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CERTIFICATE FROM THE SUPERVISORS

This is to certify that the thesis entitled **"A STUDY OF CONGRUENCES ON SEMINEARRINGS"** submitted by **Miss Kamalika Chakraborty**, who got her name registered on 05.09.2018 **(Index No: 143/18/Maths./26)** for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon her own work under the supervision of Prof. Sujit Kumar Sardar, Department of Mathematics, Jadavpur University and co-supervision of Dr. Pavel Pal, Assistant Professor, Department of Mathematics, Bankura University and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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Dedicated to

my parents

Ram Sashi Chakraborty and

Dipali Chakraborty

and

my sister Tamalika Chakraborty

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Abstract

The present work is a study of congruences on different classes of seminearrings which, among others, comprises various characterization theorems. Firstly, near-ring congruences on additively regular seminearrings have been studied. In this study a lattice isomorphism has been obtained between the set of all normal full k-ideals and that of all near-ring congruences in distributively generated additively regular seminearrings. The lattice of near-ring congruences has subsequently been studied. Since the seminearring $M(S)$, one of the most naturally arising seminearrings, of self maps of an additive semigroup S is an additively regular seminearring if and only if S is a regular semigroup, a large class of seminearrings arises naturally to be non-additively regular. Propelled by this fact, near-ring and zero-symmetric near-ring congruences on seminearrings which need not be either additively regular or distributively generated have been studied by obtaining inclusion preserving bijective correspondences between the set of all zero-symmetric near-ring congruences (near-ring congruences) and the set of all generalised strong dense reflexive k-ideals (resp., right k-ideals). Further some sufficient conditions (*viz.,* existence of left local units, being E^+-inversive) have been obtained imposition of which on the seminearrings under consideration ensures that the set of all zero-symmetric near-ring congruences (near-ring congruences) form lattices so that the above correspondences turn out to be lattice isomorphisms. A detailed study of these lattices has been accomplished alongside. Since E-inversive semigroup generalizes regular semigroup and the theory of seminearrings is greatly influenced by the development of semigroup theory, the structure theorem of full subdirect product of a semilattice and a group in terms of E-inversive semigroups motivates us to characterize full subdirect product of a bi-semilattice and a (zero-symmetric) near-ring and subdirect product of a distributive lattice and a (zero-symmetric) near-ring in terms of E^+-inversive seminearrings. To conclude the work, the relationships among various classes of E^+-inversive seminearrings have been discussed.

Contents

Introduction

A ring $(R, +, \cdot)$ consists of an abelian group $(R, +)$ and a semigroup $(R, \cdot)$ where '$\cdot$' distributes over '$+$' from both sides *i.e.,* $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in R$. For a commutative group $(U, +)$, the set of all endomorphisms of U under point-wise addition and composition becomes a natural example of a ring. Now if we take a group $(G, +)$ (not necessarily abelian), then the set $M(G)$ of all maps from G to G under point-wise addition and composition gives a different algebraic structure where $M(G)$ under addition is a group (not necessarily abelian), $M(G)$ under composition is a semigroup and composition distributes over addition only from the right side. Even if the group $(G, +)$ is abelian, in $M(G)$ composition will distribute over addition only from the right side. This algebraic structure is known as a near-ring. According to G. Pilz [91], *near-ring* $(N, +, \cdot)$ is an algebraic structure where

(i) $(N, +)$ is a group (not necessarily abelian),

(ii) $(N, \cdot)$ is a semigroup and

(iii) for all $n_1, n_2, n_3 \in N$, $(n_1 + n_2) \cdot n_3 = n_1 \cdot n_3 + n_2 \cdot n_3$, *i.e.*, '$\cdot$' distributes over '$+$' from the right side ("right distributive law").

Since '$\cdot$' distributes over '$+$' from the right side, more precisely it is called a 'right near-ring'. If (iii) is replaced by

(iii') for all $n_1, n_2, n_3 \in N$, $n_1 \cdot (n_2 + n_3) = n_1 \cdot n_2 + n_1 \cdot n_3$, *i.e.*, '$\cdot$' distributes over '$+$' from the left side ("left distributive law"),

then one gets 'left near-ring'. The theory runs completely parallel in both cases. Throughout our work 'near-ring' stands for 'right near-ring'. First step towards near-ring was an axiomatic research done by Dickson [25] in 1905 by showing the existence of fields with only one distributive law. Since then near-ring is drawing the attention of many researchers from both theoretical and practical point of views. Among the other mathematicians J. R. Clay, W. M. L. Holcombe, C. J. Maxson, S. D. Scott made some significant contributions to the theory of near-rings (*cf.* [20, 21, 40, 41, 74, 75, 100]).

Now in the natural example of a near-ring, if we replace a group $(G, +)$ (not necessarily abelian) by a semigroup $(S, +)$ (not necessarily commutative), then the set $M(S)$ of all self maps from S to S under point-wise addition and composition gives a new algebraic structure which is different from a near-ring since $M(S)$ under addition is no longer a group. This algebraic structure is known as a 'seminearring'. To be specific a *seminearring* $(S, +, \cdot)$ is an algebraic structure where $(S, +)$, $(S, \cdot)$ are semigroups and '$\cdot$' distributes over '$+$' from the right side *i.e.,* $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in S$. Since '$\cdot$' distributes over '$+$' from the right side, the above structure is more particularly a 'right distributive seminearring'. Similarly in the algebraic structure $(S, +, \cdot)$ where $(S, +)$, $(S, \cdot)$ are semigroups, if '$\cdot$' distributes over '$+$' from the left side, *i.e.,* $a \cdot (b+c) = a \cdot b + a \cdot c$ for all $a, b, c \in S$, then one gets 'left distributive seminearring'. Like the theory of near-rings, here also the seminearring theory runs completely parallel in both cases. A seminearring $(S, +, \cdot)$ is a near-ring if $(S, +)$ is a group. In our work, 'seminearring' stands for 'right distributive seminearring'. Again an algebraic structure $(S, +, \cdot)$ is said to be a '*semiring*' if $(S, +)$, $(S, \cdot)$ are semigroups and '$\cdot$' distributes over '$+$' from both sides. A seminearring $(S, +, \cdot)$ is a semiring if '$\cdot$' distributes over '$+$' from the left side as well. Therefore seminearrings generalize semirings as well as near-rings. So the development of seminearring theory takes impetus from the theory of semirings as well as the theory of near-rings.

In 1967, van Hoorn et al. introduced the notion of seminearrings in [44]. In [43], van Hoorn studied the radicals of a seminearring and found 14 radicals in a seminearring. In 1976, Hoogewijs Studied $\mathcal{I}$-congruences in [42]. Since 70's, Weinert studied related representation theorems, seminearfields, partially and fully ordered seminearrings (*cf.* [39, 105, 106, 108]). In 1981, he investigated interrelationships between seminearrings and different types of semigroups of right quotients in [107]. In 90's, the theory of seminearrings has been developed in many directions. J. Ahsan generalised the notion of 'semiring ideals' in the setting of seminearrings and called it '$\mathcal{S}$-ideals'. He characterized seminearrings in terms of $\mathcal{S}$-ideals in [2, 3]. Ahsan and Zhongkui encountered the notion of 'strongly idempotent seminearring' which is an analogue of 'fully idempotent ring' (a ring in which each ideal I of that ring is idempotent, *i.e.,* $I^2 = I$) and characterized these seminearrings (*cf.* [4]). In [6], Ayaragarnchanakul and

Mitchell established that any finite division seminearring is uniquely determined by the Zappa-Szép product of two multiplicative subgroups. Blackett showed in [11] that under the operations 'pseudosum' and 'pseudoproduct', the set of probability generating functions forms a seminearring with commutative addition and an additive identity and observed that how the algebra of seminearrings of probability generating functions helps to understand the probability theory of non-negative integer-valued random variables. Boykett extended the results of [11] and studied seminearrings of all polynomials over a commutative semifield with zero in [13] under the operations of multiplication and composition.

