{M})$ *is a retract of a direct sum* $\oplus_i P$ *of the semimodule* $_RP$*.*
(4) For every semimodule $_RM \in Ob(_R\mathcal{M})$*, there exists a surjection* $\oplus_i P \to M$ *for some direct sum* $\oplus_i P$*.*

Definition 1.3.65. [49] Let R be a semiring with identity. Then a semimodule $_RP \in Ob(_R\mathcal{M})$ is said to be a progenerator for the category $_R\mathcal{M}$ if it is a finitely generated projective generator.

Definition 1.3.66. [48, 49] Let M_R be a right R-semimodule and $_RN$ be a left R-semimodule. If F is the free $\mathbb{N}$-semimodule generated by the Cartesian product $M \times N$ and σ is the congruence on F generated by all ordered pairs having the form,

$$((m+m', n), (m, n) + (m', n)),\ ((m, n+n'), (m, n) + (m, n')) \text{ and } ((mr, n), (m, rn)),$$

with $m, m' \in M_R$, $n, n' \in {_RN}$ and $r \in R$, then the factor semimodule F/σ is defined to be the tensor product of M and N and is denoted by $M \otimes_R N$. When there is no confusion over the semiring, we denote the tensor product as $M \otimes N$ and the class containing (m, n) by $m \otimes n$.

Definition 1.3.67. [49] Two semirings R and S with identities are said to be Morita equivalent if there exists a progenerator $_RP \in Ob(_R\mathcal{M})$ for $_R\mathcal{M}$ such that $S \cong End(_RP)$ as semirings.

Theorem 1.3.68. *[49] Two semirings* R *and* S *with identities are Morita equivalent if and only if the categories* $_R\mathcal{M}$ *and* $_S\mathcal{M}$ *are equivalent categories.*

Definition 1.3.69. [50] By a Morita invariant of a semiring, we mean a property of semiring which remains unchanged under Morita equivalence.

Theorem 1.3.70. *[49] Let* R *be a semiring with identity,* $_RP \in Ob(_R\mathcal{M})$ *be a progenerator for* $_R\mathcal{M}$ *and* $S := End(_RP)$*. Then*
(1) $Q = P^* := Hom_R(P, R) \cong Hom_S(P, S)$ *as an* S*-*R*-bisemimodule,*
(2) $P \cong Hom_S(Q, S) \cong Hom_R(Q, R)$ *as an* R*-*S*-bisemimodule,*
(3) $R \cong End(P_S) \cong End(_SQ)$ *as a semiring,*
(4) $S \cong End(Q_R)$ *as a semiring and*
(5) $P_S \in Ob(\mathcal{M}_S)$*,* $Q_R \in Ob(\mathcal{M}_R)$ *and* $_SQ \in Ob(_S\mathcal{M})$ *are also progenerators for the categories* $\mathcal{M}_S$*,* $\mathcal{M}_R$ *and* $_S\mathcal{M}$ *respectively.*

Definition 1.3.71. [20] Let R and S be two semirings and $_RP_S$ and $_SQ_R$ be R-S-bisemimodule and S-R-bisemimodule respectively. The quadruple (R, P, Q, S) is called a Morita context if the set $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$ of matrices forms a semiring under matrix addition and multiplication.

This definition will make sense if we assume the existence of mappings

$$P \times Q \to R \quad \text{and} \quad Q \times P \to S \quad \text{denoted by}$$

$$(p, q) \mapsto pq \quad \text{and} \quad (q, p) \mapsto qp$$

such that for all $p_1, p_2, p \in P$; $q_1, q_2, q \in Q$; $r \in R$, $s \in S$ the following eight identities along with their dual are satisfied:

$$\begin{array}{ll}
(p_1 + p_2)q = p_1q + p_2q & p(q_1 + q_2) = pq_1 + pq_2 \\
r(pq) = (rp)q & (pq)r = p(qr) \\
(ps)q = p(sq) & (p_1q)p_2 = p_1(qp_2) \\
p0_Q = 0_R & 0_Pq = 0_R;
\end{array}$$

$$\begin{array}{ll}
(q_1 + q_2)p = q_1p + q_2p & q(p_1 + p_2) = qp_1 + qp_2 \\
s(qp) = (sq)p & (qp)s = q(ps) \\
(qr)p = q(rp) & (q_1p)q_2 = q_1(pq_2) \\
0_Qp = 0_S & q0_P = 0_S.
\end{array}$$

Sardar and Gupta redefined Morita context [81] as followed.

Definition 1.3.72. Let R and S be two semirings and ${}_RP_S$ and ${}_SQ_R$ be an R-S-bisemimodule and an S-R-bisemimodule, respectively and $\theta : P \otimes_S Q \to R$ and $\phi : Q \otimes_R P \to S$ be an R-S-bisemimodule homomorphism and an S-R-bisemimodule homomorphism, respectively, such that $\theta(p \otimes q)p' = p\phi(q \otimes p')$ and $\phi(q \otimes p)q' = q\theta(p \otimes q')$ for all $p, p' \in P$ and $q, q' \in Q$. Then the sixtuple $(R, S, P, Q, \theta, \phi)$ is called a Morita context for semirings.

Remark 1.3.73. In the rest of this section, every semiring is considered to have an identity.

Theorem 1.3.74. *[81] Let R and S be two Morita equivalent semirings. Then there exists a Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ with θ and ϕ surjective.*

Theorem 1.3.75. *[81] Let $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ be a Morita context with θ and ϕ surjective. Then*

(1) R and S are Morita equivalent semirings.

(2) $Q \cong Hom_R(P, R) \cong Hom_S(P, S)$ as an S-R-bisemimodule and $P \cong Hom_S(Q, S) \cong$

$Hom_R(Q,R)$ as an R-S-bisemimodule.
(3) $R \cong End(P_S) \cong End({}_SQ)$ as a semiring and $S \cong End({}_RP) \cong End(Q_R)$ as a semiring.
(4) ${}_RP \in Ob({}_R\mathcal{M})$, $P_S \in Ob(\mathcal{M}_S)$, $Q_R \in Ob(\mathcal{M}_R)$ and ${}_SQ \in Ob({}_S\mathcal{M})$ are progenerators for the categories ${}_R\mathcal{M}$, $\mathcal{M}_S$, $\mathcal{M}_R$ and ${}_S\mathcal{M}$ respectively.

Theorem 1.3.76. *[81] The following are equivalent for two given semirings R and S:*
(1) R and S are Morita equivalent semirings.
(2) There exists a Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ with θ and ϕ surjective.

Theorem 1.3.77. *[36] Let R and S be Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then we see that the lattice of ideals of R and the lattice of subsemimodules of P are isomorphic via the following mappings.*

$$f_1 : Id(R) \to Sub(P) \text{ and } g_1 : Sub(P) \to Id(R) \text{ are defined by}$$
$$f_1(I) := \left\{ \sum_{k=1}^{n} i_k p_k \mid p_k \in P,\ i_k \in I \text{ for all } k;\ n \in \mathbb{Z}^+ \right\},$$
$$\text{and } g_1(M) := \left\{ \sum_{k=1}^{n} \theta(p_k \otimes q_k) \mid p_k \in M,\ q_k \in Q \text{ for all } k;\ n \in \mathbb{Z}^+ \right\}$$

Moreover, this isomorphism takes finitely generated ideals to finitely generated subsemimodules and vice-versa. Similar isomorphism can be defined for other pairs of the Morita context as follows.

$$f_2 : Id(R) \to Sub(Q) \text{ and } g_2 : Sub(Q) \to Id(R) \text{ are defined by}$$
$$f_2(I) := \left\{ \sum_{k=1}^{n} q_k i_k \mid q_k \in Q,\ i_k \in I \text{ for all } k;\ n \in \mathbb{Z}^+ \right\},$$
$$\text{and } g_2(N) := \left\{ \sum_{k=1}^{n} \theta(p_k \otimes q_k) \mid p_k \in P,\ q_k \in N \text{ for all } k;\ n \in \mathbb{Z}^+ \right\}$$

We can also define $f_3 : Id(S) \to Sub(P)$, $g_3 : Sub(P) \to Id(S)$, $f_4 : Id(S) \to Sub(Q)$, $g_4 : Sub(Q) \to Id(S)$ in a similar way.

Theorem 1.3.78. *[36] Let R and S be Morita equivalent semirings via the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the lattice of k-ideals (h-ideals) of R and the lattice of k-subsemimodules (respectively h-subsemimodules) of P are isomorphic.*

Remark 1.3.79. [36] The f_is and g_is in Theorem 1.3.77 take k-ideals (h-ideals) to k-subsemimodules (respectively h-subsemimodules) and vice-versa.

Theorem 1.3.80. *[82] Let R and S be Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then we see that the lattice of ideals of R and the lattice of* *ideals of S are isomorphic. Moreover this isomorphism takes finitely generated ideals to finitely generated ideals and vice-versa.*

$$\Theta : Id(S) \to Id(R) \text{ and } \Phi : Id(R) \to Id(S) \text{ are defined by}$$
$$\Theta(J) := \left\{ \sum_{k=1}^{n} \theta(p_k j_k \otimes q_k) \mid p_k \in P, q_k \in Q, j_k \in J \text{ for all } k;\ n \in \mathbb{Z}^+ \right\},$$
$$\text{and } \Phi(I) := \left\{ \sum_{k=1}^{n} \phi(q_k i_k \otimes p_k) \mid p_k \in P, q_k \in Q, i_k \in I \text{ for all } k;\ n \in \mathbb{Z}^+ \right\}.$$

Theorem 1.3.81. *[82] If $\{A_i \mid i \in I\}$ is an arbitrary set of ideals of a semiring R, then $\Phi(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} \Phi(A_i)$. Similar results hold for the map Θ.*

Remark 1.3.82. [82] Both Θ and Φ preserve k-ideals.

1.4 Topology

Definition 1.4.1. [71] A topology on a set X is a collection τ of subsets of X having the following properties:

(1) $\varnothing$ and X are in τ.

(2) The union of the elements of any subcollection of τ is in τ.

(3) The intersection of the elements of any finite subcollection of τ is in τ.

Remark 1.4.2. A topological space is an ordered pair (X, τ) consisting of a set X and a topology τ, but we often omit specific mention of τ if no confusion will arise.

Remark 1.4.3. [71] If X is any set, the collection of all subsets of X is a topology on X and is called the discrete topology. The collection consisting of X and $\varnothing$ only is also a topology on X and is called the indiscrete topology.

Definition 1.4.4. [71] Suppose that τ and τ' are two topologies on a given set X. If $\tau \subseteq \tau'$, we say that τ' is finer than τ. We also say that τ is coarser than τ'.

Definition 1.4.5. [71] A subset U of X is said to be an open set of X if $U \in \tau$.

Definition 1.4.6. [96] If X is a topological space and $x \in X$, a neighborhood of x is a set U which contains an open set V containing x.

Definition 1.4.7. [71] If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

(1) For each $x \in X$, there is at least one basis element B containing x.

(2) If $x \in B_1 \cap B_2$ for basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

Remark 1.4.8. [71] Let $\mathcal{B}$ be a basis for a topology τ on a set X. Then τ equals the collection of all unions of elements of $\mathcal{B}$.

Remark 1.4.9. [71] Suppose X is a topological space and $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.

Definition 1.4.10. [71] A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection τ of all unions of finite intersections of elements of $\mathcal{S}$.

Definition 1.4.11. [71] Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Theorem 1.4.12. *[71] The collection*

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$, where the maps π_1 and π_2 are projections of $X \times Y$ onto X and Y, respectively.

Definition 1.4.13. [71] Let (X, τ) be a topological space. If Y is a subset of X, the collection $\tau_Y = \{Y \cap U \mid U \in \tau\}$ is a topology on Y, called the subspace topology.

Definition 1.4.14. [71] A subset A of a topological space X is said to be closed if the set $X \setminus A$ is open.

Theorem 1.4.15. *[71] Let X be a topological space. Then the following conditions hold:*

(1) $\varnothing$ and X are closed.

(2) Arbitrary intersections of closed sets are closed.

(3) Finite unions of closed sets are closed.

Remark 1.4.16. [71] One could specify a topology on a space by giving a collection of sets (to be called "closed sets") satisfying the three properties of Theorem 1.4.15, then define open sets as the complements of closed sets and proceed just as before.

Definition 1.4.17. [71] For a subset A of a topological space X, the closure of A is defined as the intersection of all closed sets containing A and is denoted by $\bar{A}$.

Definition 1.4.18. [71] Let X and Y be topological spaces. A function $f : X \to Y$ is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Remark 1.4.19. [71] To prove continuity of f it suffices to show that the inverse image of every basis element is open.

Remark 1.4.20. [71] To prove continuity of f it suffices to show that the inverse image of every subbasis element is open.

Theorem 1.4.21. *[71] Let X and Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if for every closed set B of Y, the set $f^{-1}(B)$ is closed in X.*

Definition 1.4.22. [71] Let X and Y be topological spaces and $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a homeomorphism.

Definition 1.4.23. [96] A topological space X is a T_0-space if and only if whenever x and y are distinct points in X, there is an open set containing one and not the other.

Definition 1.4.24. [96] A topological space X is a T_1-space if and only if whenever x and y are distinct points in X, there is a neighbourhood of each not containing the other.

Definition 1.4.25. [96] A topological space X is a T_2-space (Hausdorff space) if and only if whenever x and y are distinct points in X, there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.

Remark 1.4.26. Every T_2 space is T_1.

Definition 1.4.27. [96] A topological space X is a regular space if and only if whenever A is closed in X and $x \notin A$, then there are disjoint open sets U and V with $x \in U$ and $A \subseteq V$.

Definition 1.4.28. [96] A topological space X is completely regular if and only if whenever A is a closed set in X and $x \notin A$, then there is a continuous function $f : X \to \{0, 1\}$ such that $f(x) = 0$ and $f(A) = 1$.

Definition 1.4.29. [71] A collection $\mathcal{A}$ of subsets of a topological space X is said to cover X, or to be a covering of X, if the union of elements of $\mathcal{A}$ is equal to X. It is called an open covering of X if its elements are open subsets of X.

Definition 1.4.30. [71] A topological space X is said to be compact if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Chapter 2

Morita equivalence of semirings with local units

In this chapter, we aim to extend the theory of Morita equivalence of semirings to cover a wider range of semirings namely the semirings with local units, in the sense that any two elements of the semiring have a common two-sided identity. In order to develop this theory we consider the category R-Sem consisting of all unitary left R-semimodules M, i.e., semimodules ${}_RM$ such that $RM = M$, where R is a semiring with local units and call two such semirings R and S to be Morita equivalent if the categories R-Sem and S-Sem are equivalent. Since for a semiring R with identity, R-Sem coincides with the category ${}_R\mathcal{M}$ of all left R-semimodules, our notion of Morita equivalence coincides with that of semiring with identity [49]. Consequently, some of the results of Katsov et al. [49] are encompassed in their counterparts obtained here. We have arranged the chapter in the following way. Firstly we define locally projective unitary R-semimodule (*cf.* Definition 2.1.11) and present some characterizing properties of locally projective generators (*cf.* Propositions 2.1.18 - 2.1.22) in semimodule categories. Then we develop some tools to investigate some necessary and sufficient conditions for R-Sem and S-Sem to be equivalent. Analogous to the case of semirings with identity, we show that two semirings with local units R and S are Morita equivalent if and only if there exists a unitary Morita context $(R, S, P, Q, \theta, \phi)$ with θ, ϕ surjective (*cf.* Theorem 2.2.15).

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We also identify the semirings with local units that are Morita equivalent to semirings with identity (*cf.* Proposition 2.2.16). Finally, we study some properties of semirings with local units, which are preserved under Morita equivalence (*cf.* Theorem 2.3.1 - Corollary 2.3.13).

For preliminaries of category theory, semirings and semimodules, we refer, respectively, to Section 1.1, Section 1.3 of Chapter 1.

We adopt the following notion from Ánh and Márki [5].

Definition 2.0.1. Let R be a semiring and $E(R)$ be a set of idempotents of R. Then R is said to be a *semiring with local units* if every finite subset of R is contained in a subsemiring of the form eRe where $e \in E(R)$ or equivalently if for any finite number of elements $r_1, r_2, ..., r_n \in R$, there exists $e \in E(R)$ such that $er_i = r_i = r_ie$ for all $i = 1, 2, ..., n$. In this case $E(R)$ is a *set of local units (slu)* of R.

Here we give some examples of semirings with local units.

Example 2.0.2.

1. Suppose L is a distributive lattice with the least element 0 but with no greatest element[1]. Consider L together with the addition $+$ and multiplication $\cdot$ defined by $a + b = sup\{a, b\}$ and $a \cdot b = inf\{a, b\}$ respectively, for $a, b \in L$. Then $(L, +, \cdot)$ is a semiring with additive identity 0 but with no multiplicative identity. But it is a semiring with local units, as for any two elements $a, b \in L$, by the absorption law, $a \cdot (a + b) = a = (a + b) \cdot a$ and $b \cdot (a + b) = b = (a + b) \cdot b$, i.e., $a + b$ acts as the common two-sided identity of a and b.

2. Let S be a semiring with identity, X be an infinite set and $R = \{f \mid f : X \to S$ has finite support$\}$. Then R together with the operations $(f+g)(x) := f(x)+g(x)$ and $(fg)(x) := f(x)g(x)$ for $f, g \in R$ and $x \in X$ is a semiring without multiplicative identity. But it is a semiring with local units in view of the following reason. Suppose $f, g \in R$ with finite supports $supp(f)$ and $supp(g)$ respectively, define $h : X \to S$ by $h(x) = 1$ if $x \in supp(f) \cup supp(g)$ and $h(x) = 0$ otherwise, then for $x \in supp(f)$, $fh(x) = f(x)h(x) = f(x)$ and for $x \in X \smallsetminus supp(f)$, $fh(x) = f(x)h(x) = 0 \cdot h(x) = 0 = f(x)$. By a similar argument $hf = f$ and hence $fh = f = hf$ and similarly $gh = g = hg$, i.e., h acts as a two-sided identity of f and g.

[1] $(\mathbb{N}, lcm, gcd)$, $(\mathbb{N}, max, min)$, where $\mathbb{N}$ is the set of all non-negative integers, are some examples of such lattices.

Definition 2.0.3. A left R-semimodule M over R is said to be unitary if $RM = M$ i.e., for each $m \in M$, there exist $r_1, r_2, ..., r_n \in R$, $m_1, m_2, ..., m_n \in M$ such that $m = r_1m_1 + r_2m_2 + ... + r_nm_n$.

Remark 2.0.4. If R is a semiring with slu E and M is a unitary R-semimodule then for each $m \in M$, $m = r_1m_1 + r_2m_2 + ... + r_nm_n$ for some $r_1, r_2, ..., r_n \in R$, $m_1, m_2, ..., m_n \in M$. Now for $r_1, r_2, ..., r_n \in R$, there exists $e \in E$ such that $er_i = r_i$ for all $i = 1, 2, ..., n$, therefore $m = \sum_{i=1}^{n} r_im_i = \sum_{i=1}^{n} er_im_i = em$. Thus for every finite subset $M' \subset M$ there exists an $e \in E$ such that $eM' = M'$.

By R-Sem we denote the category of unitary left R-semimodules together with usual R-morphisms. Analogously we denote the category of unitary right S-semimodules (unitary R-S bisemimodules) together with usual semimodule morphisms by Sem-S (resp. R-Sem-S).

2.1 Locally projective generators

Throughout this chapter, unless otherwise mentioned, any semiring is with local units and homomorphisms of semimodules are written opposite the scalars.

Recall that [49], for any R-semimodule ${}_RP$, the trace ideal $tr(P) = \sum\limits_{q \in Hom_R(P,R)} Pq \subseteq R$.

Proposition 2.1.1. *Let R be a semiring with local units. For any semimodule $P \in Ob(R\text{-}Sem)$, the following are equivalent:*

(1) $tr(P) = R$.

(2) There exists a surjective R-morphism $\sigma : \bigoplus_I P \to R$ for some index set I.

(3) For every semimodule $M \in Ob(R\text{-}Sem)$, there exists a surjective R-morphism $\psi : \bigoplus_\Lambda P \to M$ for some index set Λ.

Proof. **(1) ⇒ (2)** Consider the family of all R-morphisms, $\sigma_\alpha : P \to R$. Now if we set $I = Hom_R(P, R)$, then the coproduct induced map $\sigma = \bigoplus_I \sigma_\alpha : \bigoplus_I P \to R$ is a surjective R-morphism since $(\bigoplus_I P)\sigma = \sum_{\sigma_\alpha \in I} P\sigma_\alpha = tr(P) = R$.

(2) ⇒ (3) Suppose there exists a surjective R-morphism $\sigma : \bigoplus_I P \to R$ for some index set I. Let $M \in Ob(R\text{-Sem})$. Then for each $m \in M$ consider the map $\rho_m : R \to M$ defined by $r \mapsto rm$. Then the coproduct induced map $\rho = \bigoplus_{m \in M} \rho_m : \bigoplus_M R \to M$ is a surjective R-morphism since $(\bigoplus_M R)\rho = \sum_{m\in M} R\rho_m = \sum_{m\in M} Rm = M$. Then the direct sum $\sigma' = \bigoplus_M \sigma : \bigoplus_M(\bigoplus_I P) \to \bigoplus_M R$ is a surjection. Hence $\psi = \sigma'\rho : \bigoplus_\Lambda P \to M$ is a surjective R-morphism, where $\Lambda = \dot{\cup}_M I$[1].

(3) $\Rightarrow$ **(2)** Follows trivially.

(2) $\Rightarrow$ **(1)** Suppose there exists a surjective R-morphism $\sigma : \bigoplus_I P \to R$ for some index set I. Consider the natural inclusions $\iota_i : P \to \bigoplus_I P$ for all $i \in I$. Now for each $i \in I$, let $\sigma_i = \iota_i\sigma$, then $R = (\bigoplus_I P)\sigma = \sum_{i\in I} P\sigma_i \subseteq \sum_{q\in Hom_R(P,R)} Pq = tr(P)$. Hence $tr(P) = R$. □

Definition 2.1.2. A semimodule $P \in Ob(R\text{-Sem})$ is said to be a generator for the category R-Sem if P satisfies the equivalent conditions of Proposition 2.1.1.

Suppose R is a semiring with local units. Let M be a unitary left R-semimodule and A be a subset of M. Then $RA = \{r_1a_1 + r_2a_2 + ... + r_na_n \mid n \in \mathbb{N},\ r_i \in R,\ a_i \in A,$ for all $i = 1, 2, ..., n\}$ is the subsemimodule generated by A. If A generates all of the semimodule M then A is a set of generators for M. A unitary R-semimodule M is said to be ***finitely generated*** if it has a finite set of generators.

We skip the proof of the following proposition as it is analogous to that of its counterpart in module theory (see [4, Proposition 10.1]).

Proposition 2.1.3. *If M is a finitely generated unitary left R-semimodule then the following hold:*

(1) For every set $\mathcal{A}$ of subsemimodules of M that spans M, there is a finite set $\mathcal{F} \subseteq \mathcal{A}$ that spans M.

(2) Every semimodule that generates M, finitely generates M.

Definition 2.1.4. Let R be a semiring with local units. A semimodule $P \in Ob(R\text{-Sem})$ is said to be projective if for a surjective R-morphism $\phi : M \to N$ and an R-morphism $\alpha : P \to N$ in R-Sem, there exists an R-morphism $\overline{\alpha} : P \to M$ satisfying $\overline{\alpha}\phi = \alpha$.

Remark 2.1.5. Notice that the above definition is analogous to the case of semiring with identity [31] (see Definition 1.3.59).

[1] $\dot{\cup}$ denotes the disjoint union

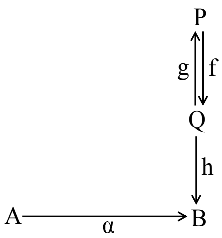
Remark 2.1.6. We see that the usual categorical definitions of generator (see Definition 1.1.36 and Remark 1.1.37) and projective object (see Definition 1.1.34) involve the notion of epimorphism, but instead of using them we use the above definitions (*cf.* Definitions 2.1.2 and 2.1.4) due to the following reasons.

(i) In R-Sem even though surjectivity implies epimorphism, the converse does not hold. So unlike the situation with modules it is usually not very easy to visualize epimorphisms in R-Sem. Hence we consider the notion of surjectivity (which coincides with the notion of epimorphism in module category) while defining generator, thus generalizing the idea of generators in module category.

(ii) While defining projective semimodules, Golan [31] used the notion of surjectivity instead of epimorphism. Also Katsov and Nam [49] used the notion of surjectivity while characterizing generator (see Theorem 1.3.64) in semimodule category ${}_R\mathcal{M}$, where R is a semiring with identity.

Lemma 2.1.7. *Retract of a projective unitary R-semimodule is projective.*

Proof. Let $P \in Ob(R\text{-Sem})$ be a projective R-semimodule (*cf.* Definition 2.1.4) and Q be a retract (see Definition 1.1.29) of P. Then there exists a retraction $f : P \to Q$ and a coretraction g such that $gf = id_Q$. Consider the following diagram in R-Sem,



where α is a surjection and h is an R-morphism. Since P is projective there exists an R-morphism $\beta : P \to A$ such that $\beta\alpha = fh$. Then $h' = g\beta$ is an R-morphism for which $h'\alpha = h$. Thus Q is projective. □

The next result is simply a restatement of Theorem 1.3.61 in the special case of the category R-Sem, where R is a semiring with local units.

Proposition 2.1.8. *If $\{P_i | \; i \in \Omega\}$ is a family of unitary left R-semimodules then $P = \bigoplus_{i\in\Omega} P_i$ is projective if and only if each P_i is projective.*

Proposition 2.1.9. *${}_RP$ is a finitely generated projective unitary semimodule if and only if there exists an idempotent $e \in R$ such that P is a retract of $(Re)^n$, $n \geq 1$.*

Proof. Suppose P is a finitely generated projective unitary semimodule. If $P = \{0_P\}$ then the zero map $\theta : Re \to P$ is a retraction in R-Sem. So we assume that $P \neq \{0_P\}$ and $\{p_1, p_2, ..., p_n\}$ is a spanning set of ${}_RP$. Then there exists $e^2 = e \in R$ such that $ep_i = p_i$ for all $i = 1, 2, ..., n$. Consider $\sigma : (Re)^n \to P$ defined by $(x_1, x_2, ..., x_n)\sigma = \sum_{i=1}^{n} x_i p_i$. Since for any $p \in P$ there exist $r_1, r_2, ..., r_n \in R$ such that $p = \sum_{i=1}^{n} r_i p_i$, $(r_1 e, r_2 e, ..., r_n e)\sigma = \sum_{i=1}^{n} r_i e p_i = \sum_{i=1}^{n} r_i p_i = p$. Thus σ is onto. Now P being projective, there exists $h : P \to (Re)^n$ such that $h\sigma = id_P$.

Conversely, suppose $\psi : (Re)^n \to P$ is a retraction in R-Sem. Let $f : A \to B$ be a surjection in R-Sem and $g : Re \to B$ be an R-morphism. Define $\overline{g} : Re \to A$ by $t \mapsto ta$, where $t \in Re$ and $a \in A$ such that $af = eg$ (if there are more than one $a \in A$ with $af = eg$ then we choose any one of them and fix it throughout). Then $\overline{g}f = g$, hence Re is projective. Therefore by Proposition 2.1.8, $(Re)^n$ is projective and from Lemma 2.1.7, ${}_RP$ is projective. Also since $(Re)^n$ has a finite spanning set $\{e_i : i = 1, 2, ..., n\}$, where each $e_i = (0_R, ..., e, ..., 0_R)$, with e in the $i-$th place for all $i = 1, 2, ..., n$, P is spanned by $\{e_i\psi : i = 1, 2, ..., n\}$. Thus ${}_RP$ is finitely generated. □

The notions introduced in the following two definitions are adopted from Ánh and Márki [5].

Definition 2.1.10. Let I be a partially ordered set such that for each $i, j \in I$ there exists $k \in I$ with $i, j \leqslant k$ and $(M_i)_{i\in I}$ a family of unitary R-semimodules. Then $(M_i)_{i\in I}$ is said to be a *direct system* if for any $i \leqslant j$ we have R-morphism $\sigma_{ij} : M_i \to M_j$ such that $\sigma_{ii} = 1_{M_i}$ for all $i \in I$ and $\sigma_{ij}\sigma_{jk} = \sigma_{ik}$ for $i \leqslant j \leqslant k$.

Moreover a direct system $(M_i)_{i\in I}$ is called a *split direct system* if for each $i \leqslant j$ in I there exists $\psi_{ji} : M_j \to M_i$ such that $\sigma_{ij}\psi_{ji} = 1_{M_i}$ and $\psi_{kj}\psi_{ji} = \psi_{ki}$ for $i \leqslant j \leqslant k$. In this case it follows that $\psi_{ii} = 1_{M_i}$.

Definition 2.1.11. A unitary R-semimodule M is said to be locally projective if it is the direct limit of a split direct system consisting of subsemimodules that are finitely generated projective, i.e., $M = \varinjlim_I M_i$ where each M_i is a finitely generated projective subsemimodule of M.

Proposition 2.1.12. *The R-semimodule ${}_RR$ is a locally projective generator.*

Proof. Let E be a set of local units of R. Define a binary relation $\leqslant$ on E by $e \leqslant f$ if and only if $ef = fe = e$. Then clearly $\leqslant$ is a partial order relation on E and R being a semiring with local units, $(E, \leqslant)$ is an upward directed set. Now for each idempotent $e \in R$ and for each pair $e, f \in R$ with $e \leqslant f$, consider the map $\psi_{fe} : Rf \to Re$ given by

$r' \mapsto r'e$, where $r' \in Rf$ and the natural inclusion maps $\sigma_e : Re \to R$ and $\sigma_{ef} : Re \to Rf$. Then $(Re)_{e\in E}$ is a split direct system in R-Sem and $R = \varinjlim_E Re$, where Re is finitely generated projective R-semimodule (as seen in the proof of Proposition 2.1.9) for each $e \in E$. Hence R is locally projective. Also for any unitary R-semimodule M and for each $m \in M$ consider the map $\rho_m : R \to M$ defined by $r \mapsto rm$, then we have $\rho = \bigoplus_{m\in M}\rho_m : \bigoplus_M R \to M$, where $(\bigoplus_M R)\rho = \sum_{m\in M} R\rho_m = \sum_{m\in M} Rm = M$, which implies that ρ is a surjection. Therefore R is a generator in R-Sem. □

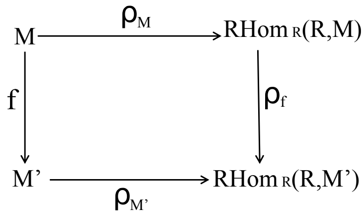
Proposition 2.1.13. *Let M be a locally projective unitary R-semimodule, then every finitely generated subsemimodule P of M is contained in a finitely generated projective subsemimodule of M.*

Proof. Let M be a locally projective unitary R-semimodule. Then there exists a split direct system (*cf.* Definition 2.1.10) $(M_i)_{i\in I}$ of finitely generated projective subsemimodules of M such that $M = \varinjlim_I M_i$. Let $M' = \dot{\cup} M_i/\rho$, where ρ on $\dot{\cup} M_i$ is given by $(x,i)\rho(y,j)$ if and only if there exists $k \in I,\ i,j \leqslant k$ such that $x\sigma_{ik} = y\sigma_{jk}$, where $i,j \in I,\ x \in M_i,\ y \in M_j$. Using the existence of $\psi_{j'i'}$ for each $i',j' \in I,\ i' \leqslant j'$, it then easily follows that $(x,i)\rho(y,j)$ if and only if $x\sigma_{ik} = y\sigma_{jk}$ for all $k \in I,\ i,j \leqslant k$. Now it is a routine matter to verify that M' together with the family of R-morphisms $\sigma_i : M_i \to M'$ given by $x \mapsto [(x,i)]_\rho$ is the direct limit of the split direct system $(M_i)_{i\in I}$. Let P be a subsemimodule of M with a finite spanning set $\{p_1, p_2, ..., p_n\}$. Then identifying M with M' we have $p_k = [(x_k, i_k)]_\rho$ for each $k = 1,2,...,n$, where $i_k \in I,\ x_k \in M_{i_k}$. Let $t \in I$ such that $i_k \leqslant t$ for all $k = 1,2,...,n$. Then for each $k = 1,2,...,n$ we have $p_k = x_k\sigma_{i_k} = x_k\sigma_{i_k t}\sigma_t \in M_t\sigma_t$. Therefore $P \subseteq M_t\sigma_t \cong M_t$, where M_t is a finitely generated projective subsemimodule of M. Hence the proof is complete. □

We observe that if R and S are semirings with local units and U_S and ${}_RV_S$ are unitary then $Hom_S(U,V)$ is a left R-semimodule by putting, for $\varphi \in Hom_S(U,V)$ and $r \in R,\ (r\varphi)(u) = r\varphi(u)$ for $u \in U$. The subsemimodule $RHom_S(U,V)$ is the largest unitary R-subsemimodule of $Hom_S(U,V)$.

Proposition 2.1.14. *Suppose R is a semiring with slu E. Then $\rho : I_{R-Sem} \to RHom_R(R,-)$ is a natural isomorphism where for each $M \in Ob(R\text{-}Sem)$, $\rho_M : M \to RHom_R(R,M)$ is given by $m \mapsto m\rho_M\ (r \mapsto rm)$. For $M' \in Ob(R\text{-}Sem)$ and $f \in Hom_R(M,M'),\ \rho_f : RHom_R(R,M) \to RHom_R(R,M')$ is given by $\gamma \mapsto \gamma f$.*

Proof. Clearly ρ_M is an R-morphism. Also the following diagram commutes:



since $r((m\rho_M)\rho_f) = r(m\rho_M f) = (rm)f = r(mf) = r((mf)\rho_{M'})$. Hence ρ is a natural transformation. For $M \in Ob(R\text{-Sem})$, let $m_1,\ m_2 \in M$, such that $m_1\rho_M = m_2\rho_M$. Now since there exists $e \in E$ such that $em_1 = m_1,\ em_2 = m_2$, we have $m_1 = e(m_1\rho_M) = e(m_2\rho_M) = m_2$. Hence ρ_M is injective. Now let $rf \in RHom_R(R, M)$ and $(r)f = m \in M$, then for any $t \in R,\ t(m\rho_M) = tm = t((r)f) = (tr)f = t(rf)$, i.e., $m\rho_M = rf$. Thus ρ is a natural isomorphism. □

Definition 2.1.15. [48, 49] Let M_R be a right R-semimodule and ${}_RN$ be a left R-semimodule. If F is the free $\mathbb{N}$-semimodule generated by the cartesian product $M \times N$ and σ is the congruence on F generated by all ordered pairs having the form $((m+m', n), (m, n)+(m', n))$, $((m, n+n'), (m, n)+(m, n'))$ and $((mr, n), (m, rn))$ with $m, m' \in M_R$, $n, n' \in {}_RN$ and $r \in R$, then the factor semimodule F/σ is defined to be the tensor product of M and N and is denoted by $M \otimes_R N$. When there is no confusion over the semiring, we denote the tensor product as $M \otimes N$ and the class containing (m, n) by $m \otimes n$.

Remark 2.1.16. Notice that the usual definition of tensor product (see Definition 1.3.66) makes no use of the identity in the semiring, hence it makes sense in our case too.

Proposition 2.1.17. *Suppose R is a semiring with slu E and $M \in Ob(R\text{-}Sem)$. Then $R \otimes M \cong M$.*

Proof. Suppose R is a semiring with slu E and M is a unitary R-semimodule. Consider the map $\mu : M \to R \otimes M$ defined by $m \mapsto e \otimes m$, where $m \in M$ and $e \in E$ such that $em = m$. First we show that the definition is independent of the choice of the idempotent e. Suppose e and f are two idempotents in R such that $em = m = fm$. Let $g \in E$ be a common identity of e and f, then $e \otimes m = ge \otimes m = g \otimes em = g \otimes m$. Similarly $f \otimes m = g \otimes m$, hence $e \otimes m = f \otimes m$. Now it is a routine matter to verify that μ is an R-morphism. Also consider the map $\psi : R \otimes M \to M$ defined by $r \otimes m \mapsto rm$, where $r \in R$ and $m \in M$. Clearly ψ is a well defined R-morphism.