'Distributively generated seminearring' is an important tool in the study of seminearrings. In 1997, Meldrum and Samman defined the notion of distributively generated seminearrings in [76] in an analogous way to the notion of 'distributively generated near-rings' [91]. Then in [32, 33, 76, 94, 95, 96, 97], the theory of distributively generated seminearrings and seminearrings of endomorphisms has been enriched. Krishna and Chatterjee investigated the algebraic structure of seminearrings in different aspects. They called a seminearring $(S, +, \cdot)$ as a 'near-semiring' where $(S, +)$ is a monoid with identity '0' satisfying $0 \cdot s = 0$ for all $s \in S$. They studied categorical representations of seminearrings, extended the result of Holcombe of near-rings to seminearrings (*cf.* [58, 59, 60]). In 2005 Neuerburg studied seminearrings of bivariate polynomials in [84]. Shabir and Ahmed [104] characterized weakly regular seminearrings and studied the topology of the space of irreducible ideals of those seminearrings, Zulfiqar [110] discussed the radicals of seminearrings and generalised several results of ring theory, Changphas and Denecke [18, 19] gave a full characterization of Green's relation on a sub-seminearring of the seminearring $Hyp\ (n)$ of all hypersubstitutions of type (n) and used seminearrings to study complexity of hypersubstitutions and lattices of varieties, Kornthorng et al. [57] introduced the notions of 'k-ideal', 'full k-ideal' and explored the lattice stricture of right full k-ideals in an additively inverse seminearring. Kumar and Krishna [62, 63, 64, 65, 66] studied affine near-semirings over Brandt semigroups. They classified the elements, cardinality of an affine near-semiring over a Brandt semigroup, characterized the Green's relations on both of its semigroup reducts. Mukherjee et al. studied various kinds of regularity in seminearrings and established analogues of some structure theorems of semigroup theory in the setting of seminearrings (*cf.* [80, 81, 82, 83, 98, 99]). In these papers congruences, near-ring congruences, additively commutative near-ring congruences on different type of seminearrings have also been studied.

Hussain et al. [48] discussed isomorphism theorems of seminearrings. Perumal, Balakrishnan, Manikandan, Senthil, Arulprakasam generalised several results of the theory of near-rings in seminearrings. In [9], Balakrishnan et al. studied left duo seminearrings, in [71, 72, 73] Manikandan et al. defined the notions of mate and mutual mate functions, mid units in duo seminearrings, strong (k, r)-seminearrings and characterized these seminearrings, in [85, 86, 87, 88] Perumal et al. studied left bipotent seminearrings, normal seminearrings, medial left bipotent seminearrings. Perumal, Senthhil et al. also studied prime ideals, minimal prime ideals in seminearrings, noetherian seminearrings, right duo seminearrings (*cf.* [102, 103]). In 2020, Khachorncharoenkul et al. [51] introduced the notion of left almost seminearrings which generalizes left almost semirings, near left almost rings and left almost rings and investigated some related properties of these seminearrings. Koppula et al. introduced the notions of prime strong ideals, perfect ideals, perfect homomorphisms in a seminearring and established some relations between them (*cf.* [55, 56]). Khan et al. [53, 54] introduced the notions of soft near-semirings, soft int-nearsemirings, soft subnear-semirings, soft ideals, soft int-ideals, idealistic soft near-semirings based on soft set theory, discussed related properties of nearsemirings and SI-nearsemirings. Then in 2022, Khan, Arif and Taouti [52] introduced the notion of group seminearrings, studied ideals and homomorphisms there.

The theory of seminearrings is not only drawing the attention of many researchers from the theoretical point of view but also from the practical point of view. We know that process algebra is an active area of research in computer science. From last century, many process algebras have been formulated, extended with data, time, mobility, probability and stochastic (see [7, 8]). A process algebra is based upon seminearring where '+' is idempotent and commutative. Seminearring is also a useful tool in the study of reversible computation [14]. It also appears in generalized linear sequential machines. In [61], the authors obtained a necessary condition to test the minimality of the machines using α-radicals. Desharnais and Struth [24], Droste et al. [26], Armstrong et al. [5], Rivas et al. [92], Jenila et al. [49] have utilized the concept of seminearring in various applications.

It is well known that the study of universal algebras in general, and that of semigroups, semirings, seminearrings etc in particular, is heavily dependent on the study of congruences. In this direction group congruences on semigroups, ring congruences on semirings play important roles. Likewise it is natural to study near-ring congruences on seminearrings. Sardar and Mukherjee initiated this study in [98] where they characterized near-ring congruences on a restricted type of additively inverse seminearrings. As a continuation of this work, we first study near-ring congruences on additively regular seminearrings. Then we characterize near-ring congruences on an arbitrary seminearring. Consequently, this study generalizes the study of additively commutative near-ring congruences on distributively generated seminearrings done in [83]. We also study the lattice structures of the set of all near-ring congruences of a seminearring. With the help of the notion of near-ring congruences, we then characterize and study seminearrings which can be decomposed as a full subdirect product of a bi-semilattice and a near-ring. We present the main work of the thesis in five chapters. We give a brief description of each chapter below:

- **Chapter 1:** Here we mainly recall some preliminary notions and results of semigroups, semirings, near-rings, seminearrings and lattices.

- **Chapter 2:** In [98, 99], Sardar and Mukherjee studied additively regular seminearrings. Among other results, they obtained in a distributively generated additively inverse seminearring S with property D, (i) an inclusion preserving bijective correspondence between the set of all near-ring congruences on S and the set of all normal full k-ideals of S (*cf.* Theorem 3.20 of [98, 99]) and (ii) the least near-ring congruence on S (*cf.* Theorem 3.22 of [98]). In the Concluding remark of [98], the authors raised a question as to whether Theorems 3.20, 3.22 can be obtained by removing the restriction *viz.,* 'property D' or 'distributively generated'. In order to address the question we found some important ideas in [67] which helped us to obtain some results on additively regular seminearring such as Theorems 2.2.1, 2.2.2, 2.2.3, 2.2.7. Using these results, we obtain an inclusion preserving bijective correspondence between the set of all normal full k-ideals and the set of all near-ring congruences in a distributively generated additively regular seminearring (*cf.* Theorem 2.2.10). The counterpart (*cf.* Corollary 2.2.12) of the above mentioned result in the setting of additively inverse seminearrings answers the question raised in [98] to a good extent because 'property D' is removed here. Then we describe the least near-ring congruence on distributively generated additively regular and additively inverse seminearrings. Theorem 2.2.17 describing the least near-ring congruence on a distributively generated additively inverse seminearring (not necessarily with property D) refines Theorem 3.22 of [98]. Then we establish that the correspondence obtained in Theorem 2.2.10 turns out to be a lattice isomorphism (*cf.* Theorem 2.3.15). This chapter concludes with the study of modularity (*cf.* Theorems 2.3.17 and 2.3.18), distributivity (*cf.* Theorem 2.3.22) and completeness (*cf.* Theorems 2.3.23 and 2.3.24) of the lattices of Theorem 2.3.15.