Now for $r \in R,\ m \in M$, we have

$$\begin{aligned}(r \otimes m)\psi\mu = (rm)\mu &= e \otimes rm, \text{ where } e \in E \text{ such that } er = r, \text{ hence } erm = rm\\ &= er \otimes m\\ &= r \otimes m.\end{aligned}$$

$$\begin{aligned}\text{Also } m\mu\psi &= (g \otimes m)\psi, \text{ where } g \in E \text{ such that } gm = m\\ &= gm = m.\end{aligned}$$

Hence μ is an isomorphism, i.e., $R \otimes M \cong M$. □

Suppose R is a semiring with slu $E(R)$ and ${}_RP$ is a unitary semimodule. Let T be a subsemiring of End_RP having local units $E(T)$ such that $TEnd_RP = T$ and $P \in Ob(\text{Sem-}T)$. Now consider the T-R bisemimodule $Q = THom_R(P, R)R$. Then define:

$$\begin{aligned}\theta : P \otimes Q &\to R \qquad \text{and} \qquad & \phi : Q \otimes P &\to T\\ p \otimes q &\mapsto pq & q \otimes p &\mapsto qp\ (p' \mapsto (p'q)p)\end{aligned}$$

It is routine to verify that the maps θ, ϕ are respectively R-R and T-T bisemimodule morphisms. Also, there is a QPQ-associativity, i.e., for any $q, q' \in Q$ and $p' \in P$, $q(p'q') = (qp')q'$ since for any $p \in P$, $p(q(p'q')) = (pq)(p'q') = ((pq)p')q' = (p(qp'))q' = p((qp')q')$ i.e., $q(pq') = (qp)q'$.

In the notations introduced above, we obtain the following results (*cf.* Propositions 2.1.18 - 2.1.22) characterizing locally projective generators, which are the counterparts of Proposition 3.7, Proposition 3.10, Theorem 3.11, Proposition 3.12, Corollary 3.13 respectively of [49] in our setting.

Proposition 2.1.18. *${}_RP$ is locally projective and Pf is finitely generated for all $f \in E(T)$ if and only if $\phi : Q \otimes P \to T$ is a surjection. Moreover, if ϕ is a surjection, then it is an isomorphism.*

Proof. For the necessary part, let $f \in E(T)$. Then since Pf is finitely generated, by Proposition 2.1.13, there exists a finitely generated projective subsemimodule P' of P such that $Pf \subseteq P'$, i.e., $Pf = Pf^2 \subseteq P'f \subseteq Pf$. Therefore $Pf = P'f$, hence it is projective (since $P'f$ being a retract of P' is projective). Therefore by Proposition 2.1.9, there exists a retraction $\sigma : (Re)^n \to Pf$ for some $n \in \mathbb{N}$, $e^2 = e \in R$ with coretraction $\psi : Pf \to (Re)^n$, i.e., $\psi\sigma = id_{Pf}$. Consider $e_i \in (Re)^n$ with e as the i-th coordinate and all others being 0_R for each $i = 1, 2, ..., n$, then for the canonical projections $\pi_i : (Re)^n \to Re$ we have $\sum_{i=1}^{n} x\pi_i e_i = x$ for all $x \in (Re)^n$. Let $p_i = e_i\sigma$

and $\alpha_i = \pi\psi\pi_i$ for each $i = 1, 2, ..., n$, where $\pi : {}_RP \to_R Pf$ is given by $p \mapsto pf$. Now if we put $q_i = f\alpha_i e \in THom_R(P, R)R = Q$, for all $i = 1, 2, ..., n$. Then for any $p \in P$, we have $pq_i = p(f\alpha_i e) = ((pf)\alpha_i)e = ((pf)(\pi\psi\pi_i))e = (pf)(\psi\pi_i)$ for all $i = 1, 2, ..., n$. Therefore for any $p \in P$,

$$p\sum_{i=1}^{n} q_i p_i = \sum_{i=1}^{n} p(q_i p_i) = \sum_{i=1}^{n} (pq_i)p_i = \sum_{i=1}^{n} ((pf)(\psi\pi_i))(e_i\sigma)$$
$$= \left(\sum_{i=1}^{n} (pf)\psi\pi_i e_i\right)\sigma = (pf)\psi\sigma = pf,$$

i.e., $f = \sum_{i=1}^{n} q_i p_i$. Now for any $t \in T$ there exists an idempotent $f = \sum_{i=1}^{n} q_i p_i$ such that $t = ft$. Then we have $t = ft = \sum_{i=1}^{n} q_i p_i t = \phi\left(\sum_{i=1}^{n} q_i \otimes p_i t\right)$. Thus ϕ is onto.

Conversely, for any idempotent $f \in T$, there exist $p_i \in P,\ q_i \in Q$ for $i = 1, 2, ..., n$ such that $\phi\left(\sum_{i=1}^{n} q_i \otimes p_i\right) = \sum_{i=1}^{n} q_i p_i = f$. Let $e \in E(R)$ such that $q_i e = q_i$ for all $i = 1, 2, ..., n$. Then we define $\alpha : (Re)^n \to Pf$ by $(x_1, x_2, ..., x_n) \mapsto \sum_{i=1}^{n} x_i p_i f$ and $\beta : Pf \to (Re)^n$ by $y \mapsto (yq_1, yq_2, ..., yq_n)$. Then for $y \in Pf$,

$$y\beta\alpha = (yq_1, yq_2, ..., yq_n)\alpha = \sum_{i=1}^{n} (yq_i)p_i f = \sum_{i=1}^{n} ((yq_i)p_i)f$$
$$= \sum_{i=1}^{n} y(q_i p_i)f = y\left(\sum_{i=1}^{n} q_i p_i\right) f = yf^2 = y,$$

i.e., $\beta\alpha = id_{Pf}$. Hence Pf being a retract of $(Re)^n$ is finitely generated projective (by Proposition 2.1.9). Also, $P = \varinjlim_R Pf$ (can be proved along the same lines as Proposition 2.1.12). Therefore ${}_RP$ is locally projective.

Now let ϕ be a surjection and $\phi\left(\sum_{i=1}^{m} q_i \otimes p_i\right) = \phi\left(\sum_{j=1}^{n} q'_j \otimes p'_j\right)$. Since P_T is unitary there exists $f \in E(T)$ such that $p_i f = p_i,\ p'_j f = p'_j$ for all $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. Now by the surjectivity of $\phi,\ f = \sum_{k=1}^{l} y_k x_k$, where $x_k \in P,\ y_k \in Q$ for all $k = 1, 2, ..., l$. Then we have

$$\sum_{i=1}^{m} q_i \otimes p_i = \sum_{i=1}^{m} q_i \otimes p_i \left(\sum_{k=1}^{l} y_k x_k\right) = \sum_{i,k} q_i \otimes p_i(y_k x_k) = \sum_{i,k} q_i \otimes (p_i y_k)x_k$$
$$= \sum_{i,k} q_i(p_i y_k) \otimes x_k = \sum_{i,k} (q_i p_i)y_k \otimes x_k = \sum_{k}\left(\sum_{i} q_i p_i\right) y_k \otimes x_k$$
$$= \sum_{k}\left(\sum_{j} q'_j p'_j\right) y_k \otimes x_k = \cdots = \sum_{j=1}^{n} q'_j \otimes p'_j,$$

which proves that ϕ is injective. Hence ϕ is an isomorphism. □

Proposition 2.1.19. *${}_RP$ is a generator for R-Sem if and only if $\theta : P \otimes Q \to R$ is a surjection. Moreover, if θ is a surjection, then it is an isomorphism.*

Proof. For the necessary part, since ${}_RP$ is a generator, ${}_RR$ is a sum of homomorphic images of P, i.e., every $r \in R$ can be written as $r = \sum_{i=1}^{n} p_i\sigma_i,\ p_i \in P,\ \sigma_i \in Hom_R(P, R)$ for all $i = 1, 2, ..., n$. Now, since P_T is unitary, there exists $f \in E(T)$ such that $p_i f = p_i$ for all $i = 1, 2, ..., n$. Also there exists $e \in E(R)$ such that $re = r$. Therefore we have

$$\begin{aligned} r = &\left(\sum_{i=1}^{n} p_i\sigma_i\right) e = \sum_{i=1}^{n} (p_i\sigma_i)e = \sum_{i=1}^{n} p_i(\sigma_i e) \\ =&\sum_{i=1}^{n} (p_i f)(\sigma_i e) = \sum_{i=1}^{n} p_i(f\sigma_i e) = \theta\left(\sum_{i=1}^{n} p_i \otimes f\sigma_i e\right), \end{aligned}$$

where $f\sigma_i e \in THom_R(P, R)R = Q$. Therefore θ is onto.

Conversely, let θ be a surjection. Then $R = \sum_{q\in Q} Pq \subseteq \sum_{q\in Hom_R(P,R)} Pq = tr(P)$. Therefore $R = tr(P)$. Hence ${}_RP$ is a generator for R-Sem.

Now if we assume θ to be surjective, then the injectivity of θ can be proved in a manner similar to that of ϕ in Proposition 2.1.18. □

Combining the above two results we obtain the following result.

Proposition 2.1.20. *${}_RP$ is a locally projective generator and ${}_RPf$ is finitely generated for all $f \in E(T)$ if and only if $\phi : Q \otimes P \to T$ and $\theta : P \otimes Q \to R$ are T-T and R-R isomorphisms respectively.*

Proposition 2.1.21. *Let ${}_RP$ be a locally projective generator for R-Sem and ${}_RPf$ be finitely generated for all $f \in E(T)$. Then the following hold:*

(1) $R \cong (End_T P)R \cong REnd_T Q$ as semirings.

(2) $Q := THom_R(P, R)R \cong Hom_T(P, T)R$ as T-R-bisemimodules.

(3) $P \cong RHom_T(Q, T)$ as R-T-bisemimodules.

(4) $P \cong (Hom_R(Q, R))T$ as R-T-bisemimodules.

(5) $T \cong (End_R Q)T$ as semirings.

Proof. (1) Consider the map $\sigma : R \to End_T P$ defined by $\sigma(r)(p) := rp$, where $r \in R,\ p \in P$. For any $r_1, r_2 \in R,\ p \in P,\ \sigma(r_1 + r_2)p = (r_1 + r_2)p = r_1p + r_2p = \sigma(r_1)(p)+\sigma(r_2)(p) = (\sigma(r_1)+\sigma(r_2))p$, i.e., $\sigma(r_1+r_2) = \sigma(r_1)+\sigma(r_2)$. Also $\sigma(r_1r_2)(p) = (r_1r_2)p = r_1(r_2p) = \sigma(r_1)\sigma(r_2)(p)$, i.e., $\sigma(r_1r_2) = \sigma(r_1)\sigma(r_2)$. Thus σ is a semiring morphism. Now let $\sigma(r_1) = \sigma(r_2)$ for some $r_1, r_2 \in R$. Therefore $r_1 p = r_2 p$ for all $p \in P$. Suppose $e \in E(R)$ such that $r_1 = r_1 e,\ r_2 = r_2 e$. Now using Proposition 2.1.19, there exist $p_k \in P,\ q_k \in Q$ for $k = 1, 2, ..., n$ such that $\sum_{k=1}^{n} p_k q_k = e$. Therefore, $r_1 = r_1 e = r_1 \sum_{k=1}^{n} p_k q_k = \sum_{k=1}^{n} (r_1 p_k) q_k = \sum_{k=1}^{n} (r_2 p_k) q_k = r_2 \sum_{k=1}^{n} p_k q_k = r_2 e = r_2$. Hence σ is injective. Therefore identifying R with the subsemiring $\sigma(R)$ of $End_T P$, let $\psi \in (End_T P)R$, then there exists an idempotent $e' = \sum_{i=1}^{m} p'_i q'_i \in R$, such that $\psi e' = \psi$. Then for any $p \in P$, we have

$$\begin{aligned}\psi(p) = (\psi e')p = \psi(e'p) = \psi\left(\sum_{i=1}^{m} (p'_i q'_i) p\right) = \psi\left(\sum_{i=1}^{m} p'_i (q'_i p)\right) \\ = \sum_{i=1}^{m} \psi(p'_i)(q'_i p) = \sum_{i=1}^{m} (\psi(p'_i) q'_i) p = \sigma\left(\sum_{i=1}^{m} \psi(p'_i) q'_i\right)(p),\end{aligned}$$

i.e., $\psi = \sigma\left(\sum_{i=1}^{m} (\psi(p'_i) q'_i)\right)$. Thus $R \cong (End_T P)R$ as semirings. Similarly, considering the map $\xi : R \to End_T Q$ defined by $\xi(r)(q) := qr$ we can show that $R \cong REnd_T Q$ as semirings.

(2) Define the map $\lambda : Q \to Hom_T(P, T)R$ by $\lambda(q)(p) := qp$, where $q \in Q,\ p \in P$. For $q \in Q$ there exists $e' \in E(R)$ such that $qe' = q$, therefore using the QRP-associativity $(\lambda(q)e')p = \lambda(q)(e'p) = q(e'p) = (qe')p = qp = \lambda(q)(p)$, i.e., $\lambda(q) = \lambda(q)e' \in Hom_T(P, T)R$. That λ is a monoid morphism follows from the fact that ϕ is a monoid morphism and using the QRP-associativity we have $(t\lambda(q)r)(p) = t\lambda(q)(rp) = t(q(rp)) = t((qr)p) = (tqr)p = \lambda(tqr)(p)$. Thus λ is a T-R morphism. For $q, q' \in Q$, let $\lambda(q) = \lambda(q')$, then for any $p \in P,\ qp = q'p$. Suppose $e^2 = e \in R$ such that $q = qe,\ q' = q'e$. Now, in view of Proposition 2.1.19, there exist $p_k \in P,\ q_k \in Q$ for $k = 1, 2, ..., n$ such that $\sum_{k=1}^{n} p_k q_k = e$. Therefore, $q = qe = q\sum_{k=1}^{n} p_k q_k = \sum_{k=1}^{n} (qp_k) q_k = \sum_{k=1}^{n} (q'p_k) q_k = q'\sum_{k=1}^{n} p_k q_k = q'e = q'$. Let $\varphi \in Hom_T(P, T)R$. Then there exists $e' = \sum_{i=1}^{m} p'_i q'_i \in E(R)$, such that $\varphi e' = \varphi$. Then using TQP-associativity, for any $p \in P$, we have

$$\begin{aligned}\varphi(p) =& (\varphi e')p = \varphi(e'p) = \varphi\left(\sum_{i=1}^{m} (p'_i q'_i) p\right) = \varphi\left(\sum_{i=1}^{m} p'_i (q'_i p)\right) \\ =& \sum_{i=1}^{m} \varphi(p'_i)(q'_i p) = \sum_{i=1}^{m} (\varphi(p'_i) q'_i) p = \lambda\left(\sum_{i=1}^{m} \varphi(p'_i) q'_i\right)(p),\end{aligned}$$

i.e., $\varphi = \lambda\left(\sum_{i=1}^{m} (\varphi(p'_i) q'_i)\right)$. Thus λ is an isomorphism.

(3),(4) can be proved in a manner similar to (2) and (5) can be proved along the same lines as (1). $\square$

Proposition 2.1.22. *Let ${}_RP$ be a locally projective generator for R-Sem and ${}_RPf$ be finitely generated for all $f \in E(T)$. Then ${}_TQ \in Ob(T\text{-}Sem)$, $P_T \in Ob(Sem\text{-}T)$, $Q_R \in Ob(Sem\text{-}R)$ are locally projective generators for their respective categories.*

Proof. Suppose that ${}_RP \in Ob(R\text{-Sem})$ is a locally projective generator for R-Sem and ${}_RPf$ is finitely generated for all $f^2 = f \in T$. Then by Proposition 2.1.21, identifying P with $RHom_T(Q, T)$ and R with $REnd_TQ$ and using the fact that θ and ϕ are isomorphisms (Proposition 2.1.20) and finally applying Proposition 2.1.20 to ${}_TQ$, we have that ${}_TQ \in Ob(T\text{-Sem})$ is a locally projective generator. Similarly P_T, Q_R can be proved to be locally projective generators for their respective categories. □

2.2 Morita equivalence and Morita context

Definition 2.2.1. Let R, S be two semirings with local units. We call R and S to be Morita equivalent if the categories R-Sem and S-Sem are equivalent, i.e., there exist additive functors $F : R\text{-Sem} \to S\text{-Sem}$ and $G : S\text{-Sem} \to R\text{-Sem}$ such that F and G are mutually inverse equivalence functors.

In what follows by equivalence functors, we mean additive equivalence functors. In this section, we are going to characterize Morita equivalence for semirings with local units (*cf.* Theorem 2.2.15). In order to achieve this, we first obtain some results below.

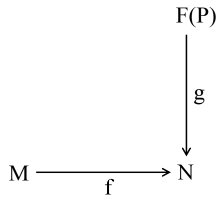
Definition 2.2.2. A unitary bisemimodule ${}_RP_S$ is said to be faithfully balanced if the canonical homomorphisms $S \to End_RP$ and $R \to End_SP$ given by $s \mapsto \rho_s(p \mapsto ps)$ and $r \mapsto \lambda_r(p \mapsto rp)$ respectively, where $s \in S,\ r \in R,\ p \in P$, are injective and identifying R and S with the corresponding subsemirings of endomorphisms of P, $SEnd_RP = S$ and $(End_SP)R = R$.

The following result is analogous to the case of categories of semimodules over a semiring with identity [49] and can be proved in a similar manner.

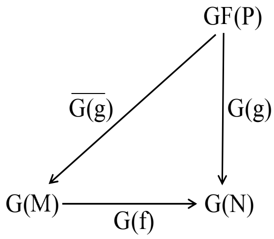
Lemma 2.2.3. *Let $F : R\text{-}Sem \rightleftarrows S\text{-}Sem : G$ be an equivalence of the categories R-Sem and S-Sem, and θ be a surjection in R-Sem. Then $F(\theta)$ is a surjection in S-Sem.*

Lemma 2.2.4. *Let $F : R\text{-}Sem \rightleftarrows S\text{-}Sem : G$ be an equivalence of the categories R-Sem and S-Sem, and ${}_RP \in Ob(R\text{-}Sem)$ be projective. Then $F(P) \in Ob(S\text{-}Sem)$ is projective, too.*

Proof. Consider the following diagram in S-Sem



where f is a surjection. Applying the functor G to the above diagram and using the fact that $GF(P) \cong P$ and that P is projective in R-Sem, we have the following diagram in R-Sem:



where $G(f)$ is a surjection by Lemma 2.2.3 and therefore $\overline{G(g)}$ exists making the above diagram commutative. Then applying the functor F to this diagram and using the fact that $FG \cong I_{S-Sem}$, we obtain $\overline{g} : F(P) \to M$ such that $\overline{g}f = g$. Hence the proof. □

Lemma 2.2.5. *Let $F : R$-Sem $\rightleftarrows S$-Sem $: G$ be an equivalence of the categories R-Sem and S-Sem, and ${}_RP \in Ob(R$-Sem$)$ be a generator for R-Sem. Then $F(P) \in Ob(S$-Sem$)$ is a generator for S-Sem.*

Proof. Let $N \in Ob(S$-Sem$)$. Since P is a generator, there exists a surjection $\alpha : \bigoplus_I P \to G(N)$ for some non-empty index set I. By Lemma 2.2.3, $F(\alpha) : F(\bigoplus_I P) \to FG(N)$ is a surjection where $FG(N) \cong N$. Also F and G being mutually inverse equivalence functors, by Theorem 1.1.23, G is the right adjoint of F. Then by the dual of Theorem 1.1.25, F preserves direct limits, hence preserves coproducts (see Remark 1.1.21), i.e., $F(\bigoplus_I P) \cong \bigoplus_I F(P)$. Thus N is a homomorphic image of a direct sum of copies of $F(P)$. Hence $F(P)$ is a generator for S-Sem. □

We skip the proof of Lemma 2.2.6 and Lemma 2.2.7 as they can be proved along the same lines as in the case of module theory [4].

Lemma 2.2.6. *Let $F : R$-Sem $\to S$-Sem be a categorical equivalence. Then for each $M, M' \in Ob(R$-Sem$)$ the restriction of F to $Hom_R(M, M')$, $F : Hom_R(M, M') \to Hom_S(F(M), F(M'))$ is a monoid isomorphism. In particular $F : End_R(M) \to End_S(F(M))$ is a semiring isomorphism.*

Lemma 2.2.7. *Let $F : R\text{-}Sem \to S\text{-}Sem$ be an equivalence of the categories R-Sem and S-Sem, and let ${}_RP \in Ob(R\text{-}Sem)$ be finitely generated. Then $F(P) \in Ob(S\text{-}Sem)$ is finitely generated, too.*

Theorem 2.2.8. *Let $F : R\text{-}Sem \rightleftarrows S\text{-}Sem : G$ be an equivalence of the categories R-Sem and S-Sem, and let ${}_RP \in Ob(R\text{-}Sem)$ be a locally projective generator. Then $F(P) \in Ob(S\text{-}Sem)$ is a locally projective generator, too.*

Proof. By Theorem 1.1.23, G is the right adjoint of F. Then by the dual of Theorem 1.1.25, F preserves direct limits. Using this fact together with Lemmas 2.2.4, 2.2.5 and 2.2.7 we obtain the result. □

In the following proposition, we observe the adjointness of the tensor functor and Hom functor between the categories of unitary semimodules. It is a routine verification so we omit the proof.

Proposition 2.2.9. *Let R, S be semirings with local units and ${}_SA_R \in Ob(S\text{-}Sem\text{-}R)$, ${}_RB \in Ob(R\text{-}Sem)$, ${}_SC \in Ob(S\text{-}Sem)$. Then*

$$\begin{aligned} \varphi : Hom_S(A \otimes B, C) &\to Hom_R(B, RHom_S(A, C)) \qquad \textit{given by} \\ \alpha &\mapsto \alpha' : B \to RHom_S(A, C) \\ &\qquad b \mapsto b\alpha' : A \to C \\ &\qquad\qquad a \mapsto (a \otimes b)\alpha \end{aligned}$$

is a bijective mapping natural in ${}_SA_R, {}_RB, {}_SC$. In particular, the functor $RHom_S(A, -)$ is right adjoint to the functor $A \otimes -$.

Lemma 2.2.10. *Every surjective morphism in R-Sem is a coequalizer of some pair of homomorphisms.*

Proof. Suppose $\gamma : A \to B$ is a surjective morphism in R-Sem. Let us define $M = \{(a_1, a_2) \in A \times A \mid a_1\gamma = a_2\gamma\}$. Then $(0, 0) \in M$ is non-empty. Also $M \in Ob(R\text{-Sem})$ follows from the fact that γ is an R-morphism and $A \in Ob(R\text{-Sem})$. Now consider the natural projections $M \underset{p_2}{\overset{p_1}{\rightrightarrows}} A$. We claim that $\gamma = coeq(p_1, p_2)$ (see Definition 1.1.14). Clearly $p_1\gamma = p_2\gamma$, since whenever $(a_1, a_2) \in M$, $(a_1, a_2)p_1\gamma = a_1\gamma = a_2\gamma = (a_1, a_2)p_2\gamma$. Let us consider a morphism $\gamma' : A \to B'$ with $p_1\gamma' = p_2\gamma'$. Then define $f : B \to B'$ by $b \mapsto a\gamma'$, where $a \in A$ such that $a\gamma = b$. To prove that f is well defined, we see that if $a' \in A$ such that $a'\gamma = b = a\gamma$, then $(a', a) \in M$ and $a'\gamma' = (a', a)p_1\gamma' = (a', a)p_2\gamma' = a\gamma'$. Therefore we have $f : B \to B'$ such that $\gamma f = \gamma'$ and hence the claim is proved. □

The next result is the counterpart of Theorem 4.5 of [49] in our setting.

Theorem 2.2.11. *For a functor F : R-Sem $\to$ S-Sem the following statements are equivalent.*

(1) F has a right adjoint.

(2) F preserves direct limits.

(3) There exists a unitary S-R-bisemimodule Q such that the functors $Q\otimes -$: R-Sem $\to$ S-Sem and F are naturally isomorphic.

Proof. **(1) $\Rightarrow$ (2)** and **(3) $\Rightarrow$ (1)** follow from the right analogue of Theorem 1.1.25 and Proposition 2.2.9 respectively.

(2) $\Rightarrow$ (3) Let $Q := F(R) \in Ob(S\text{-Sem})$. Then F induces a right R-semimodule structure on Q with the R-action given by $Q \times R \to Q$ by $(q, r) \mapsto qF(\rho_r)$, where $\rho_r : R \to R$ is given by $x \mapsto xr$. In order to show that Q_R is unitary, suppose $q \in Q$. Now $Q = F\left(\bigcup_{e\in E(R)} Re\right) = \bigcup_{e\in E(R)} F(Re)$ (since R is a semiring with local units, union coincides in this formula with direct limit and by the hypothesis F preserves direct limits). Therefore $q \in F(Re)$ for some idempotent $e \in R$. Then we have $qe = qF(\rho_e) = q$ (since $\rho_e = 1_{Re}$ implies that $F(\rho_e) = 1_{F(Re)}$). Thus Q is a unitary S-R-bisemimodule. Let $X \in Ob(R\text{-Sem})$, then R being a generator (*cf.* Proposition 2.1.12) there exists a surjection $\gamma : \bigoplus_I R \to X$, for some direct sum $\bigoplus_I R$ in R-Sem. By Lemma 2.2.10, $\gamma = coeq(\alpha, \beta)$ for some $\alpha, \beta : M \to \bigoplus_I R$, where $M \in Ob(R\text{-Sem})$. Again R being a generator there exists $\tau : \bigoplus_J R \to M$, for some direct sum $\bigoplus_J R$ in R-Sem. Then we have,

$$\bigoplus_J R \overset{\tau\alpha}{\underset{\tau\beta}{\rightrightarrows}} \bigoplus_I R \xrightarrow{\ \gamma\ } X$$

where $\gamma = coeq(\tau\alpha, \tau\beta)$ (since τ is surjective, hence an epimorphism). Now applying the functors F and $Q \otimes -$ to the above diagram and using the fact that both these functors preserve coproducts (direct sums), we obtain the following commutative diagram:

$$\begin{array}{ccccc} \bigoplus_J F(R) & \rightrightarrows & \bigoplus_I F(R) & \longrightarrow & F(X) \\ \| & & \| & & \\ \bigoplus_J Q & \rightrightarrows & \bigoplus_I Q & \longrightarrow & Q\otimes X \end{array}$$

Now since $F(R) = Q$, the commutativity of the above diagram induces an isomorphism $\eta_X : F(X) \to Q \otimes X$. We consider the class of isomorphisms $\eta := \{\eta_X : F(X) \to Q \otimes X \mid X \in Ob(R\text{-Sem})\}$. It can be verified that η is a natural isomorphism (see Definition 1.1.9) between the functors F and $Q \otimes -$. Hence the proof. ☐

Theorem 2.2.12. *Let R and S be Morita equivalent semirings with local units via inverse equivalences $F : R\text{-Sem} \to S\text{-Sem}$ and $G : S\text{-Sem} \to R\text{-Sem}$. Set $P = G(S)$ and $Q = F(R)$. Then the following hold:*

(1) ${}_RP_S, {}_SQ_R$ are unitary faithfully balanced bisemimodules.

(2) ${}_RP, P_S, {}_SQ, Q_R$ are locally projective generators.

(3) $F \cong Q \otimes -,\ G \cong P \otimes -$.

(4) $F \cong SHom_R(P, -),\ G \cong RHom_S(Q, -)$.

(5) ${}_RP_S \cong RHom_S(Q, S) \cong (Hom_R(Q, R))S$ and ${}_SQ_R \cong SHom_R(P, R) \cong Hom_S(P, S)R$.

Proof. Let $G(S) = P$, then G being an equivalence functor using Lemma 2.2.6 we have $End_SS \cong End_RP$ as semirings. By Proposition 2.1.14, $S \cong SEnd_SS$ as semirings. Since P is a right End_RP-semimodule, identifying S with the subsemiring $SEnd_SS$ of End_SS, P can be considered as a right S-semimodule with the action $P \times S \to P$ given by $(p, s) \mapsto pG(\rho_s)$, where $\rho_s : S \to S$ is given by $t \mapsto ts$. That P_S is unitary follows similarly as in the proof of Theorem 2.2.11. Thus P is a unitary R-S-bisemimodule. Now since S is a locally projective generator, by Theorem 2.2.8, ${}_RP = G(S)$ is a locally projective generator. In view of Lemma 2.2.7, $Pf = G(Sf)$ is a finitely generated left S-semimodule for all $f \in E(S)$ and $S \cong SEnd_SS \cong SEnd_RP$ as semirings. Since ${}_RP$ is a locally projective generator with Pf finitely generated for all $f^2 = f \in S$, using (1) of Proposition 2.1.21 we have $R \cong (End_SP)R$ as semirings. Hence ${}_RP_S$ is a faithfully balanced bisemimodule. Similarly $Q = F(R)$ is a unitary faithfully balanced S-R-bisemimodule. Hence (1) is proved.

Since F and G are mutually inverse equivalence functors, they are adjoint to each other (see Theorem 1.1.23). Therefore using Theorem 2.2.11, we obtain $F \cong Q \otimes -$. Similarly $G \cong P \otimes -$. By Proposition 2.2.9, $Q \otimes -$ is left adjoint to $RHom_S(Q, -)$ and $P \otimes -$ is left adjoint to $SHom_R(P, -)$. Then by uniqueness of adjoint functors upto natural isomorphism (see Theorem 1.1.24), we obtain $F \cong Q \otimes - \cong SHom_R(P, -)$

and $G \cong P \otimes - \cong RHom_S(Q, -)$. This proves (3) and (4).

Now using (4) we obtain, $P = G(S) \cong RHom_S(Q, S)$ as R-S-bisemimodule and $Q = F(R) \cong SHom_R(P, R)$ as S-R-bisemimodule. Since by (1), Q_R is unitary, using Proposition 2.1.21 we obtain, $Q = QR \cong SHom_R(P, R)R \cong Hom_S(P, S)R$ as S-R-bisemimodule and also $P \cong (Hom_R(Q, R))S$ as R-S-bisemimodule, which proves (5). Now (2) clearly follows from Proposition 2.1.22. □

Definition 2.2.13. [81] Let R and S be two semirings and ${}_RP_S$ and ${}_SQ_R$ be an R-S-bisemimodule and an S-R-bisemimodule, respectively and $\theta : P \otimes_S Q \to R$ and $\phi : Q \otimes_R P \to S$ be an R-S-bisemimodule homomorphism and an S-R-bisemimodule homomorphism, respectively, such that $\theta(p \otimes q)p' = p\phi(q \otimes p')$ and $\phi(q \otimes p)q' = q\theta(p \otimes q')$ for all $p, p' \in P$ and $q, q' \in Q$. Then the sixtuple $(R, S, P, Q, \theta, \phi)$ is called a Morita context for semirings.
Moreover, we say that a Morita context is unitary if ${}_RP_S$ and ${}_SQ_R$ are unitary bisemimodules.

Remark 2.2.14. Notice that the usual definition of a Morita context for semirings (see Definition 1.3.72) makes no use of the identities of the semirings, hence it makes sense in our case.

Theorem 2.2.15. *Let R and S be semirings with local units. Then the following are equivalent:*

(1) R and S are Morita equivalent.

(2) There exists a faithfully balanced unitary bisemimodule ${}_RP_S$ such that ${}_RP$ is a locally projective generator and ${}_RPf$ is finitely generated for all $f \in E(S)$.

(3) There exists a unitary Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ with surjective θ, ϕ.

(4) There exists a unitary Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ with bijective θ, ϕ.

Proof. **(1) ⇒ (2)** Let $P := G(S)$. Then the proof follows from Theorem 2.2.12.

(2) ⇒ (3) Suppose there exists a unitary bisemimodule ${}_RP_S$ such that ${}_RP$ is a locally projective generator and ${}_RPf$ is finitely generated for all $f \in E(S)$ and $S \cong SEnd_RP$ as semirings. Let $Q = SHom_R(P, R)R$. Then define:

$$\theta : P \otimes Q \to R \quad \text{and} \quad \phi : Q \otimes P \to S$$
$$p \otimes q \mapsto pq \qquad\qquad q \otimes p \mapsto qp \ (p' \mapsto (p'q)p)$$

It is routine to verify that the maps θ, ϕ are respectively R-R and S-S morphisms. For any $p' \in P$,

$$\begin{aligned} p'(\phi(q \otimes p)q') =& p'((qp)q') = (p'(qp))q' = ((p'q)p)q' \\ =& (p'q)(pq') = p'(q(pq')) = p'(q\theta(p \otimes q')), \end{aligned}$$

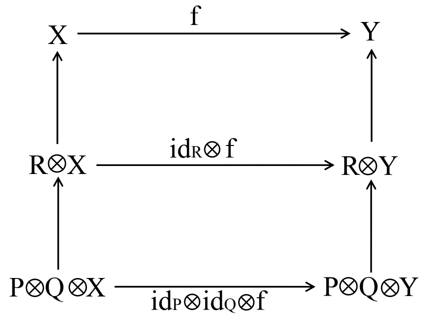
i.e., $\phi(q \otimes p)q' = q\theta(p \otimes q')$. Also $\theta(p \otimes q)p' = (pq)p' = p(qp') = p\phi(q \otimes p')$. Consequently, $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ is a Morita context. By hypothesis, ${}_RP$ is a locally projective generator and ${}_RPf$ is finitely generated for all $f \in E(S)$ and $S \cong SEnd_RP$ as semirings. Hence using Proposition 2.1.19 and Proposition 2.1.18, we get that θ, ϕ are surjections.

(3) ⇒ (4) Suppose $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ is a unitary Morita context with surjective θ, ϕ. Let $\theta\left(\sum_{i=1}^{m} p_i \otimes q_i\right) = \theta\left(\sum_{j=1}^{n} p'_j \otimes q'_j\right)$, where $p_i, p'_j \in P,\ q_i, q'_j \in Q$ for all $i = 1, 2, ..., m,\ j = 1, 2, ..., n$. Since Q_R is unitary, there exists an idempotent $e \in R$ such that $q_i e = q_i,\ q'_j e = q'_j$ for all $i = 1, 2, ..., m,\ j = 1, 2, ..., n$. Now by the surjectivity of $\theta,\ e = \theta(\sum_{k=1}^{l} x_k \otimes y_k)$, where $x_k \in P,\ y_k \in Q$ for all $k = 1, 2, ..., l$. Therefore we have

$$\begin{aligned} \sum_{i=1}^{m} p_i \otimes q_i =& \sum_{i=1}^{m} p_i \otimes q_i \theta\left(\sum_{k=1}^{l} x_k \otimes y_k\right) = \sum_{i,k} p_i \otimes q_i \theta(x_k \otimes y_k) \\ =& \sum_{i,k} p_i \otimes \phi(q_i \otimes x_k) y_k = \sum_{i,k} p_i \phi(q_i \otimes x_k) \otimes y_k \\ =& \sum_k \sum_i \theta(p_i \otimes q_i) x_k \otimes y_k = \sum_{k=1}^{l} \theta\left(\sum_{i=1}^{m} p_i \otimes q_i\right) x_k \otimes y_k \\ =& \sum_{k=1}^{l} \theta\left(\sum_{j=1}^{n} p'_j \otimes q'_j\right) x_k \otimes y_k = \cdots = \sum_{j=1}^{n} p'_j \otimes q'_j, \end{aligned}$$

which proves that θ is injective. Similarly ϕ is also injective.

(4) ⇒ (1) Let $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ be a unitary Morita context with bijective θ, ϕ. Then $P \otimes_S Q \cong R$ and $Q \otimes_R P \cong S$. Therefore for every $M \in Ob(R\text{-Sem})$, $P \otimes_S (Q \otimes_R M) \cong (P \otimes_S Q) \otimes_R M \cong R \otimes_R M \cong M$ (*cf.* Proposition 2.1.17). Now we consider the class of isomorphisms $\eta = \{\eta_X : P \otimes_S (Q \otimes_R X) \to_R X |\ X \in Ob(R\text{-Sem})\}$. Then η is a natural isomorphism between the identity functor I_{R-Sem} and the functor $P \otimes_S (Q \otimes_R -)$ as for all ${}_RX, {}_RY \in Ob(R\text{-Sem})$ and $f \in Hom_R(X, Y)$ the following diagram commutes:



Then $P \otimes_S (Q \otimes_R -) \cong I_{R-Sem}$. Similarly $Q \otimes_R (P \otimes_S -) \cong I_{S-Sem}$. Thus $P \otimes_S - : S$ -Sem $\to R$-Sem : $Q \otimes_R -$ is an equivalence of the categories R-Sem and S-Sem. □

Analogous to Corollary 4.3 of [1], we have the following proposition.

Proposition 2.2.16. *Let R be a semiring with slu. Then the following are equivalent:*

(1) R is Morita equivalent to a semiring with identity.

(2) There exists an idempotent $e \in R$ such that $R = ReR$.

Proof. **(1) ⇒ (2)** Suppose R is Morita equivalent to a semiring S with identity via inverse equivalences $F : R$-Sem $\rightleftarrows S$-Sem : G. Let $P = G(S)$. Since S is a finitely generated projective generator, ${}_RP$ also is a finitely generated projective generator. Now ${}_RP$ being a finitely generated projective unitary R-semimodule, by Proposition 2.1.9, there exists a surjective R-morphism $\sigma : (Re)^m \to P$ for some idempotent $e \in R$ and $m \in \mathbb{N}$ which implies that Re is a generator for R-Sem. Also since for any $r \in R$, Rr is finitely generated, using Proposition 2.1.3 there exists a surjective R-morphism $\psi : (Re)^n \to Rr$ for some $n \in \mathbb{N}$. Therefore there exists $(r_1, r_2, ..., r_n) \in (Re)^n$ such that $r = (r_1, r_2, ..., r_n)\psi = r_1e((e, 0_R, ..., 0_R)\psi) + ... + r_ne((0_R, ..., 0_R, e)\psi) \in ReRr \subseteq ReR$, which is true for any $r \in R$. Therefore $R = ReR$.

(2) ⇒ (1) Let $P = Re$. Then clearly P is a finitely generated projective unitary R-semimodule. Also for any $M \in Ob(R$-Sem$)$, for each $m \in M$ consider the map ρ_m : $P \to M$ defined by $y \mapsto ym$, where $y \in P,\ m \in M$. Then $\rho = \bigoplus_{m\in M}\rho_m : \bigoplus_M P \to M$, where $(\bigoplus_M P)\rho = \sum_{m\in M} P\rho_m = PM = P(RM) = (PR)M = (ReR)M = RM = M$, which implies that ρ is a surjection. Thus P is a finitely generated projective generator hence a locally projective generator for R-Sem. Now if we take $S = End_RP = End_R(Re) = eRe$, then using (2) of Theorem 2.2.15, R and $S = eRe$ are Morita equivalent semirings. □

2.3 Morita invariant properties

In this section, we discuss some properties of semirings with local units which remain invariant under Morita equivalence. The results obtained here are counterparts of the results of [36] in the setting of semirings with local units and their proofs are mostly similar to the ones presented there with some of them requiring slight modifications. For the convenience of the readers, here we include the proofs in detail.

Theorem 2.3.1. *Let R and S be Morita equivalent semirings with local units via the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then R is additively cancellative if and only if P is additively cancellative.*

Proof. Let R be additively cancellative and $a, b, c \in P$ such that $a + c = b + c$. Then for any $q_k \in Q$ and $n \in \mathbb{Z}^+$, using additive cancellativity of R, we have

$$\begin{aligned} & \sum_{k=1}^{n}\theta((a+c)\otimes q_k) = \sum_{k=1}^{n}\theta((b+c)\otimes q_k) \\ \Rightarrow\ & \sum_{k=1}^{n}\theta(a\otimes q_k) + \sum_{k=1}^{n}\theta(c\otimes q_k) = \sum_{k=1}^{n}\theta(b\otimes q_k) + \sum_{k=1}^{n}\theta(c\otimes q_k) \\ \Rightarrow\ & \sum_{k=1}^{n}\theta(a\otimes q_k) = \sum_{k=1}^{n}\theta(b\otimes q_k). \end{aligned}$$

Therefore for any $p_l \in P$ and $m \in \mathbb{Z}^+$

$$\begin{aligned} & \sum_{l=1}^{m}\sum_{k=1}^{n}\theta(a\otimes q_k)p_l = \sum_{l=1}^{m}\sum_{k=1}^{n}\theta(b\otimes q_k)p_l \\ \Rightarrow\ & a\sum_{l=1}^{m}\sum_{k=1}^{n}\phi(q_k\otimes p_l) = b\sum_{l=1}^{m}\sum_{k=1}^{n}\phi(q_k\otimes p_l). \end{aligned}$$

Let $f \in E(S)$ such that $a = af,\ b = bf$. Then we choose p_l, q_k, m, n in such a manner that we can write

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\phi(q_k\otimes p_l) = f.$$

Therefore, in particular, we have $a = b$. Hence P is additively cancellative.

Again let P be additively cancellative and $a, b, c \in R$ such that $a + c = b + c$. Then for any $p_k \in P$ and $n \in \mathbb{Z}^+$, using additive cancellativity of P, we have

$$\begin{aligned} & \sum_{k=1}^{n}(a+c)p_k = \sum_{k=1}^{n}(b+c)p_k \\ \Rightarrow\ & \sum_{k=1}^{n}ap_k + \sum_{k=1}^{n}cp_k = \sum_{k=1}^{n}bp_k + \sum_{k=1}^{n}cp_k \\ \Rightarrow\ & \sum_{k=1}^{n}ap_k = \sum_{k=1}^{n}bp_k. \end{aligned}$$

Therefore for any $q_l \in Q$ and $m \in \mathbb{Z}^+$

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(ap_k \otimes q_l) = \sum_{l=1}^{m}\sum_{k=1}^{n}\theta(bp_k \otimes q_l)$$
$$\Rightarrow \quad a\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_k \otimes q_l) = b\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_k \otimes q_l).$$

Let $e \in E(R)$ such that $a = ae,\ b = be$. Then we choose p_k, q_l, m, n in such a manner that we can write

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_k \otimes q_l) = e.$$

Therefore, in particular, we have $a = b$. Hence R is additively cancellative whence the proof. □

Theorem 2.3.2. *Let R and S be Morita equivalent semirings with local units via the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then R is additively idempotent if and only if P is additively idempotent.*

Proof. Let R be additively idempotent and $a \in P$. Then for any $q_k \in Q$ and $n \in \mathbb{Z}^+$, $\sum_{k=1}^{n}\theta(a \otimes q_k) \in R$. Since R is additively idempotent, we have

$$\sum_{k=1}^{n}\theta(a \otimes q_k) + \sum_{k=1}^{n}\theta(a \otimes q_k) = \sum_{k=1}^{n}\theta(a \otimes q_k)$$
$$\Rightarrow \quad \sum_{k=1}^{n}\theta((a+a) \otimes q_k) = \sum_{k=1}^{n}\theta(a \otimes q_k).$$

Therefore for any $p_l \in P$ and $m \in \mathbb{Z}^+$

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\theta((a+a) \otimes q_k)p_l = \sum_{l=1}^{m}\sum_{k=1}^{n}\theta(a \otimes q_k)p_l$$
$$\Rightarrow \quad (a+a)\sum_{l=1}^{m}\sum_{k=1}^{n}\phi(q_k \otimes p_l) = a\sum_{l=1}^{m}\sum_{k=1}^{n}\phi(q_k \otimes p_l).$$

Let $f \in E(S)$ such that $a = af$. Then we choose p_l, q_k, m, n in such a manner that we can write

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\phi(q_k \otimes p_l) = f.$$

Therefore, in particular, we have $a + a = a$. Hence P is additively idempotent.

Again let P be additively idempotent and $a \in R$. Then for any $p_k \in P$ and $n \in \mathbb{Z}^+$, $\sum_{k=1}^{n}ap_k \in P$. Since P is additively idempotent, we have

$$\sum_{k=1}^{n}ap_k + \sum_{k=1}^{n}ap_k = \sum_{k=1}^{n}ap_k$$
$$\Rightarrow \quad \sum_{k=1}^{n}(a+a)p_k = \sum_{k=1}^{n}ap_k.$$

Therefore for any $q_l \in Q$ and $m \in \mathbb{Z}^+$

$$\begin{aligned} \sum_{l=1}^{m}\sum_{k=1}^{n}\theta((a+a)p_k \otimes q_l) &= \sum_{l=1}^{m}\sum_{k=1}^{n}\theta(ap_k \otimes q_l) \\ \Rightarrow \quad (a+a)\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_k \otimes q_l) &= a\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_k \otimes q_l). \end{aligned}$$

Let $e \in E(R)$ such that $a = ae$. Then we choose p_k, q_l, m, n in such a manner that we can write

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_k \otimes q_l) = e.$$

Therefore, in particular, we have $a + a = a$. Hence R is additively idempotent whence the proof. □

Theorem 2.3.3. *Let R and S be Morita equivalent semirings with local units via the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then R is additively regular if and only if P is additively regular.*

Proof. Let R be additively regular and $a \in P$. Then for any $q_k \in Q$ and $n \in \mathbb{Z}^+$, $\sum_{k=1}^{n}\theta(a \otimes q_k) \in R$. Since R is additively regular, there exists $b \in R$ such that

$$\sum_{k=1}^{n}\theta(a \otimes q_k) + b + \sum_{k=1}^{n}\theta(a \otimes q_k) = \sum_{k=1}^{n}\theta(a \otimes q_k).$$

Therefore for any $p_l \in P$ and $m \in \mathbb{Z}^+$

$$\begin{aligned} \sum_{l=1}^{m}\left(\sum_{k=1}^{n}\theta(a \otimes q_k) + b + \sum_{k=1}^{n}\theta(a \otimes q_k)\right)p_l &= \sum_{l=1}^{m}\sum_{k=1}^{n}\theta(a \otimes q_k)p_l \\ \Rightarrow \quad a\sum_{l=1}^{m}\sum_{k=1}^{n}\phi(q_k \otimes p_l) + a' + a\sum_{l=1}^{m}\sum_{k=1}^{n}\phi(q_k \otimes p_l) &= a\sum_{l=1}^{m}\sum_{k=1}^{n}\phi(q_k \otimes p_l) \text{ where } a' = \sum_{l=1}^{m}bp_l. \end{aligned}$$

Let $f \in E(S)$ such that $a = af$. Then we choose p_l, q_k, m, n in such a manner that we can write

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\phi(q_k \otimes p_l) = f.$$

Therefore, in particular, we have $a + a' + a = a$. Hence P is additively regular.

Again let P be additively cancellative and $a \in R$. Then for any $p_k \in P$ and $n \in \mathbb{Z}^+$, $\sum_{k=1}^{n}ap_k \in P$. Since P is additively regular, there exists $b \in P$

$$\sum_{k=1}^{n}ap_k + b + \sum_{k=1}^{n}ap_k = \sum_{k=1}^{n}ap_k.$$

Therefore for any $q_l \in Q$ and $m \in \mathbb{Z}^+$

$$\sum_{l=1}^{m}\theta\left(\left(\sum_{k=1}^{n}ap_k + b + \sum_{k=1}^{n}ap_k\right)\otimes q_l\right) = \sum_{l=1}^{m}\sum_{k=1}^{n}\theta(ap_k \otimes q_l)$$
$$\Rightarrow \quad a\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_k \otimes q_l) + a' + a\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_k \otimes q_l) = a\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_k \otimes q_l) \text{ where } a' = \sum_{l=1}^{m}\theta(b \otimes q_l).$$

Let $e \in E(R)$ such that $a = ae$. Then we choose p_k, q_l, m, n in such a manner that we can write

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_k \otimes q_l) = e.$$

Therefore, in particular, we have $a + a' + a = a$. Hence R is additively regular whence the proof. □

Theorem 2.3.4. *Let R and S be Morita equivalent semirings with local units via the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then R is zero-sum free if and only if P is zero-sum free.*

Proof. Let R be zero-sum free and $a, b \in P$ such that $a + b = 0_P$. Then for any $q_k \in Q$ and $n \in \mathbb{Z}^+$, using zero-sum freeness of R, we have

$$\sum_{k=1}^{n}\theta((a+b)\otimes q_k) = \sum_{k=1}^{n}\theta(0_P \otimes q_k)$$
$$\Rightarrow \quad \sum_{k=1}^{n}\theta(a \otimes q_k) + \sum_{k=1}^{n}\theta(b \otimes q_k) = 0_R$$
$$\Rightarrow \quad \sum_{k=1}^{n}\theta(a \otimes q_k) = \sum_{k=1}^{n}\theta(b \otimes q_k) = 0_R.$$

Therefore for any $p_l \in P$ and $m \in \mathbb{Z}^+$

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(a \otimes q_k)p_l = \sum_{l=1}^{m}\sum_{k=1}^{n}\theta(b \otimes q_k)p_l = \sum_{l=1}^{m}0_R p_l$$
$$\Rightarrow \quad a\sum_{l=1}^{m}\sum_{k=1}^{n}\phi(q_k \otimes p_l) = b\sum_{l=1}^{m}\sum_{k=1}^{n}\phi(q_k \otimes p_l) = 0_P.$$

Let $f \in E(S)$ such that $a = af,\ b = bf$. Then we choose p_l, q_k, m, n in such a manner that we can write

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\phi(q_k \otimes p_l) = f.$$

Therefore, in particular, we have $a = b = 0_P$. Hence P is zero-sum free.

Again let P be zero-sum free and $a, b \in R$ such that $a + b = 0_R$. Then for any $p_k \in P$ and $n \in \mathbb{Z}^+$, using zero-sum freeness of P, we have

$$\begin{aligned} & \sum_{k=1}^{n}(a+b)p_k = \sum_{k=1}^{n}0_R p_k \\ \Rightarrow\ & \sum_{k=1}^{n}ap_k + \sum_{k=1}^{n}bp_k = 0_P \\ \Rightarrow\ & \sum_{k=1}^{n}ap_k = \sum_{k=1}^{n}bp_k = 0_P. \end{aligned}$$

Therefore for any $q_l \in Q$ and $m \in \mathbb{Z}^+$

$$\begin{aligned} & \sum_{l=1}^{m}\sum_{k=1}^{n}\theta(ap_k \otimes q_l) = \sum_{l=1}^{m}\sum_{k=1}^{n}\theta(bp_k \otimes q_l) = \sum_{l=1}^{m}\theta(0_P \otimes q_l) \\ \Rightarrow\ & a\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_k \otimes q_l) = b\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_k \otimes q_l) = 0_R. \end{aligned}$$

Let $e \in E(R)$ such that $a = ae,\ b = be$. Then we choose p_k, q_l, m, n in such a manner that we can write

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_k \otimes q_l) = e.$$

Therefore, in particular, we have $a = b = 0_R$. Hence R is zero-sum free whence the proof. □

Theorem 2.3.5. *Let R and S be Morita equivalent semirings with local units via the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the lattice $Id(R)$ of ideals of R and the lattice $Sub(P)$ of subsemimodules of P are isomorphic. Moreover, the isomorphism takes finitely generated ideals to finitely generated subsemimodules and vice-versa.*

Proof. Let us define

$$f : Id(R) \to Sub(P) \quad \text{and} \quad g : Sub(P) \to Id(R)$$

by

$$f(I) := \left\{ \sum_{k=1}^{n} i_k p_k \mid p_k \in P,\ i_k \in I \text{ for all } k;\ n \in \mathbb{Z}^+ \right\},$$

and

$$g(N) := \left\{ \sum_{k=1}^{n} \theta(p_k \otimes q_k) \mid p_k \in N,\ q_k \in Q \text{ for all } k;\ n \in \mathbb{Z}^+ \right\},$$

respectively. Then clearly $f(I)$ and $g(N)$ are closed under addition. Now let $\sum_{k=1}^{n} i_k p_k \in f(I),\ \sum_{k=1}^{n}\theta(p_k \otimes q_k) \in g(N),\ r, r' \in R$ and $s \in S$. Then using the fact that I is an ideal of R and P_S is a semimodule, we have

$$r\left(\sum_{k=1}^{n} i_k p_k\right)s = \sum_{k=1}^{n}(ri_k)(p_k s) \in f(I).$$

Therefore $f(I)$ is a subsemimodule of P. Using the fact that θ is an R-R-bisemimodule homomorphism, N is a subsemimodule of P and Q_R is a semimodule, we have

$$r\left(\sum_{k=1}^{n}\theta(p_k\otimes q_k)\right)r' = \sum_{k=1}^{n}\theta(rp_k\otimes q_kr') \in g(N).$$

Therefore $g(N)$ is an ideal of R.

Again for any ideal I of R, we have

$$\begin{aligned} g(f(I)) &= \left\{\sum_{l=1}^{m}\theta(p_l\otimes q_l) \mid p_l\in f(I),\ q_l\in Q\right\}\\ &= \left\{\sum_{l=1}^{m}\theta\left(\sum_{k=1}^{n}i_{lk}p_{lk}\otimes q_l\right) \mid p_{lk}\in P,\ i_{lk}\in I,\ q_l\in Q\right\}\\ &= \left\{\sum_{l=1}^{m}\sum_{k=1}^{n}i_{lk}\theta(p_{lk}\otimes q_l) \mid p_{lk}\in P,\ i_{lk}\in I,\ q_l\in Q\right\}\subseteq I. \end{aligned}$$

For the reverse inclusion, take $r\in I$. Let $e\in E(R)$ such that $r=re$. Then by the surjectivity of θ, $e=\sum_{k=1}^{m'}\theta(x_k\otimes y_k)$, where $x_k\in P,\ y_k\in Q$ for all $k=1,2,\ldots,m'$. Then

$$r = re = r\sum_{k=1}^{m'}\theta(x_k\otimes y_k) = \sum_{k=1}^{m'}\theta(rx_k\otimes y_k)\in g(f(I)).$$

Now for any subsemimodule N of P, we have

$$\begin{aligned} f(g(N)) &= \left\{\sum_{k=1}^{n}i_kp_k \mid i_k\in g(N),\ p_k\in P\right\}\\ &= \left\{\sum_{k=1}^{n}\sum_{l=1}^{m}\theta(p'_{kl}\otimes q_{kl})p_k \mid p_k\in P,\ p'_{kl}\in N,\ q_{kl}\in Q\right\}\\ &= \left\{\sum_{k=1}^{n}\sum_{l=1}^{m}p'_{kl}\phi(q_{kl}\otimes p_k) \mid p_k\in P,\ p'_{kl}\in N,\ q_{kl}\in Q\right\}\subseteq N. \end{aligned}$$

For the reverse inclusion, take $p\in N$. Let $f\in E(S)$ such that $p=pf$. Then by the surjectivity of ϕ, $f=\sum_{k=1}^{n'}\phi(y_k\otimes x_k)$, where $x_k\in P,\ y_k\in Q$ for all $k=1,2,\ldots,n'$. Then

$$p = pf = p\sum_{k=1}^{n'}\phi(y_k\otimes x_k) = \sum_{k=1}^{n'}\theta(p\otimes y_k)x_k\in f(g(N)).$$

Consequently, f and g are mutually inverse maps. It follows from the definitions that f and g preserve inclusion. Hence f and g are lattice isomorphisms.

Let $I\in Id(R)$ be finitely generated by $A=\{a_1,a_2,\ldots,a_t\}$. Then

$$f(I) = \left\{\sum_{l=1}^{n}r_lb_lp_l \mid b_l\in A,\ r_l\in R,\ p_l\in P\right\}.$$

Let $e \in E(R)$ such that $a_i e = a_i$ for all $a_i \in A$. Then by the surjectivity of θ, $e = \sum_{k=1}^{m'} \theta(x_k \otimes y_k)$, where $x_k \in P,\ y_k \in Q$ for all $k = 1, 2, \ldots, m'$. Then

$$r_l b_l p_l = r_l b_l \left(\sum_{k=1}^{m'} \theta(x_k \otimes y_k) \right) p_l = \sum_{k=1}^{m'} r_l b_l x_k \phi(y_k \otimes p_l).$$

Hence $f(I) \subseteq \langle B \rangle$, where $B = \{a_i x_k \mid i = 1, 2, \ldots, t;\ k = 1, 2, \ldots, m'\}$. Also clearly $\langle B \rangle \subseteq f(I)$, i.e., $f(I) = \langle B \rangle$.

Now let $N \in Sub(P)$ be finitely generated by $A = \{a_1, a_2, \ldots, a_t\}$. Then

$$g(N) = \left\{ \sum_{l=1}^{n} \theta(r_l b_l \otimes q_l) \mid b_l \in A,\ r_l \in R,\ q_l \in Q \right\}.$$

Let $f \in E(S)$ such that $a_i f = a_i$ for all $a_i \in A$. Then by the surjectivity of ϕ, $f = \sum_{k=1}^{n'} \phi(y_k \otimes x_k)$, where $x_k \in P,\ y_k \in Q$ for all $k = 1, 2, \ldots, n'$. Then

$$\begin{aligned} \theta(r_l b_l \otimes q_l) = \theta\left(r_l b_l \left(\sum_{k=1}^{n'} \phi(y_k \otimes x_k) \right) \otimes q_l \right) &= \sum_{k=1}^{n'} r_l \theta(\theta(b_l \otimes y_k) x_k \otimes q_l) \\ &= \sum_{k=1}^{n'} r_l \theta(b_l \otimes y_k) \theta(x_k \otimes q_l). \end{aligned}$$

Hence $g(N) \subseteq \langle B \rangle$, where $B = \{\theta(a_i \otimes y_k) \mid i = 1, 2, \ldots, t;\ k = 1, 2, \ldots, n'\}$. Also clearly $\langle B \rangle \subseteq g(N)$, i.e., $g(N) = \langle B \rangle$. Hence the proof. □

Theorem 2.3.6. *Let R and S be Morita equivalent semirings with local units via the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the lattice of k-ideals of R and the lattice of k-subsemimodules of P are isomorphic.*

Proof. In view of the proof of Theorem 2.3.5 it is sufficient to prove that $f(I)$ is a k-subsemimodule of P for any k-ideal I of R and $g(N)$ is a k-ideal of R for any k-subsemimodule N of P.

Let I be a k-ideal of R and $x \in f(I)$ and $y \in P$ such that $x + y \in f(I)$. Then for any $q_k \in Q$ and $n \in \mathbb{Z}^+$

$$\sum_{k=1}^{n} \theta(x \otimes q_k) \in g(f(I)) = I \quad \text{and} \quad \sum_{k=1}^{n} \theta((x + y) \otimes q_k) \in g(f(I)) = I.$$

Now

$$\sum_{k=1}^{n} \theta((x + y) \otimes q_k) = \sum_{k=1}^{n} \theta(x \otimes q_k) + \sum_{k=1}^{n} \theta(y \otimes q_k).$$

Since I is a k-ideal of R, $\sum_{k=1}^{n} \theta(y \otimes q_k) \in I$. Therefore for any $p_l \in P$ and $m \in \mathbb{Z}^+$

$$\sum_{l=1}^{m} \sum_{k=1}^{n} \theta(y \otimes q_k) p_l \in f(I) \Rightarrow y \sum_{l=1}^{m} \sum_{k=1}^{n} \phi(q_k \otimes p_l) \in f(I).$$

Let $f \in E(S)$ such that $yf = y$. Then we choose p_l, q_k, m, n in such a manner that we can write

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\phi(q_k \otimes p_l) = f.$$

Therefore $y = yf \in f(I)$. Hence $f(I)$ is a k-subsemimodule of P.

Let N be a k-subsemimodule of P and $a \in g(N)$ and $b \in R$ such that $a+b \in g(N)$. Then for any $p_l \in P$ and $m \in \mathbb{Z}^+$

$$\sum_{l=1}^{m}(a+b)p_l \in f(g(N)) = N \quad \text{and} \quad \sum_{l=1}^{m}ap_l \in f(g(N)) = N.$$

Now

$$\sum_{l=1}^{m}(a+b)p_l = \sum_{l=1}^{m}ap_l + \sum_{l=1}^{m}bp_l.$$

Since N is a k-subsemimodule of P, $\sum_{l=1}^{m}bp_l \in N$. Therefore for any $q_k \in Q$ and $n \in \mathbb{Z}^+$

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(bp_l \otimes q_k) \in g(N) \Rightarrow b\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_l \otimes q_k) \in g(N).$$

Let $e \in E(R)$ such that $be = b$. Then we choose p_l, q_k, m, n in such a manner that we can write

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_l \otimes q_k) = e.$$

Therefore $b = be \in g(N)$. Hence $g(N)$ is a k-ideal of R. □

Theorem 2.3.7. *Let R and S be Morita equivalent semirings with local units via the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the lattice of h-ideals of R and the lattice of h-subsemimodules of P are isomorphic.*

Proof. In view of the proof of Theorem 2.3.5 it is sufficient to prove that $f(I)$ is an h-subsemimodule of P for any h-ideal I of R and $g(N)$ is an h-ideal of R for any h-subsemimodule N of P.

Let I be an h-ideal of R and $y, y' \in f(I)$ and $x, z \in P$ such that $x + y + z = y' + z$. Then for any $q_k \in Q$ and $n \in \mathbb{Z}^+$, $\sum_{k=1}^{n}\theta(y \otimes q_k)$, $\sum_{k=1}^{n}\theta(y' \otimes q_k) \in g(f(I)) = I$. Now

$$\begin{aligned} &\sum_{k=1}^{n}\theta((x+y+z) \otimes q_k) = \sum_{k=1}^{n}\theta((y'+z) \otimes q_k) \\ \Rightarrow\ &\sum_{k=1}^{n}\theta(x \otimes q_k) + \sum_{k=1}^{n}\theta(y \otimes q_k) + \sum_{k=1}^{n}\theta(z \otimes q_k) = \sum_{k=1}^{n}\theta(y' \otimes q_k) + \sum_{k=1}^{n}\theta(z \otimes q_k). \end{aligned}$$

Since I is an h-ideal of R, $\sum_{k=1}^{n}\theta(x \otimes q_k) \in I$. Therefore for any $p_l \in P$ and $m \in \mathbb{Z}^+$

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(x \otimes q_k)p_l \in f(I) \Rightarrow x\sum_{l=1}^{m}\sum_{k=1}^{n}\phi(q_k \otimes p_l) \in f(I).$$

Let $f \in E(S)$ such that $x = xf$. Then we choose p_l, q_k, m, n in a way so that

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\phi(q_k \otimes p_l) = f.$$

Therefore $x = xf \in f(I)$. Hence $f(I)$ is an h-subsemimodule of P.

Let N be an h-subsemimodule of P and $b, b' \in g(N)$ and $a, c \in R$ such that $a+b+c = b'+c$. Then for any $p_l \in P$ and $m \in \mathbb{Z}^+$, $\sum_{l=1}^{m} bp_l$, $\sum_{l=1}^{m} b'p_l \in f(g(N)) = N$. Now

$$\begin{aligned} &\sum_{l=1}^{m}(a+b+c)p_l = \sum_{l=1}^{m}(b'+c)p_l \\ \Rightarrow \quad &\sum_{l=1}^{m}ap_l + \sum_{l=1}^{m}bp_l + \sum_{l=1}^{m}cp_l = \sum_{l=1}^{m}b'p_l + \sum_{l=1}^{m}cp_l. \end{aligned}$$

Since N is an h-subsemimodule of P, $\sum_{l=1}^{m} ap_l \in N$. Therefore for any $q_k \in Q$ and $n \in \mathbb{Z}^+$

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(ap_l \otimes q_k) \in g(N) \;\Rightarrow\; a\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_l \otimes q_k) \in g(N).$$

Let $e \in E(R)$ such that $a = ae$. Then we choose p_l, q_k, m, n in a way so that

$$\sum_{l=1}^{m}\sum_{k=1}^{n}\theta(p_l \otimes q_k) = e.$$

Therefore $a = ae \in g(N)$. Hence $g(N)$ is an h-ideal of R. □

The following result is an obvious corollary of Theorems 2.3.5, 2.3.6, 2.3.7.

Corollary 2.3.8. *Let R and S be Morita equivalent semirings with local units via the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then R is ideal-simple (k-ideal-simple, h-ideal-simple) if and only if P is subsemimodule-simple (respectively k-subsemimodule-simple, h-subsemimodule-simple).*

In view of Theorems 2.3.5, 2.3.6 and 2.3.7, we obtain the following result.

Remark 2.3.9. f and g preserve k-closure and h-closure.

The following result is the counterpart of Theorem 2.8 of [36] in the present setting.

Theorem 2.3.10. *Let R and S be Morita equivalent semirings with local units via the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then R is Noetherian if and only if P is Noetherian.*

Proof. Suppose R is a Noetherian semiring. Let us consider an ascending chain of subsemimodules of P, namely,

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots \subseteq M_k \subseteq M_{k+1} \subseteq \ldots$$

Then by Theorem 2.3.5,

$$g(M_1) \subseteq g(M_2) \subseteq g(M_3) \subseteq \ldots \subseteq g(M_k) \subseteq g(M_{k+1}) \subseteq \ldots$$

is an ascending chain of ideals in R. Since R is Noetherian, there exists $n \in \mathbb{Z}^+$ such that

$$g(M_n) = g(M_{n+1}) = g(M_{n+2}) = \ldots \quad .$$

Again applying Theorem 2.3.5, f being the inverse lattice isomorphism of g, it follows that

$$M_n = M_{n+1} = M_{n+2} = \ldots \quad .$$

Hence P is a Noetherian semimodule.

Conversely, suppose P is a Noetherian semimodule. Let us consider an ascending chain of ideals of R, namely,

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq I_{k+1} \subseteq \ldots \quad .$$

Then by Theorem 2.3.5,

$$f(I_1) \subseteq f(I_2) \subseteq f(I_3) \subseteq \ldots \subseteq f(I_k) \subseteq f(I_{k+1}) \subseteq \ldots$$

is an ascending chain of subsemimodules in P. Since P is Noetherian, there exists $n \in \mathbb{Z}^+$ such that

$$f(I_n) = f(I_{n+1}) = f(I_{n+2}) = \ldots \quad .$$

Again applying Theorem 2.3.5, g being the inverse lattice isomorphism of f, it follows that

$$I_n = I_{n+1} = I_{n+2} = \ldots \quad .$$

Hence R is a Noetherian semiring. □

Remark 2.3.11. The above proof is exactly the same as that of [36, Theorem 2.8] as it does not require the existence of local units or identity of the semiring.

Theorem 2.3.12. *Let R and S be Morita equivalent semirings with local units via the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the lattices $Con(R)$ and $Con(P)$ of congruences of R and P respectively are isomorphic. Moreover, the isomorphism takes Bourne congruences to Bourne congruences, Iizuka congruences to Iizuka congruences, and ring congruences to module congruences and vice-versa.*

Proof. Let us define

$$\alpha : Con(R) \to Con(P) \quad \text{by} \quad \alpha(\rho) := \alpha_\rho^{tr}$$

and

$$\beta : Con(P) \to Con(R) \quad \text{by} \quad \beta(\sigma) := \beta_\sigma^{tr},$$

where

$$\alpha_\rho = \left\{ \left(\sum_{k=1}^{n} r_k p_k, \sum_{k=1}^{n} r'_k p_k \right) \mid (r_k, r'_k) \in \rho,\ p_k \in P \text{ for all } k;\ n \in \mathbb{Z}^+ \right\}$$

and

$$\beta_\sigma = \left\{ \left(\sum_{k=1}^{n} \theta(p_k \otimes q_k), \sum_{k=1}^{n} \theta(p'_k \otimes q_k) \right) \mid (p_k, p'_k) \in \sigma,\ q_k \in Q \text{ for all } k;\ n \in \mathbb{Z}^+ \right\}.$$

Let $p \in P$. Then there exists $e \in E(R)$ such that $p = ep$. Also let $r = \sum_{k=1}^{n} \theta(p_k \otimes q_k) \in R$. Then, using the reflexivity of ρ and σ, we have

$$(p, p) = (ep, ep) \in \alpha_\rho \text{ and } (r, r) = \left(\sum_{k=1}^{n} \theta(p_k \otimes q_k), \sum_{k=1}^{n} \theta(p_k \otimes q_k) \right) \in \beta_\sigma.$$

So α_ρ and β_σ are reflexive. Symmetricity of α_ρ and β_σ follows from that of ρ and σ. Let $(\sum_{k=1}^{n} r_k p_k, \sum_{k=1}^{n} r'{}_k p_k) \in \alpha_\rho$. Then

$$\left(\sum_{k=1}^{n} r_k p_k + p, \sum_{k=1}^{n} r'_k p_k + p \right) = \left(\sum_{k=1}^{n} r_k p_k + ep, \sum_{k=1}^{n} r'_k p_k + ep \right) \in \alpha_\rho,$$

which follows from the definition of α_ρ and the fact that $(e, e) \in \rho$. Therefore α_ρ is compatible with addition. Now to show that β_σ is compatible with addition we take $(\sum_{k=1}^{n} \theta(p_k \otimes q_k), \sum_{k=1}^{n} \theta(p'_k \otimes q_k)) \in \beta_\sigma$ and $\sum_{l=1}^{m} \theta(p''_l \otimes q''_l) \in R$. Then

$$\left(\sum_{k=1}^{n} \theta(p_k \otimes q_k) + \sum_{l=1}^{m} \theta(p''_l \otimes q''_l), \sum_{k=1}^{n} \theta(p'_k \otimes q_k) + \sum_{l=1}^{m} \theta(p''_l \otimes q''_l) \right) \in \beta_\sigma.$$

The last step follows from the definition of β_σ and the fact that $(p''_l, p''_l) \in \sigma$ for all $l = 1, 2, \ldots, m$. Again let $(\sum_{k=1}^{n} r_k p_k, \sum_{k=1}^{n} r'_k p_k) \in \alpha_\rho,\ r \in R$ and $s \in S$. Then

$$\left(r \sum_{k=1}^{n} r_k p_k, r \sum_{k=1}^{n} r'_k p_k \right) = \left(\sum_{k=1}^{n} (r r_k) p_k, \sum_{k=1}^{n} (r r'_k) p_k \right) \in \alpha_\rho$$

since $(r_k, r'_k) \in \rho$ implies $(r r_k, r r'_k) \in \rho$. Also, by the definition of α_ρ

$$\left(\left(\sum_{k=1}^{n} r_k p_k \right) s, \left(\sum_{k=1}^{n} r'_k p_k \right) s \right) = \left(\sum_{k=1}^{n} r_k (p_k s), \sum_{k=1}^{n} r'_k (p_k s) \right) \in \alpha_\rho.$$

To show the compatibility of β_σ, let $(\sum_{k=1}^{n}\theta(p_k \otimes q_k), \sum_{k=1}^{n}\theta(p'_k \otimes q_k)) \in \beta_\sigma$ and $r \in R$. Then

$$\left(r\sum_{k=1}^{n}\theta(p_k \otimes q_k), r\sum_{k=1}^{n}\theta(p'_k \otimes q_k)\right) = \left(\sum_{k=1}^{n}\theta(rp_k \otimes q_k), \sum_{k=1}^{n}\theta(rp'_k \otimes q_k)\right) \in \beta_\sigma$$

since $(p_k, p'_k) \in \sigma$ implies $(rp_k, rp'_k) \in \sigma$. Also, by the definition of β_σ

$$\left(\sum_{k=1}^{n}\theta(p_k \otimes q_k)r, \sum_{k=1}^{n}\theta(p'_k \otimes q_k)r\right) = \left(\sum_{k=1}^{n}\theta(p_k \otimes q_k r), \sum_{k=1}^{n}\theta(p'_k \otimes q_k r)\right) \in \beta_\sigma.$$

It follows that α_ρ^{tr} and β_σ^{tr} are congruences on P and R respectively.