• **Chapter 3:** So far our knowledge goes, the study of near-ring congruences on seminearrings was accomplished in [17, 83, 99]. Results obtained in [17, 83, 99] mainly connect, via inclusion preserving bijective correspondence, (*i*) near-ring congruences with normal full k-ideals in a distributively generated additively inverse seminearring with property D (*cf.* Theorem 3.20 [99]), (*ii*) near-ring congruences with normal full k-ideals in a distributively generated additively regular seminearring (*cf.* Theorem 2.2.10), (*iii*) additively commutative near-ring congruences with normal subseminearrings in a zero-symmetric distributively generated seminearring (*cf.* Theorem 1.5.24 (*i.e.,* Theorem 3.6 [83])). In each of these results, seminearrings, under consideration, are assumed to be distributively generated. Our goal in this chapter is to make similar study without putting any restriction on the seminearring under consideration. In this chapter we consider a seminearring S without any restriction (*i.e.,* neither distributively generated nor additively regular) and establish that the set of all near-ring congruences on S and the set $\{I \subseteq S | I$ is a strong, dense, reflexive and closed additive subsemigroup of S with $IS \subseteq I\}$ are in an inclusion preserving bijective correspondence (*cf.* Theorem 3.1.8). We also show that in a seminearring S, the set of all zero-symmetric near-ring congruences on S and the set $\{I \subseteq S | I$ is a strong, dense, reflexive and closed additive subsemigroup of S with SI, $IS \subseteq I\}$ are in an inclusion preserving bijective correspondence (*cf.* Theorem 3.1.10). We then obtain the analogue of Theorems 3.1.8 and 3.1.10 in the setting of distributively generated seminearrings (*cf.* Theorem 3.2.3).

• **Chapter 4:** As a continuation of the work done in **Chapter 3**, in this chapter, we aim to extend the bijections, established in the previous chapter (*cf.* Theorems 3.1.8 and 3.1.10), to lattice isomorphisms. In this chapter, we first call a subset I of a seminearring S a 'generalised strong dense reflexive (right) k-ideal' if I is a strong, dense, reflexive and closed additive subsemigroup of S such that (respectively, $IS \subseteq I$) $IS, SI \subseteq I$ and explain the motivation behind considering a 'generalised strong dense reflexive (right) k-ideal' to be a suitable structure to get connected (via inclusion preserving bijections) with the near-ring congruences. Then we establish that in a seminearring with left local units, the set of all generalised strong dense reflexive k-ideals and the set of all zero-symmetric near-ring congruences become lattices under set inclusions (*cf.* Theorems 4.2.15 and 4.2.17) and these lattices become isomorphic (*cf.* Theorem 4.2.18). Thereafter with the help of this lattice isomorphism, we study the modularity and distributivity of these lattices in a seminearring with left local units (*cf.* Theorems 4.2.19, 4.2.22 and 4.2.26). Then in Theorem 4.3.9 we establish that in an E^+-inversive seminearring, the bijections between (i) the set of all near-ring congruences and the set of all generalised strong dense reflexive right k-ideals and (ii) the set of all zero-symmetric near-ring congruences and the set of all generalised strong dense reflexive k-ideals turn out to be lattice isomorphisms. We also study the modularity, distributivity and completeness of the above-mentioned lattices in an E^+-inversive seminearring (*cf.* Theorem 4.3.10 and Theorem 4.3.12).

• **Chapter 5:** After studying near-ring congruences in various types of seminearrings, in this chapter we mainly aim to find some sort of analogues of some important structure theorems for E-inversive semigroups in the setting of seminearrings using the notion of near-ring congruences. This idea was motivated by the fact that one nice aspect of studying seminearrings is to obtain semigroup theoretic analogues in the setting of seminearrings. Major part of this study is devoted to obtain analogues of some structure theorems of semigroups, in general for regular semigroups, in particular for completely regular and for clifford semigroups ([80, 81, 82, 98]). This trend of development of seminearring theory together with a structure theorem and the fact that E-inversive semigroup occupy an important part in the semigroup theory motivate the outcome of this chapter. In order to find an analogue of the famous structure theorem,"a semigroup is a full subdirect product of a semilattice and a group if and only if it is an E-inversive sturdy semilattice of cancellative monoids" (Theorem 14 [78] *i.e.,* Theorem 1.1.33), in the setting of seminearrings, here we characterize the seminearrings which are full subdirect products of a bi-semilattice and a near-ring as the E^+-inversive seminearrings which are strong bi-semilattice of additively cancellative seminearrings (*cf.* Theorem 5.1.12). Then we obtain some variants of Theorem 5.1.12 viz., Theorems 5.1.16, 5.1.17 and 5.1.18 (this variation occurs due to the replacement of bi-semilattice by distributive lattice and near-ring by zero-symmetric near-ring). Each of these four theorems is not only an analogue of Mitsch's Theorem 14 [78] (*cf.* Theorem 1.1.33) in our setting, but also an analogue of Ghosh's Theorem 2.3 [29] (*cf.* Theorem 1.3.14) on semirings. On the other hand, Ghosh obtained a different characterization in Theorem 2.10 [29] (*cf.* Theorem 1.3.15) of the class of semirings characterized in Theorem 2.3 [29] (*cf.* Theorem 1.3.14). This motivates us to make an attempt for obtaining an analogue of Ghosh's Theorem 2.10 [29] (*cf.* Theorem 1.3.15) in our setting. In this attempt we have been able to obtain Theorem 5.1.27 and Theorem 5.1.29 which respectively provide different characterizations of the classes of seminearrings determined in Theorem 5.1.16 and Theorem 5.1.18. We conclude this chapter with the sketch (*cf.* Theorem 5.2.1, Theorem 5.2.6) of the relationships of the classes of E^+-inversive seminearrings characterized here with the classes of additively regular seminearrings characterized in [81, 82, 98].

To conclude the introduction it is relevant to mention that in the process of accomplishing the whole work of this thesis we mainly take impetus from ideal theory of semigroups, congruences on semigroups, regularity of semigroups, ideal theory of semirings, congruences on semirings and theory of lattices for which, among others, we have consulted [1, 22, 23, 35, 37, 38, 45, 46, 77, 90].

CHAPTER 1

PRELIMINARIES

1 Preliminaries

In this chapter certain basic definitions and results are presented for their use in the sequel or for some historical connections.

1.1 Semigroups

We recall some preliminary notions of semigroup theory from [22, 45, 46, 68, 77, 89, 90].

Definition 1.1.1. (i) A non-empty set S together with a binary operation is called a *groupoid*.

(ii) A groupoid S satisfying the associative law is a *semigroup*. A semigroup having only one element is *trivial*.

(iii) If a semigroup S contains an element 1 with the property that, for all x in S,

$$x1 = 1x = x,$$

we say that 1 is an *identity element* of S, and that S is a *semigroup with identity* or (more usually) a *monoid*. A semigroup S has at most one identity element. If S has no identity element then we can adjoin an element 1 to S to form a monoid. We define

$$1s = s1 = s \text{ for all } s \in S \text{ and } 11 = 1,$$

then $S \cup \{1\}$ becomes a monoid. Now we define

$$S^1 = S \text{ if } S \text{ has an identity element and } S \cup \{1\} \text{ otherwise.}$$

We refer to S^1 as the *monoid obtained from* S *by adjoining an identity if necessary.*

(iv) A non-empty subset A of a semigroup $(S, \cdot)$ is called a *subsemigroup* of S if $A^2 \subseteq A$.

In the rest of this section, for our convenience we will consider the binary operation of a semigroup as 'addition' and denote a semigroup by $(S, +)$. Now we recall the following definitions from [31, 36, 46] .

Definition 1.1.2. *(i)* A subsemigroup A of a semigroup $(S, +)$ is called a *dense subsemigroup* of S if for each $s \in S$ there exist $x, y \in S$ such that $s + x$, $y + s \in A$.
(ii) In a semigroup $(S, +)$, let us define the *closure operator* $w : P(S) \to P(S)$ by $A \mapsto Aw$, where $P(S)$ denotes the power set of S and $Aw = \{s \in S :$ there exists $a \in A$ such that $a + s \in A\}$. Then a subsemigroup I of S is said to be a *closed subsemigroup* if $Iw = I$.
(iii) A subsemigroup I of a semigroup $(S, +)$ is said to be a *reflexive subsemigroup* if for $a, b \in S$, $a + b \in I$ implies $b + a \in I$.

Remark 1.1.3. If T is a subsemigroup of $(S, +)$, $T \subseteq Tw$. If $(T, +)$ and $(Tw, +)$ are subsemigroups of a semigroup $(S, +)$ then Tw is a closed subsemigroup of S.

Now we recall the definitions of congruence, homomorphism and results related with them.

Definition 1.1.4. In a semigroup $(S, +)$, an equivalence relation σ on S is said to be a *congruence* if $(a, b) \in \sigma$ implies $(a + c, b + c)$, $(c + a, c + b) \in \sigma$ for all $a, b, c \in S$.

Definition 1.1.5. Let ρ be a congruence on a semigroup $(S, +)$. Then

$$S/\rho := \{[s] : [s] \text{ is the congruence class of } s \text{ under } \rho \text{ where } s \in S\}$$

forms a semigroup with respect to '+' defined by

$$[x] + [y] = [x + y] \text{ for all } x, y \in S.$$

A congruence ρ on a semigroup $(S, +)$ is said to be a *group congruence* on S if $(S/\rho, +)$ becomes a group.

Definition 1.1.6. Let $(S, +)$ and $(S_1, \oplus)$ be two semigroups. Then a function $f : S \to S_1$ is said to be a *semigroup homomorphism* if

$$f(x + y) = f(x) \oplus f(y) \text{ for all } x, y \in S.$$

If $(S, +)$ is a monoid, *i.e.,* a semigroup with zero (0) (where 0 is the identity element of $(S, +)$) then for a semigroup morphism f of S, *kernel of f* is the set $\{s \in S : f(s) = f(0)\}$ and is denoted by *ker f*.

Remark 1.1.7. Let $(S, +)$ be a semigroup.

(i) Let T be a semigroup and $\phi : S \to T$ be a semigroup homomorphism. Then the relation

$$\rho_\phi = \{(a, b) \in S \times S : \phi(a) = \phi(b)\}$$

is a congruence on S.

(ii) Let ρ be a congruence on $(S, +)$. Then S/ρ is a semigroup with respect to the operation defined in Definition 1.1.5 and $\lambda_\rho : S \to S/\rho$, defined by $\lambda(a) = [a]_\rho$, is a semigroup homomorphism.

Definition 1.1.8. For any two binary relations ρ, σ on a non-empty set S

$$\rho \circ \sigma := \{(x, y) \in S \times S |\ (x, z) \in \rho \text{ and } (z, y) \in \sigma \text{ for some } z \in S\}$$

is again a binary relation on S.

Definition 1.1.9. Let ρ be a relation on a non-empty set S. Then the *transitive closure of* ρ, denoted by ρ^∞, is defined as follows

$$\rho^\infty := \bigcup_{n=1}^{\infty} \rho^n.$$

Result 1.1.10. If ρ is a relation on a non-empty set S then ρ^∞ is the smallest transitive relation on S containing ρ.

Theorem 1.1.11. *Let ρ and σ be two equivalence relations on a non-empty set S (congruences on a semigroup $(S, +)$) and $\rho \vee \sigma$ denote the smallest equivalence relation on S (the smallest congruence on $(S, +)$) containing both ρ and σ. If $a, b \in S$, then $(a, b) \in \rho \vee \sigma$ if and only if for some $n \in \mathbb{N}$ there exist elements $x_1, x_2, ..., x_{2n-1}$ in S such that*

$$(a, x_1) \in \rho,\ (x_1, x_2) \in \sigma,\ (x_2, x_3) \in \rho, \ldots, (x_{2n-1}, b) \in \sigma.$$

Remark 1.1.12. Theorem 1.1.11 says effectively that

$$\rho \vee \sigma = (\rho \circ \sigma)^{\infty}.$$

Corollary 1.1.13. *Let ρ and σ be two equivalence relations on a non-empty set S (congruences on a semigroup $(S, +)$) such that $\rho \circ \sigma = \sigma \circ \rho$. Then*

$$\rho \vee \sigma = \rho \circ \sigma.$$

Definition 1.1.14. Let $(S, +)$ be a semigroup. Then $a \in S$ is said to be an *idempotent element* of S if $a + a = a$.

Definition 1.1.15. If a is an element of a semigroup $(S, +)$, we say that a' is an *inverse* of a if

$$a + a' + a = a \text{ and } a' + a + a' = a'.$$

Definition 1.1.16. Let $(S, +)$ be a semigroup. Then S is said to be

(*i*) an *E-inversive semigroup* if for each $a \in S$, there exists $x \in S$ such that $a + x \in E(S)$, the set of all idempotents of S,

(*ii*) a *regular semigroup* if for each $a \in S$, there exists $x \in S$ such that $a = a + x + a$,

(*iii*) a *band* if every element of S is idempotent,

(*iv*) a *left (right) normal band* if S is a band in which $a + b + c = a + c + b$ (resp., $a + b + c = b + a + c$) for all $a, b, c \in S$,

(*v*) a *normal band* if S is a band in which $a+b+c+a = a+c+b+a$ for all $a, b, c \in S$,

(*v*) a *semilattice* if S is a commutative band,

(*vi*) an *inverse semigroup* if every a in S possesses a unique inverse *i.e.*, if for every $a \in S$ there exists a unique element a^* in S such that

$$a + a^* + a = a, a^* + a + a^* = a^*,$$

(*vii*) an *E-semigroup* if $E(S)$ is a subsemigroup of S where $E(S)$ denotes the set of all idempotents of S, (*vii*) an *E-unitary semigroup* if $e + a,\ b + f \in E(S)$ imply $a, b \in E(S)$ for every $e, f \in E(S)$ and $a, b \in S$ where $E(S)$ denotes the set of all idempotents of S.

Observation 1.1.17. [78] Let S be a semigroup. If S is E-inversive, then it can be shown that for each $a \in S$, there exists $x \in S$ such that $a + x,\ x + a \in E(S)$, the set of all idempotents of S. Consequently, it can be said that S is E-inversive if and only if $I(a) = \{x \in S : a + x, x + a \in E(S)\} \neq \emptyset$ for every $a \in S$. A regular semigroup (whence an inverse semigroup) is E-inversive.

Theorem 1.1.18. *The following statements about a semigroup S are equivalent:*

(a) S is an inverse semigroup;

(b) S is regular and idempotent elements commute.

Notation 1.1.19. Throughout this thesis, in a semigroup $(S, +)$,

(*i*) $E(S)$ denotes the set of all idempotents of S,

(*ii*) $V(a)$ denotes the set of all inverses of a, *i.e.,* $V(a) = \{x \in S : a + x + a = a$ and $x + a + x = x\}$,

(*iii*) if S is an inverse semigroup, for each $a \in S$, a^* denotes the unique element of S satisfying

$$a + a^* + a = a, a^* + a + a^* = a^*.$$

Definition 1.1.20. (*i*) A subsemigroup I of a semigroup $(S, +)$ is said to be a *full subsemigroup* if $E(S) \subseteq I$.
(*ii*) In a regular semigroup $(S, +)$, a subsemigroup T of S is called *self-conjugate* if $x + t + x' \in T$ for all $t \in T$, for all $x \in S$ and for all $x' \in V(x)$ (*cf.* Notation 1.1.19).

Theorem 1.1.21 (Theorem 1 [67]). *Let $(S, +)$ be a regular semigroup and H be a full, self-conjugate subsemigroup of $(S, +)$. Then the relation*

$$\beta_H = \{(a, b) \in S \times S : x + a = b + y \text{ for some } x, y \in H\}$$

is a group congruence on $(S, +)$.
The least group congruence on S is given by $\sigma = \beta_U$ where U is the intersection of all full, self-conjugate subsemigroups of S.