Clearly the maps α and β preserve the order relation of congruences. It remains to prove that they are mutually inverse. In order to prove that $\rho \subseteq \beta(\alpha(\rho)) = \beta^{tr}_{\alpha^{tr}_\rho}$ we show that $\rho \subseteq \beta_{\alpha^{tr}_\rho}$. To do this let $(r, r') \in R \times R$. Then there exists $e = \sum_{k=1}^{n'}\theta(p_k \otimes q_k) \in E(R)$ such that $r = re,\ r' = r'e$. Then

$$\begin{aligned} (r, r') &= \left(r\sum_{k=1}^{n'}\theta(p_k \otimes q_k), r'\sum_{k=1}^{n'}\theta(p_k \otimes q_k)\right) \\ &= \left(\sum_{k=1}^{n'}\theta(rp_k \otimes q_k), \sum_{k=1}^{n'}\theta(r'p_k \otimes q_k)\right). \end{aligned}$$

Therefore if $(r, r') \in \rho$ then

$$(rp_k, r'p_k) \in \alpha_\rho \subseteq \alpha_\rho^{tr} \text{ for all } k = 1, 2, ..., n'.$$

Consequently,

$$(\theta(rp_k \otimes q_k), \theta(r'p_k \otimes q_k)) \in \beta_{\alpha^{tr}_\rho} \text{ for all } k = 1, 2, ..., n'.$$

Hence by additive compatibility of $\beta_{\alpha^{tr}_\rho}$, $(r, r') \in \beta_{\alpha^{tr}_\rho}$. To prove $\beta^{tr}_{\alpha^{tr}_\rho} \subseteq \rho$ it is sufficient to show that $\beta_{\alpha^{tr}_\rho} \subseteq \rho$. Let

$$\left(\sum_{k=1}^{n}\theta(p_k \otimes q_k), \sum_{k=1}^{n}\theta(p'_k \otimes q_k)\right) \in \beta_{\alpha^{tr}_\rho}.$$

Then $(p_k, p'_k) \in \alpha_\rho^{tr}$ for all $k = 1, 2, ..., n$. Therefore for each k there exists $p_{k_i} \in S,\ i = 0, 1, ..., t$ such that

$$p_k = p_{k_0}\ \alpha_\rho\ p_{k_1}\ \alpha_\rho\ p_{k_2}\ \alpha_\rho\ \ldots\ \alpha_\rho\ p_{k_t} = p'_k.$$

By the definition of α_ρ for each $k = 1, 2, ..., n$ and for every $i = 0, 1, ..., t-1$ there exist $(r_{k_{i_l}}, r'_{k_{i_l}}) \in \rho,\ p''_{k_{i_l}} \in P,\ l = 1, 2, ..., m$ such that

$$(p_{k_i}, p_{k_{i+1}}) = \left(\sum_{l=1}^{m} r_{k_{i_l}} p''_{k_{i_l}}, \sum_{l=1}^{m} r'_{k_{i_l}} p''_{k_{i_l}}\right).$$

Now, using multiplicative compatibility of ρ, we have

$$\begin{aligned}\sum_{k=1}^{n}\theta(p_k\otimes q_k) &= \sum_{k=1}^{n}\theta\left(\sum_{l=1}^{m} r_{k0_l}p''_{k0_l}\otimes q_k\right)\\ &= \sum_{k=1}^{n}\sum_{l=1}^{m} r_{k0_l}\theta(p''_{k0_l}\otimes q_k)\\ &\rho \quad \sum_{k=1}^{n}\sum_{l=1}^{m} r'_{k0_l}\theta(p''_{k0_l}\otimes q_k)\\ &= \sum_{k=1}^{n}\sum_{l=1}^{m}\theta(r'_{k0_l}p''_{k0_l}\otimes q_k)\\ &= \sum_{k=1}^{n}\theta(p_{k_1}\otimes q_k).\end{aligned}$$

Repeating this process t times and using transitivity of ρ we obtain

$$\left(\sum_{k=1}^{n}\theta(p_k\otimes q_k),\sum_{k=1}^{n}\theta(p'_k\otimes q_k)\right)\in\rho.$$

Hence $\beta(\alpha(\rho))=\rho$ for every $\rho\in Con(R)$. In order to prove that $\sigma\subseteq\alpha(\beta(\sigma))=\alpha^{tr}_{\beta^{tr}_\sigma}$ we show that $\sigma\subseteq\alpha_{\beta^{tr}_\sigma}$. To do this let $(p,p')\in P\times P$. Then there exists $f=\sum_{k=1}^{n'}\phi(q_k\otimes p_k)\in E(S)$ such that $p=pf,\ p'=p'f$. Then

$$\begin{aligned}(p,p') &= \left(p\sum_{k=1}^{n'}\phi(q_k\otimes p_k),p'\sum_{k=1}^{n'}\phi(q_k\otimes p_k)\right)\\ &= \left(\sum_{k=1}^{n'}\theta(p\otimes q_k)p_k,\sum_{k=1}^{n'}\theta(p'\otimes q_k)p_k\right).\end{aligned}$$

Therefore if $(p,p')\in\sigma$ then

$$(\theta(p\otimes q_k),\theta(p'\otimes q_k))\in\beta_\sigma\subseteq\beta^{tr}_\sigma\ \text{ for all } k=1,2,...,n'.$$

Consequently,

$$(\theta(p\otimes q_k)p_k,\theta(p'\otimes q_k)p_k)\in\alpha_{\beta^{tr}_\sigma}\ \text{ for all } k=1,2,...,n'.$$

Hence by additive compatibility of $\alpha_{\beta^{tr}_\sigma}$, $(p,p')\in\alpha_{\beta^{tr}_\sigma}$. To prove $\alpha^{tr}_{\beta^{tr}_\sigma}\subseteq\sigma$ it is sufficient to show that $\alpha_{\beta^{tr}_\sigma}\subseteq\sigma$. Let

$$\left(\sum_{k=1}^{n}r_kp''_k,\sum_{k=1}^{n}r'_kp''_k\right)\in\alpha_{\beta^{tr}_\sigma}.$$

Then $(r_k,r'_k)\in\beta^{tr}_\sigma$ for all $k=1,2,...,n$. Therefore for each k there exists $r_{k_i}\in R,\ i=0,1,...,t$ such that

$$r_k=r_{k_0}\ \beta_\sigma\ r_{k_1}\ \beta_\sigma\ r_{k_2}\ \beta_\sigma\ ...\ \beta_\sigma\ r_{k_t}=r'_k.$$

By the definition of β_σ for each $k = 1, 2, ..., n$ and for every $i = 0, 1, ..., t-1$ there exist $(p_{k_{i_l}}, p'_{k_{i_l}}) \in \sigma,\ q_{k_{i_l}} \in Q,\ l = 1, 2, ..., m$ such that

$$(r_{k_i}, r_{k_{i+1}}) = \left(\sum_{l=1}^{m} \theta(p_{k_{i_l}} \otimes q_{k_{i_l}}), \sum_{l=1}^{m} \theta(p'_{k_{i_l}} \otimes q_{k_{i_l}}) \right).$$

Now, using compatibility of σ with semimodule action, we have

$$\begin{aligned} \sum_{k=1}^{n} r_k p''_k &= \sum_{k=1}^{n}\sum_{l=1}^{m} \theta(p_{k_{0_l}} \otimes q_{k_{0_l}}) p''_k \\ &= \sum_{k=1}^{n}\sum_{l=1}^{m} p_{k_{0_l}} \phi(q_{k_{0_l}} \otimes p''_k) \\ &\sigma\ \sum_{k=1}^{n}\sum_{l=1}^{m} p'_{k_{0_l}} \phi(q_{k_{0_l}} \otimes p''_k) \\ &= \sum_{k=1}^{n}\sum_{l=1}^{m} \theta(p'_{k_{0_l}} \otimes q_{k_{0_l}}) p''_k = \sum_{k=1}^{n} r_{k_1} p''_k. \end{aligned}$$

Repeating this process t times and using transitivity of σ we obtain

$$\left(\sum_{k=1}^{n} r_k p''_k, \sum_{k=1}^{n} r'_k p''_k \right) \in \sigma.$$

Hence $\alpha(\beta(\sigma)) = \sigma$ for every $\sigma \in Con(P)$ whence the lattice isomorphism is proved. That these lattice isomorphisms preserve Bourne congruence and Iizuka congruence can be proved along the same line as in Theorem 2.10 of [36], hence is skipped.

Now for the proof regarding ring congruence, let $\rho \in Con(R)$ be a ring congruence. To prove α_ρ^{tr} is a module congruence it is sufficient to show that any element of P/α_ρ^{tr} have an additive inverse. Let $[p]_{\alpha_\rho^{tr}} \in P/\alpha_\rho^{tr}$. Then for any $q_k \in Q,\ k = 1, 2, ..., n;\ n \in \mathbb{Z}^+$ we consider $\left[\sum_{k=1}^{n} \theta(p \otimes q_k)\right]_\rho$ in R/ρ. Since ρ is a ring congruence on R, there exists $[r']_\rho \in R/\rho$ such that

$$\left[\sum_{k=1}^{n} \theta(p \otimes q_k) \right]_\rho + [r']_\rho = [0_R]_\rho \text{ whence } \left[\sum_{k=1}^{n} \theta(p \otimes q_k) + r' \right]_\rho = [0_R]_\rho.$$

Therefore for any $p_l \in P,\ l = 1, 2, ..., m;\ m \in \mathbb{Z}^+$,

$$\begin{aligned} &\left(\sum_{l=1}^{m} \left(\sum_{k=1}^{n} \theta(p \otimes q_k) + r' \right) p_l, 0_P \right) \in \alpha_\rho \subseteq \alpha_\rho^{tr} \\ \Rightarrow\ &\left(p \sum_{l=1}^{m}\sum_{k=1}^{n} \phi(q_k \otimes p_l) + p', 0_P \right) \in \alpha_\rho^{tr} \quad \text{where } p' = \sum_{l=1}^{m} r' p_l. \end{aligned}$$

The arbitrariness of p_l, q_k, m, n allow us to choose those such that $\sum_{l=1}^{m}\sum_{k=1}^{n} \phi(q_k \otimes p_l) = f \in E(S)$ such that $pf = p$. Therefore $(p + p', 0_P) \in \alpha_\rho^{tr}$. Consequently,

$[p]_{\alpha_\rho^{tr}} + [p']_{\alpha_\rho^{tr}} = [0_P]_{\alpha_\rho^{tr}}$.

Again let $\sigma \in Con(P)$ be a module congruence. To prove β_σ^{tr} is a ring congruence it is sufficient to show that any element of R/β_σ^{tr} have an additive inverse. Let $[r]_{\beta_\sigma^{tr}} \in R/\beta_\sigma^{tr}$. Then for any $p_k \in P,\ k = 1, 2, ..., n;\ n \in \mathbb{Z}^+$ we consider $[\sum_{k=1}^{n} rp_k]_\sigma \in P/\sigma$. Since σ is a module congruence on P, there exists $[p']_\sigma \in P/\sigma$ such that

$$\left[\sum_{k=1}^{n} rp_k\right]_\sigma + [p']_\sigma = [0_P]_\sigma \text{ whence } \left[\sum_{k=1}^{n} rp_k + p'\right]_\sigma = [0_P]_\sigma.$$

Therefore for any $q_l \in Q,\ l = 1, 2, ..., m;\ m \in \mathbb{Z}^+$,

$$\left(\sum_{l=1}^{m} \theta\left(\left(\sum_{k=1}^{n} rp_k + p'\right) \otimes q_l\right), 0_R\right) \in \beta_\sigma \subseteq \beta_\sigma^{tr}$$
$$\Rightarrow \left(r\sum_{l=1}^{m}\sum_{k=1}^{n} \theta(p_k \otimes q_l) + r', 0_R\right) \in \beta_\sigma^{tr} \text{ where } r' = \sum_{l=1}^{m} \theta(p' \otimes q_l).$$

The arbitrariness of p_k, q_l, m, n allow us to choose those such that $\sum_{l=1}^{m}\sum_{k=1}^{n} \theta(p_k \otimes q_l) = e \in E(R)$ such that $re = r$. Therefore $(r + r', 0_R) \in \beta_\sigma^{tr}$. Consequently, $[r]_{\beta_\sigma^{tr}} + [r']_{\beta_\sigma^{tr}} = [0_R]_{\beta_\sigma^{tr}}$. Hence the proof. □

The following result is an obvious corollary of the above theorem.

Corollary 2.3.13. *Let R and S be Morita equivalent semirings with local units via the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then R is (Bourne, Iizuka, ring) congruence-simple if and only if P is (Bourne, Iizuka, module) congruence-simple.*

Remark 2.3.14. All the above results in Section 2.3 investigate relationship between R and P. But similar relationship can be established between R and Q, S and P, S and Q i.e., Theorem 2.3.1 - Corollary 2.3.13 have their counterparts for other pairs of the components of Morita equivalent semirings with local units. Since a semiring with identity is also a semiring with local units, Theorems 2.2.12, 2.2.15 include some of the results of Theorems 4.8, 4.6 of [81] (see Theorems 1.3.75 and 1.3.74).

Chapter 3

On some Morita invariant radicals of semirings

In this chapter, we study the Morita invariance of certain radicals of semirings with identity such as prime radical (*cf.* Theorem 3.1.9), strongly prime radical (*cf.* Theorem 3.2.10), uniformly strongly prime radical (*cf.* Theorem 3.3.11) and Levitzki radical (*cf.* Theorem 3.4.10). In 1975, Handelman and Lawrence [38] introduced the notion of right strongly prime ring motivated by the notion of primitive group ring and characterized them. A ring R is said to be (right) strongly prime if for each non-zero element r of R, there is a finite subset $S(r)$ (right insulator for r) of R such that for $t \in R$, $\{rst \mid s \in S(r)\} = \{0_R\}$ implies $t = 0_R$. Later in the year 1987, D. M. Olson [76] introduced the notion of uniformly strongly prime ring and uniformly strongly prime ideals of a ring. A ring R is said to be uniformly strongly prime if the same insulator may be chosen for each non-zero element of R. In order to investigate the validity of these concepts of ring theory in the settings of a semiring, T. K. Dutta and M. L. Das generalized the notion of (right) strongly prime rings and uniformly strongly prime rings to (right) strongly prime semirings [21] and uniformly strongly prime semirings [22] respectively. Hebisch and Weinert [41] studied several radicals of semirings, including strongly prime radical and uniformly strongly prime radical. On the other hand, Barbut [10] introduced the Levitzki radical for semiring as the sum of locally nilpotent ideals of the semiring.

The results of the first three sections of this chapter are based on the work of the following paper: M. Das and S. K. Sardar, *On some Morita invariant radicals of semirings*, Discussiones Mathematicae - General Algebra and Applications, Vol. 43 (To be published).

In our venture of studying the Morita invariance of the said radicals of semirings, firstly, we introduce the notion of (right) strongly prime subsemimodules, uniformly strongly prime subsemimodules, locally nilpotent subsemimodules of a bisemimodule by using the nice interplay between various components of a Morita context [82, 36]. Then we observe that if R and S are Morita equivalent semirings with identity via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ then there exists a one-to-one inclusion-preserving correspondence between the set of all prime ((right) strongly prime, uniformly strongly prime, locally nilpotent) ideals of R and the set of all prime (resp. (right) strongly prime, uniformly strongly prime, locally nilpotent) subsemimodules of P. Similar correspondence can also be established between R and Q, S and P, S and Q, which in turn result in a one-to-one inclusion-preserving correspondence between the set of all prime ((right) strongly prime, uniformly strongly prime, locally nilpotent) ideals of R and S. In addition, with the help of these correspondences, we prove that structures like prime radical, strongly prime radical, uniformly strongly prime radical, and Levitzki radical of semirings are preserved under Morita equivalence.

If R and S are two semirings with identity, ${}_RP_S$ and ${}_SQ_R$ are R-S-bisemimodule and S-R-bisemimodule respectively, and $\theta : P \otimes Q \to R$ and $\phi : Q \otimes P \to S$ are respectively R-R-bisemimodule homomorphism and S-S-bisemimodule homomorphism such that $\theta(p \otimes q)p' = p\phi(q \otimes p')$ and $\phi(q \otimes p)q' = q\theta(p \otimes q')$ for all $p, p' \in P$ and $q, q' \in Q$ then the sixtuple $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ is called a Morita context [81] for semirings. Recall that, two semirings R, S are Morita equivalent if and only if there exists a Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ with θ and ϕ surjective (see Theorem 1.3.76). Throughout this chapter every semiring is considered to have an identity, unless mentioned otherwise.

Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then for subsets $X \subseteq P$ and $Y \subseteq Q$ we write

$$\theta(X \otimes Y) = \left\{ \sum_{k=1}^{n} \theta(p_k \otimes q_k) \mid p_k \in X,\ q_k \in Y \text{ for all } k;\ n \in \mathbb{Z}^+ \right\} \text{ and}$$

$$\phi(Y \otimes X) = \left\{ \sum_{k=1}^{n} \phi(q_k \otimes p_k) \mid q_k \in Y,\ p_k \in X \text{ for all } k;\ n \in \mathbb{Z}^+ \right\}.$$

Also for subsets $U \subseteq R,\ V \subseteq S,\ X \subseteq P,\ Y \subseteq Q$ we write,

$$UX = \left\{ \sum_{k=1}^{n} r_k p_k \mid r_k \in U,\ p_k \in X \text{ for all } k;\ n \in \mathbb{Z}^+ \right\},$$

similarly we define $XV,\ YU,\ VY$.

Recall that (see Theorem 1.3.77), if R and S are Morita equivalent semirings with identity via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$, then the lattice of ideals of R and the lattice of subsemimodules of P are isomorphic. Moreover, this isomorphism takes k-ideals to k-subsemimodules and vice-versa (see Remark 1.3.79). The isomorphisms are given below and the same notations denoting them are used throughout the chapter without mentioning explicitly.

$$f_1 : Id(R) \to Sub(P) \text{ and } g_1 : Sub(P) \to Id(R) \text{ are defined by}$$
$$f_1(I) := \left\{ \sum_{k=1}^{n} i_k p_k \mid p_k \in P,\ i_k \in I \text{ for all } k;\ n \in \mathbb{Z}^+ \right\} = IP, \text{ and}$$
$$g_1(M) := \left\{ \sum_{k=1}^{n} \theta(p_k \otimes q_k) \mid p_k \in M,\ q_k \in Q \text{ for all } k;\ n \in \mathbb{Z}^+ \right\} = \theta(M \otimes Q)$$

Similar isomorphism can be defined for other pairs of the Morita context as follows.

$$f_2 : Id(R) \to Sub(Q) \text{ and } g_2 : Sub(Q) \to Id(R) \text{ are defined by}$$
$$f_2(I) := \left\{ \sum_{k=1}^{n} q_k i_k \mid q_k \in Q,\ i_k \in I \text{ for all } k;\ n \in \mathbb{Z}^+ \right\} = QI, \text{ and}$$
$$g_2(N) := \left\{ \sum_{k=1}^{n} \theta(p_k \otimes q_k) \mid p_k \in P,\ q_k \in N \text{ for all } k;\ n \in \mathbb{Z}^+ \right\} = \theta(P \otimes N)$$

Also $f_3 : Id(S) \to Sub(P)$, $g_3 : Sub(P) \to Id(S)$, $f_4 : Id(S) \to Sub(Q)$, $g_4 : Sub(Q) \to Id(S)$ can be defined in a similar way. Again in Theorem 1.3.80, we see that the lattice of ideals of R and the lattice of ideals of S are isomorphic via the following lattice isomorphisms.

$$\Theta : Id(S) \to Id(R) \text{ and } \Phi : Id(R) \to Id(S) \text{ are defined by}$$
$$\Theta(J) := \left\{ \sum_{k=1}^{n} \theta(p_k j_k \otimes q_k) \mid p_k \in P,\ q_k \in Q,\ j_k \in J \text{ for all } k;\ n \in \mathbb{Z}^+ \right\} = \theta(PJ \otimes Q)$$
$$\Phi(I) := \left\{ \sum_{k=1}^{n} \phi(q_k i_k \otimes p_k) \mid p_k \in P,\ q_k \in Q,\ i_k \in I \text{ for all } k;\ n \in \mathbb{Z}^+ \right\} = \phi(QI \otimes P)$$

Throughout this chapter, 1_R and 1_S denote respectively the identity elements of the Morita equivalent semirings R and S of the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ and also we take $1_R = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v})$, $1_S = \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u})$ (existence of such $\bar{p_v},\ \bar{q_v},\ \tilde{q_u},\ \tilde{p_u}$ is guaranteed since θ and ϕ are surjective).

For preliminaries of semirings and semimodules, we refer to Section 1.3 of Chapter 1.

3.1 Prime Radical

Definition 3.1.1. [31] A proper ideal I of a semiring R is called prime ideal if for ideals $A,\ B$ of R, $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$.

Definition 3.1.2. [16] Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. A subsemimodule M of P is said to be a prime subsemimodule if for subsemimodules A, B of P, $\theta(A \otimes Q)B \subseteq M$ implies either $A \subseteq M$ or $B \subseteq M$.

Definition 3.1.3. Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. A subsemimodule N of Q is said to be a prime subsemimodule if for subsemimodules A, B of Q, $\phi(A \otimes P)B \subseteq N$ implies either $A \subseteq N$ or $B \subseteq N$.

Proposition 3.1.4. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_1 : Id(R) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all prime ideals of R and the set of all prime subsemimodules of P.*

Proof. Let I be a prime ideal of R and A and B be subsemimodules of P such that $\theta(A \otimes Q)B \subseteq f_1(I)$. Then using the fact that f_1 and g_1 are mutually inverse lattice isomorphisms and I is a prime ideal, we have,

$$\begin{aligned}
&\theta(\theta(A \otimes Q)B \otimes Q) \subseteq \theta(f_1(I) \otimes Q) \\
\text{i.e.,}\quad &\theta(A \otimes Q)\theta(B \otimes Q) \subseteq g_1(f_1(I)) = I \\
\text{i.e.,}\quad &\theta(A \otimes Q) \subseteq I \text{ or } \theta(B \otimes Q) \subseteq I \\
\text{i.e.,}\quad &g_1(A) \subseteq I \text{ or } g_1(B) \subseteq I \\
\text{i.e.,}\quad &A = f_1(g_1(A)) \subseteq f_1(I) \text{ or } B = f_1(g_1(B)) \subseteq f_1(I)
\end{aligned}$$

Hence $f_1(I)$ is a prime subsemimodule of P.

Conversely, let M be a prime subsemimodule of P and I and J be ideals of R such that $IJ \subseteq g_1(M)$. Then using the fact that θ is surjective, i.e., $\theta(P \otimes Q) = R$ and M is a prime subsemimodule, we have,

$$\begin{aligned}
&I\theta(P \otimes Q)J = IRJ \subseteq IJ \subseteq g_1(M) \\
\text{i.e.,}\quad &\theta(IP \otimes Q)J \subseteq g_1(M) \\
\text{i.e.,}\quad &\theta(IP \otimes Q)JP \subseteq g_1(M)P = f_1(g_1(M)) = M \\
\text{i.e.,}\quad &IP \subseteq M \text{ or } JP \subseteq M \\
\text{i.e.,}\quad &f_1(I) \subseteq M \text{ or } f_1(J) \subseteq M \\
\text{i.e.,}\quad &I = g_1(f_1(I)) \subseteq g_1(M) \text{ or } J = g_1(f_1(J)) \subseteq g_1(M)
\end{aligned}$$

Therefore $g_1(M)$ is a prime ideal of R. Since f_1 and g_1 are mutually inverse maps, the proof follows. ∎

Analogously we obtain the following result.

Proposition 3.1.5. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_4 : Id(S) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all prime ideals of S and the set of all prime subsemimodules of Q.*

Proposition 3.1.6. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_2 : Id(R) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all prime ideals of R and the set of all prime subsemimodules of Q.*

Proof. Let I be a prime ideal of R and C and D be subsemimodules of Q such that $\phi(C \otimes P)D \subseteq f_2(I)$. Then we have,

$$\begin{aligned} g_2(C)g_2(D) = \theta(P \otimes C)\theta(P \otimes D) &\subseteq \theta(\theta(P \otimes C)P \otimes D) \subseteq \theta(P\phi(C \otimes P) \otimes D) \\ &\subseteq \theta(P \otimes \phi(C \otimes P)D) \subseteq \theta(P \otimes f_2(I)) = g_2(f_2(I)) = I \end{aligned}$$

$$\begin{aligned} \text{Since } I \text{ is a prime ideal, we have, } \quad & g_2(C) \subseteq I \quad \text{or} \quad g_2(D) \subseteq I \\ \text{i.e.,} \quad & C = f_2(g_2(C)) \subseteq f_2(I) \quad \text{or} \quad D = f_2(g_2(D)) \subseteq f_2(I) \end{aligned}$$

Hence $f_2(I)$ is a prime subsemimodule of Q.

Conversely, let N be a prime subsemimodule of Q and I and J be ideals of R such that $IJ \subseteq g_2(N)$. Then using the fact that f_2 and g_2 are mutually inverse lattice isomorphisms and N is a prime subsemimodule, we have,

$$\begin{aligned} & I\theta(P \otimes Q)J = IRJ \subseteq IJ \subseteq g_2(N) \\ \text{i.e.,} \quad & I\theta(P \otimes QJ) \subseteq g_2(N) \\ \text{i.e.,} \quad & QI\theta(P \otimes QJ) \subseteq Qg_2(N) = f_2(g_2(N)) = N \\ \text{i.e.,} \quad & \phi(QI \otimes P)QJ \subseteq N \\ \text{i.e.,} \quad & QI \subseteq N \quad \text{or} \quad QJ \subseteq N \\ \text{i.e.,} \quad & f_2(I) \subseteq N \quad \text{or} \quad f_2(J) \subseteq N \\ \text{i.e.,} \quad & I = g_2(f_2(I)) \subseteq g_2(N) \quad \text{or} \quad J = g_2(f_2(J)) \subseteq g_2(N) \end{aligned}$$

Therefore $g_2(N)$ is a prime ideal of R. Since f_2 and g_2 are mutually inverse lattice isomorphisms, the proof follows. □

Analogously we obtain the following result.

Proposition 3.1.7. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_3 : Id(S) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all prime ideals of S and the set of all prime subsemimodules of P.*

Although [82, Theorem 2.8] gives a direct proof of the following result, we can prove it using Proposition 3.1.4 and Proposition 3.1.7.

Theorem 3.1.8. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $\Theta : Id(S) \to Id(R)$ defines a one-to-one inclusion preserving correspondence between the set of all prime ideals of S and the set of all prime ideals of R.*

Proof. Let J be a prime ideal of S. Then from Proposition 3.1.7, $f_3(J) = PJ$ is a prime subsemimodule of P and therefore, from the proof of Proposition 3.1.4 we see that, $g_1(PJ)$ is a prime ideal of R. Since $\Theta(J) = \theta(PJ \otimes Q) = g_1(PJ)$, therefore $\Theta(J)$ is a prime ideal of R. Analogously we can prove that for any prime ideal I of R, $\Phi(I)$ is a prime ideal of S. Since Θ and Φ are mutually inverse lattice isomorphisms, the proof follows. □

Theorem 3.1.9. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then $\Theta : Id(S) \to Id(R)$ maps the prime radical $(\beta(S))$ of S, to the prime radical $(\beta(R))$ of R, i.e., $\Theta(\beta(S)) = \beta(R)$.*

Proof. Let $\mathcal{C}_P(R)$ and $\mathcal{C}_P(S)$ be the collection of all prime ideals of R and S respectively. Then using Theorem 1.3.81, Theorem 3.1.8 we have,

$$\Theta(\beta(S)) = \Theta\left(\bigcap_{J \in \mathcal{C}_P(S)} J\right) = \bigcap_{J \in \mathcal{C}_P(S)} \Theta(J) \supseteq \bigcap_{I \in \mathcal{C}_P(R)} I = \beta(R)$$

Similarly we have $\Phi(\beta(R)) \supseteq \beta(S)$. Since Θ and Φ are mutually inverse lattice isomorphisms, we have $\beta(R) \supseteq \Theta(\beta(S))$. Hence, $\Theta(\beta(S)) = \beta(R)$. □

3.2 Strongly Prime Radical

Definition 3.2.1. [21] An ideal I of a semiring R is said to be a (right) strongly prime ideal of R if for every r in R with $r \notin I$, there exists a finite subset $F \subseteq \langle r \rangle$ (ideal generated by r) such that for $r' \in R$, $Fr' \subseteq I$ implies that $r' \in I$.

Definition 3.2.2. Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. A subsemimodule M of P is said to be a (right) strongly prime subsemimodule if for every element p of P with $p \notin M$ there exist finite subsets $X \subseteq \langle p \rangle$ (subsemimodule generated by p) and $Y \subseteq Q$ such that for $p' \in P$, $\theta(X \otimes Y)p' \subseteq M$ implies that $p' \in M$.

Definition 3.2.3. Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. A subsemimodule N of Q is said to be a (right) strongly prime subsemimodule if for every element q of Q with $q \notin N$ there exist finite subsets $Y \subseteq \langle q \rangle$ (subsemimodule generated by q) and $X \subseteq P$ such that for $q' \in Q$, $\phi(Y \otimes X)q' \subseteq N$ implies that $q' \in N$.

Proposition 3.2.4. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_1 : Id(R) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all (right) strongly prime ideals of R and the set of all (right) strongly prime subsemimodules of P.*

Proof. Let I be a (right) strongly prime ideal of R and $p \notin f_1(I) = IP$ for some $p \in P$. Then there exists $k \in \{1, 2, ..., m'\}$ such that $\theta(p \otimes \tilde{q_k}) \notin I$, otherwise $p = p1_S = p\sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u}) = \sum_{u=1}^{m'} \theta(p \otimes \tilde{q_u})\tilde{p_u} \in IP$ - a contradiction. Since $\theta(p \otimes \tilde{q_k}) \notin I$, therefore by hypothesis there exists a finite subset $F \subseteq \langle \theta(p \otimes \tilde{q_k}) \rangle$ such that for $r' \in R$, $Fr' \subseteq I$ implies that $r' \in I$. Let $Y = \{\bar{q_v} \mid v = 1, 2, ..., n'\} \subseteq Q$ and $X = \{r\bar{p_v} \mid r \in F, v = 1, 2, ..., n'\}$. Then both Y and X are finite subsets of Q and P respectively. Since every element of X is of the form $r\bar{p_v}$ for some $r \in F$, i.e., $r = \sum_{i=1}^{l} r_i\theta(p \otimes \tilde{q_k})r'_i$, for some $l \in \mathbb{Z}^+$, where $r_i, r'_i \in R$ for all $i = 1, 2, ..., l$, therefore $r\bar{p_v} = \sum_{i=1}^{l} r_i\theta(p \otimes \tilde{q_k})r'_i\bar{p_v} = \sum_{i=1}^{l} r_ip\phi(\tilde{q_k} \otimes r'_i\bar{p_v}) \in RpS = \langle p \rangle$, i.e., $X \subseteq \langle p \rangle$.

Suppose $p' \in P$ such that $\theta(X \otimes Y)p' \subseteq f_1(I) = IP$. Let $r \in F$ and $q \in Q$. Then using the fact that f_1 and g_1 are mutually inverse maps we have,

$$\begin{aligned} r\theta(p' \otimes q) = r1_R\theta(p' \otimes q) &= r\sum_{v=1}^{n'}\theta(\bar{p_v} \otimes \bar{q_v})\theta(p' \otimes q) \\ &= \theta\left(\sum_{v=1}^{n'}\theta(r\bar{p}_v \otimes \bar{q_v})p' \otimes q\right) \\ &\in \theta(\theta(X \otimes Y)p' \otimes q) \subseteq \theta(f_1(I) \otimes Q) = g_1(f_1(I)) = I. \end{aligned}$$

Since every element of $F\theta(p' \otimes q)$ is a finite sum of elements of the form $r\theta(p' \otimes q)$ for some $r \in F$, therefore we see that $F\theta(p' \otimes q) \subseteq I$. Then by our hypothesis we have

$\theta(p' \otimes q) \in I$, which is true for all $q \in Q$, in particular for all $\tilde{q_u}$, where $u = 1, 2, ..., m'$. Therefore $p' = p'1_S = p' \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u}) = \sum_{u=1}^{m'} \theta(p' \otimes \tilde{q_u})\tilde{p_u} \in IP = f_1(I)$. Hence $f_1(I)$ is a (right) strongly prime subsemimodule of P.

Conversely, let M be a (right) strongly prime subsemimodule of P and $r \in R$ such that $r \notin g_1(M) = \theta(M \otimes Q)$. Then there exists $k \in \{1, 2, ..., n'\}$ such that $r\bar{p_k} \notin M$, otherwise $r = r1_R = r \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}) = \sum_{v=1}^{n'} \theta(r\bar{p_v} \otimes \bar{q_v}) \in \theta(M \otimes Q) = g_1(M)$ - a contradiction. Since $r\bar{p_k} \notin M$, therefore there exist finite subsets $X \subseteq \langle r\bar{p_k} \rangle$ and $Y \subseteq Q$ such that for $p' \in P$, $\theta(X \otimes Y)p' \subseteq M$ implies that $p' \in M$. Let $F = \{\theta(x \otimes y) \mid x \in X,\ y \in Y\}$. Then clearly F is a finite subset of R and for any $\theta(x \otimes y) \in F$ we have, $\theta(x \otimes y) \in \theta(\langle r\bar{p_k} \rangle \otimes Q) = \theta(R(r\bar{p_k})S \otimes Q) \subseteq Rr\theta(\bar{p_k}S \otimes Q) \subseteq RrR = \langle r \rangle$, i.e., $F \subseteq \langle r \rangle$.

Suppose $r' \in R$ such that $Fr' \subseteq g_1(M) = \theta(M \otimes Q)$. Let $x \in X,\ y \in Y$ and $p \in P$. Then using the fact that f_1 and g_1 are mutually inverse maps we have, $\theta(x \otimes y)(r'p) \in F(r'p) = (Fr')p \subseteq g_1(M)P = f_1(g_1(M)) = M$. Since every element of the set $\theta(X \otimes Y)(r'p)$ is a finite sum of elements of the form $\theta(x \otimes y)r'p$ for some $x \in X,\ y \in Y$, therefore we see that $\theta(X \otimes Y)r'p \subseteq M$. Then by our hypothesis we have $r'p \in M$, which is true for all $p \in P$, in particular for all $\bar{p_v}$, where $v = 1, 2, ..., n'$. Therefore $r' = r'1_R = r' \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}) = \sum_{v=1}^{n'} \theta(r'\bar{p_v} \otimes \bar{q_v}) \in \theta(M \otimes Q) = g_1(M)$. Thus $g_1(M)$ is a (right) strongly prime ideal of R. Since f_1 and g_1 are mutually inverse lattice isomorphisms, the proof follows. □

Analogously we obtain the following result.

Proposition 3.2.5. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_4 : Id(S) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all (right) strongly prime ideals of S and the set of all (right) strongly prime subsemimodules of Q.*

Proposition 3.2.6. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_2 : Id(R) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all (right) strongly prime ideals of R and the set of all (right) strongly prime subsemimodules of Q.*

Proof. Let I be a (right) strongly prime ideal of R and $q \notin f_2(I) = QI$ for some $q \in Q$. Then there exists $k \in \{1, 2, ..., m'\}$ such that $\theta(\tilde{p_k} \otimes q) \notin I$, otherwise $q = 1_Sq = \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u})q = \sum_{u=1}^{m'} \tilde{q_u}\theta(\tilde{p_u} \otimes q) \in QI$ - a contradiction. Since $\theta(\tilde{p_k} \otimes q) \notin I$, therefore by hypothesis there exists a finite subset $F \subseteq \langle \theta(\tilde{p_k} \otimes q) \rangle$ such that for $r' \in R$, $Fr' \subseteq I$ implies that $r' \in I$. Let $Y = \{\bar{q_v}r \mid r \in F,\ v = 1, 2, ..., n'\} \subseteq Q$

and $X = \{\tilde{p_u} \mid u = 1, 2, ..., m'\}$. Then both Y and X are finite subsets of Q and P respectively. Since every element of Y is of the form $\bar{q_v}r$ for some $r \in F$, i.e., $r = \sum_{i=1}^{l} r_i\theta(\tilde{p_k} \otimes q)r'_i$ for some $l \in \mathbb{Z}^+$, where $r_i, r'_i \in R$ for all $i = 1, 2, ..., l$, therefore $\bar{q_v}r = \bar{q_v}\sum_{i=1}^{l} r_i\theta(\tilde{p_k} \otimes q)r'_i = \sum_{i=1}^{l} \phi(\bar{q_v} \otimes r_i\tilde{p_k})qr'_i \in SqR = \langle q \rangle$, i.e., $Y \subseteq \langle q \rangle$.

Suppose $q' \in Q$ such that $\phi(Y \otimes X)q' \subseteq f_2(I) = QI$. Let $r \in F$ and $u \in \{1, 2, ..., m'\}$. Then using the fact that f_2 and g_2 are mutually inverse maps we have,

$$\begin{aligned} r\theta(\tilde{p_u} \otimes q') = 1_R r\theta(\tilde{p_u} \otimes q') &= \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v})r\theta(\tilde{p_u} \otimes q') \\ &= \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}r\theta(\tilde{p_u} \otimes q')) = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \phi(\bar{q_v}r \otimes \tilde{p_u})q') \\ &\in \theta(P \otimes \phi(Y \otimes X)q') \subseteq \theta(P \otimes f_2(I)) = g_2(f_2(I)) = I. \end{aligned}$$

Since every element of $F\theta(\tilde{p_u} \otimes q')$ is a finite sum of elements of the form $r\theta(\tilde{p_u} \otimes q')$ for some $r \in F$, therefore we see that $F\theta(\tilde{p_u} \otimes q') \subseteq I$. Then by our hypothesis we have $\theta(\tilde{p_u} \otimes q') \in I$, which is true for all $\tilde{p_u}$, where $u = 1, 2, ..., m'$. Therefore $q' = 1_S q' = \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u})q' = \sum_{u=1}^{m'} \tilde{q_u}\theta(\tilde{p_u} \otimes q') \in QI = f_2(I)$. Hence $f_2(I)$ is a (right) strongly prime subsemimodule of Q.

Conversely, let N be a (right) strongly prime subsemimodule of Q and $r \in R$ such that $r \notin g_2(N) = \theta(P \otimes N)$. Then there exists $k \in \{1, 2, ..., n'\}$ such that $\bar{q_k}r \notin N$, otherwise $r = 1_R r = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v})r = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}r) \in \theta(P \otimes N) = g_2(N)$ - a contradiction. Since $\bar{q_k}r \notin N$, therefore there exist finite subsets $X \subseteq P$, $Y \subseteq \langle \bar{q_k}r \rangle$ such that for $q' \in Q$, $\phi(Y \otimes X)q' \subseteq N$ implies that $q' \in N$. Let $F = \{\theta(\tilde{p_u} \otimes y)\theta(x \otimes \bar{q_v}) \mid y \in Y,\ x \in X,\ u = 1, 2, ..., m',\ v = 1, 2, ..., n'\}$. Then clearly F is a finite subset of R and since $y \in \langle \bar{q_k}r \rangle$, $y = \sum_{i=1}^{l} s_i(\bar{q_k}r)r_i$ for some $l \in \mathbb{Z}^+$, where $s_i \in S$, $r_i \in R$ for all $i = 1, 2, ..., l$, therefore for any element of F, $\theta(\tilde{p_u} \otimes y)\theta(x \otimes \bar{q_v}) = \theta\left(\tilde{p_u} \otimes \sum_{i=1}^{l} s_i(\bar{q_k}r)r_i\right)\theta(x \otimes \bar{q_v}) = \sum_{i=1}^{l} \theta(\tilde{p_u} \otimes s_i\bar{q_k})rr_i\theta(x \otimes \bar{q_v}) \in \langle r \rangle$, i.e., $F \subseteq \langle r \rangle$.

Suppose $r' \in R$ such that $Fr' \subseteq g_2(N) = \theta(P \otimes N)$. Let $x \in X$, $y \in Y$ and $v \in \{1, 2, ..., n'\}$. Then using the fact that f_2 and g_2 are mutually inverse maps we have,

$$\begin{aligned} \phi(y \otimes x)\bar{q_v}r' = 1_S\phi(y \otimes x)\bar{q_v}r' &= \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u})\phi(y \otimes x)\bar{q_v}r' \\ &= \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u})y\theta(x \otimes \bar{q_v})r' = \sum_{u=1}^{m'} \tilde{q_u}\theta(\tilde{p_u} \otimes y)\theta(x \otimes \bar{q_v})r' \\ &\in QFr' \subseteq Qg_2(N) = f_2(g_2(N)) = N. \end{aligned}$$

Since every element of the set $\phi(Y \otimes X)(\bar{q_v}r')$ is a finite sum of elements of the form $\phi(y \otimes x)\bar{q_v}r'$ for some $x \in X,\ y \in Y$, therefore we see that $\phi(Y \otimes X)(\bar{q_v}r') \subseteq N$. Then by our hypothesis we have $\bar{q_v}r' \in N$, which is true for all $\bar{q_v}$, where $v = 1, 2, ..., n'$. Therefore $r' = 1_R r' = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v})r' = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}r') \in \theta(P \otimes N) = g_2(N)$. Thus $g_2(N)$ is a (right) strongly prime ideal of R. Since f_2 and g_2 are mutually inverse lattice isomorphisms, the proof follows. □

Analogously we obtain the following result.

Proposition 3.2.7. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_3 : Id(S) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all (right) strongly prime ideals of S and the set of all (right) strongly prime subsemimodules of P.*

Theorem 3.2.8. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $\Theta : Id(S) \to Id(R)$ defines a one-to-one inclusion preserving correspondence between the set of all (right) strongly prime ideals of S and the set of all (right) strongly prime ideals of R.*

Proof. Let J be a (right) strongly prime ideal of S. Then from Proposition 3.2.7, $f_3(J) = PJ$ is a (right) strongly prime subsemimodule of P and therefore, from the proof of Proposition 3.2.4 we see that, $g_1(PJ)$ is a (right) strongly prime ideal of R. Since $\Theta(J) = \theta(PJ \otimes Q) = g_1(PJ)$, therefore $\Theta(J)$ is a (right) strongly prime ideal of R. Analogously we can prove that for any (right) strongly prime ideal I of R, $\Phi(I)$ is a (right) strongly prime ideal of S. Hence the proof follows in view of the fact that Θ and Φ are mutually inverse lattice isomorphisms. □

Definition 3.2.9. [41] For a semiring R, the (right) strongly prime radical is defined to be the intersection of all (right) strongly prime k-ideals of R.

Theorem 3.2.10. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then $\Theta : Id(S) \to Id(R)$ maps the (right) strongly prime radical $(SP(S))$ of S to the (right) strongly prime radical $(SP(R))$ of R, i.e., $\Theta(SP(S)) = SP(R)$.*

Proof. Let $\mathcal{C}_{SP}(R)$ and $\mathcal{C}_{SP}(S)$ be the collection of all (right) strongly prime k-ideals of R and S respectively. Then using Theorem 3.2.8 and Theorem 1.3.81 and the fact that Θ preserves k-ideals we have,

$$\Theta(SP(S)) = \Theta\left(\bigcap_{J \in \mathcal{C}_{SP}(S)} J\right) = \bigcap_{J \in \mathcal{C}_{SP}(S)} \Theta(J) \supseteq \bigcap_{\mathcal{C}_{SP}(R)} I = SP(R)$$

Similarly we have $\Phi(SP(R)) \supseteq SP(S)$. Since Θ and Φ are mutually inverse lattice isomorphisms, we have $SP(R) \supseteq \Theta(SP(S))$. Hence, $\Theta(SP(S)) = SP(R)$. □

3.3 Uniformly Strongly Prime Radical

Definition 3.3.1. [22] An ideal I of a semiring R is said to be a uniformly strongly prime ideal of R if and only if there exists a finite subset F of R such that for $r', r'' \in R$, $r'Fr'' \subseteq I$ implies that $r' \in I$ or $r'' \in I$.

Definition 3.3.2. Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. A subsemimodule M of P is said to be a uniformly strongly prime subsemimodule if there exist finite subsets X and Y of P and Q respectively such that for $p', p'' \in P$, $\theta(p' \otimes Y)\theta(X \otimes Y)p'' \subseteq M$ implies that $p' \in M$ or $p'' \in M$.

Definition 3.3.3. Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. A subsemimodule N of Q is said to be a uniformly strongly prime subsemimodule if there exist finite subsets Y and X of Q and P respectively such that for $q', q'' \in Q$, $\phi(q' \otimes X)\phi(Y \otimes X)q'' \subseteq N$ implies that $q' \in N$ or $q'' \in N$.

Lemma 3.3.4. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the following statements are equivalent for a subsemimodule $M \subseteq P$.*

(a) M is a uniformly strongly prime subsemimodule of P.

(b) There exist finite subsets X of P and Y', Y'' of Q such that for $p', p'' \in P$, $\theta(p' \otimes Y')\theta(X \otimes Y'')p'' \subseteq M$ implies that $p' \in M$ or $p'' \in M$.

Proof. Clearly **(a)** $\Rightarrow$ **(b)**.

(b) $\Rightarrow$ **(a)** Suppose $Y = Y' \cup Y''$, then clearly Y is a finite subset of Q. Let p', $p'' \in P$ such that $\theta(p' \otimes Y)\theta(X \otimes Y)p'' \subseteq M$. Then $\theta(p' \otimes Y')\theta(X \otimes Y'')p'' \subseteq \theta(p' \otimes Y)\theta(X \otimes Y)p'' \subseteq M$ and hence from (b) we get $p' \in M$ or $p'' \in M$. Consequently, M is a uniformly strongly prime subsemimodule of P. □

Proposition 3.3.5. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_1 : Id(R) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all uniformly strongly prime ideals of R and the set of all uniformly strongly prime subsemimodules of P.*

Proof. Let I be a uniformly strongly prime ideal of R. Then there exists a finite subset $F \subseteq R$ such that for $r', r'' \in R$, $r'Fr'' \subseteq I$ implies that $r' \in I$ or $r'' \in I$. Suppose $X = \{r\bar{p_v} \mid r \in F,\ v = 1, 2, ..., n'\}$, $Y' = \{\tilde{q_u} \mid u = 1, 2, ..., m'\}$, $Y'' = \{\bar{q_v} \mid v = 1, 2, ..., n'\}$. Since F is finite, clearly X is a finite subset of P, also both Y', Y'' are finite subsets of Q.

Let $p', p'' \in P$ such that $\theta(p' \otimes Y')\theta(X \otimes Y'')p'' \subseteq f_1(I) = IP$ and $p' \notin IP$. Then there exists $k \in \{1, 2, ..., m'\}$ such that $\theta(p' \otimes \tilde{q_k}) \notin I$, otherwise $p' = p'1_S = p' \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u}) = \sum_{u=1}^{m'} \theta(p' \otimes \tilde{q_u})\tilde{p_u} \in IP$ - a contradiction. Now for any $r \in F$, $q \in Q$ we have,

$$\begin{aligned}\theta(p' \otimes \tilde{q_k})r\theta(p'' \otimes q) = \theta(p' \otimes \tilde{q_k})r1_R\theta(p'' \otimes q) &= \theta(p' \otimes \tilde{q_k})r\sum_{v=1}^{n'}\theta(\bar{p_v} \otimes \bar{q_v})\theta(p'' \otimes q) \\ = \theta(p' \otimes \tilde{q_k})\sum_{v=1}^{n'}\theta(r\bar{p_v} \otimes \bar{q_v})\theta(p'' \otimes q) &= \sum_{v=1}^{n'}\theta(\theta(p' \otimes \tilde{q_k})\theta(r\bar{p_v} \otimes \bar{q_v})p'' \otimes q) \\ \in \theta(\theta(p' \otimes Y')\theta(X \otimes Y'')p'' \otimes q) &\subseteq \theta(f_1(I) \otimes Q) = g_1(f_1(I)) = I\end{aligned}$$

This is true for all $r \in F$. Therefore $\theta(p' \otimes \tilde{q_k})F\theta(p'' \otimes q) \subseteq I$. Now since $\theta(p' \otimes \tilde{q_k}) \notin I$, therefore by our hypothesis $\theta(p'' \otimes q) \in I$, which is true for all $q \in Q$, in particular for all $\tilde{q_u}$, $u = 1, 2, ..., m'$. So we get $p'' = p''1_S = p'' \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u}) = \sum_{u=1}^{m'} \theta(p'' \otimes \tilde{q_u})\tilde{p_u} \in IP$. Hence by Lemma 3.3.4, $f_1(I)$ is a uniformly strongly prime subsemimodule of P.

Conversely, let M be a uniformly strongly prime subsemimodule of P. Then there exist finite subsets $X \subseteq P$ and $Y \subseteq Q$ such that for $p', p'' \in P$, $\theta(p' \otimes Y)\theta(X \otimes Y)p'' \subseteq M$ implies that $p' \in M$ or $p'' \in M$. Let $F = \{\theta(\bar{p_v} \otimes y')\theta(x \otimes y'') \mid x \in X,\ y', y'' \in Y,\ v = 1, 2..., n'\}$. Then clearly F is a finite subset of R.

Suppose $r', r'' \in R$ such that $r'Fr'' \subseteq g_1(M) = \theta(M \otimes Q)$ and $r' \notin \theta(M \otimes Q)$, then there exists $k \in \{1, 2, ..., n'\}$ such that $r'\bar{p_k} \notin M$, otherwise $r' = r'1_R = r' \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}) = \sum_{v=1}^{n'} \theta(r'\bar{p_v} \otimes \bar{q_v}) \in \theta(M \otimes Q)$ - a contradiction. Now for any $y', y'' \in Y$, $x \in X$ and $p \in P$, using the fact that f_1 and g_1 are mutually inverse maps we have, $\theta(r'\bar{p_k} \otimes y')\theta(x \otimes y'')r''p = r'\theta(\bar{p_k} \otimes y')\theta(x \otimes y'')r''p \in r'Fr''p \subseteq g_1(M)P = f_1(g_1(M)) = M$. Since every element of $\theta(r'\bar{p_k} \otimes Y)\theta(X \otimes Y)r''p$ is a finite sum of elements of the form $\theta(r'\bar{p_k} \otimes y')\theta(x \otimes y'')r''p$ for some $x \in X$, $y', y'' \in Y$, therefore $\theta(r'\bar{p_k} \otimes Y)\theta(X \otimes Y)r''p \subseteq M$. As $r'\bar{p_k} \notin M$, by our hypothesis $r''p \in M$, which is true for all $p \in P$, in particular for all $\bar{p_v}$, where $v = 1, 2, ..., n'$. Therefore $r'' = r''1_R = r'' \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}) = \sum_{v=1}^{n'} \theta(r''\bar{p_v} \otimes \bar{q_v}) \in \theta(M \otimes Q) = g_1(M)$. Thus $g_1(M)$ is a uniformly strongly prime ideal of R. Since f_1 and g_1 are mutually inverse lattice isomorphisms, the proof follows. □

Analogously we obtain the following result.

Proposition 3.3.6. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_4 : Id(S) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all uniformly strongly prime ideals of S and the set of all uniformly strongly prime subsemimodules of Q.*

Proposition 3.3.7. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_2 : Id(R) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all uniformly strongly prime ideals of R and the set of all uniformly strongly prime subsemimodules of Q.*

Proof. Let I be a uniformly strongly prime ideal of R. Then there exists a finite subset $F \subseteq R$ such that for $r', r'' \in R$, $r'Fr'' \subseteq I$ implies that $r' \in I$ or $r'' \in I$. Suppose $X' = \{r\bar{p_v} \mid r \in F,\ v = 1, 2, ..., n'\}$, $X'' = \{\tilde{p_u} \mid u = 1, 2, ..., m'\}$, $Y = \{\bar{q_v} \mid v = 1, 2, ..., n'\}$. Since F is finite, X' is a finite subset of P, also both X'' and Y are finite subsets of P and Q respectively.

Let $q', q'' \in Q$ such that $\phi(q' \otimes X')\phi(Y \otimes X'')q'' \subseteq f_2(I) = QI$ and $q' \notin QI$. Then there exists $k \in \{1, 2, ..., m'\}$ such that $\theta(\tilde{p_k} \otimes q') \notin I$, otherwise $q' = 1_S q' = \sum\limits_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u})q' = \sum\limits_{u=1}^{m'} \tilde{q_u}\theta(\tilde{p_u} \otimes q') \in QI$ - a contradiction. Now for any $r \in F,\ u \in \{1, 2, ..., m'\}$ we have,

$$\begin{aligned}
\theta(\tilde{p_k} \otimes q')r\theta(\tilde{p_u} \otimes q'') &= \theta(\tilde{p_k} \otimes q')r1_R\theta(\tilde{p_u} \otimes q'') = \theta(\tilde{p_k} \otimes q')r\sum_{v=1}^{n'}\theta(\bar{p_v} \otimes \bar{q_v})\theta(\tilde{p_u} \otimes q'') \\
&= \theta(\tilde{p_k} \otimes q')\sum_{v=1}^{n'}\theta(r\bar{p_v} \otimes \bar{q_v})\theta(\tilde{p_u} \otimes q'') = \sum_{v=1}^{n'}\theta(\tilde{p_k} \otimes q'\theta(r\bar{p_v} \otimes \bar{q_v})\theta(\tilde{p_u} \otimes q'')) \\
&= \sum_{v=1}^{n'}\theta(\tilde{p_k} \otimes \phi(q' \otimes r\bar{p_v})\bar{q_v}\theta(\tilde{p_u} \otimes q'')) = \sum_{v=1}^{n'}\theta(\tilde{p_k} \otimes \phi(q' \otimes r\bar{p_v})\phi(\bar{q_v} \otimes \tilde{p_u})q'') \\
&\in \theta(P \otimes \phi(q' \otimes X')\phi(Y \otimes X'')q'') \subseteq \theta(P \otimes f_2(I)) = g_2(f_2(I)) = I
\end{aligned}$$

This is true for all $r \in F$. Therefore $\theta(\tilde{p_k} \otimes q')F\theta(\tilde{p_u} \otimes q'') \subseteq I$. Now since $\theta(\tilde{p_k} \otimes q') \notin I$, therefore by our hypothesis $\theta(\tilde{p_u} \otimes q'') \in I$, which is true for all $\tilde{p_u}$, $u = 1, 2, ..., m'$. So we get $q'' = 1_S q'' = \sum\limits_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u})q'' = \sum\limits_{u=1}^{m'} \tilde{q_u}\theta(\tilde{p_u} \otimes q'') \in QI$. Hence by Q analogue of Lemma 3.3.4, $f_2(I)$ is a uniformly strongly prime subsemimodule of Q.

Conversely, let N be a uniformly strongly prime subsemimodule of Q. Then there exist finite subsets $X \subseteq P$ and $Y \subseteq Q$ such that for $q', q'' \in Q$, $\phi(q' \otimes X)\phi(Y \otimes X)q'' \subseteq N$ implies that $q' \in N$ or $q'' \in N$. Let $F = \{\theta(x' \otimes y)\theta(x'' \otimes \bar{q_v}) \mid x', x'' \in X,\ y \in Y,\ v = 1, 2..., n'\}$. Then clearly F is a finite subset of R.

Suppose $r', r'' \in R$ such that $r'Fr'' \subseteq g_2(N) = \theta(P \otimes N)$ and $r' \notin \theta(P \otimes N)$, then there exists $k \in \{1, 2, ..., n'\}$ such that $\bar{q_k} r' \notin N$, otherwise $r' = 1_R r' = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}) r' = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v} r') \in \theta(P \otimes N)$ - a contradiction. Now for any $x', x'' \in X$, $y \in Y$ and $v \in \{1, 2..., n'\}$, using the fact that f_2 and g_2 are mutually inverse maps we have,

$$\begin{aligned}\phi(\bar{q_k} r' \otimes x')\phi(y \otimes x'')\bar{q_v} r'' &= \phi(\bar{q_k} r' \otimes x'\phi(y \otimes x''))\bar{q_v} r'' = \phi(\bar{q_k} r' \otimes \theta(x' \otimes y)x'')\bar{q_v} r'' \\ &= \bar{q_k} r'\theta(\theta(x' \otimes y)x'' \otimes \bar{q_v})r'' = \bar{q_k} r'\theta(x' \otimes y)\theta(x'' \otimes \bar{q_v})r'' \\ &\in \bar{q_k} r'Fr'' \subseteq Qg_2(N) = f_2(g_2(N)) = N.\end{aligned}$$

Since every element of $\phi(\bar{q_k} r' \otimes X)\phi(Y \otimes X)\bar{q_v} r''$ is a finite sum of elements of the form $\phi(\bar{q_k} r' \otimes x')\phi(y \otimes x'')\bar{q_v} r''$ for some $x', x'' \in X$, $y \in Y$, therefore $\phi(\bar{q_k} r' \otimes X)\phi(Y \otimes X)\bar{q_v} r'' \subseteq N$. As $\bar{q_k} r' \notin N$, by our hypothesis $\bar{q_v} r'' \in N$, which is true for all $v = 1, 2, ..., n'$. Therefore $r'' = 1_R r'' = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}) r'' = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v} r'') \in \theta(P \otimes N) = g_2(N)$. Thus $g_2(N)$ is a uniformly strongly prime ideal of R. This completes the proof as f_2 and g_2 are mutually inverse lattice isomorphisms. □

Analogously we obtain the following result.

Proposition 3.3.8. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_3 : Id(S) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all uniformly strongly prime ideals of S and the set of all uniformly strongly prime subsemimodules of P.*

Theorem 3.3.9. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $\Theta : Id(S) \to Id(R)$ defines a one-to-one inclusion preserving correspondence between the set of all uniformly strongly prime ideals of S and the set of all uniformly strongly prime ideals of R.*

Proof. Let J be a uniformly strongly prime ideal of S. Then from Proposition 3.3.8, $f_3(J) = PJ$ is a uniformly strongly prime subsemimodule of P and therefore, from the proof of Proposition 3.3.5 we see that, $g_1(PJ)$ is a uniformly strongly prime ideal of R. Since $\Theta(J) = \theta(PJ \otimes Q) = g_1(PJ)$, therefore $\Theta(J)$ is a uniformly strongly prime ideal of R. Analogously we can prove that for any uniformly strongly prime ideal I of R, $\Phi(I)$ is a uniformly strongly prime ideal of S. In view of the fact that Θ and Φ are mutually inverse lattice isomorphisms, the proof follows. □

Definition 3.3.10. [41] For a semiring R, the uniformly strongly prime radical is defined to be the intersection of all uniformly strongly prime k-ideals of R.

Theorem 3.3.11. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then $\Theta : Id(S) \to Id(R)$ maps the uniformly strongly prime radical $(USP(S))$ of S to the uniformly strongly prime radical $(USP(R))$ of R, i.e., $\Theta(USP(S)) = USP(R)$.*

Proof. Let $\mathcal{C}_{USP}(R)$ and $\mathcal{C}_{USP}(S)$ be the collection of all uniformly strongly prime k-ideals of R and S respectively. Then using Theorem 3.3.9 and Theorem 1.3.81 and the fact that Θ preserves k-ideals we have,

$$\Theta(USP(S)) = \Theta\left(\bigcap_{J\in\mathcal{C}_{USP}(S)} J\right) = \bigcap_{J\in\mathcal{C}_{USP}(S)} \Theta(J) \supseteq \bigcap_{I\in\mathcal{C}_{USP}(R)} I = USP(R)$$

Similarly we have $\Phi(USP(R)) \supseteq USP(S)$. Since Θ and Φ are mutually inverse lattice isomorphisms, we have $USP(R) \supseteq \Theta(USP(S))$. Hence, $\Theta(USP(S)) = USP(R)$. □

3.4 Levitzki Radical

Definition 3.4.1. [10] An ideal I of a semiring R is said to be locally nilpotent if for every finite set $F \subseteq I$ there exists a positive integer n such that $F^n = \{0_R\}$.

Definition 3.4.2. Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. A subsemimodule M of P is said to be a locally nilpotent subsemimodule if for any finite set $X \subseteq M$ and any finite set $Y \subseteq Q$, there exists a positive integer n such that $\theta(X \otimes Y)^{n-1}X = \{0_P\}$.

Definition 3.4.3. Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. A subsemimodule N of Q is said to be a locally nilpotent subsemimodule if for any finite set $Y \subseteq N$ and any finite set $X \subseteq P$, there exists a positive integer n such that $\phi(Y \otimes X)^{n-1}Y = \{0_Q\}$.

Proposition 3.4.4. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_1 : Id(R) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all locally nilpotent ideals of R and the set of all locally nilpotent subsemimodules of P.*

Proof. Suppose I is a locally nilpotent ideal of R. Let $X \subseteq f_1(I) = IP$ and $Y \subseteq Q$ be finite sets. Then $F := \{\theta(x \otimes y) \mid x \in X,\ y \in Y\}$ is finite. Also $F \subseteq \theta(X \otimes Y) \subseteq \theta(IP \otimes Q) = I\theta(P \otimes Q) = I$. Therefore there exists $n \in \mathbb{Z}^+$ such that $F^n = \{0_R\}$. Let

$x_1, x_2, ..., x_n, x_{n+1} \in X,\ y_1, y_2, ..., y_n \in Y$. Then

$$\begin{aligned} \theta(x_1 \otimes y_1)\theta(x_2 \otimes y_2)\cdots\theta(x_n \otimes y_n)x_{n+1} &\in FF\cdots Fx_{n+1} \\ &= F^n x_{n+1} \\ &= \{0_R\}x_{n+1} \\ &= \{0_P\}. \end{aligned}$$

Since every element of $\theta(X \otimes Y)^n X$ is a finite sum of elements of the form $\theta(x_1 \otimes y_1)\theta(x_2 \otimes y_2)\cdots\theta(x_n \otimes y_n)x_{n+1}$, therefore $\theta(X \otimes Y)^n X = \{0_P\}$. Hence $f_1(I)$ is locally nilpotent.

Conversely, suppose M is a locally nilpotent subsemimodule of P and F is a finite subset of $g_1(M) = \theta(M \otimes Q)$. Let $X = \{r\bar{p_v} \mid r \in F,\ v = 1, 2, ..., n'\}$ and $Y = \{\bar{q_v} \mid v = 1, 2, ..., n'\}$. Then clearly X and Y are finite subsets of P and Q respectively. In particular, $X \subseteq FP \subseteq g_1(M)P = f_1(g_1(M)) = M$. Therefore there exists $n \in \mathbb{Z}^+$, such that $\theta(X \otimes Y)^{n-1}X = \{0_P\}$. Hence $\theta(X \otimes Y)^n = \theta(X \otimes Y)^{n-1}\theta(X \otimes Y) = \theta(\theta(X \otimes Y)^{n-1}X \otimes Y) = \{0_R\}$. We claim that $F^n = \{0_R\}$. Let $r_i \in F$ for all $i = 1, 2, ..., n$. Then,

$$\begin{aligned} r_1 r_2 \cdots r_n &= r_1 1_R r_2 1_R \cdots r_n 1_R \\ &= r_1 \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}) r_2 \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}) \cdots r_n \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}) \\ &= \sum_{v=1}^{n'} \theta(r_1\bar{p_v} \otimes \bar{q_v}) \sum_{v=1}^{n'} \theta(r_2\bar{p_v} \otimes \bar{q_v}) \cdots \sum_{v=1}^{n'} \theta(r_n\bar{p_v} \otimes \bar{q_v}) \\ &\in \theta(X \otimes Y)\theta(X \otimes Y)\cdots\theta(X \otimes Y) \\ &= \theta(X \otimes Y)^n = \{0_R\}. \end{aligned}$$

Since every element of F^n is a finite sum of elements of the form $r_1 r_2 \cdots r_n$, where each $r_i \in F$, for all $i = 1, 2, ..., n$, therefore $F^n = \{0_R\}$. Hence $g_1(M)$ is locally nilpotent. $\square$

Analogously we have the following result.

Proposition 3.4.5. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_4 : Id(S) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all locally nilpotent ideals of S and the set of all locally nilpotent subsemimodules of Q.*

Proposition 3.4.6. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_2 : Id(R) \to Sub(Q)$ defines a one-to-one* *inclusion preserving correspondence between the set of all locally nilpotent ideals of R and the set of all locally nilpotent subsemimodules of Q.*

Proof. Suppose I is a locally nilpotent ideal of R. Let $Y \subseteq f_2(I) = QI$ and $X \subseteq P$ be finite sets. Then $F := \{\theta(x \otimes y) \mid x \in X,\ y \in Y\}$ is finite. Also $F \subseteq \theta(X \otimes Y) \subseteq \theta(P \otimes QI) = \theta(P \otimes Q)I = I$. Therefore there exists $n \in \mathbb{Z}^+$ such that $F^n = \{0_R\}$. Let $x_1, x_2, ..., x_n \in X,\ y_1, y_2, ..., y_{n+1} \in Y$. Then,

$$\begin{aligned}
\phi(y_1 \otimes x_1)\phi(y_2 \otimes x_2)\cdots\phi(y_n \otimes x_n)y_{n+1} = &= \phi(y_1 \otimes x_1)\cdots y_n\theta(x_n \otimes y_{n+1}) \\
&= \cdots \\
&= y_1\theta(x_1 \otimes y_2)\theta(x_2 \otimes y_3)\cdots\theta(x_n \otimes y_{n+1}) \\
&\in y_1FF\cdots F \\
&= y_1F^n = y_1\{0_R\} = \{0_Q\}.
\end{aligned}$$

Since every element of $\phi(Y \otimes X)^nY$ is a finite sum of elements of the form $\phi(y_1 \otimes x_1)\phi(y_2 \otimes x_2)\cdots\phi(y_n \otimes x_n)y_{n+1}$, therefore $\phi(Y \otimes X)^nY = \{0_Q\}$. Hence $f_2(I)$ is locally nilpotent.

Conversely, suppose N is a locally nilpotent subsemimodule of Q and F is a finite subset of $g_2(N) = \theta(P \otimes N)$. Let $Y = \{\bar{q_v}r \mid r \in F,\ v = 1, 2, ..., n'\}$ and $X = \{\bar{p_v} \mid v = 1, 2, ..., n'\}$. Then clearly X and Y are finite subsets of P and Q respectively. In particular, $Y \subseteq QF \subseteq Qg_2(N) = f_2(g_2(N)) = N$. Therefore there exists $n \in \mathbb{Z}^+$ such that $\phi(Y \otimes X)^{n-1}Y = \{0_Q\}$. We claim that $\theta(X \otimes Y)^n = \{0_R\}$. Let $x_i \in X,\ y_i \in Y$, for all $i = 1, 2, ..., n$. Then,

$$\begin{aligned}
\theta(x_1 \otimes y_1)\theta(x_2 \otimes y_2)\cdots\theta(x_n \otimes y_n) &= \theta(x_1 \otimes y_1\theta(x_2 \otimes y_2)\theta(x_3 \otimes y_3)\cdots\theta(x_n \otimes y_n)) \\
&= \theta(x_1 \otimes \phi(y_1 \otimes x_2)y_2\theta(x_3 \otimes y_3)\cdots\theta(x_n \otimes y_n)) \\
&= \theta(x_1 \otimes \phi(y_1 \otimes x_2)\phi(y_2 \otimes x_3)y_3\cdots\theta(x_n \otimes y_n)) \\
&= \cdots \\
&= \theta(x_1 \otimes \phi(y_1 \otimes x_2)\phi(y_2 \otimes x_3)\cdots y_n) \\
&\in \theta(X \otimes \phi(Y \otimes X)^{n-1}Y) \\
&= \theta(X \otimes \{0_Q\}) = \{0_R\}.
\end{aligned}$$

Since any element of $\theta(X \otimes Y)^n$ is a finite sum of elements of the form $\theta(x_1 \otimes y_1)\theta(x_2 \otimes$

$y_2) \cdots \theta(x_n \otimes y_n)$, therefore $\theta(X \otimes Y)^n = \{0_R\}$. Let $r_i \in F$ for all $i = 1, 2, ..., n$. Then,

$$\begin{aligned} r_1 r_2 \cdots r_n &= 1_R r_1 1_R r_2 \cdots 1_R r_n \\ &= \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}) r_1 \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}) r_2 \cdots \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}) r_n \\ &= \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v} r_1) \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v} r_2) \cdots \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v} r_n) \\ &\in \theta(X \otimes Y)\theta(X \otimes Y) \cdots \theta(X \otimes Y) \\ &= \theta(X \otimes Y)^n = \{0_R\}. \end{aligned}$$

Since every element of F^n is a finite sum of elements of the form $r_1 r_2 \cdots r_n$, where each $r_i \in F$ for all $i = 1, 2, ..., n$, therefore $F^n = \{0_R\}$. Hence $g_2(N)$ is locally nilpotent. □

Analogously we have the following result.

Proposition 3.4.7. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_3 : Id(S) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all locally nilpotent ideals of S and the set of all locally nilpotent subsemimodules of P.*

Theorem 3.4.8. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $\Theta : Id(S) \to Id(R)$ defines a one-to-one inclusion preserving correspondence between the set of all locally nilpotent ideals of S and the set of all locally nilpotent ideals of R.*

Proof. Let J be a locally nilpotent ideal of S. Then from Proposition 3.4.7, $f_3(J) = PJ$ is a locally nilpotent subsemimodule of P and therefore, from the proof of Proposition 3.4.4 we see that, $g_1(PJ)$ is a locally nilpotent ideal of R. Since $\Theta(J) = \theta(PJ \otimes Q) = g_1(PJ)$, therefore $\Theta(J)$ is a locally nilpotent ideal of R. Analogously we can prove that for any locally nilpotent ideal I of R, $\Phi(I)$ is a locally nilpotent ideal of S. Since Θ and Φ are mutually inverse lattice isomorphisms, the proof follows. □

Definition 3.4.9. [10] For a semiring R, the Levitzki radical is defined to be the sum of all locally nilpotent ideals of R.

Theorem 3.4.10. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then $\Theta : Id(S) \to Id(R)$ maps the Levitzki radical $(\mathcal{L}(S))$ of S, to the Levitzki radical $(\mathcal{L}(R))$ of R, i.e., $\Theta(\mathcal{L}(S)) = \mathcal{L}(R)$.*

Proof. Let $\mathcal{C}_L(R)$ and $\mathcal{C}_L(S)$ be the collection of all locally nilpotent ideals of R and S respectively. Then using Theorem 3.4.8 and the fact that Θ is a lattice isomorphism we have,

$$\Theta(\mathcal{L}(S)) = \Theta\left(\sum_{J\in\mathcal{C}_L(S)} J\right) = \sum_{J\in\mathcal{C}_L(S)} \Theta(J) \subseteq \sum_{\mathcal{C}_L(R)} I = \mathcal{L}(R)$$

By similar argument we have $\Phi(\mathcal{L}(R)) \subseteq \mathcal{L}(S)$. Since Θ and Φ are mutually inverse lattice isomorphisms, we have $\mathcal{L}(R) \subseteq \Theta(\mathcal{L}(S))$. Hence, $\Theta(\mathcal{L}(S)) = \mathcal{L}(R)$. ☐

Chapter 4

Topology on the prime spectrum of a semimodule related to a Morita context

The interplay between the algebraic properties of a given ring and the properties of topology defined on its prime spectrum has been studied intensively in the literature. In 1945, Jacobson [47] showed that the set of primitive ideals of an arbitrary ring can be made into a topological space by means of closure operator defined in terms of intersection and inclusion relations among ideals of the ring. Later McCoy [66] observed that the same method can be used without modification to introduce a topology in the set of prime ideals in a ring. Several other literatures [56, 29, 99] can also be found on the study of topological properties of the prime spectra of arbitrary rings. Commutative rings are generally given Zariski topology [7] on its prime spectrum, in which a set of prime ideals is closed if and only if it is the set of all prime ideals that contain a fixed ideal. On the other hand, there are several works on the topology defined on the prime spectra of modules over commutative rings [65] as well as non-commutative rings [92]. For a semiring with identity, Golan [31] proved that its prime spectrum, endowed with the Zariski topology, is a quasicompact T_0 space. For a commutative semiring with nonzero identity, Peña et al. [74] proved that its prime spectrum equipped with Zariski topology is a spectral space and also investigated the separation axioms of the topological space. While for semimodules over semirings, Atani et al. [6] defined a very strong multiplication semimodule M over a commutative semiring R and studied Zariski topology defined on the k-prime spectrum consisting of the prime k-subsemimodules of M. Later, Han et al. [37] defined top semimodule over a semiring, analogous to the notion of a top module (i.e., module whose spectrum of prime submodules attains a Zariski topology [65]) and studied some of its topological properties along with several other results regarding multiplication semimodules over commutative semirings. In this chapter, motivated by the recent development in the study of Morita context of semirings [82, 36, 16], we make an attempt to investigate some properties of the topology on the prime spectrum of a semimodule P related to a Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ for semirings. First we prove that if R and S are two Morita equivalent semirings via the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$, then the R-S bisemimodule P is a top bisemimodule (*cf.* Theorem 4.1.5). Then we study some separation axioms, compactness, and condition for the irreducibility of a closed subset of the prime spectrum of P equipped with Zariski topology, while incorporating some of the results of [37]. Then we obtain a homeomorphism between the prime spectrums of P and R, both equipped with Zariski topology (*cf.* Theorem 4.1.18). Finally we observe that if R and S are two Morita equivalent semirings then there is a homeomorphism between the prime spectrums of R and S, both equipped with Zariski topology (*cf.* Theorem 4.1.20).

Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then for subsets $C \subseteq P$ and $D \subseteq Q$ we write

$$\theta(C \otimes D) = \left\{ \sum_{k=1}^{n} \theta(p_k \otimes q_k) \mid p_k \in C,\ q_k \in D \text{ for all } k;\ n \in \mathbb{Z}^+ \right\} \text{ and}$$
$$\phi(D \otimes C) = \left\{ \sum_{k=1}^{n} \phi(q_k \otimes p_k) \mid q_k \in D,\ p_k \in C \text{ for all } k;\ n \in \mathbb{Z}^+ \right\}.$$

Throughout this chapter unless stated otherwise 1_R and 1_S denote respectively the identity elements of the Morita equivalent semirings R and S of the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ and also we take $1_R = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v})$, $1_S = \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u})$ (existence of such $\bar{p_v},\ \bar{q_v},\ \tilde{q_u},\ \tilde{p_u}$ is guaranteed since θ and ϕ are surjective).

For preliminaries on semirings and semimodules, we refer to Section 1.3 and for preliminaries on topology we refer to Section 1.4 of Chapter 1.

4.1 Main Results

Definition 4.1.1. [16] Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. A subsemimodule M of P is said to be prime subsemimodule if for subsemimodules A, B of P, $\theta(A \otimes Q)B \subseteq M$ implies either $A \subseteq M$ or $B \subseteq M$.

Suppose R and S are two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ and $Spec(P)$ is the collection of all prime subsemimodules of P.

For any subset X of P, we put,

$$\mathbb{V}(X) = \{M \in Spec(P) \mid X \subseteq M\}$$

As a partial analogue of [37, Lemma 3.1], we have the following result in our settings.

Lemma 4.1.2. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then for any subsemimodule M of P, the following hold:*

(1) If $\varnothing \neq X \subseteq Y \subseteq M$, then $\mathbb{V}(Y) \subseteq \mathbb{V}(X)$.

(2) If $\varnothing \neq X_\lambda \subseteq M$ for all $\lambda \in \Lambda$, then $\bigcap_{\lambda\in\Lambda} \mathbb{V}(X_\lambda) = \mathbb{V}(\bigcup_{\lambda\in\Lambda} X_\lambda)$.

(3) If $\varnothing \neq X \subseteq M$, then $\mathbb{V}(X) = \mathbb{V}(\langle X \rangle)$.

(4) If X and Y are nonempty subsets of P, then $\mathbb{V}(X) \cup \mathbb{V}(Y) \subseteq \mathbb{V}(X \cap Y)$.

(5) $\mathbb{V}(0_P) = Spec(P)$ and $\mathbb{V}(P) = \varnothing$.