Theorem 1.1.22 (Theorem 3 [67]). *In a regular semigroup $(S,+)$, the mapping $H \to (H) := \{(a,b) \in S \times S : a + b' \in H$ for some $b' \in V(b)\}$ is a one-to-one, inclusion-preserving mapping of the set of all self-conjugate, full and closed subsemigroups onto the set of all group congruences on S.*

Theorem 1.1.23 (Theorem 4 [67]). *For any group congruence τ on a regular semigroup $(S,+)$, say $\tau = \beta_H$ where H is a self-conjugate, full subsemigroup of S, the following are equivalent.*

(i) $a\tau b$,

(ii) $a + x + b' \in H$ *for some* $x \in H$ *and some (all)* $b' \in V(b)$,

(iii) $a' + x + b \in H$ *for some* $x \in H$ *and some (all)* $a' \in V(a)$,

(iv) $b + x + a' \in H$ *for some* $x \in H$ *and some (all)* $a' \in V(a)$,

(v) $b' + x + a \in H$ *for some* $x \in H$ *and some (all)* $b' \in V(b)$,

(vi) $a + x = y + b$ *for some* $x, y \in H$,

(vii) $x + a = b + y$ *for some* $x, y \in H$,

(viii) $H + a + H \cap H + b + H \neq \emptyset$.

Corollary 1.1.24 (Corollary 1 [67]). *Let σ denote the least group congruence on a regular semigroup $(S,+)$. Then the following are equivalent.*

(i) $a\sigma b$,

(ii) $a + u + b' \in U$ *for some* $u \in U$ *and some (all)* $b' \in V(b)$,

(iii) $a' + u + b \in U$ *for some* $u \in U$ *and some (all)* $a' \in V(a)$,

(iv) $b + u + a' \in U$ *for some* $u \in U$ *and some (all)* $a' \in V(a)$,

(v) $b' + u + a \in U$ *for some* $u \in U$ *and some (all)* $b' \in V(b)$,

(vi) $a + u = v + b$ *for some* $u, v \in U$,

(vii) $u + a = b + v$ *for some* $u, v \in U$,

(viii) $U + a + U \cap U + b + U \neq \emptyset$

where U is the least member of the set of all self-conjugate, full subsemigroups of S, i.e., the intersection of all self-conjugate, full subsemigroups of S.

Theorem 1.1.25. *[77] If $(S,+)$ is an inverse semigroup with semilattice of idempotents $E(S)$, then the relation*

$$\sigma := \{(x,y) \in S \times S : x+e = y+e \text{ for some } e \in E(S)\}$$

is the minimum group congruence on S.

Now we recall some results from [31] related with group congruences on an arbitrary semigroup which guide us to study near-ring congruences on seminearrings.

Lemma 1.1.26 (Lemma 2.2 [31])**.** *Let ρ_1, ρ_2 be two group congruences on a semigroup $(S,+)$. Then $\rho_1 \subseteq \rho_2$ if and only if $\{x \in S : (x, x+x) \in \rho_1\} \subseteq \{x \in S : (x, x+x) \in \rho_2\}$.*

Lemma 1.1.27 (Lemma 2.3 [31])**.** *Let B be a non-empty subset of a semigroup $(S,+)$. Consider four relations on S :*

(i) $\rho_{1,B} = \{(a,b) \in S \times S :$ *there exists* $x \in S$ *such that* $a+x,\ b+x \in B\}$,

(ii) $\rho_{2,B} = \{(a,b) \in S \times S :$ *there exist* $x, y \in B$ *such that* $a+x = y+b\}$,

(iii) $\rho_{3,B} = \{(a,b) \in S \times S :$ *there exists* $x \in S$ *such that* $x+a,\ x+b \in B\}$,

(iv) $\rho_{4,B} = \{(a,b) \in S \times S :$ *there exist* $x, y \in B$ *such that* $x+a = b+y\}$.

If B is a dense and reflexive subsemigroup of S, then

$$\rho_{1,B} = \rho_{2,B} = \rho_{3,B} = \rho_{4,B}.$$

If B is a dense, reflexive subsemigroup of S, then we denote the above four relations by ρ_B.

Remark 1.1.28. In a semigroup $(S,+)$, a closed and dense subsemigroup $(I,+)$ is always a full subsemigroup since for any $e \in E(S)$, there exists $x \in S$ such that $e+x \in I$ whence $e+(e+x)$, $e+x \in I$.

So in view of the above Remark, we rewrite Theorem 2.4 [31] with a slight modification.

Theorem 1.1.29. *Let B be a dense and reflexive subsemigroup of a semigroup $(S,+)$. Then the relation*

$$\rho_B = \{(a,b) \in S \times S : \text{there exists } x \in S \text{ such that } a+x,\ b+x \in B\}$$

is a group congruence on S. Moreover, $B \subseteq Bw = \{x \in S : (x, x+x) \in \rho_B\}$*. If* B *is closed, then* $B = \{x \in S : (x, x+x) \in \rho_B\}$*.*

Conversely, if σ *is a group congruence on* S*, then there exists a dense, reflexive and closed subsemigroup* N *of* S *such that the relation* ρ_N *coincides with* σ *where*

$$\rho_N = \{(a,b) \in S \times S : \text{there exists } x \in S \text{ such that } a+x,\ b+x \in N\}.$$

In fact, $N = \{x \in S : (x, x+x) \in \sigma\}$*. Thus there exists an inclusion preserving bijection between the set of all dense, reflexive and closed subsemigroups of* S *and the set of all group congruences on* S*.*

Definition 1.1.30. Let ρ be a congruence on a semigroup S. If S/ρ is a semilattice then ρ is called a *semilattice congruence* on S. In such a case S is a *semilattice* $Y = S/\rho$ *of semigroups* S_α, $\alpha \in Y$, where S_α are the ρ-classes, or briefly a *semilattice of semigroups* S_α.

Definition 1.1.31. Let Y be a semilattice. Suppose for each $\alpha \in Y$ there is a semigroup S_α such that $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$. For each pair $\alpha, \beta \in Y$, $\alpha \geq \beta$, let $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ be a homomorphism satisfying the following conditions:

(*i*) $\phi_{\alpha,\alpha} = i_{S_\alpha}$ where $\alpha \in Y$ and i_{S_α} denotes the identity morphism of S_α,

(*ii*) $\phi_{\alpha,\beta} \circ \phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ if $\alpha > \beta > \gamma$ (here functions are written from right).

On the set $\bigcup\limits_{\alpha \in Y} S_\alpha$, a multiplication is defined by

$$a * b = (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta})$$

for $a \in S_\alpha, b \in S_\beta$. This multiplication '$*$' is associative and the new multiplication coincides with the given one on each S_α. The semigroup so defined is denoted by $[Y; S_\alpha, \phi_{\alpha,\beta}] = S$ and is a *strong semilattice* Y *of semigroups* S_α determined by the homomorphisms $\phi_{\alpha,\beta}$ or briefly a *strong semilattice of semigroups* S_α.

A strong semilattice of semigroups $S = [Y; S_\alpha, \phi_{\alpha,\beta}]$ is said to be a *sturdy semilattices of semigroups* if each $\phi_{\alpha,\beta}$ is an injective morphism.

Definition 1.1.32. A semigroup isomorphic with a subsemigroup H of the direct product of two semigroups S and T is called a *subdirect product* of S and T if the two projections $\pi_1 : H \to S$, $\pi_1(s,t) = s$ and $\pi_2 : H \to T$, $\pi_2(s,t) = t$ are surjective.

A subdirect product H of a semigroup S and a group G is called *full* if $(e, 1) \in H$ for every $e \in E(S)$ and 1 is the identity of the group G.

Theorem 1.1.33 (Theorem 14 [78]). *For a semigroup S the following are equivalent:*

(i) S is a full subdirect product of a semilattice and a group,

(ii) S is an E-inversive sturdy semilattice (cf. Definition 1.1.31) of cancellative monoids,

(iii) S is an E-inversive sturdy semilattice of monoids with a single idempotent.

1.2 Lattices and related structures

For the following preliminaries on lattices, mainly [23, 37] are consulted.