Proof. The proofs follow immediately from the definition of $\mathbb{V}(X)$ for any subset X of M. □

Lemma 4.1.3. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then for any subsemimodules M and N of P, $\mathbb{V}(M) \cup \mathbb{V}(N) = \mathbb{V}(M \cap N)$.*

Proof. For any subsemimodules M and N of P we have,

$$\begin{aligned} &\theta(M \otimes Q)N \subseteq RN \subseteq N \\ \text{and} \quad &\theta(M \otimes Q)N \subseteq M\phi(Q \otimes N) \subseteq MS \subseteq M \\ \text{i.e.,} \quad &\theta(M \otimes Q)N \subseteq M \cap N. \end{aligned}$$

Then from Lemma 4.1.2(1), we see that $\mathbb{V}(M \cap N) \subseteq \mathbb{V}(\theta(M \otimes Q)N)$. Again if $L \in \mathbb{V}(\theta(M \otimes Q)N)$, then $\theta(M \otimes Q)N \subseteq L$. Now L being a prime subsemimodule (see Definition 4.1.1) of P, either $M \subseteq L$ or $N \subseteq L$, i.e., $L \in \mathbb{V}(M) \cup \mathbb{V}(N)$. Therefore $\mathbb{V}(M \cap N) \subseteq \mathbb{V}(\theta(M \otimes Q)N) \subseteq \mathbb{V}(M) \cup \mathbb{V}(N)$. Now using Lemma 4.1.2(4) we have, $\mathbb{V}(M \cap N) \subseteq \mathbb{V}(\theta(M \otimes Q)N) \subseteq \mathbb{V}(M) \cup \mathbb{V}(N) \subseteq \mathbb{V}(M \cap N)$, i.e., $\mathbb{V}(M) \cup \mathbb{V}(N) = \mathbb{V}(M \cap N)$. □

We adopt the following notion from [37].

Definition 4.1.4. An R-S-bisemimodule P is called a top bisemimodule if for any subsemimodules M and N of P, there exists a subsemimodule L of P such that $\mathbb{V}(M) \cup \mathbb{V}(N) = \mathbb{V}(L)$.

Theorem 4.1.5. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then P is a top bisemimodule.*

Proof. The proof follows directly using Lemma 4.1.3 and Definition 4.1.4. □

In view of Lemma 4.1.2 and Lemma 4.1.3, we see that if R, S are two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$, then the collection $\{\mathbb{V}(X) \mid \varnothing \neq X \subseteq P\}$ of subsets of $Spec(P)$ satisfies the properties of closed sets in a topological space (see Theorem 1.4.15). The resulting topology is called the ***Zariski topology*** on $Spec(P)$. Also by Lemma 4.1.2(3), we can say that any closed set is of the form $\mathbb{V}(M)$ for some subsemimodule M of P, whereas every open set is of the form $Spec(P) \setminus \mathbb{V}(M)$ and is denoted by $\mathbb{D}(M)$.

Remark 4.1.6. In what follows, whenever considering $Spec(P)$ as a topological space, we mean $Spec(P)$ together with the Zariski topology without mentioning the topology explicitly.

As a consequence of Theorem 4.1.5, the following results (*cf.* Lemma 4.1.7, Lemma 4.1.8, Theorem 4.1.9) are analogous to [37, Lemma 3.3], [37, Lemma 3.4] and [37, Theorem 3.1] respectively, in our settings. We omit the proofs as they are similar to that of [37].

Lemma 4.1.7. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the collection $\{\mathbb{D}(p) \mid p \in P\}$ is a base for the Zariski topology on $Spec(P)$.*

Lemma 4.1.8. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. If $\varnothing \neq Y \subseteq Spec(P)$, then the following hold:*

(1) $Y \subseteq \mathbb{V}(\bigcap_{P_i \in Y} P_i)$,

(2) $\overline{Y} = \mathbb{V}(\bigcap_{P_i \in Y} P_i)$.

Theorem 4.1.9. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then $Spec(P)$ is a T_0-space.*

Although Theorem 3.2 of [37] gives a proof of the following result, we can prove it in the following way as well.

Theorem 4.1.10. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then $Spec(P)$ is a T_1-space if and only if no prime subsemimodule of P is contained in any other prime subsemimodule of P.*

Proof. Let $Spec(P)$ be a T_1-space and M_1 and M_2 be two distinct elements of $Spec(P)$. Then each of M_1 and M_2 has a neighbourhood not containing the other. Since M_1 and M_2 are two arbitrary prime subsemimodules of P, this implies that no prime subsemimodule of P is contained in any other prime subsemimodule of P.

Conversely, suppose that no prime subsemimodule of P is contained in any other prime subsemimodule of P and M_1 and M_2 are two distinct elements of $Spec(P)$. Then by our hypothesis, $M_1 \nsubseteq M_2$, $M_2 \nsubseteq M_1$, i.e., there exist $p_1, p_2 \in P$ such that $p_1 \in M_1 \setminus M_2$, $p_2 \in M_2 \setminus M_1$. Thus we have $M_1 \in \mathbb{D}(p_2)$ but $M_2 \notin \mathbb{D}(p_2)$, and $M_2 \in \mathbb{D}(p_1)$ but $M_1 \notin \mathbb{D}(p_1)$, i.e., each of M_1 and M_2 has a neighbourhood not containing the other. Hence $Spec(P)$ is a T_1-space. ☐

Theorem 4.1.11. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then $Spec(P)$ is a Hausdorff space if and only if for any distinct pair of elements M', M'' of $Spec(P)$, there exist $p', p'' \in P$ such that $p' \notin M''$, $p'' \notin M'$ and there does not exist any element M of $Spec(P)$ such that $p' \notin M$ and $p'' \notin M$.*

Proof. Let $Spec(P)$ be a Hausdorff space. Then for any two distinct elements M', M'' of $Spec(P)$, there exist basic open sets $\mathbb{D}(p')$ and $\mathbb{D}(p'')$ such that $M' \in \mathbb{D}(p'')$, $M'' \in \mathbb{D}(p')$ and $\mathbb{D}(p') \cap \mathbb{D}(p'') = \varnothing$. Thus we have $p' \notin M''$ and $p'' \notin M'$, now if possible, let $M \in Spec(P)$ such that $p' \notin M$, $p'' \notin M$. Then $M \in \mathbb{D}(p') \cap \mathbb{D}(p'')$ - a contradiction since $\mathbb{D}(p')$ and $\mathbb{D}(p'')$ are disjoint. Thus there does not exist any element $M \in Spec(P)$ such that $p' \notin M$, $p'' \notin M$.

Conversely, let us suppose that the given condition holds and M', M'' are two distinct elements of $Spec(P)$, then by our hypothesis there exist $p', p'' \in P$ such that $p' \notin M''$, $p'' \notin M'$ and there does not exist any element M of $Spec(P)$ such that $p' \notin M$ and $p'' \notin M$. Thus we have $M' \in \mathbb{D}(p'')$, $M'' \in \mathbb{D}(p')$ and $\mathbb{D}(p') \cap \mathbb{D}(p'') = \varnothing$. Hence $Spec(P)$ is a Hausdorff space. ☐

Corollary 4.1.12. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. If $Spec(P)$ is a Hausdorff space, then no proper prime subsemimodule contains any other proper prime subsemimodule. If $Spec(P)$ contains more than* *one element, then there exist $p', p'' \in P$ such that $Spec(P) = \mathbb{D}(p') \cup \mathbb{D}(p'') \cup \mathbb{V}(M)$, where M is the subsemimodule generated by p', p''.*

Proof. Let $Spec(P)$ be a Hausdorff space, then clearly $Spec(P)$ is a T_1-space. Hence by Theorem 4.1.10 no proper prime subsemimodule contains any other proper prime subsemimodule. Now let M', M'' be a distinct pair of elements of $Spec(P)$. Since $Spec(P)$ is a Hausdorff space, there exist basic open sets $\mathbb{D}(p')$ and $\mathbb{D}(p'')$ such that $M' \in \mathbb{D}(p')$, $M'' \in \mathbb{D}(p'')$ and $\mathbb{D}(p') \cap \mathbb{D}(p'') = \varnothing$. Let M be the subsemimodule generated by p' an p''. Then M is the smallest subsemimodule containing both p' and p''. Let $N \in Spec(P)$. Then either (i) $p' \in N$, $p'' \notin N$ or (ii) $p' \notin N$, $p'' \in N$ or (iii) p', $p'' \in N$. The case $p' \notin N$, $p'' \notin N$ is not possible, since $\mathbb{D}(p') \cap \mathbb{D}(p'') = \varnothing$. Now in the first case, $N \in \mathbb{D}(p'') \subseteq \mathbb{D}(p') \cup \mathbb{D}(p'') \cup \mathbb{V}(M)$, in the second case $N \in \mathbb{D}(p') \subseteq \mathbb{D}(p') \cup \mathbb{D}(p'') \cup \mathbb{V}(M)$ and in the third case $M \subseteq N$, i.e., $N \in \mathbb{V}(M) \subseteq \mathbb{D}(p') \cup \mathbb{D}(p'') \cup \mathbb{V}(M)$. So we find that $Spec(P) \subseteq \mathbb{D}(p') \cup \mathbb{D}(p'') \cup \mathbb{V}(M)$. Again, clearly $\mathbb{D}(p') \cup \mathbb{D}(p'') \cup \mathbb{V}(M) \subseteq Spec(P)$. Hence $Spec(P) = \mathbb{D}(p') \cup \mathbb{D}(p'') \cup \mathbb{V}(M)$. □

Theorem 4.1.13. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then $Spec(P)$ is a regular space if and only if for any $M \in Spec(P)$ and $p \in P \setminus M$, there exist a subsemimodule M' of P and an element $p' \in P$ such that $M \in \mathbb{D}(p') \subseteq \mathbb{V}(M') \subseteq \mathbb{D}(p)$.*

Proof. Let $Spec(P)$ be a regular space and $M \in Spec(P)$ and $p \in P \setminus M$. Then $M \in \mathbb{D}(p)$ and $\mathbb{V}(p) = Spec(P) \setminus \mathbb{D}(p)$ is a closed set not containing M. Since $Spec(P)$ is a regular space, there exist disjoint open sets $\mathbb{D}(M')$ and $\mathbb{D}(M'')$ for some subsemimodules M' and M'' of P such that $\mathbb{V}(p) \subseteq \mathbb{D}(M')$ and $M \in \mathbb{D}(M'')$. Since $\mathbb{V}(p) \subseteq \mathbb{D}(M')$, therefore $\mathbb{V}(M') \subseteq \mathbb{D}(p)$. Again since $M \in \mathbb{D}(M'')$, therefore $M'' \nsubseteq M$, i.e., there exists $p' \in M'' \setminus M$, so $M \in \mathbb{D}(p')$. We claim to prove that $\mathbb{D}(p') \subseteq \mathbb{V}(M')$. Since $\mathbb{D}(M')$ and $\mathbb{D}(M'')$ are disjoint, $\mathbb{D}(M') \subseteq Spec(P) \setminus \mathbb{D}(M'') = \mathbb{V}(M'')$. Now let $N \in \mathbb{D}(M') \subseteq \mathbb{V}(M'')$, i.e., $M'' \subseteq N$, i.e., $p' \in N$, i.e., $N \in \mathbb{V}(p')$. This implies that $\mathbb{D}(M') \subseteq \mathbb{V}(p')$, i.e., $\mathbb{D}(p') \subseteq \mathbb{V}(M')$ and thus $M \in \mathbb{D}(p') \subseteq \mathbb{V}(M') \subseteq \mathbb{D}(p)$.

Conversely, let us suppose that the given condition holds. Let $M \in Spec(P)$ and $\mathbb{V}(N)$ be any closed set not containing M. Since $M \notin \mathbb{V}(N)$, we have $N \nsubseteq M$, so there exists $p \in N \setminus M$. Now by our hypothesis, there exists a subsemimodule M' of P and $p' \in P$ such that $M \in \mathbb{D}(p') \subseteq \mathbb{V}(M') \subseteq \mathbb{D}(p)$. Since $p \in N$, then clearly $\mathbb{D}(p) \cap \mathbb{V}(N) = \varnothing$. Therefore $\mathbb{V}(N) \subseteq Spec(P) \setminus \mathbb{D}(p) \subseteq Spec(P) \setminus \mathbb{V}(M') = \mathbb{D}(M')$. Since $\mathbb{D}(p') \subseteq \mathbb{V}(M')$, therefore $\mathbb{D}(p') \cap \mathbb{D}(M') = \varnothing$. Thus we have disjoint open sets $\mathbb{D}(p')$ and $\mathbb{D}(M')$ containing M and $\mathbb{V}(N)$ respectively. Consequently, $Spec(P)$ is a regular space. □

Theorem 4.1.14. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then $Spec(P)$ is a compact space.*

Proof. Suppose that $\{\mathbb{D}(p) \mid p \in P\}$ is an open cover of $Spec(P)$ consisting of basic open sets. Since for any $p \in P$, we have $p = p1_S = p\sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u}) = \sum_{u=1}^{m'} \theta(p \otimes \tilde{q_u})\tilde{p_u}$, i.e., P is generated by $\{\tilde{p_u} \mid u = 1, 2, ..., m'\}$. Let $M \in Spec(P)$, then there exists at least one $\tilde{p_k}$ for some $k \in \{1, 2, ..., m'\}$ such that $\tilde{p_k} \notin M$ since M is a proper subsemimodule of P. So $M \in \mathbb{D}(\tilde{p_k})$ and hence $Spec(P) \subseteq \bigcup_{u=1}^{m'} \mathbb{D}(\tilde{p_u})$, therefore the open cover $\{\mathbb{D}(p) \mid p \in P\}$ has a finite subcover $\{\mathbb{D}(\tilde{p_u}) \mid u = 1, 2, ..., m'\}$ and $Spec(P)$ is compact. □

Definition 4.1.15. Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. $Spec(P)$ is said to be irreducible if for any decomposition $Spec(P) = \mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_1, \mathcal{C}_2$ are closed subsets of $Spec(P)$, we have either $Spec(P) = \mathcal{C}_1$ or $Spec(P) = \mathcal{C}_2$.

As a consequence of Theorem 4.1.5, the following result is a direct analogue of [37, Theorem 3.3] in our settings. But in view of Definition 4.1.1 of prime subsemimodule of P, we can prove it in the following manner.

Theorem 4.1.16. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ and X be a closed subset of $Spec(P)$. Then X is irreducible if and only if $\bigcap_{P_i \in X} P_i$ is a prime subsemimodule of P.*

Proof. Suppose X is irreducible. Let M, N be two subsemimodules of P such that $\theta(M \otimes Q)N \subseteq \bigcap_{P_i \in X} P_i$, then $\theta(M \otimes Q)N \subseteq P_i$ for all $P_i \in X$. Since each P_i is prime, we have $M \subseteq P_i$ or $N \subseteq P_i$. Thus we have, for each $P_i \in X$, either $P_i \in \mathbb{V}(M)$ or $P_i \in \mathbb{V}(N)$. Hence $X = (X \cap \mathbb{V}(M)) \cup (X \cap \mathbb{V}(N))$. Since X is irreducible and $(X \cap \mathbb{V}(M))$, $(X \cap \mathbb{V}(N))$ are closed, it follows that either $X = (X \cap \mathbb{V}(M))$ or $X = (X \cap \mathbb{V}(N))$ and hence $X \subseteq \mathbb{V}(M)$ or $X \subseteq \mathbb{V}(N)$. It follows that $M \subseteq \bigcap_{P_i \in X} P_i$ or $N \subseteq \bigcap_{P_i \in X} P_i$. Consequently, $\bigcap_{P_i \in X} P_i$ is a prime subsemimodule of P.
Conversely, suppose that X is a closed subset of $Spec(P)$ and $\bigcap_{P_i \in X} P_i$ is a prime subsemimodule of P. Let $X = X_1 \cup X_2$, where X_1 and X_2 are closed subsets of $Spec(P)$. Then we have

$$\bigcap_{P_i \in X} P_i = \bigcap_{P_i \in X_1 \cup X_2} P_i = \left(\bigcap_{P_i \in X_1} P_i\right) \cap \left(\bigcap_{P_i \in X_2} P_i\right)$$

Since $\bigcap_{P_i \in X_1} P_i = M'(say)$ and $\bigcap_{P_i \in X_2} P_i = M''(say)$ are subsemimodules of P, we have

$$\theta(M' \otimes Q)M'' \subseteq RM'' \subseteq M''$$
$$\text{and } \theta(M' \otimes Q)M'' \subseteq M'\phi(Q \otimes M'') \subseteq M'S \subseteq M'$$
$$\text{i.e., } \theta(M' \otimes Q)M'' \subseteq M' \cap M'' = \left(\bigcap_{P_i \in X_1} P_i\right) \cap \left(\bigcap_{P_i \in X_2} P_i\right) = \bigcap_{P_i \in X} P_i.$$

Since $\bigcap_{P_i \in X} P_i$ is a prime subsemimodule of P, therefore either $\bigcap_{P_i \in X_1} P_i = M' \subseteq \bigcap_{P_i \in X} P_i$ or $\bigcap_{P_i \in X_2} P_i = M'' \subseteq \bigcap_{P_i \in X} P_i$, i.e., either $\bigcap_{P_i \in X_1} P_i = \bigcap_{P_i \in X} P_i$ or $\bigcap_{P_i \in X_2} P_i = \bigcap_{P_i \in X} P_i$. Without loss of generality let us suppose that $\bigcap_{P_i \in X_1} P_i = \bigcap_{P_i \in X} P_i$, then for any $N \in X$, we have $\bigcap_{P_i \in X_1} P_i \subseteq N$. Now using Lemma 4.1.8 and the fact that X_1 is closed, we see that $N \in \overline{X_1} = X_1$ ie., $X = X_1$. □

Now let us briefly recall [31] the construction of Zariski topology on $Spec(R)$, the set of all prime ideals of a semiring R. For each ideal I of R, $\mathbb{V}(I) = \{H \in Spec(R) \mid I \subseteq H\}$ and $\mathbb{D}(I) = Spec(R) \setminus \mathbb{V}(I)$. Then $Zar(R) = \{\mathbb{V}(I) \mid I \text{ is an ideal of } R\}$ is the family of closed sets for the Zariski topology on $Spec(R)$.

Proposition 4.1.17. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then there exists an inclusion preserving bijection between $Spec(R)$ and $Spec(P)$.*

Proof. Let R and S be Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then in Theorem 1.3.77 we see that the lattice of ideals of R and the lattice of subsemimodules of P are isomorphic, where the lattice isomorphisms,

$$f : Id(R) \to Sub(P) \text{ and } g : Sub(P) \to Id(R) \text{ are defined by}$$
$$f(I) := \left\{\sum_{k=1}^{n} i_k p_k \mid p_k \in P, i_k \in I \text{ for all } k;\ n \in \mathbb{Z}^+\right\} = IP \text{ and,}$$
$$g(M) := \left\{\sum_{k=1}^{n} \theta(p_k \otimes q_k) \mid p_k \in M, q_k \in Q \text{ for all } k;\ n \in \mathbb{Z}^+\right\} = \theta(M \otimes Q)$$

In Proposition 3.1.4, we see that the given mapping takes prime ideals to prime subsemimodules and vice versa. It follows that $f' := f|_{Spec(R)} : Spec(R) \to Spec(P)$ and $g' := g|_{Spec(P)} : Spec(P) \to Spec(R)$ are mutually inverse mappings. Consequently f' is an inclusion preserving bijection from $Spec(R)$ to $Spec(P)$. □

Theorem 4.1.18. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then there exists a homeomorphism between $Spec(R)$ and $Spec(P)$.*

Proof. By Proposition 4.1.17, $f' := f|_{Spec(R)} : Spec(R) \to Spec(P)$ and $g' := g|_{Spec(P)} : Spec(P) \to Spec(R)$ are mutually inverse mappings. In order to prove the continuity of f', let X be any closed subset of $Spec(P)$, then $X = \mathbb{V}(M)$ for some subsemimodule M of P. Let $I \in f'^{-1}(\mathbb{V}(M))$, then $f'(I) \in \mathbb{V}(M)$, i.e., $f(I) \in \mathbb{V}(M)$, i.e., $M \subseteq f(I)$, i.e., $g(M) \subseteq I$ (since f and g are inclusion preserving mutually inverse mappings), i.e., $I \in \mathbb{V}(g(M))$, therefore $f'^{-1}(\mathbb{V}(M)) \subseteq \mathbb{V}(g(M))$. The reverse inclusion follows analogously. Thus $f'^{-1}(\mathbb{V}(M)) = \mathbb{V}(g(M))$, which is closed in $Spec(R)$, hence f' is continuous. Again for any closed subset $\mathbb{V}(I)$ of $Spec(P)$, where I is an ideal of R, we can prove in a similar manner that $g'^{-1}(\mathbb{V}(I)) = \mathbb{V}(f(I))$, which is closed in $Spec(P)$, hence g' is continuous. Since f' and g' are mutually inverse mappings such that both of them are continuous, therefore $Spec(R)$ and $Spec(P)$ are homeomorphic. □

Analogously we have the following theorem.

Theorem 4.1.19. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then there exists a homeomorphism between $Spec(S)$ and $Spec(P)$.*

Combining Theorems 4.1.18 and 4.1.19 we obtain the following result.

Theorem 4.1.20. *Let R, S be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then there exists a homeomorphism between $Spec(R)$ and $Spec(S)$.*

Chapter 5

On some Morita invariants of monoids

In this chapter, we prove that if S and T are Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$, then there exists a one-to-one inclusion preserving correspondence between the set of all (right) strongly prime (uniformly strongly prime, nil, nilpotent) ideals of S and the set of all (right) strongly prime (resp. uniformly strongly prime, nil, nilpotent) sub-biacts of P (*cf.* Propositions 5.1.4, 5.1.12, 5.2.4, 5.2.12). Similar correspondences are established between T and Q (*cf.* Propositions 5.1.5, 5.1.13, 5.2.5, 5.2.13), S and Q (*cf.* Propositions 5.1.6, 5.1.14, 5.2.6, 5.2.14), T and P (*cf.* Propositions 5.1.7, 5.1.15, 5.2.7, 5.2.15), which in turn result in one-to-one inclusion preserving correspondence between the set of all (right) strongly prime (uniformly strongly prime, nil, nilpotent) ideals of S and T (*cf.* Theorems 5.1.8, 5.1.16, 5.2.8, 5.2.16).

A six-tuple $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ is known as a Morita context of monoids [90, 84], where S, T are monoids, ${}_SP_T$ and ${}_TQ_S$ are biacts, and $\theta : {}_S(P \otimes_T Q)_S \to {}_SS_S$ and $\phi : {}_T(Q \otimes_S P)_T \to {}_TT_T$ are biact homomorphisms such that for every $p, p' \in P$ and $q, q' \in Q$, $\theta(p \otimes q)p' = p\phi(q \otimes p')$ and $\phi(q \otimes p)q' = q\theta(p \otimes q')$. As a simple consequence of Remark 1.2.38 we see that S and T are Morita equivalent monoids if and only if there exists a Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ with θ and ϕ surjective.

Let S and T be Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then as a consequence of Theorem 1.2.42, when the semigroups are replaced by monoids, we see that the lattice of ideals of S and the lattice of sub-biacts of P are isomorphic via the following mappings (see Remark 1.2.43).

$$f_1 : Id(S) \to Sub(P) \;\text{ and }\; g_1 : Sub(P) \to Id(S) \;\text{ are defined by}$$
$$f_1(I) := \{ip \mid p \in P,\ i \in I\} = IP,$$
$$g_1(M) := \{\theta(m \otimes q) \mid m \in M,\ q \in Q\} = \theta(M \otimes Q)$$

For the other pairs of the Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$, similar isomorphism can be defined as follows.

$$f_2 : Id(S) \to Sub(Q) \;\text{ and }\; g_2 : Sub(Q) \to Id(S) \;\text{ are defined by}$$
$$f_2(I) := \{qi \mid q \in Q,\ i \in I\} = QI,$$
$$g_2(N) := \{\theta(p \otimes n) \mid p \in P,\ n \in N\} = \theta(P \otimes N)$$

The mappings $f_3 : Id(T) \to Sub(P)$, $g_3 : Sub(P) \to Id(T)$, $f_4 : Id(T) \to Sub(Q)$, $g_4 : Sub(Q) \to Id(T)$ are defined in an analogous manner. Again, as a consequence of Theorem 1.2.41 we see that the lattice of ideals of S and the lattice of ideals of T are isomorphic via the following mappings.

$$\Theta : Id(T) \to Id(S) \;\text{ and }\; \Phi : Id(S) \to Id(T) \;\text{ are defined by}$$
$$\Theta(J) := \{\theta(pj \otimes q) \mid p \in P,\ q \in Q,\ j \in J\} = \theta(PJ \otimes Q)$$
$$\Phi(I) := \{\phi(qi \otimes p) \mid p \in P,\ q \in Q,\ i \in I\} = \phi(QI \otimes P)$$

Throughout this chapter unless stated otherwise 1_S and 1_T denote respectively the identity elements of the Morita equivalent monoids S and T of the Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ and also we take $1_S = \theta(\bar{p} \otimes \bar{q})$, $1_T = \phi(\tilde{q} \otimes \tilde{p})$ (existence of such $\bar{p},\ \bar{q},\ \tilde{q},\ \tilde{p}$ is guaranteed since θ and ϕ are surjective).

For preliminaries of monoids and S-acts, we refer to Section 1.2 of Chapter 1.

5.1 Strongly prime and Uniformly strongly prime sub-biacts

In this section, we define (right) strongly prime and uniformly strongly prime sub-biact of a monoid act related to a Morita context of monoids and investigate the correspondence between the set of all right strongly prime (uniformly strongly prime) sub-biacts and the set of all right strongly prime (resp. uniformly strongly prime) ideals of a pair of biact and monoid connected via Morita context.

Definition 5.1.1. [18] An ideal I of a monoid (semigroup) S is called a (right) strongly prime ideal if for every a in S with $a \notin I$, there is a finite set $F \subseteq \langle a \rangle$ such that for $b \in S$, $Fb \subseteq I$ implies that $b \in I$.

Definition 5.1.2. Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. A sub-biact M of P is said to be a (right) strongly prime sub-biact if for every element p of P with $p \notin M$ there exist finite subsets $X \subseteq \langle p \rangle$ (sub-biact generated by p) and $Y \subseteq Q$ such that for $p' \in P$, $\theta(X \otimes Y)p' \subseteq M$ implies that $p' \in M$.

Definition 5.1.3. Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. A sub-biact N of Q is said to be a (right) strongly prime sub-biact if for every element q of Q with $q \notin N$ there exist finite subsets $Y \subseteq \langle q \rangle$ (sub-biact generated by q) and $X \subseteq P$ such that for $q' \in Q$, $\phi(Y \otimes X)q' \subseteq N$ implies that $q' \in N$.

Proposition 5.1.4. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $f_1 : Id(S) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all (right) strongly prime ideals of S and the set of all (right) strongly prime sub-biacts of P.*

Proof. Let I be a (right) strongly prime ideal of S and $p \notin f_1(I) = IP$ for some $p \in P$. Then clearly $\theta(p \otimes \tilde{q}) \notin I$, otherwise $p = p1_T = p\phi(\tilde{q} \otimes \tilde{p}) = \theta(p \otimes \tilde{q})\tilde{p} \in IP$ - a contradiction. Since $\theta(p \otimes \tilde{q}) \notin I$, therefore by hypothesis there exists a finite subset $F \subseteq \langle \theta(p \otimes \tilde{q}) \rangle$ such that for $s' \in S$, $Fs' \subseteq I$ implies that $s' \in I$. Let $Y = \{\bar{q}\} \subseteq Q$ and $X = \{s\bar{p} \mid s \in F\}$. Then both Y and X are finite subsets of Q and P respectively. Since every element of X is of the form $s\bar{p}$ for some $s \in F$, i.e., $s = a\theta(p \otimes \tilde{q})b$, where $a, b \in S$, therefore $s\bar{p} = a\theta(p \otimes \tilde{q})b\bar{p} = ap\phi(\tilde{q} \otimes b\bar{p}) \in SpT = \langle p \rangle$, i.e., $X \subseteq \langle p \rangle$.

Suppose $p' \in P$ such that $\theta(X \otimes Y)p' \subseteq f_1(I) = IP$. Let $s \in F$. Then using the fact that f_1 and g_1 are mutually inverse maps we have,

$$\begin{aligned} s\theta(p' \otimes \tilde{q}) &= s1_S\theta(p' \otimes \tilde{q}) = s\theta(\bar{p} \otimes \bar{q})\theta(p' \otimes \tilde{q}) \\ &= \theta(s\bar{p} \otimes \bar{q})\theta(p' \otimes \tilde{q}) = \theta(\theta(s\bar{p} \otimes \bar{q})p' \otimes \tilde{q}) \\ &\in \theta(\theta(X \otimes Y)p' \otimes \tilde{q}) \subseteq \theta(f_1(I) \otimes Q) = g_1(f_1(I)) = I. \end{aligned}$$

Therefore we see that $F\theta(p' \otimes \tilde{q}) \subseteq I$. Then by our hypothesis we have $\theta(p' \otimes \tilde{q}) \in I$. Therefore $p' = p'1_T = p'\phi(\tilde{q} \otimes \tilde{p}) = \theta(p' \otimes \tilde{q})\tilde{p} \in IP = f_1(I)$. Hence $f_1(I)$ is a (right) strongly prime sub-biact of P.

Conversely, let M be a (right) strongly prime sub-biact of P and $s \in S$ such that $s \notin g_1(M) = \theta(M \otimes Q)$. Then clearly $s\bar{p} \notin M$, otherwise $s = s1_S = s\theta(\bar{p} \otimes \bar{q}) = \theta(s\bar{p} \otimes \bar{q}) \in \theta(M \otimes Q) = g_1(M)$ - a contradiction. Since $s\bar{p} \notin M$, therefore there exist finite subsets $X \subseteq \langle s\bar{p} \rangle$ and $Y \subseteq Q$ such that for $p' \in P$, $\theta(X \otimes Y)p' \subseteq M$ implies that $p' \in M$. Let $F = \{\theta(x \otimes y) \mid x \in X,\ y \in Y\}$. Then clearly F is a finite subset of S and for any $\theta(x \otimes y) \in F$ we have, $\theta(x \otimes y) \in \theta(\langle s\bar{p}\rangle \otimes Q) = \theta(S(s\bar{p})T \otimes Q) \subseteq Ss\theta(\bar{p}T \otimes Q) \subseteq SsS = \langle s\rangle$, i.e., $F \subseteq \langle s\rangle$.

Suppose $s' \in S$ such that $Fs' \subseteq g_1(M) = \theta(M \otimes Q)$. Let $x \in X,\ y \in Y$. Then using the fact that f_1 and g_1 are mutually inverse maps we have, $\theta(x \otimes y)(s'\bar{p}) \in F(s'\bar{p}) = (Fs')\bar{p} \subseteq g_1(M)P = f_1(g_1(M)) = M$. Therefore we see that $\theta(X \otimes Y)s'\bar{p} \subseteq M$. Then by our hypothesis we have $s'\bar{p} \in M$. Therefore $s' = s'1_S = s'\theta(\bar{p} \otimes \bar{q}) = \theta(s'\bar{p} \otimes \bar{q}) \in \theta(M \otimes Q) = g_1(M)$. Thus $g_1(M)$ is a (right) strongly prime ideal of S. Since f_1 and g_1 are mutually inverse lattice isomorphisms, the proof follows. □

Analogously we obtain the following result.

Proposition 5.1.5. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $f_4 : Id(T) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all (right) strongly prime ideals of T and the set of all (right) strongly prime sub-biacts of Q.*

Proposition 5.1.6. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $f_2 : Id(S) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all (right) strongly prime ideals of S and the set of all (right) strongly prime sub-biacts of Q.*

Proof. Let I be a (right) strongly prime ideal of S and $q \notin f_2(I) = QI$ for some $q \in Q$. Then clearly $\theta(\tilde{p} \otimes q) \notin I$, otherwise $q = 1_Tq = \phi(\tilde{q} \otimes \tilde{p})q = \tilde{q}\theta(\tilde{p} \otimes q) \in QI$ - a contradiction. Since $\theta(\tilde{p} \otimes q) \notin I$, therefore by hypothesis there exists a finite subset $F \subseteq \langle\theta(\tilde{p} \otimes q)\rangle$ such that for $s' \in S$, $Fs' \subseteq I$ implies that $s' \in I$. Let $Y = \{\bar{q}s \mid s \in F\} \subseteq Q$ and $X = \{\tilde{p}\}$. Then both Y and X are finite subsets of Q and P respectively. Since every element of Y is of the form $\bar{q}s$ for some $s \in F$, i.e., $s = a\theta(\tilde{p} \otimes q)b$ for some $a, b \in S$, therefore $\bar{q}s = \bar{q}a\theta(\tilde{p} \otimes q)b = \phi(\bar{q} \otimes a\tilde{p})qb \in TqS = \langle q\rangle$, i.e., $Y \subseteq \langle q\rangle$.

Suppose $q' \in Q$ such that $\phi(Y \otimes X)q' \subseteq f_2(I) = QI$. Let $s \in F$. Then using the fact that f_2 and g_2 are mutually inverse maps we have,

$$\begin{aligned} s\theta(\tilde{p} \otimes q') &= 1_Ss\theta(\tilde{p} \otimes q') = \theta(\bar{p} \otimes \bar{q})s\theta(\tilde{p} \otimes q') \\ &= \theta(\bar{p} \otimes \bar{q}s\theta(\tilde{p} \otimes q')) = \theta(\bar{p} \otimes \phi(\bar{q}s \otimes \tilde{p})q') \\ &\in \theta(P \otimes \phi(Y \otimes X)q') \subseteq \theta(P \otimes f_2(I)) = g_2(f_2(I)) = I. \end{aligned}$$

Therefore we see that $F\theta(\tilde{p} \otimes q') \subseteq I$. Then by our hypothesis we have $\theta(\tilde{p} \otimes q') \in I$. Therefore $q' = 1_Tq' = \phi(\tilde{q} \otimes \tilde{p})q' = \tilde{q}\theta(\tilde{p} \otimes q') \in QI = f_2(I)$. Hence $f_2(I)$ is a (right) strongly prime sub-biact of Q.

Conversely, let N be a (right) strongly prime sub-biact of Q and $s \in S$ such that $s \notin g_2(N) = \theta(P \otimes N)$. Then clearly $\bar{q}s \notin N$, otherwise $s = 1_S s = \theta(\bar{p} \otimes \bar{q})s = \theta(\bar{p} \otimes \bar{q}s) \in \theta(P \otimes N) = g_2(N)$ - a contradiction. Since $\bar{q}s \notin N$, therefore there exist finite subsets $X \subseteq P,\ Y \subseteq \langle \bar{q}s \rangle$ such that for $q' \in Q$, $\phi(Y \otimes X)q' \subseteq N$ implies that $q' \in N$. Let $F = \{\theta(\tilde{p} \otimes y)\theta(x \otimes \bar{q}) \mid y \in Y,\ x \in X\}$. Then clearly F is a finite subset of S and since $y \in \langle \bar{q}s \rangle,\ y = c(\bar{q}s)a$ for some $c \in T,\ a \in S$, therefore for any element of F, $\theta(\tilde{p} \otimes y)\theta(x \otimes \bar{q}) = \theta(\tilde{p} \otimes c(\bar{q}s)a)\theta(x \otimes \bar{q}) = \theta(\tilde{p} \otimes c\bar{q})sa\theta(x \otimes \bar{q}) \in \langle s \rangle$, i.e., $F \subseteq \langle s \rangle$.

Suppose $s' \in S$ such that $Fs' \subseteq g_2(N) = \theta(P \otimes N)$. Let $x \in X,\ y \in Y$. Then using the fact that f_2 and g_2 are mutually inverse maps we have,

$$\begin{aligned}\phi(y \otimes x)\bar{q}s' &= 1_T\phi(y \otimes x)\bar{q}s' = \phi(\tilde{q} \otimes \tilde{p})\phi(y \otimes x)\bar{q}s' \\ &= \phi(\tilde{q} \otimes \tilde{p})y\theta(x \otimes \bar{q})s' = \tilde{q}\theta(\tilde{p} \otimes y)\theta(x \otimes \bar{q})s' \\ &\in QFs' \subseteq Qg_2(N) = f_2(g_2(N)) = N.\end{aligned}$$

Therefore we see that $\phi(Y \otimes X)(\bar{q}s') \subseteq N$. Then by our hypothesis we have $\bar{q}s' \in N$. Therefore $s' = 1_S s' = \theta(\bar{p} \otimes \bar{q})s' = \theta(\bar{p} \otimes \bar{q}s') \in \theta(P \otimes N) = g_2(N)$. Thus $g_2(N)$ is a (right) strongly prime ideal of S. Since f_2 and g_2 are mutually inverse lattice isomorphisms, the proof follows. □

Analogously we obtain the following result.

Proposition 5.1.7. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $f_3 : Id(T) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all (right) strongly prime ideals of T and the set of all (right) strongly prime sub-biacts of P.*

Theorem 5.1.8. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $\Theta : Id(T) \to Id(S)$ defines a one-to-one inclusion preserving correspondence between the set of all (right) strongly prime ideals of T and the set of all (right) strongly prime ideals of S.*

Proof. Let J be a (right) strongly prime ideal of T. Then from Proposition 5.1.7, $f_3(J) = PJ$ is a (right) strongly prime sub-biact of P and therefore from the proof of Proposition 5.1.4 we see that, $g_1(PJ)$ is a (right) strongly prime ideal of S. Since $\Theta(J) = \theta(PJ \otimes Q) = g_1(PJ)$, therefore $\Theta(J)$ is a (right) strongly prime ideal of S. Analogously we can prove that for any (right) strongly prime ideal I of S, $\Phi(I)$ is a (right) strongly prime ideal of T. Hence the proof follows in view of the fact that Θ and Φ are mutually inverse lattice isomorphisms. □

Definition 5.1.9. [19] A proper ideal I of a monoid (semigroup) S is called a uniformly strongly prime ideal of S, if there exists a finite subset F of S such that for $x, y \in S$, $xFy \subseteq I$ implies that $x \in I$ or $y \in I$.

Definition 5.1.10. Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. A sub-biact M of P is said to be a uniformly strongly prime sub-biact if there exist finite subsets X and Y of P and Q respectively such that for $p', p'' \in P$, $\theta(p' \otimes Y)\theta(X \otimes Y)p'' \subseteq M$ implies that $p' \in M$ or $p'' \in M$.

Definition 5.1.11. Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. A sub-biact N of Q is said to be a uniformly strongly prime sub-biact if there exist finite subsets Y and X of Q and P respectively such that for $q', q'' \in P$, $\phi(q' \otimes X)\phi(Y \otimes X)q'' \subseteq N$ implies that $q' \in N$ or $q'' \in N$.

Proposition 5.1.12. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $f_1 : Id(S) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all uniformly strongly prime ideals of S and the set of all uniformly strongly prime sub-biacts of P.*

Proof. Let I be a uniformly strongly prime ideal of S. Then there exists a finite subset $F \subseteq S$ such that for $s', s'' \in S$, $s'Fs'' \subseteq I$ implies that $s' \in I$ or $s'' \in I$. Suppose $X = \{s\bar{p} \mid s \in F\}$, $Y = \{\tilde{q}, \bar{q}\}$. Since F is finite, clearly X is a finite subset of P.

Let $p', p'' \in P$ such that $\theta(p' \otimes Y)\theta(X \otimes Y)p'' \subseteq f_1(I) = IP$ and $p' \notin IP$. Then clearly $\theta(p' \otimes \tilde{q}) \notin I$, otherwise $p' = p'1_T = p'\phi(\tilde{q} \otimes \tilde{p}) = \theta(p' \otimes \tilde{q})\tilde{p} \in IP$ - a contradiction. Now for any $s \in F$ we have,

$$\begin{aligned}
\theta(p' \otimes \tilde{q})s\theta(p'' \otimes \tilde{q}) &= \theta(p' \otimes \tilde{q})s1_S\theta(p'' \otimes \tilde{q}) = \theta(p' \otimes \tilde{q})s\theta(\bar{p} \otimes \bar{q})\theta(p'' \otimes \tilde{q}) \\
&= \theta(p' \otimes \tilde{q})\theta(s\bar{p} \otimes \bar{q})\theta(p'' \otimes \tilde{q}) = \theta(\theta(p' \otimes \tilde{q})\theta(s\bar{p} \otimes \bar{q})p'' \otimes \tilde{q}) \\
&\in \theta(\theta(p' \otimes Y)\theta(X \otimes Y)p'' \otimes \tilde{q}) \subseteq \theta(f_1(I) \otimes Q) = g_1(f_1(I)) = I
\end{aligned}$$

Therefore $\theta(p' \otimes \tilde{q})F\theta(p'' \otimes \tilde{q}) \subseteq I$. Now since $\theta(p' \otimes \tilde{q}) \notin I$, therefore by our hypothesis $\theta(p'' \otimes \tilde{q}) \in I$. So we get $p'' = p''1_T = p''\phi(\tilde{q} \otimes \tilde{p}) = \theta(p'' \otimes \tilde{q})\tilde{p} \in IP$. Hence $f_1(I)$ is a uniformly strongly prime sub-biact of P.

Conversely, let M be a uniformly strongly prime sub-biact of P. Then there exist finite subsets $X \subseteq P$ and $Y \subseteq Q$ such that for $p', p'' \in P$, $\theta(p' \otimes Y)\theta(X \otimes Y)p'' \subseteq M$ implies that $p' \in M$ or $p'' \in M$. Let $F = \{\theta(\bar{p} \otimes y')\theta(x \otimes y'') \mid x \in X,\ y', y'' \in Y\}$. Then clearly F is a finite subset of S.

Suppose $s', s'' \in S$ such that $s'Fs'' \subseteq g_1(M) = \theta(M \otimes Q)$ and $s' \notin \theta(M \otimes Q)$, then clearly $s'\bar{p} \notin M$, otherwise $s' = s'1_S = s'\theta(\bar{p} \otimes \bar{q}) = \theta(s'\bar{p} \otimes \bar{q}) \in \theta(M \otimes Q)$ - a contradiction. Now for any $y', y'' \in Y$, $x \in X$ and $p \in P$, using the fact that f_1 and g_1 are mutually inverse maps we have, $\theta(s'\bar{p} \otimes y')\theta(x \otimes y'')s''\bar{p} = s'\theta(\bar{p} \otimes y')\theta(x \otimes y'')s''\bar{p} \in s'Fs''\bar{p} \subseteq g_1(M)P = f_1(g_1(M)) = M$. Therefore $\theta(s'\bar{p} \otimes Y)\theta(X \otimes Y)s''\bar{p} \subseteq M$. As $s'\bar{p} \notin M$, by our hypothesis $s''\bar{p} \in M$. Therefore $s'' = s''1_S = s''\theta(\bar{p} \otimes \bar{q}) = \theta(s''\bar{p} \otimes \bar{q}) \in \theta(M \otimes Q) = g_1(M)$. Thus $g_1(M)$ is a uniformly strongly prime ideal of S. Since f_1 and g_1 are mutually inverse lattice isomorphisms, the proof follows. □

Analogously we obtain the following result.

Proposition 5.1.13. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $f_4 : Id(T) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all uniformly strongly prime ideals of T and the set of all uniformly strongly prime sub-biacts of Q.*

Proposition 5.1.14. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $f_2 : Id(S) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all uniformly strongly prime ideals of S and the set of all uniformly strongly prime sub-biacts of Q.*

Proof. Let I be a uniformly strongly prime ideal of S. Then there exists a finite subset $F \subseteq S$ such that for $s', s'' \in S$, $s'Fs'' \subseteq I$ implies that $s' \in I$ or $s'' \in I$. Suppose $X = \{s\bar{p} \mid s \in F\} \cup \{\tilde{p}\}$, $Y = \{\bar{q}\}$. Since F is finite, X is a finite subset of P.

Let $q', q'' \in Q$ such that $\phi(q' \otimes X)\phi(Y \otimes X)q'' \subseteq f_2(I) = QI$ and $q' \notin QI$. Then clearly $\theta(\tilde{p} \otimes q') \notin I$, otherwise $q' = 1_Tq' = \phi(\tilde{q} \otimes \tilde{p})q' = \tilde{q}\theta(\tilde{p} \otimes q') \in QI$ - a contradiction. Now for any $s \in F$, we have,

$$\begin{aligned}
\theta(\tilde{p} \otimes q')s\theta(\tilde{p} \otimes q'') &= \theta(\tilde{p} \otimes q')s1_S\theta(\tilde{p} \otimes q'') = \theta(\tilde{p} \otimes q')s\theta(\bar{p} \otimes \bar{q})\theta(\tilde{p} \otimes q'') \\
&= \theta(\tilde{p} \otimes q')\theta(s\bar{p} \otimes \bar{q})\theta(\tilde{p} \otimes q'') = \theta(\tilde{p} \otimes q'\theta(s\bar{p} \otimes \bar{q})\theta(\tilde{p} \otimes q'')) \\
&= \theta(\tilde{p} \otimes \phi(q' \otimes s\bar{p})\bar{q}\theta(\tilde{p} \otimes q'')) = \theta(\tilde{p} \otimes \phi(q' \otimes s\bar{p})\phi(\bar{q} \otimes \tilde{p})q'') \\
&\in \theta(P \otimes \phi(q' \otimes X)\phi(Y \otimes X)q'') \subseteq \theta(P \otimes f_2(I)) = g_2(f_2(I)) = I
\end{aligned}$$

Therefore $\theta(\tilde{p} \otimes q')F\theta(\tilde{p} \otimes q'') \subseteq I$. Now since $\theta(\tilde{p} \otimes q') \notin I$, therefore by our hypothesis $\theta(\tilde{p} \otimes q'') \in I$. So we get $q'' = 1_Tq'' = \phi(\tilde{q} \otimes \tilde{p})q'' = \tilde{q}\theta(\tilde{p} \otimes q'') \in QI$. Hence $f_2(I)$ is a uniformly strongly prime sub-biact of Q.

Conversely, let N be a uniformly strongly prime sub-biact of Q. Then there exist finite subsets $X \subseteq P$ and $Y \subseteq Q$ such that for $q', q'' \in Q$, $\phi(q' \otimes X)\phi(Y \otimes X)q'' \subseteq N$

implies that $q' \in N$ or $q'' \in N$. Let $F = \{\theta(x' \otimes y)\theta(x'' \otimes \bar{q}) \mid x', x'' \in X,\ y \in Y\}$. Then clearly F is a finite subset of S.

Suppose $s', s'' \in S$ such that $s'Fs'' \subseteq g_2(N) = \theta(P \otimes N)$ and $s' \notin \theta(P \otimes N)$, then clearly $\bar{q}s' \notin N$, otherwise $s' = 1_S s' = \theta(\bar{p} \otimes \bar{q})s' = \theta(\bar{p} \otimes \bar{q}s') \in \theta(P \otimes N)$ - a contradiction. Now for any $x', x'' \in X,\ y \in Y$, using the fact that f_2 and g_2 are mutually inverse maps we have,

$$\begin{aligned}\phi(\bar{q}s' \otimes x')\phi(y \otimes x'')\bar{q}s'' &= \phi(\bar{q}s' \otimes x'\phi(y \otimes x''))\bar{q}s'' = \phi(\bar{q}s' \otimes \theta(x' \otimes y)x'')\bar{q}s'' \\ &= \bar{q}s'\theta(\theta(x' \otimes y)x'' \otimes \bar{q})s'' = \bar{q}s'\theta(x' \otimes y)\theta(x'' \otimes \bar{q})s'' \\ &\in \bar{q}s'Fs'' \subseteq Qg_2(N) = f_2(g_2(N)) = N.\end{aligned}$$

Therefore $\phi(\bar{q}s' \otimes X)\phi(Y \otimes X)\bar{q}s'' \subseteq N$. As $\bar{q}s' \notin N$, by our hypothesis $\bar{q}s'' \in N$. Therefore $s'' = 1_S s'' = \theta(\bar{p} \otimes \bar{q})s'' = \theta(\bar{p} \otimes \bar{q}s'') \in \theta(P \otimes N) = g_2(N)$. Thus $g_2(N)$ is a uniformly strongly prime ideal of S. This completes the proof as f_2 and g_2 are mutually inverse lattice isomorphisms. ☐

Analogously we obtain the following result.

Proposition 5.1.15. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $f_3 : Id(T) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all uniformly strongly prime ideals of T and the set of all uniformly strongly prime sub-biacts of P.*

Theorem 5.1.16. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $\Theta : Id(T) \to Id(S)$ defines a one-to-one inclusion preserving correspondence between the set of all uniformly strongly prime ideals of T and the set of all uniformly strongly prime ideals of S.*

Proof. Let J be a uniformly strongly prime ideal of T. Then from Proposition 5.1.15, $f_3(J) = PJ$ is a uniformly strongly prime sub-biact of P and therefore from the proof of Proposition 5.1.12 we see that, $g_1(PJ)$ is a uniformly strongly prime ideal of S. Since $\Theta(J) = \theta(PJ \otimes Q) = g_1(PJ)$, therefore $\Theta(J)$ is a uniformly strongly prime ideal of S. Analogously we can prove that for any uniformly strongly prime ideal I of S, $\Phi(I)$ is a uniformly strongly prime ideal of T. In view of the fact that Θ and Φ are mutually inverse lattice isomorphisms, the proof follows. ☐

5.2 Nil and Nilpotent sub-biacts

In this section, we define nil and nilpotent sub-biacts and investigate the correspondence between the set of all nil (nilpotent) sub-biacts and the set of all nil (resp.

nilpotent) ideals of a pair of biact and monoid related via Morita context of monoids. Throughout this section, we consider all the monoids and biacts to have kernel (see Definition 1.2.8 and Definition 1.2.17).

Definition 5.2.1. [13] An element x of a semigroup (monoid) S is said to be nilpotent if $x^n \in K_S$ for some $n \in \mathbb{Z}^+$. An ideal I of S is said to be a nil ideal of S provided every element of I is nilpotent.

Definition 5.2.2. Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. An element $p \in P$ is said to be nilpotent if for each $q \in Q$, there exists $n \in \mathbb{Z}^+$ such that $\theta(p \otimes q)^n p \in K_P$. A sub-biact M of P is said to be a nil sub-biact of P provided every element of M is nilpotent.

Definition 5.2.3. Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. An element $q \in Q$ is said to be nilpotent if for each $p \in P$ there exists $n \in \mathbb{Z}^+$ such that $\phi(q \otimes p)^n q \in K_Q$. A sub-biact N of Q is said to be a nil sub-biact of Q provided every element of N is nilpotent.

Proposition 5.2.4. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $f_1 : Id(S) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all nil ideals of S and the set of all nil sub-biacts of P.*

Proof. Let I be a nil ideal of S and $ip \in f_1(I) = IP$ for some $i \in I,\ p \in P$. Then for any $q \in Q$, $\theta(ip \otimes q) \in \theta(IP \otimes Q) = I\theta(P \otimes Q) = I$. Therefore there exists $k \in \mathbb{Z}^+$ such that $\theta(ip \otimes q)^k \in K_S$. Then, $\theta(ip \otimes q)^k ip \in K_S IP \subseteq K_S P = f_1(K_S) = K_P$. Hence $f_1(I)$ is a nil sub-biact of P.

Conversely, let M be a nil sub-biact of P and $x \in g_1(M) = \theta(M \otimes Q)$. Then clearly $x = \theta(m \otimes q)$, for some $m \in M,\ q \in Q$. Since $m \in M$, therefore there exists $k = k(q) \in \mathbb{Z}^+$ such that $\theta(m \otimes q)^k m \in K_P$. Then we have, $x^{k+1} = \theta(m \otimes q)^{k+1} = \theta(m \otimes q)^k \theta(m \otimes q) = \theta(\theta(m \otimes q)^k m \otimes q) \in \theta(K_P \otimes Q) = g_1(K_P) = K_S$. Hence $g_1(M)$ is a nil ideal of S. Since f_1 and g_1 are mutually inverse lattice isomorphisms, the proof follows. ∎

Analogously we obtain the following result.

Proposition 5.2.5. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $f_4 : Id(T) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all nil ideals of T and the set of all nil sub-biacts of Q.*

Proposition 5.2.6. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $f_2 : Id(S) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all nil ideals of S and the set of all nil sub-biacts of Q.*

Proof. Let I be a nil ideal of S and $qi \in f_2(I) = QI$ for some $q \in Q,\ i \in I$. Then for any $p \in P$, $\theta(p \otimes qi) \in \theta(P \otimes QI) = \theta(P \otimes Q)I = I$. Therefore there exists $k \in \mathbb{Z}^+$ such that $\theta(p \otimes qi)^k \in K_S$. Then we have,

$$\begin{aligned} qi\theta(p \otimes qi)^k &\in QIK_S \subseteq QK_S = f_2(K_S) = K_Q \\ \text{i.e., } qi\theta(p \otimes qi)\theta(p \otimes qi) \cdots \theta(p \otimes qi) &\in K_Q \\ \text{i.e., } \phi(qi \otimes p)qi\theta(p \otimes qi) \cdots \theta(p \otimes qi) &\in K_Q \\ \text{i.e., } \phi(qi \otimes p)^k qi &\in K_Q \end{aligned}$$

Hence $f_2(I)$ is a nil sub-biact of Q.

Conversely, let N be a nil sub-biact of Q and $x \in g_2(N) = \theta(P \otimes N)$. Then clearly $x = \theta(p \otimes n)$, for some $p \in P,\ n \in N$. Since $n \in N$, therefore there exists $k = k(p) \in \mathbb{Z}^+$ such that $\phi(n \otimes p)^k n \in K_Q$. Then we have,

$$\begin{aligned} \phi(n \otimes p)\phi(n \otimes p) \cdots \phi(n \otimes p)n = \phi(n \otimes p)^k n &\in K_Q \\ \text{i.e., } \phi(n \otimes p)\phi(n \otimes p) \cdots n\theta(p \otimes n) &\in K_Q \\ \cdots \\ \text{i.e., } n\theta(p \otimes n)^k &\in K_Q \end{aligned}$$

Therefore, $x^{k+1} = \theta(p \otimes n)^{k+1} = \theta(p \otimes n)\theta(p \otimes n)^k = \theta(p \otimes n\theta(p \otimes n)^k) \in \theta(P \otimes K_Q) = g_2(K_Q) = K_S$. Hence $g_2(N)$ is a nil ideal of S. Since f_2 and g_2 are mutually inverse lattice isomorphisms, the proof follows. □

Analogously we obtain the following result.

Proposition 5.2.7. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $f_3 : Id(T) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all nil ideals of T and the set of all nil sub-biacts of P.*

Although [84, Theorem 8] gives a direct proof of the following result, we can prove it using Proposition 5.2.4 and Proposition 5.2.7.

Theorem 5.2.8. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $\Theta : Id(T) \to Id(S)$ defines a one-to-one inclusion preserving correspondence between the set of all nil ideals of T and the set of all nil ideals of S.*

Proof. Let J be a nil ideal of T. Then from Proposition 5.2.7, $f_3(J) = PJ$ is a nil sub-biact of P and therefore from the proof of Proposition 5.2.4 we see that, $g_1(PJ)$ is a nil ideal of S. Since $\Theta(J) = \theta(PJ \otimes Q) = g_1(PJ)$, therefore $\Theta(J)$ is a nil ideal of S. Analogously we can prove that for any nil ideal I of S, $\Phi(I)$ is a nil ideal of T. Hence the proof follows in view of the fact that Θ and Φ are mutually inverse lattice isomorphisms. □

Definition 5.2.9. [88] An ideal I of a semigroup (monoid) S is called a nilpotent ideal of S if $I^n \subseteq K_S$ for some $n \in \mathbb{Z}^+$.

Definition 5.2.10. Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. A sub-biact M of P is said to be a nilpotent sub-biact of P if $\theta(M \otimes Q)^k M \subseteq K_P$ for some $k \in \mathbb{Z}^+$.

Definition 5.2.11. Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. A sub-biact N of Q is said to be a nilpotent sub-biact of Q if $\phi(N \otimes P)^k N \subseteq K_Q$ for some $k \in \mathbb{Z}^+$.

Proposition 5.2.12. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $f_1 : Id(S) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all nilpotent ideals of S and the set of all nilpotent sub-biacts of P.*

Proof. Let I be a nilpotent ideal of S. Then $I^k \subseteq K_S$ for some $k \in \mathbb{Z}^+$. Therefore,

$$\begin{aligned}\theta(f_1(I) \otimes Q)^k f_1(I) &= \theta(IP \otimes Q)^k IP = (I\theta(P \otimes Q))^k IP \\ &\subseteq I^k(IP) \subseteq K_S P = f_1(K_S) = K_P.\end{aligned}$$

Hence $f_1(I)$ is a nilpotent sub-biact of P.

Conversely, let M be a nilpotent sub-biact of P. Then $\theta(M \otimes Q)^k M \subseteq K_P$ for some $k \in \mathbb{Z}^+$. Therefore, $g_1(M)^{k+1} = \theta(M \otimes Q)^{k+1} = \theta(M \otimes Q)^k \theta(M \otimes Q) = \theta(\theta(M \otimes Q)^k M \otimes Q) \in \theta(K_P \otimes Q) = g_1(K_P) = K_S$. Hence $g_1(M)$ is a nilpotent ideal of S. Since f_1 and g_1 are mutually inverse lattice isomorphisms, the proof follows. □

Analogously we obtain the following result.

Proposition 5.2.13. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $f_4 : Id(T) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all nilpotent ideals of T and the set of all nilpotent sub-biacts of Q.*

Proposition 5.2.14. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $f_2 : Id(S) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all nilpotent ideals of S and the set of all nilpotent sub-biacts of Q.*

Proof. Let I be a nilpotent ideal of S. Then $I^k \subseteq K_S$ for some $k \in \mathbb{Z}^+$. Therefore,

$$\begin{aligned}
\phi(f_2(I) \otimes P)^k f_2(I) &= \phi(QI \otimes P)^k QI \\
&= \phi(QI \otimes P)\phi(QI \otimes P) \cdots \phi(QI \otimes P)QI \\
&= \phi(QI \otimes P)\phi(QI \otimes P) \cdots QI\theta(P \otimes QI) \\
&= \cdots \\
&= QI\theta(P \otimes QI)^k \\
&= QI(\theta(P \otimes Q)I)^k \\
&\subseteq QI(I)^k \subseteq (QI)K_S \subseteq QK_S = f_2(K_S) = K_Q.
\end{aligned}$$

Hence $f_2(I)$ is a nilpotent sub-biact of Q.

Conversely, let N be a nilpotent sub-biact of Q. Then $\phi(N \otimes P)^k N \subseteq K_Q$ for some $k \in \mathbb{Z}^+$. Therefore we see that,

$$\begin{aligned}
&\phi(N \otimes P)\phi(N \otimes P) \cdots \phi(N \otimes P)N = \phi(N \otimes P)^k N \in K_Q \\
\text{i.e., } &\phi(N \otimes P)\phi(N \otimes P) \cdots N\theta(P \otimes N) \in K_Q \\
&\cdots \\
\text{i.e., } &N\theta(P \otimes N)^k \in K_Q.
\end{aligned}$$

Therefore, $g_2(N)^{k+1} = \theta(P \otimes N)^{k+1} = \theta(P \otimes N)\theta(P \otimes N)^k = \theta(P \otimes N\theta(P \otimes N)^k) \in \theta(P \otimes K_Q) = g_2(K_Q) = K_S$. Hence $g_2(N)$ is a nilpotent ideal of S. Since f_2 and g_2 are mutually inverse lattice isomorphisms, the proof follows. □

Analogously we obtain the following result.

Proposition 5.2.15. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $f_3 : Id(T) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all nilpotent ideals of T and the set of all nilpotent sub-biacts of P.*

Although [84, Theorem 8] gives a direct proof of the following result we can prove it using Proposition 5.2.12 and Proposition 5.2.15.

Theorem 5.2.16. *Let S, T be two Morita equivalent monoids via Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the mapping $\Theta : Id(T) \to Id(S)$ defines a one-to-one inclusion preserving correspondence between the set of all nilpotent ideals of T and the set of all nilpotent ideals of S.*

Proof. Let J be a nilpotent ideal of T. Then from Proposition 5.2.15, $f_3(J) = PJ$ is a nilpotent sub-biact of P and therefore from the proof of Proposition 5.2.12 we see that, $g_1(PJ)$ is a nilpotent ideal of S. Since $\Theta(J) = \theta(PJ \otimes Q) = g_1(PJ)$, therefore $\Theta(J)$ is a nilpotent ideal of S. Analogously we can prove that for any nilpotent ideal I of S, $\Phi(I)$ is a nilpotent ideal of T. Hence the proof follows in view of the fact that Θ and Φ are mutually inverse lattice isomorphisms. □

Chapter 6

On Categorical Properties of Topological S-Acts

There have been various works on topological semigroups and their structures, a lot of which was initiated by A. D. Wallace in the year 1953 [94]. Aspects of topological semigroups as well as topological acts over topological semigroups can be found in [44, 46, 55, 60, 73]. In this chapter, we are concerned about the topological acts over a topological monoid from a categorical point of view. Before we make an outline of our current work, we must point out that the main objective, that led us to work on the problem of this chapter, has been to build the Morita theory for topological monoids analogous to the existing theory of Morita equivalence for monoids [53]. Our plan of work involved transferring results of Morita equivalence of monoids [53] to topological monoids. In order to accomplish this, first, we consider the category S-Top of all topological S-acts over a topological monoid (S, τ_S) and identify the product, coproduct, and characterize projective objects, free objects, and generators in the category. But our work has only been partly successful since we were unable to identify the tensor product in this category, which is generally considered to be one of the necessary tools required to develop the Morita theory. However, if one manages to overcome the problem, the results obtained in this paper might help initiate the study of Morita theory for topological monoids.

This chapter is based on the work of the following paper:
M. Das, S. K. Sardar and S. Gupta, *On Categorical Properties of Topological S-Acts*, Southeast Asian Bulletin of Mathematics, Vol. 46, No. 1, pp. 1-14 (2022).

As mentioned earlier, in this chapter we investigate some of the categorical aspects of S-Top. Previously Khosravi conducted some studies on the category of topological S-acts in [51, 52]. He introduced the notions of free topological S-acts over a topological space, over a set as well as over an S-act [51]. Then by using the notion of free topological S-acts over S-acts he characterized projective topological S-acts. In [52], he considered the category S-CReg of Hausdorff completely regular topological S-acts, where S is a completely regular Hausdorff topological monoid and studied the coproduct, free objects over completely regular space and characterized the projective objects in this category. He also characterized the algebraic and topological structure of a projective topological S-act for an arbitrary topological monoid S. In this chapter, we identify the product (*cf.* Proposition 6.1.2), coproduct (*cf.* Proposition 6.1.4) in the category of topological S-acts. Then we revisit (*cf.* Proposition 6.1.8) the result of Khosravi [51, Proposition 3.9] for the construction of free topological S-act over a set and observe its general structure (*cf.* Corollary 6.1.11). We define indecomposable topological S-act, which is more general than what is meant by Khosravi [52], and observe that every topological S-act has a unique decomposition into indecomposable topological subacts (*cf.* Definition 6.1.18 and Theorem 6.1.22). Then we study projective topological S-act and revisit (*cf.* Theorem 6.1.26) one characterization [52, Theorem 2.2] of it. Finally, we define generator in the category of topological S-acts and obtain some of its characterization (*cf.* Theorem 6.1.30), which are analogous to [53, Theorem 2.3.16].

For preliminaries on category theory, monoids and acts, topology we refer, respectively, to Section 1.1, Section 1.2, Section 1.4 of Chapter 1.

Below we recall the definitions of topological monoid and topological S-act from [73].

Definition 6.0.1. [73] A monoid S with a topology τ_S is a topological monoid if the multiplication $S \times S \to S$ is (jointly) continuous in both the variables, i.e., if $st \in U \in \tau_S$ for some $s, t \in S$, then there exist $V \in \tau_S$ containing s and $W \in \tau_S$ containing t such that $VW \subseteq U$.

Definition 6.0.2. [73] For a topological monoid (S, τ_S), a left S-act A with a topology τ_A is said to be a left topological S-act if the action $S \times A \to A$ is (jointly) continuous, i.e., if $sa \in X \in \tau_A$ for some $s \in S$, $a \in A$ then there exist $U \in \tau_S$ containing s and $Y \in \tau_A$ containing a such that $UY \subseteq X$. Analogously right topological S-act is defined.

Here we give some usual examples of (left) topological S-acts.

Example 6.0.3. (1) (S, τ_S) itself is a topological S-act, where the S-action is given by monoid multiplication.

(2) Any S-act A together with the indiscrete topology is a topological S-act.

(3) Let (A, τ_A) be a topological S-act. Then any subact B of A together with the subspace topology τ_B is also a topological S-act.

For further notion and examples of topological S-acts we refer to [73, 51, 52].

Remark 6.0.4. [51] For a topological monoid (S, τ_S), we denote the category of all left topological S-acts together with continuous S-maps by S-Top. Analogously we denote the category of right topological S-acts together with continuous S-maps by Top-S.

6.1 Categorical properties of topological S-acts

We begin this section by producing a canonical example of left topological S-act. In the subsequent discussion, by an S-act we mean a left S-act, and by a topological S-act we mean a left topological S-act (*cf.* Definition 6.0.2) unless mentioned otherwise.

Example 6.1.1. Let (S, τ_S) be a topological monoid. For any non-empty set X, consider $S^X = \{f \mid f : X \to S\}$ with product topology τ together with left S-action defined as

$$\begin{aligned} &S \times S^X \to S^X \\ &(s, f) \mapsto sf \ (x \mapsto sf(x)). \end{aligned}$$

Then S^X endowed with the product topology τ is a topological S-act. In order to prove this, let $sf \in O \in \tau$, for some $s \in S,\ f \in S^X$. Then there exist $U_{\alpha_i} \in \tau_S$, where $\alpha_i \in X,\ i = 1, 2, ..., n$, for some $n \in \mathbb{N}$ such that $sf \in \bigcap_{i=1}^{n} \Pi_{\alpha_i}^{-1}(U_{\alpha_i}) \subseteq O$, where $\Pi_{\alpha_i} : S^X \to S,\ i = 1, 2, ..., n$ are natural projection maps. Therefore,

$$\begin{aligned} &\Pi_{\alpha_i}(sf) \in U_{\alpha_i} \\ \Rightarrow\ &(sf)(\alpha_i) \in U_{\alpha_i} \\ \Rightarrow\ &sf(\alpha_i) \in U_{\alpha_i} \end{aligned}$$

for all $i = 1, 2, ..., n$. Then for each $i = 1, 2, ..., n$, there exist $V_{\alpha_i}, W_{\alpha_i} \in \tau_S$ with $s \in V_{\alpha_i},\ f(\alpha_i) \in W_{\alpha_i}$ such that $V_{\alpha_i} W_{\alpha_i} \subseteq U_{\alpha_i}$. Thus we have,

$$\begin{aligned} &s \in \bigcap_{i=1}^{n} V_{\alpha_i} = V \in \tau_S \quad \text{and} \\ &f \in \bigcap_{i=1}^{n} \Pi_{\alpha_i}^{-1}(W_{\alpha_i}) = W \in \tau. \end{aligned}$$

Now since $\Pi_{\alpha_i}(VW) = VW_{\alpha_i} \subseteq V_{\alpha_i}W_{\alpha_i} \subseteq U_{\alpha_i}$ for all $i = 1, 2, ..., n$, therefore denoting $\bigcap_{i=1}^{n} \Pi_{\alpha_i}^{-1}(U_{\alpha_i})$ as U we have $VW \subseteq U$ with $s \in V \in \tau_S,\ f \in W \in \tau$. Hence (S^X, τ) is a topological S-act.

The following result describes the product in the category of topological S-acts.

Proposition 6.1.2. *Let $(A_\alpha, \tau_\alpha)_{\alpha\in\Lambda}$ be a collection of topological S-acts. Suppose $\prod_{\alpha\in\Lambda} A_\alpha$ is the product of $(A_\alpha)_{\alpha\in\Lambda}$ in S-Act with canonical projections, $p_\beta : \prod_{\alpha\in\Lambda} A_\alpha \to A_\beta$ for $\beta \in \Lambda$. Then $(\prod_{\alpha\in\Lambda} A_\alpha, \prod_{\alpha\in\Lambda} \tau_\alpha)$ is the product of the family $(A_\alpha, \tau_\alpha)_{\alpha\in\Lambda}$ in S-Top, where $\prod_{\alpha\in\Lambda} \tau_\alpha$ is the product topology on $\prod_{\alpha\in\Lambda} A_\alpha$.*

Proof. Suppose $\times_{\alpha\in\Lambda} A_\alpha$ is the cartesian product of the family $(A_\alpha)_{\alpha\in\Lambda}$ of S-acts with projections $p_\beta : \times_{\alpha\in\Lambda} A_\alpha \to A_\beta$ defined by $p_\beta((x_\alpha)_{\alpha\in\Lambda}) := x_\beta$, where $\beta \in \Lambda,\ (x_\alpha)_{\alpha\in\Lambda} \in \times_{\alpha\in\Lambda} A_\alpha$. Then we know from [53] that this cartesian product together with the S-action defined on it as componentwise multiplication by elements of S is the product of $(A_\alpha)_{\alpha\in\Lambda}$ in S-Act and is denoted by $\prod_{\alpha\in\Lambda} A_\alpha$.

Let $sx \in U \in \prod_{\alpha\in\Lambda} \tau_\alpha$, where $s \in S,\ x = (x_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} A_\alpha$. Then there exist $U_{\alpha_i} \in \tau_{\alpha_i}$, where $\alpha_i \in \Lambda,\ i = 1, 2, ..., n$, for some $n \in \mathbb{N}$ such that $sx \in \bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i}) \subseteq U$. Therefore we see that for all $i = 1, 2, ..., n,\ p_{\alpha_i}(sx) \in U_{\alpha_i}$ which implies $sx_{\alpha_i} \in U_{\alpha_i}$. Then for each $i = 1, 2, ..., n$, there exist $V_{\alpha_i} \in \tau_S$ and $W_{\alpha_i} \in \tau_{\alpha_i}$ with $s \in V_{\alpha_i},\ x_{\alpha_i} \in W_{\alpha_i}$ such that $V_{\alpha_i}W_{\alpha_i} \subseteq U_{\alpha_i}$. Thus we have

$$s \in \bigcap_{i=1}^{n} V_{\alpha_i} = V \in \tau_S \quad \text{and} \quad x \in \bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(W_{\alpha_i}) = W \in \prod_{\alpha\in\Lambda} \tau_\alpha.$$

Now since $p_{\alpha_i}(VW) \subseteq VW_{\alpha_i} \subseteq V_{\alpha_i}W_{\alpha_i} \subseteq U_{\alpha_i}$ for all $i = 1, 2, ..., n$, therefore denoting $\bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i})$ as U we have $VW \subseteq U$, where $s \in V \in \tau_S,\ x \in W \in \prod_{\alpha\in\Lambda} \tau_\alpha$. Hence $(\prod_{\alpha\in\Lambda} A_\alpha, \prod_{\alpha\in\Lambda} \tau_\alpha)$ is a topological S-act.

Let (Q, τ_Q) be a topological S-act and $f_\alpha : Q \to A_\alpha$ be a family of morphisms for all $\alpha \in \Lambda$. Define $f : Q \to \prod_{\alpha\in\Lambda} A_\alpha$ by $f(x) = (f_\alpha(x))_\alpha$. Now for $U_\alpha \in \tau_\alpha$, $x \in f^{-1}(p_\alpha^{-1}(U_\alpha))$ if and only if $f(x)(\alpha) \in U_\alpha$ if and only if $x \in f_\alpha^{-1}(U_\alpha)$. Therefore the continuity of f_α implies that $f^{-1}(p_\alpha^{-1}(U_\alpha)) = f_\alpha^{-1}(U_\alpha) \in \tau_Q$. Hence f is a continuous S-map from (Q, τ_Q) to $(\prod_{\alpha\in\Lambda} A_\alpha, \prod_{\alpha\in\Lambda} \tau_\alpha)$ such that $p_\alpha f = f_\alpha$ for all $\alpha \in \Lambda$.
Again let $g : Q \to \prod_{\alpha\in\Lambda} A_\alpha$ be another continuous S-map such that $p_\alpha g = f_\alpha$ holds for all $\alpha \in \Lambda$. Then for $y \in Q,\ p_\alpha g(y) = f_\alpha(y)$ for all $\alpha \in \Lambda$, which in turn implies that $g(y) = (f_\alpha(y))_\alpha = f(y)$. Therefore $f = g$. This completes the proof. □

Notation 6.1.3. In what follows we write $\prod\limits_{\alpha\in\Lambda}(A_\alpha, \tau_\alpha)$ for $(\prod\limits_{\alpha\in\Lambda} A_\alpha, \prod\limits_{\alpha\in\Lambda}\tau_\alpha)$. If $(A_\alpha, \tau_\alpha) = (A, \tau)$ for all $\alpha \in \Lambda$ then we use the notation $\prod\limits_{\Lambda}(A, \tau)$ for $\prod\limits_{\alpha\in\Lambda}(A_\alpha, \tau_\alpha)$.

The following result describes the coproduct in the category of topological S-acts.

Proposition 6.1.4. *Let $(A_\alpha, \tau_\alpha)_{\alpha\in\Lambda}$ be a collection of topological S-acts. Suppose $\coprod\limits_{\alpha\in\Lambda} A_\alpha$ is the coproduct of $(A_\alpha)_{\alpha\in\Lambda}$ in S-Act with canonical injections $\iota_\beta : A_\beta \to \coprod\limits_{\alpha\in\Lambda} A_\alpha$ for $\beta \in \Lambda$. Then $(\coprod\limits_{\alpha\in\Lambda} A_\alpha, \coprod\limits_{\alpha\in\Lambda}\tau_\alpha)$ is the coproduct of the family $(A_\alpha, \tau_\alpha)_{\alpha\in\Lambda}$ in S-Top, where $\coprod\limits_{\alpha\in\Lambda}\tau_\alpha$ is the disjoint union topology* [1] *on $\coprod\limits_{\alpha\in\Lambda} A_\alpha$.*

Proof. Suppose $\dot{\bigcup\limits_{\alpha\in\Lambda}} A_\alpha$ is the disjoint union of the family $(A_\alpha)_{\alpha\in\Lambda}$ of S-acts with injections $\iota_\beta : A_\beta \to \dot{\bigcup\limits_{\alpha\in\Lambda}} A_\alpha$ defined by $\iota_\beta(a) := (a, \beta)$, where $\beta \in \Lambda,\ a \in A_\beta$. Then we know from [53] that the disjoint union together with the S-action defined on it as

$$S \times \dot{\bigcup_{\alpha\in\Lambda}} A_\alpha \to \dot{\bigcup_{\alpha\in\Lambda}} A_\alpha$$
$$(s, (a, \beta)) \mapsto (sa, \beta)$$

is the coproduct of $(A_\alpha)_{\alpha\in\Lambda}$ in S-Act and is denoted by $\coprod\limits_{\alpha\in\Lambda} A_\alpha$.