Definition 1.2.1. Let P be a set. An *order* (or *partially order*) on P is a binary relation $\leq$ on P such that, for all $x, y, z \in P$,

(i) $x \leq x$ (known as reflexivity)

(ii) $x \leq y$ and $y \leq x$ imply $x = y$ (known as antisymmetry)

(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$ (known as transitivity).

A set P equipped with an order relation $\leq$ is said to be an *ordered set* (or *partially ordered set* or *poset*).

Definition 1.2.2. Let P be an ordered set. We say that P has a bottom element if there exists $\perp \in P$ (called *bottom*) with the property that $\perp \leq x$ for all $x \in P$.

Dually P has a top element if there exists $\chi \in P$ (called *top*) with the property that $\chi \geq x$ for all $x \in P$.

Definition 1.2.3. [23] Let P and Q be two ordered sets. Then a map $\phi : P \to Q$ is said to be

(i) an *order preserving* if $x \leq y$ in $P \Rightarrow \phi(x) \leq \phi(y)$ in Q,

(ii) an *order-embedding* if $x \leq y$ in P if and only if $\phi(x) \leq \phi(y)$ in Q,

(iii) an *order-isomorphism* if it is an order-embedding which maps P onto Q and P and Q are called order-isomorphic.

Definition 1.2.4. [23] Let L be an ordered set and $S \subseteq L$. An element $x \in L$ is an *upper bound* of S if $s \leq x$ for all $s \in S$. A *lower bound* is defined dually. x is called the *least upper bound of S (supremum of S)* and denoted by $\sup S$ if x is an upper bound of S and $x \leq y$ for all upper bounds y of S. Dually, x is called the *greatest lower bound of S (infimum of S)* and denoted by $\inf S$ if x is a lower bound of S and $x \geq y$ for all lower bounds y of S.

Definition 1.2.5. [23] Let L be a non-empty ordered set. Then L is called a *lattice* if $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for any two elements x, y in L.

Notation 1.2.6. We often write $a \wedge b$ or *meet* of a, b instead of $\inf\{a, b\}$ and $a \vee b$ or *join* of a, b instead of $\sup\{a, b\}$.

Definition 1.2.7. Let L be a lattice and $\emptyset \neq M \subseteq L$. Then M is called a *sublattice* of L if $a, b \in M$ implies $a \vee b$, $a \wedge b \in M$.

Theorem 1.2.8. *[37] (a) Let $(V, \leq)$ be a lattice. Considering $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$ as binary operations on V, we obtain the following:*

(1) $(V, \vee)$ and $(V, \wedge)$ are semilattices (cf. Definition 1.1.16).

(2) $a \vee b = b \Leftrightarrow a \wedge b = a$ for all $a, b \in V$.

(b) Conversely, let $(V, \vee, \wedge)$ be a non-empty set with binary operations $\vee$ and $\wedge$ which satisfy (1) and (2). Then

$$a \leq b \Leftrightarrow a \vee b = b \text{ for all } a, b \in V$$

or equivalently

$$a \leq b \Leftrightarrow a \wedge b = a \text{ for all } a, b \in V$$

defines a relation '$\leq$' on V for which $(V, \leq)$ is a lattice satisfying $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$ for all $a, b \in V$.

Definition 1.2.9. [23] Let L and K be two lattices. A mapping $f : L \to K$ is said to be a *lattice homomorphism* if f is both join-preserving and meet-preserving *i.e.*, for all $a, b \in L$,

$$f(a \vee b) = f(a) \vee f(b) \text{ and } f(a \wedge b) = f(a) \wedge f(b).$$

A bijective lattice homomorphism is called a *lattice isomorphism.*

Proposition 1.2.10. *[23] Let L and K be two lattices and $f : L \to K$ be a map.*

(i) Then the following are equivalent.

(a) f is order preserving.

(b) $f(a \vee b) \geq f(a) \vee f(b)$ for all $a, b \in L$.

(c) $f(a \wedge b) \leq f(a) \wedge f(b)$ for all $a, b \in L$. In particular, if f is a homomorphism, then f is order preserving.

(ii) f is a lattice isomorphism if and only if it is an order-isomorphism.

Lemma 1.2.11. *[23] Let L be a lattice and $a, b, c \in L$. Then*

(i) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$, and dually,

(ii) $a \geq c \Rightarrow a \wedge (b \vee c) \geq (a \wedge b) \vee c$, and dually.

Definition 1.2.12. [23] A lattice L is said to be a *modular lattice* if

$$a \geq c \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee c$$

for all $a, b, c \in L$.

Definition 1.2.13. [23] A lattice L is said to be a *distributive lattice* if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

or equivalently

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

for all $a, b, c \in L$.

Definition 1.2.14. [23] A lattice L is said to be a *complete lattice* if the join (supremum), $\bigvee S$, and the meet (infimum), $\bigwedge S$, exist for every subset S of L.

Remark 1.2.15. Every finite lattice is complete.

Theorem 1.2.16. *[23] Let P be a non-empty ordered set. Then the following are equivalent.*

(i) P is a complete lattice.

(ii) $\bigwedge S$ exists in P for every subset S of P.

(iii) P has a top element and $\bigwedge S$ exists in P for every non-empty subset S of P.

1.3 Semirings

Semiring is a generalization of both rings and distributive lattices. The first mathematical structure we encounter 'the set of all natural numbers' under usual addition and multiplication is a semiring. Other semirings arise naturally in diverse areas of mathematics such as combinatorics, functional analysis, topology, graph theory, optimization theory. In the literature of semiring theory, various definitions of semirings followed by several authors are found. Glazek [34], Hebisch and Weinert [38] consider the definition of 'semiring' in the following form.

Definition 1.3.1. [38] An algebraic structure $(R, +, \cdot)$ is said to be a *semiring* if it satisfies the following axioms:

(1) $(R, +)$ is a semigroup (not necessarily commutative),

(2) $(R, \cdot)$ is a semigroup (not necessarily commutative),

(3) multiplication distributes over addition from each side, *i.e.,* for all $a, b, c \in R$

$$a \cdot (b + c) = a \cdot b + a \cdot c \text{ and } (a + b) \cdot c = a \cdot c + b \cdot c.$$

$(R, +, \cdot)$ is said to be an *additively commutative semiring* if $(R, +)$ is commutative and a *multiplicatively commutative semiring* if $(R, \cdot)$ is commutative. $(R, +, \cdot)$ is said to be a *commutative semiring* if both $(R, +)$ and $(R, \cdot)$ are commutative.

A subset A of a semiring R is said to be a *subsemiring* if A is closed under '+' and '·', *i.e.,* $(A, +)$, $(A, \cdot)$ are subsemigroups.

We present below two other definitions of semirings from Golan [35], Hebisch and Weinert [37].

Definition 1.3.2. [35] A *semiring* is a non-empty set R on which two binary operations, say '+' and '·' are defined such that $(R, +)$ is a commutative semigroup with identity element 0, $(R, \cdot)$ is a semigroup with identity element 1, '·' distributes over '+' from each side and $0 \cdot r = 0 = r \cdot 0$ for all $r \in R$.

Definition 1.3.3. [37] A non-empty set R along with two binary operations, say '+' and '·' is said to be a *semiring* if $(R, +)$ is a commutative semigroup, $(R, \cdot)$ is a semigroup and '·' distributes over '+' from each side.

Remark 1.3.4. A semiring in the sense of [35] and [37] are *additively commutative* in the sense of [38]. In the present thesis, unless otherwise mentioned, a semiring is assumed to be in the sense of [38] *i.e.,* Definition 1.3.1.

Definition 1.3.5. An additively commutative semiring $(S, +, \cdot)$ is said to be a *hemiring* if $(S, +)$ is a monoid with identity element 0 and $a \cdot 0 = 0 \cdot a = 0$ for all $a \in S$. An additively cancellative hemiring is called a *halfring.*

Definition 1.3.6. Let $(S, +, \cdot)$ be a semiring. An element e of S is called an *additive idempotent* if $e + e = e$.
An element a of S is called an *idempotent* if $a + a = aa = a$.

Definition 1.3.7. [38] In a semiring $(S, +, \cdot)$, a subset $I \subseteq S$ is said to be a *left ideal (right ideal)* of S if

(i) $(I, +)$ is a subsemigroup of $(S, +)$ and

(ii) $s \cdot a \in I$ (resp., $a \cdot s \in I$) for all $s \in S$ and $a \in I$.

If I is a left as well as a right ideal of S then I is called an *ideal* of S.

Definition 1.3.8. [38] Let A be an ideal of an additively commutative semiring $(S, +, \cdot)$. Then

$$\overline{A} := \{\overline{a} \in S | \overline{a} + a \in A \text{ for some } a \in A\}$$

defines an ideal of $(S, +, \cdot)$ satisfying $A \subseteq \overline{A}$ and $\overline{\overline{A}} = \overline{A}$, called the *$k$-closure of* A. In particular, if $A = \overline{A}$ holds, then A is called a *k-ideal* of S. Therefore an ideal I of an additively commutative semiring S is a k-ideal if for $a \in I, x \in S$, $a + x \in I \Leftrightarrow x \in I$.

Definition 1.3.9. [10] A semiring S is called a *distributive lattice D of rings $R_\alpha(\alpha \in D)$* if S admits a congruence δ such that $D = S/\delta$ is a distributive lattice and each δ-class R_α is a subring of S. More generally a semiring S is called a *distributive lattice D of semirings S_α $(\alpha \in D)$* if S admits a congruence δ such that $D = S/\delta$ is a distributive lattice and each δ-class S_α is a subsemiring of S.

Definition 1.3.10. [29]Let D be a distributive lattice and $\{S_\alpha : \alpha \in D\}$ be a family of pair wise disjoined semirings which are indexed by the elements of D. For each $(\alpha \leq \beta)$ in D, we now define a semiring monomorphism $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ satisfying the following conditions: (1) $\phi_{\alpha,\alpha} = I_{S_\alpha}$, where I_{S_α} denotes the identity mapping on S_α, (2) $\phi_{\beta,\gamma}\phi_{\alpha,\beta}$ $= \phi_{\alpha,\gamma}$, if $\alpha \leq \beta \leq \gamma$, (3) $\phi_{\alpha,\gamma}(S_\alpha)\phi_{\beta,\gamma}(S_\beta) \subseteq \phi_{\alpha\beta,\gamma}(S_{\alpha\beta})$, if $\alpha + \beta \leq \gamma$.

On $S = \cup S_\alpha$ (the disjoint union of S_α's) we define $\oplus$ and $\odot$ as follows: (4) $a \oplus b = \phi_{\alpha,\alpha+\beta}(a) + \phi_{\beta,\alpha+\beta}(b)$ and (5) $a \odot b = c \in S_{\alpha\beta}$ such that $\phi_{\alpha\beta,\alpha+\beta}(c) = \phi_{\alpha,\alpha+\beta}(a) \cdot \phi_{\beta,\alpha+\beta}(b)$ where $a \in S_\alpha$, $b \in S_\beta$. We denote the above system by $\langle D, S_\alpha, \phi_{\alpha,\beta}\rangle$. This is a semiring and we call it a *strong distributive lattice D of semirings S_α, $\alpha \in D$.*

Example 1.3 [29] shows that a distributive lattice of rings may not be a strong distributive lattice of rings but a strong distributive lattice of rings is always a distributive lattice of rings.

Definition 1.3.11. [29] An additively commutative semiring $(S, +, \cdot)$ is said to be an *E-inversive semiring* if its additive reduct $(S, +)$ is an E-inversive semigroup (*cf.* Definition 1.1.16).

Definition 1.3.12. A semiring isomorphic with a subsemiring H of the direct product of two semirings S and T is called a *subdirect product* of S and T if the two projections $\pi_1 : H \to S$, $\pi_1(s, t) = s$ and $\pi_2 : H \to T$, $\pi_2(s, t) = t$ are surjective.

A subdirect product H of a semiring S and a ring R is called *full* if $(e, 0) \in H$ for every additive idempotent $e \in S$ and 0 is the identity of $(R, +)$.

Remark 1.3.13. A subdirect product of a distributive lattice and a ring is always a full subdirect product (*cf.* Lemma 2.2 [29]).

Theorem 1.3.14 (Theorem 2.3 [29])**.** *The following conditions on a semiring $(S, +, \cdot)$ are equivalent.*

(1) *S is a subdirect product of a distributive lattice and a ring.*

(2) *S is an E-inversive strong distributive lattice of halfrings.*

(3) *S is an E-inversive strong distributive lattice of hemirings, each of which contains a single additive idempotent.*

Theorem 1.3.15 (Theorem 2.10 [29])**.** *A semiring S is a subdirect product of a distributive lattice and a ring if and only if S is an E-inversive semiring satisfying the following properties:*

(*i*) $ef = fe$ *for all* $e, f \in E^+(S)$ *where* $E^+(S) = \{e \in S : e + e = e\}$,

(*ii*) $e^2 = e$ *for all* $e \in E^+(S)$,

(*iii*) $a + ae = a$ *for all* $a \in S$, *for all* $e \in E^+(S)$, (*iv*) *if* $a \in S$ *be such that* $a + b = b$ *for some* $b \in S$*, then* $a + a = a$*,*

(*v*) *if* $a, b \in S$ *be such that* $a^0 = b^0$ *and* $I(a) = I(b)$ *then* $a = b$ *where* $I(a) = \{x \in S : a + x \in E^+(S)\}$ *and* $a^0 = \{a(a + x) : x \in I(a)\}$*.*

1.4 Near-rings

We recall some preliminary notions of near-rings mainly from [91].

Definition 1.4.1. [91] An algebraic structure $(N, +, \cdot)$ is said to be a *near-ring* if it satisfies the following conditions :

(1) $(N, +)$ is a group (not necessarily abelian),

(2) $(N, \cdot)$ is a semigroup (not necessarily commutative) and

(3) for all $a, b, c \in N$, $(a + b) \cdot c = a \cdot c + b \cdot c$ ('right distributive law').

Since in (3), '$\cdot$' distributes over '+' from the right side, it can be called as a '*right near-ring*'. Similarly, if '$\cdot$' distributes over '+' from the left side, then we get a left near-ring. To be specific, an algebraic structure $(N, +, \cdot)$ is said to be a *left near-ring* if N satisfies (1) and (2) of Definition 1.4.1 and $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in N$.

Remark 1.4.2. Throughout the present thesis '*near-ring*' stands for '*right near-ring*'.

Example 1.4.3. [91] Let G be an additively written (but not necessarily abelian) group. Then the following sets of mappings are near-rings under point wise addition and composition of functions:

(i) $M(G) := \{f : G \to G\}$,

(ii) $M_0(G) := \{f : G \to G | f(0) = 0\}$,

(iii) $M_c(G) := \{f : G \to G | f \text{ is constant}\}$.

Definition 1.4.4. [91] A near-ring $(N, +, \cdot)$ is called a *zero-symmetric near-ring* if $a \cdot 0 = 0$ for all $a \in N$.

It can be noted that from the definition of near-ring $0 \cdot a = 0$ for all $a \in N$. In Example 1.4.3, $M_0(G)$ is a zero-symmetric near-ring but $M(G)$ is not a zero-symmetric near-ring.

Definition 1.4.5. [91] Let $(N, +, \cdot)$ be a near-ring. Then a subgroup M of $(N, +)$ is called a *subnear-ring* of N if $m_1 m_2 \in M$ for all $m_1, m_2 \in M$.