Let $s(a, \beta) \in U \in \coprod\limits_{\alpha\in\Lambda}\tau_\alpha$ for some $s \in S,\ (a, \beta) \in \coprod\limits_{\alpha\in\Lambda} A_\alpha$. Then $(sa, \beta) \in U$, i.e., $sa \in \iota_\beta^{-1}(U) \in \tau_\beta$. Now (A_β, τ_β) being a topological S-act there exist $V \in \tau_S$ containing s and $W_\beta \in \tau_\beta$ containing a such that $VW_\beta \subseteq \iota_\beta^{-1}(U) = U_\beta$ (say). Denoting $\iota_\beta(W_\beta)$ as W, we have $(a, \beta) \in W \in \coprod\limits_{\alpha\in\Lambda}\tau_\alpha$ such that $VW = \iota_\beta(VW_\beta) \subseteq U$. Hence $(\coprod\limits_{\alpha\in\Lambda} A_\alpha, \coprod\limits_{\alpha\in\Lambda}\tau_\alpha)$ is a topological S-act.

Let (Q, τ_Q) be a topological S-act and $f_\alpha : A_\alpha \to Q$ be a family of morphisms for all $\alpha \in \Lambda$. Define $f : \coprod\limits_{\alpha\in\Lambda} A_\alpha \to Q$ by $f((a, \alpha)) = f_\alpha(a)$, where $\alpha \in \Lambda,\ a \in A_\alpha$. Clearly f is an S-map. Now let $m \in f^{-1}(V) \subseteq \coprod\limits_{\alpha\in\Lambda} A_\alpha$. Therefore $m = (a, \beta)$ for some $\beta \in \Lambda,\ a \in A_\beta$. Now $(a, \beta) \in f^{-1}(V)$ implies that $f_\beta(a) \in V$ whence $a \in f_\beta^{-1}(V)$, i.e., $m \in \iota_\beta(f_\beta^{-1}(V))$. So $f^{-1}(V) \subseteq \bigcup\limits_{\alpha\in\Lambda}\iota_\alpha(f_\alpha^{-1}(V))$. The reverse inclusion follows in a similar manner. Thus $f^{-1}(V) = \bigcup\limits_{\alpha\in\Lambda}\iota_\alpha(f_\alpha^{-1}(V))$, which is clearly open in $\coprod\limits_{\alpha\in\Lambda} A_\alpha$. Thus we have a continuous S-map f such that $f\iota_\alpha = f_\alpha$ for all $\alpha \in \Lambda$.

Let $g : \coprod\limits_{\alpha\in\Lambda} A_\alpha \to Q$ be another continuous S-map such that $g\iota_\alpha = f_\alpha$ holds for all $\alpha \in \Lambda$, i.e., for any $a \in A_\alpha,\ g\iota_\alpha(a) = f_\alpha(a)$ for all $\alpha \in \Lambda$. Therefore $g(a, \alpha) = f_\alpha(a)$ which implies that $g = f$. This completes the proof. ∎

[1] $\coprod\limits_{\alpha\in\Lambda}\tau_\alpha$ is defined to be the finest topology on $\dot{\bigcup\limits_{\alpha\in\Lambda}} A_\alpha$ such that each $\iota_\beta : A_\beta \to \dot{\bigcup\limits_{\alpha\in\Lambda}} A_\alpha$ is continuous.

Notation 6.1.5. In what follows we write $\coprod\limits_{\alpha\in\Lambda}(A_\alpha,\tau_\alpha)$ for $(\coprod\limits_{\alpha\in\Lambda}A_\alpha, \coprod\limits_{\alpha\in\Lambda}\tau_\alpha)$. If $(A_\alpha,\tau_\alpha)=(A,\tau)$ for all $\alpha\in\Lambda$ then we use the notation $\coprod\limits_{\Lambda}(A,\tau)$ for $\coprod\limits_{\alpha\in\Lambda}(A_\alpha,\tau_\alpha)$.

Remark 6.1.6. The coproduct, described in the above proposition for S-Top when restricted to S-CReg (the category of completely regular Hausdorff S-acts), is the same as that of Khosravi [52] which is explained below.

Suppose $(A_\alpha,\tau_\alpha)_{\alpha\in\Lambda}$ is a family of topological S-acts in the category [52] S-CReg of completely regular Hausdorff S-acts with continuous S-maps between them as morphisms, where S is a Hausdorff completely regular topological monoid and (A,τ) is the coproduct of $(A_\alpha,\tau_\alpha)_{\alpha\in\Lambda}$ in S-Top. Let F be a closed subset of A and $(a,\beta)\in A\setminus F$ for some $\beta\in\Lambda,\ a\in A_\beta$. Now since $\iota_\beta^{-1}(F)$ is closed in (A_β,τ_β), there exists a continuous map $f_\beta : A_\beta\to\mathbb{R}$ such that $f_\beta(\iota_\beta^{-1}(F))=1$ and $f_\beta(a)=0$, and for every $\alpha\in\Lambda,\ \alpha\neq\beta$ define $f_\alpha : A_\alpha\to\mathbb{R}$ by $f_\alpha(x)=1$ for all $x\in A_\alpha$. Now consider the mapping $f : A\to\mathbb{R}$ given by $(y,\alpha)\mapsto f_\alpha(y),\ \alpha\in\Lambda,\ y\in A_\alpha$. Then clearly f is a continuous real valued function such that $f(F)=1,\ f((a,\beta))=0$. Therefore (A,τ) is completely regular. Now for $(x,\alpha),(y,\gamma)\in A$ with $\alpha\neq\gamma$ in Λ there exist open sets $\iota_\alpha(A_\alpha),\iota_\gamma(A_\gamma)\in\tau$ such that $\iota_\alpha(A_\alpha)\cap\iota_\gamma(A_\gamma)=\varnothing$. Again for $(m,\alpha),(n,\alpha)\in A$ with $m\neq n$ there exist $U_\alpha,V_\alpha\in\tau_\alpha$ containing m,n respectively such that $U_\alpha\cap V_\alpha=\varnothing$. Therefore $\iota_\alpha(U_\alpha),\iota_\alpha(V_\alpha)\in\tau$ such that $\iota_\alpha(U_\alpha)\cap\iota_\alpha(V_\alpha)=\varnothing$. Hence (A,τ) is a completely regular Hausdorff S-act and thus is the coproduct of the family $(A_\alpha,\tau_\alpha)_{\alpha\in\Lambda}$ in S-CReg.

Definition 6.1.7. [51] Let (S,τ_S) be a topological monoid. A topological S-act (F,τ_F) together with a map $\iota : X\to F$ is said to be a free topological S-act over a given set X if for any topological S-act (A,τ_A) and for any mapping $\sigma : X\to A$, there exists a unique continuous S-map $\overline{\sigma} : (F,\tau_F)\to(A,\tau_A)$ such that $\overline{\sigma}\iota=\sigma$.

We recall from [53] that for a monoid S, the free S-act over a set X is the S-act $S\times(S\times X)\to S\times X,\ (s,(t,x))\mapsto(st,x)$ for $t,s\in S$ and $x\in X$ together with the map $\iota : X\to S\times X,\ x\mapsto(1_S,x)$. From now on we denote this act as $F(X)$. Now by providing a direct proof we revisit the following result of Khosravi [51, Proposition 3.9].

Proposition 6.1.8. [51] *Let (S,τ_S) be a topological monoid and X be a set. Then the free topological S-act on the set X is $F(X)$ with the topology $\tau_{S\times X}$* [2] *where τ_X in the definition of $\tau_{S\times X}$ is the discrete topology.*

[2] $\tau_{S\times X}$ is the product topology on $S\times X$.

Proof. Consider the one-one map $\iota : X \to F(X)$ defined by $x \mapsto (1_S, x)$ and for a topological S-act (A, τ_A) consider a function $\sigma : X \to A$. We define $\overline{\sigma} : (F(X), \tau_{S\times X}) \to (A, \tau_A)$ by $\overline{\sigma}((s, x)) = s\sigma(x)$. Clearly $\overline{\sigma}$ is an S-map. Let $s\sigma(x) \in U \in \tau_A$. Then (A, τ_A) being a topological S-act, there exist $V \in \tau_S$, $W \in \tau_A$ such that $s \in V$, $\sigma(x) \in W$ and $VW \subseteq U$. Thus there exist $V \in \tau_S$ containing s and $\sigma^{-1}(W) \in \tau_X$ containing x such that $\overline{\sigma}(V \times \sigma^{-1}(W)) \subseteq VW \subseteq U$. Hence $\overline{\sigma}$ is a continuous S-map such that $\overline{\sigma}\iota(x) = \overline{\sigma}((1_S, x)) = \sigma(x)$, i.e., $\overline{\sigma}\iota = \sigma$. □

Proposition 6.1.9. *Let (S, τ_S) be a topological monoid and X be a non-empty set. Then $\coprod_X (S, \tau_S)$ (cf. Notation 6.1.5) together with the map $f : X \to \coprod_X (S, \tau_S)$ defined by $f(x) := (1_S, x)$, is free over X in S-Top.*

Proof. Let (A, τ_A) be a topological S-act and $g : X \to A$ be a mapping. We define $\overline{g} : \coprod_X (S, \tau_S) \to (A, \tau_A)$ by $\overline{g}((s, x)) := sg(x)$. Clearly $\overline{g}$ is an S-map. Let $V \in \tau_A$, $t \in \iota_x^{-1}(\overline{g}^{-1}(V))$ where for $x \in X$, $\iota_x : (S, \tau_S) \to \coprod_X (S, \tau_S)$ is the natural injection given by $s \mapsto (s, x)$. Then $tg(x) \in V$, which implies that there exist $U_t \in \tau_S$ and $W \in \tau_A$ with $t \in U_t$, $g(x) \in W$ such that $U_tW \subseteq V$. Let $s \in U_t$. Then $\overline{g}((s, x)) = sg(x) \in U_tW \subseteq V$ which implies $(s, x) \in \overline{g}^{-1}(V)$, i.e., $s \in \iota_x^{-1}(\overline{g}^{-1}(V))$. Thus for every $t \in \iota_x^{-1}(\overline{g}^{-1}(V))$, there exists $U_t \in \tau_S$ such that $t \in U_t \subseteq \iota_x^{-1}(\overline{g}^{-1}(V))$. Hence $\iota_x^{-1}(\overline{g}^{-1}(V))$ is open in S implying the continuity of the S-map $\overline{g}$ such that for $x \in X$, $\overline{g}f(x) = \overline{g}((1_S, x)) = g(x)$.

Let h be another continuous S-map such that $hf = g$. Then we have, for all $x \in X$,

$$\begin{aligned} & hf(x) = \overline{g}f(x) \\ \text{i.e., } & h((1_S, x)) = \overline{g}((1_S, x)) \\ \text{i.e., } & sh((1_S, x)) = s\overline{g}((1_S, x)) \\ \text{i.e., } & h((s, x)) = \overline{g}((s, x)) \\ \text{i.e., } & h = \overline{g}. \end{aligned}$$

This completes the proof. □

Remark 6.1.10. It follows from the above result that any topological monoid (S, τ_S) is a free topological S-act.

Corollary 6.1.11. *Let (S, τ_S) be a topological monoid. A topological S-act (F, τ_F) is free over a set X if and only if it is isomorphic to $\coprod_X (S, \tau_S)$.*

Proof. In view of Definition 6.1.7 and Proposition 6.1.9, the result follows from the categorical fact that free object over a set in a category is unique up to isomorphism. □

Proposition 6.1.12. *For any topological S-act (A, τ_A) there exists a free topological S-act (F, τ_F) such that (A, τ_A) is an epimorphic image of (F, τ_F).*

Proof. Let $(F(A), \tau)$ be the free topological S-act over the set A where $\iota : A \to F(A)$ is given by $\iota(a) = (1_S, a)$. Then by Definition 6.1.7, for the identity map $id_A : A \to A$, there exists a continuous S-map $f : (F(A), \tau) \to (A, \tau_A)$ such that $f\iota = id_A$. Now f being a surjective continuous S-map is an epimorphism. Hence (A, τ_A) is an epimorphic image of a free topological S-act. □

Definition 6.1.13. A topological S-act (P, τ_P) is projective in S-Top category, if for any epimorphism $\pi : (A, \tau_A) \to (B, \tau_B)$ between two topological S-acts $(A, \tau_A), (B, \tau_B)$ and any morphism $\varphi : (P, \tau_P) \to (B, \tau_B)$, there exists a morphism $\overline{\varphi} : (P, \tau_P) \to (A, \tau_A)$ such that $\varphi = \pi\overline{\varphi}$.

Proposition 6.1.14. *Every free topological S-act is projective.*

Proof. It is well-known [53] that in a concrete category if epimorphisms are surjective, then every free object is projective (see Remark 1.1.35). We prove here that in S-Top epimorphisms are surjective, which in turn proves the result.

Let $f : (A, \tau_A) \to (B, \tau_B)$ be an epimorphism in S-Top. Define the relation θ on B by $x\theta y$ if and only if either $x = y$ or $x, y \in Imf$. Then for $x \neq y$ in B, $x\theta y$ implies that there exist $m, n \in A$ such that $x = f(m)$, $y = f(n)$. Therefore for $s \in S$, $sx = f(sm)$, $sy = f(sn)$, which implies that $sx\theta sy$. Hence θ is a congruence on B and B/θ together with the indiscrete topology τ is a topological S-act where the action is defined as

$$S \times B/\theta \to B/\theta$$
$$(s, [x]_\theta) \mapsto [sx]_\theta.$$

Now define

$$g : B \to B/\theta \quad \text{and} \quad h : B \to B/\theta \quad \text{by}$$
$$x \mapsto [x]_\theta \quad \text{and} \quad x \mapsto [f(c)]_\theta \quad \text{for some fixed } c \in A.$$

Since τ is indiscrete, both the S-maps are continuous such that $gf(a) = [f(a)]_\theta = [f(c)]_\theta = hf(a)$, for all $a \in A$. Therefore we have $gf = hf$, which implies that $g = h$, since f is an epimorphism. Thus for any $x \in B$, $[x]_\theta = g(x) = h(x) = [f(c)]_\theta$, which implies $B = Imf$. Hence f is surjective. □

We recall below one result on projective topological S-acts from [52, Proof of Lemma 2.1] for its immediate use in Example 6.1.17.

Proposition 6.1.15. [52] *For any idempotent $e \in S$, Se together with the subspace topology τ_{Se} is a projective topological S-act .*

Remark 6.1.16. That the converse of Proposition 6.1.14 is not true is illustrated in the following example.

Example 6.1.17. Consider the topological monoid $(\mathbb{Z}, \tau_{dis})$, where $\mathbb{Z}$ is the multiplicative monoid and τ_{dis} is the discrete topology. Then in view of Proposition 6.1.15, $(\{0\}, \tau_{\{0\}})$ is a projective topological $\mathbb{Z}$-act where $\tau_{\{0\}} = \{\varnothing, \{0\}\}$. But we show below that it is not free. Suppose it is free over a set X with corresponding mapping $\iota : X \to \{0\}$ defined by $x \mapsto 0$ for all $x \in X$. Consider the topological $\mathbb{Z}$-act $(\mathbb{Z}, \tau_{dis})$ and a map $f : X \to \mathbb{Z}$ given by $x \mapsto 1$ for all $x \in X$. Then there exists continuous $\mathbb{Z}$-map $\overline{f} : (\{0\}, \tau_{\{0\}}) \to (\mathbb{Z}, \tau_{dis})$ such that $\overline{f}\iota = f$ which implies that $\overline{f}(0) = 1$ - a contradiction since $\overline{f}$ is a $\mathbb{Z}$-map. Hence $(\{0\}, \tau_{\{0\}})$ is not free.

Definition 6.1.18. We call a topological S-act (A, τ_A) decomposable if there is an indexed set Λ of cardinality at least two and non-empty closed proper subacts X_i of $A,\ i \in \Lambda$ such that $A = \underset{i\in\Lambda}{\cup} X_i$ and for each pair $i, j \in \Lambda$, with $i \neq j,\ X_i \cap X_j = \varnothing$. In this case $A = \underset{i\in\Lambda}{\cup} X_i$ is called a decomposition of (A, τ_A). Otherwise, (A, τ_A) is called indecomposable. A subact B of A is said to be indecomposable if (B, τ_B) is an indecomposable topological S-act, where τ_B is the induced topology.

Remark 6.1.19. Recall that [53] an S-act A is called decomposable in S-Act if there exist two subacts $B, C \subseteq A$ such that $A = B \cup C$ and $B \cap C = \varnothing$. Otherwise, A is called indecomposable. We call a topological S-act (A, τ_A) algebraically indecomposable if the underlying S-act A is indecomposable in S-Act[3]. Clearly, every algebraically indecomposable topological S-act is indecomposable. But the converse is not true, which is evident from the following example.

Example 6.1.20. Let us consider the topological multiplicative monoid $(\mathbb{N}, \eta)$, where η is the discrete topology and the topological $\mathbb{N}$-act $(\mathbb{Z}, \tau)$ with τ as the indiscrete topology and the action given by

$$\begin{aligned} \mathbb{N} \times \mathbb{Z} &\to \mathbb{Z} \\ (n, a) &\mapsto na. \end{aligned}$$

Here $(\mathbb{Z}, \tau)$ is indecomposable since it has no non-empty closed proper subact. But there are subacts $\mathbb{Z}^+ \cup \{0\}, \mathbb{Z}^-$ such that $\mathbb{Z} = (\mathbb{Z}^+ \cup \{0\}) \cup \mathbb{Z}^-$. Hence $(\mathbb{Z}, \tau)$ is algebraically decomposable.

[3]This notion is called indecomposable topological S-act by Khosravi [52].

Lemma 6.1.21. *For topological S-act (A, τ_A), let $(A_i)_{i\in I}$ be subacts of A such that (A_i, τ_i) (τ_i's are subspace topologies) are indecomposable topological S-acts. Then $\bigcup_{i\in I} A_i$ equipped with the subspace topology τ^* is an indecomposable topological S-act whenever $\bigcap_{i\in I} A_i \neq \varnothing$.*

Proof. Clearly $(\bigcup_{i\in I} A_i, \tau^*)$ is a topological S-act. Let $\bigcup_{i\in I} A_i = \bigcup_{\alpha\in\Lambda} X_\alpha$ be a decomposition of $(\bigcup_{i\in I} A_i, \tau^*)$, where X_α's are non-empty closed proper subacts in $\bigcup_{i\in I} A_i$. Take $x \in \bigcap_{i\in I} A_i$ with $x \in X_\beta$ for some $\beta \in \Lambda$. Then for $k \in I$, $A_k = \bigcup_{\alpha\in\Lambda}(A_k \cap X_\alpha)$, where $(A_k \cap X_\alpha)$ is a closed subact of A_k for all $\alpha \in \Lambda$. But since (A_k, τ_k) is indecomposable, it follows that $A_k \cap X_\alpha = \varnothing$ for all $\alpha \in \Lambda$, $\alpha \neq \beta$. This is true for all $k \in I$. Therefore $\bigcup_{i\in I} A_i = X_\beta$ - a contradiction. Hence the proof. ☐

Theorem 6.1.22. *Every topological S-act (A, τ_A) has a unique decomposition into indecomposable subacts.*

Proof. Take $a \in A$. Since the cyclic S-act Sa is indecomposable in S-Act [53], Sa equipped with subspace topology τ_{Sa} induced by τ_A is indecomposable in S-Top. Let $Sub(A)$ be the collection of all subacts of A. Then by Lemma 6.1.21, we get that $U_a = \cup\{V \in Sub(A) \mid (V, \tau_V)$ is indecomposable and $a \in V\}$ (where τ_V is the subspace topology on V) together with the subspace topology τ_a induced by τ_A is indecomposable topological S-act.

Let $\overline{U_a}$ denote the closure of U_a in (A, τ_A). We claim to prove that $\overline{U_a}$ is an indecomposable subact of (A, τ_A). For this, let $s \in S$, $b \in \overline{U_a}$ and U be an open set in A containing sb. Then (A, τ_A) being a topological S-act there exists $W \in \tau_A$ containing b such that $sW \subseteq U$. Now $b \in W \in \tau_A$ implies that there exists some $y \in W \cap U_a$ such that $sy \in U_a \cap sW \subseteq U_a \cap U$, i.e., $U_a \cap U \neq \varnothing$. Hence $sb \in \overline{U_a}$. Now if $\overline{U_a} = \bigcup_{i\in I} X_i$, where X_i's are closed proper subacts of $\overline{U_a}$, then $U_a = \bigcup_{i\in I}(X_i \cap U_a)$. But since U_a is indecomposable we must have $U_a = X_k \cap U_a$ for some $k \in I$, which in turn implies that $\overline{U_a} = X_k$ - a contradiction. Thus $\overline{U_a}$ together with the induced topology is an indecomposable topological S-act containing a. Therefore $U_a = \overline{U_a}$, i.e., U_a is closed.

For $x, y \in A$, we get that $U_x = U_y$ or $U_x \cap U_y = \varnothing$. Indeed, $z \in U_x \cap U_y$ implies $U_x, U_y \subseteq U_z$. Thus $x \in U_x \subseteq U_z$, $y \in U_y \subseteq U_z$, i.e., $U_z \subseteq U_x \cap U_y$. Therefore $U_x = U_y = U_z$. Denote by A' a representative subset of elements $x \in A$ with respect to the equivalence relation $\sim$ defined by $x \sim y$ if and only if $U_x = U_y$. Then $A = \bigcup_{x\in A'} U_x$ is a decomposition of A in indecomposable subacts.

Now for uniqueness, let $A = \bigcup_{\alpha\in B} V_\alpha$ be another decomposition of (A, τ_A) into indecomposable subacts. Then there exists at least one U_y for some $y \in A'$, such that

$U_y \neq V_\alpha$ for all $\alpha \in B$. Now $U_y = A \cap U_y = \underset{\alpha \in B}{\cup}(V_\alpha \cap U_y)$. For $a \in V_\beta \cap U_y$ for some $\beta \in B$ implies $V_\beta \subseteq U_a = U_y$. By hypothesis we have $U_y \neq V_\beta$ therefore for $\alpha \in B$, either $V_\alpha \cap U_y = \varnothing$ or $V_\alpha \subsetneq U_y$. Let $J = \{\alpha \in B \mid V_\alpha \subsetneq U_y\}$. It is evident that J is a non-empty, non-singleton set such that $U_y = \underset{\alpha \in J}{\cup} V_\alpha$, where V_α is indecomposable subact for all $\alpha \in J$. Thus we have a decomposition of the topological S-act (U_y, τ_y) - a contradiction. Hence $A = \underset{x \in A'}{\cup} U_x$ is the unique decomposition of A into indecomposable subacts. □

Theorem 6.1.23. *For any indecomposable projective topological S-act (P, τ) there exists an idempotent $e \in S$ such that (P, τ) is isomorphic to (Se, τ_{Se}), where τ_{Se} is the subspace topology.*

Proof. For any $p \in P$, consider the continuous S-map $\sigma_p : (S, \tau_S) \to (P, \tau)$ defined by $s \mapsto sp$. Then there exists a continuous S-map

$$\sigma = \coprod_{p \in P} \sigma_p : \coprod_{p \in P} (S_p, \tau_p) \to (P, \tau) \qquad ((S_p, \tau_p) = (S, \tau_S))$$
$$(s, p) \mapsto \sigma_p(s)$$

such that $Im\sigma = P$. Therefore (P, τ) being projective there exists a continuous S-map $\gamma : (P, \tau) \to \coprod_{p \in P} (S_p, \tau_p)$ such that $\sigma\gamma = id_P$. Consider $(\gamma(P), \tau^*)$, where τ^* is the subspace topology, i.e., $\tau^* = \{U \cap \gamma(P) \mid U \in \coprod_{p \in P} \tau_p\}$. Then $V \in \tau^*$ implies that $V = V' \cap \gamma(P)$ for some $V' \in \coprod_{p \in P} \tau_p$, which implies that $\gamma^{-1}(V) = \gamma^{-1}(V') \in \tau$. Hence $\gamma : (P, \tau) \to (\gamma(P), \tau^*)$ is continuous and also $\sigma^* = \sigma|_{\gamma(P)} : (\gamma(P), \tau^*) \to (P, \tau)$ is continuous such that $\sigma^*\gamma = \sigma\gamma = id_P$ and $\gamma\sigma^* = id_{\gamma(P)}$. Hence $(\gamma(P), \tau^*)$ is isomorphic to (P, τ) and thus is indecomposable. Now consider the injections $\iota_p : S_p \to \coprod_{x \in P} S_x$ defined by $s \mapsto (s, p)$. Then we have an algebraic decomposition of $\gamma(P)$ as follows :

$$\gamma(P) = \underset{x \in P}{\cup}(\gamma(P) \cap \iota_x(S)) = \underset{x \in P}{\cup} A_x. \tag{6.1}$$

Then for any $p \in P$,

$$\gamma(P) \setminus A_p = \underset{x \in P \setminus \{p\}}{\cup} A_x = \gamma(P) \cap \left(\underset{x \in P \setminus \{p\}}{\cup} \iota_x(S) \right) \in \tau^*.$$

Also $SA_p = S(\gamma(P) \cap \iota_p(S)) \subseteq (\gamma(P) \cap \iota_p(S)) = A_p$. Therefore A_p is a closed subact of $\gamma(P)$ for all $p \in P$. Now since $(\gamma(P), \tau^*)$ is indecomposable, therefore $\gamma(P) \subseteq \iota_m(S)$ for a unique $m \in P$. So we have, $P = id_P(P) = \sigma\gamma(P) \subseteq \sigma\iota_m(S) = \sigma_m(S) = Sm \subseteq P$, i.e., $P = Sm$.

Now for the epimorphism, $\sigma_m : (S, \tau_S) \to (P, \tau)$, there exists a continuous S-map $\varphi : (P, \tau) \to (S, \tau_S)$ such that $\sigma_m\varphi = id_P$. Denote $\varphi(m) = e \in S$. Since $m =$

$id_P(m) = \sigma_m\varphi(m) = \sigma_m(e) = em$, we have $e = \varphi(m) = \varphi(em) = e\varphi(m) = e^2$. Again $\varphi(P) = \varphi(Sm) = S\varphi(m) = Se$.

Also (P, τ) is isomorphic to $\varphi(P)$ together with subspace topology. Therefore (P, τ) is isomorphic to (Se, τ_{Se}). □

Remark 6.1.24. As mentioned earlier, that by indecomposable topological S-acts Khosravi [52] meant the topological S-acts, which are algebraically indecomposable and obtained a characterization [52, Lemma 2.1] similar to that of Theorem 6.1.23, which we recall below.

Theorem 6.1.25. [52] *Any indecomposable projective S-space P is cyclic and there exists $e^2 = e \in S$ such that P is topologically isomorphic to Se.*

Khosravi [52, Theorem 2.2] proved the following result using Theorem 6.1.25. But it can be proved using our result given in Theorem 6.1.23.

Theorem 6.1.26. *A topological S-act (P, τ_P) is projective if and only if $(P, \tau_P) = \coprod_{i\in I}(P_i, \tau_i)$ where each (P_i, τ_i) is isomorphic to (Se_i, τ_{Se_i}) for some idempotent $e_i \in S$ together with subspace topology τ_{Se_i}, $i \in I$.*

To conclude the chapter, we introduce the notion of generator in the category S-Top and characterize it (*cf.* Theorem 6.1.30), which is a partial analogue of [53, Theorem 2.3.16] (see Theorem 1.2.27 for details).

Definition 6.1.27. A topological S-act (G, τ_G) is said to be a generator in S-Top if for $f,\ g : (X, \tau_X) \to (Y, \tau_Y)$ in S-Top with $f \neq g$ there exists a continuous S-map $\alpha : (G, \tau_G) \to (X, \tau_X)$ such that $f\alpha \neq g\alpha$.

Remark 6.1.28. Suppose (S, τ_S) is a topological monoid and $(X, \tau_X), (Y, \tau_Y)$ are topological S-acts. Then for notational convenience we denote the set of all continuous S-maps from (X, τ_X) to (Y, τ_Y) by $C(X, Y)$ when there is no ambiguity regarding the topology of X and Y.

Before giving a characterization of generators in S-Top, we recall the following Lemma from [53].

Lemma 6.1.29. *[53] Suppose $\mathcal{C}$ is an arbitrary category and $G \in \mathcal{C}$ is a generator in $\mathcal{C}$. If for every $X \in \mathcal{C}$ there exists $X \coprod X$ in $\mathcal{C}$ such that the injections $u_1, u_2 : X \to X \coprod X$ are different, then $Hom_{\mathcal{C}}(G, X) \neq \varnothing$ for all $X \in \mathcal{C}$, where $Hom_{\mathcal{C}}(G, X)$ denotes the set of all morphisms from G to X in $\mathcal{C}$.*

Theorem 6.1.30. *Suppose (S, τ_S) is a topological monoid. For $(G, \tau_G) \in S$-Top the following conditions are equivalent.*

(i) (G, τ_G) *is a generator in S-Top.*

(ii) *Every* $(X, \tau_X) \in S$*-Top is an epimorphic image of* $\coprod_{C(G,X)} (G, \tau_G)$.

(iii) *For every* $(X, \tau_X) \in S$*-Top there exists a set I such that* (X, τ_X) *is an epimorphic image of* $\coprod_I (G, \tau_G)$.

(iv) *There exists an epimorphism* $\pi : (G, \tau_G) \to (S, \tau_S)$.

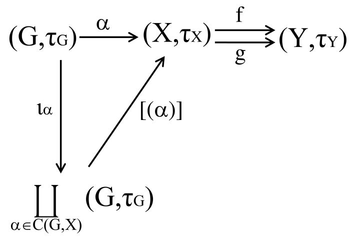
(v) (S, τ_S) *is a retract of* (G, τ_G).

(vi) *There exists* $\psi^2 = \psi \in C(G, G)$ *such that* $\psi(G)$ *is topologically isomorphic to* (S, τ_S).

Proof. **(i) ⇒ (ii)**

Suppose $(X, \tau_X), (Y, \tau_Y) \in S$-Top and $f, g : (X, \tau_X) \to (Y, \tau_Y)$ are continuous S-maps such that $f \neq g$. We already have from Lemma 6.1.29 that $C(G, X) \neq \varnothing$.

Now consider the following diagram in S-Top



where ι_α are the canonical injections into $\coprod_{C(G,X)} (G, \tau_G)$ and $[(\alpha)]$ is coproduct induced.

By (i) there exists $\beta \in C(G, X)$ such that $f\beta \neq g\beta$. Therefore if we assume that $f[(\alpha)] = g[(\alpha)]$ then we have $f[(\alpha)]\iota_\beta = g[(\alpha)]\iota_\beta$ which implies that $f\beta = g\beta$ - a contradiction. This proves that $[(\alpha)]$ is an epimorphism.

(ii) ⇒ (iii)

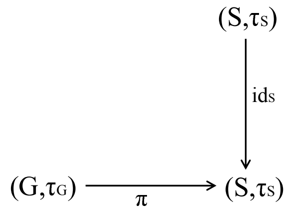
Follows trivially.

(iii) ⇒ (iv)

Let $f : \coprod_{i \in I} (G_i, \tau_i) \to (S, \tau_S)$ be an epimorphism, where $(G_i, \tau_i) = (G, \tau_G)$ for all $i \in I$. Since epimorphisms are surjective in S-Top (*cf.* proof of Proposition 6.1.14) there exists $(g,k) \in \coprod_{i\in I}(G_i,\tau_i)$ such that $k \in I, g \in G_k$ and $f((g,k)) = 1_S$. Therefore for any $s \in S$, $s = s.1_S = s.f((g,k)) = f((sg,k)) = f\iota_k(sg)$, where $\iota_k : (G_k,\tau_k) \to \coprod_{i\in I}(G_i,\tau_i)$ denotes the canonical injection. Then $\pi = f\iota_k : (G_k,\tau_k) \to (S,\tau_S)$ is a surjection and also being the composition of two continuous S-maps is a continuous S-map. Thus $\pi : (G,\tau_G) \to (S,\tau_S)$ is an epimorphism in S-Top.

(iv) ⇒ (v)

Consider the following diagram in S-Top



In view of Remark 6.1.10 and Proposition 6.1.14, (S,τ_S) is projective so there exists a continuous S-map $\gamma : (S,\tau_S) \to (G,\tau_G)$ such that $\pi\gamma = id_S$. Hence the proof.

(v) ⇒ (vi)

Let $\pi : (G,\tau_G) \to (S,\tau_S)$ be a retraction in S-Top. Then there exists a continuous S-map $\gamma : (S,\tau_S) \to (G,\tau_G)$ such that $\pi\gamma = id_S$. Then clearly $\psi = \gamma\pi \in C(G,G)$ is an idempotent and since $\gamma(1_S) \in G$ we get that $\gamma(1_S) = \gamma(\pi\gamma(1_S)) = (\gamma\pi)\gamma(1_S) \in \psi(G)$ i.e., $S\gamma(1_S) \subseteq \psi(G) = \gamma\pi(G) = \gamma(S) = S\gamma(1_S)$. Thus $\gamma(S) = \psi(G)$. Also since γ is a coretraction, (S,τ_S) is isomorphic to $(\gamma(S),\tau_{\gamma(S)})$, where $\tau_{\gamma(S)}$ is the subspace topology induced from τ_G. Hence $\psi(G)$ is topologically isomorphic to (S,τ_S).

(vi) ⇒ (iv)

Follows clearly since $\psi(G)$ is topologically isomorphic to (S,τ_S).

(iv) ⇒ (i)

Consider $f,g : (X,\tau_X) \to (Y,\tau_Y)$ in S-Top with $f \neq g$. Then there exists $x \in X$ such that $f(x) \neq g(x)$. In view of Proposition 6.1.9 (S,τ_S) is a free topological S-act over any singleton set $\{t\}$ so we consider the following diagram:

$$\begin{array}{ccc} \{t\} & \xrightarrow{\gamma} & (X,\tau_X) \\ \psi \downarrow & & \\ (S,\tau_S) & & \end{array}$$

where $\gamma(t) = x,\ \psi(t) = 1_S$. Then there exists $\overline{\gamma} : (S, \tau_S) \to (X, \tau_X)$ in S-Top such that $\overline{\gamma}\psi = \gamma$ i.e., $\overline{\gamma}(1_S) = x$. Then we have $\overline{\gamma}\pi : (G, \tau_G) \to (X, \tau_X)$ such that $f(\overline{\gamma}\pi) \neq g(\overline{\gamma}\pi)$, since π is an epimorphism. Hence the proof. $\square$

Some Remarks and Scope of Further Study

• In view of **Chapter 2**, one can further extend the theory of Morita equivalence for semirings to idempotent semirings, analogous to [27]. Also attempt can be made to investigate if there is a generalization of the concept of Morita equivalence for semirings similar to the notion of Morita like equivalence for xst-rings [97].

• In view of **Chapter 3**, one can introduce the concepts of right semiregular subsemimodule, quasi-regular subsemimodule of a semimodule, analogous to their counterparts in semiring theory and check whether these notions remain invariant under the maps f_is and g_is, using which one can further investigate if Jacobson radical is preserved under Morita equivalence of semirings. Also one can investigate the validity of the results for semirings without identity.

• In **Chapter 4**, we have topologized the prime spectrum of a semimodule P related to a Morita context of semirings and studied the interplay between the properties of the space and the algebraic properties of P. A similar study can be accomplished with the set of all maximal subsemimodules of P.

• In **Chapter 5**, we introduce terms like (right) strongly prime sub-biacts, uniformly strongly prime sub-biacts, nil sub-biacts, nilpotent sub-biacts of a monoid act related to a Morita context of monoids and observe their invariance under the maps f_is and g_is. Similar studies can be attempted for semigroups with weak local units.

• In **Chapter 6**, we study various categorical aspects of the category S-Top of topological S-acts for a topological monoid S. The results obtained in the chapter may be considered to be some of the necessary tools required to initiate the study of Morita equivalence of topological monoids whose counterpart for monoids and semigroups has been a topic of sustained research interest, which is evident from various works mentioned in [53] and [90, 35, 83].

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