Definition 1.4.6. [91] Let $(N, +, \cdot)$ be a near-ring and $I \subseteq N$. Then I is called an *ideal* of N if

(*i*) $(I, +)$ is a normal subgroup of $(N, +)$,

(*ii*) $IN \subseteq I$, *i.e., in* $\in I$ for all $i \in I$ and for all $n \in N$,

(*iii*) for all $n, n' \in N$ and for all $i \in I$, $n(n' + i) - nn' \in I$.

Normal subgroups R of $(N, +)$ satisfying condition (*ii*) are called *right ideals* of N while normal subgroups L of $(N, +)$ satisfying condition (*iii*) are called *left ideals*.

Definition 1.4.7. [91] In a near-ring $(N, +, \cdot)$ an element d is said to be a *distributive element* if $d \cdot (a + b) = d \cdot a + d \cdot b$ for all $a, b \in N$.

Let $(N, +, \cdot)$ be a near-ring and $N_d := \{d \in N | d \text{ is distributive}\}$. Then N is called a *distributively generated near-ring* if N_d generates the group $(N, +)$.

Example 1.4.8. [91] Let $(G, +)$ be a group (not necessarily abelian). Consider the set of all endomorphisms on G, denoted by $End(G)$. Then

$$\langle End(G) \rangle := \{\sum_{i=1}^{n} \sigma_i e_i | n \in \mathbb{N}, \sigma_i \in \{-1, 1\}, e_i \in End(G)\}$$

is a subnear-ring of $M(G)$, distributively generated by $(End(G), \cdot)$ and called the *endomorphism near-ring on G*. It can be verified that if G is not an abelian group then $\langle End(G) \rangle$ is not a ring.

Remark 1.4.9. Let $(N, +, \cdot)$ be a distributively generated near-ring. Then

(*i*) N is a zero-symmetric near-ring and

(*ii*) N is a ring if and only if $(N, +)$ is abelian.

1.5 Seminearrings

Various versions of the definitions of seminearrings are prevalent in the literature. We present below three versions, one from Hoorn et al. [44], one from Krishna [58] and one from Weinert [108].

Definition 1.5.1. An algebraic structure $(S, +, \cdot)$ is said to be a *(right distributive) seminearring* if it satisfies the following axioms:

(1) $(S, +)$ is a semigroup (not necessarily commutative),

(2) $(S, \cdot)$ is a semigroup (not necessarily commutative),

(3) $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in S$ ("right distributive law").

Remark 1.5.2. If we replace (3) by (3′), where

$$(3') : \ a \cdot (b + c) = a \cdot b + a \cdot c \text{ for all } a, b, c \in S,$$

then one gets *left distributive seminearring.* In [108], Weinert mention this definition. The theory of left distributive seminearrings runs parallel to that of right distributive seminearrings.

Notation 1.5.3. Throughout our work, unless mentioned otherwise, the term '*seminearring*' will stand for *'(right distributive) seminearring'*. In a seminearring $(S, +, \cdot)$, for $a, b \in S$, 'ab' will stand for '$a \cdot b$'.

A seminearring $(S, +, \cdot)$ is a near-ring if $(S, +)$ is a group. Again a seminearring is semiring if it satisfies (3′) as well. Therefore seminearrings generalize semirings as well as near-rings.

Definition 1.5.4. A seminearring $(S, +, \cdot)$ is said to be *with zero (0)* if $(S, +)$ is a monoid with identity element 0 and $0s = 0$ for all $s \in S$.

A seminearring S with zero is said to be *zero-symmetric* if $s0 = 0$ for all $s \in S$.

A subset M of a seminearring $(S, +, \cdot)$ is called a *subseminearring* of S if $(M, +)$ is a subsemigroup of $(S, +)$ and $m_1 m_2 \in M$ for all $m_1, m_2 \in M$.

Remark 1.5.5. The notion of seminearring was first introduced by Hoorn et al. in [44]. In [44], the authors mainly considered zero-symmetric left distributive seminearring. In [58], the author considered right distributive seminearring with zero and called it *near-semiring.*

Example 1.5.6. [58, 91] Let S be an additively written (but not necessarily commutative) semigroup. Then the following sets of mappings are seminearrings under point wise addition '+' and composition '$\circ$' of functions:

(i) $M(S) := \{f : S \to S\}$,

(ii) $M_c(S) := \{f : S \to S | f \text{ is constant}\}$,

(iii) $M_0(S) := \{f : S \to S | f(0) = 0\}$ where $(S, +)$ is a monoid with identity element 0.

If $(S, +)$ is a monoid, $(M_0(S), +, \circ)$ is a zero-symmetric seminearring but $(M(S), +, \circ)$ is a seminearring with zero but not a zero-symmetric seminearring. If $(S, +)$ is a semigroup without identity element, $(M(S), +, \circ)$ is a seminearring without zero.

Definition 1.5.7. [76, 108] In a seminearring $(S, +, \cdot)$, an element d is said to be a *distributive element* if $d(a + b) = da + db$ for all $a, b \in S$.

A seminearring $(S, +, \cdot)$ is said to be a *distributively generated seminearring* if S contains a multiplicative subsemigroup $(D, \cdot)$ of distributive elements which generates $(S, +)$.

Remark 1.5.8. In a distributively generated seminearring $(S, +, \cdot)$ any element can be expressed as a finite sum of distributive elements of S, *i.e.,* for each $a \in S$, $a = \sum_{i=1}^{n} d_i$ where d_i is a distributive element for $1 \leq i \leq n$.

Example 1.5.9. [96] For a semigroup $(S, +)$, let $End(S)$ denote the set of all endomorphisms of S. Since $(S, +)$ is not commutative, $End(S)$ need not be closed with respect to the point wise addition of functions. But $End(S)$ forms a semigroup with respect to the composition of functions. Then

$$\langle End(S) \rangle := \{\sum_{i=1}^{n} f_i | i \in \mathbb{N}, f_i \in End(S)\}$$

is a subseminearring of $M(S)$, the seminearring of all self maps of S. Clearly each element of $End(S)$ is a distributive element in $M(S)$ and so from the construction it is clear that $\langle End(S) \rangle$ is distributively generated by $End(S)$. So $\langle End(S) \rangle$ forms a distributively generated seminearring with respect to point wise addition and composition of functions.

Remark 1.5.10. In view of (ii) of Remark 1.4.9, a distributively generated seminearring $(S, +, \cdot)$ is a semiring if and only if S is an additively commutative seminearring (*i.e.,* $(S, +)$ is commutative).

Definition 1.5.11. [2] A subset I of a seminearring $(S, +, \cdot)$ is said to be a *right (left)* $\mathcal{S}$*-ideal* if

(i) for all $x, y \in I$, $x + y \in I$,

(ii) for all $x \in I$ and for all $s \in S$, $xs \in I$ (resp., $sx \in I$).

I is said to be an *$\mathcal{S}$-ideal* if it is both a left and a right $\mathcal{S}$-ideal.

Definition 1.5.12. In a seminearring $(S, +, \cdot)$, a *right (left) k-ideal* I is a right (left) $\mathcal{S}$-ideal of S such that whenever $a + b \in I$, $a, b \in S$, then $a \in I$ if and only if $b \in I$. I is said to be a *k-ideal* if it is both right and left k-ideal.

Definition 1.5.13. In a seminearring $(S, +, \cdot)$, an element x is said to be an *additive idempotent* if $x + x = x$ and a *multiplicative idempotent* if $x^2 = x$.

Notation 1.5.14. The set of all additive idempotents in a seminearring S is denoted by $E^+(S)$ and the set of all multiplicative idempotents in S is denoted by $E^\times(S)$.

Definition 1.5.15. [42] A seminearring $(S, +, \cdot)$ is called

(i) an *additively regular seminearring* if $(S, +)$ is a regular semigroup,

(ii) an *additively inverse seminearring* if $(S, +)$ is an inverse semigroup,

(iii) a *multiplicatively regular seminearring* if $(S, \cdot)$ is a regular semigroup.

Proposition 1.5.16. *[98] Let $(S, +)$ be a semigroup. Then*

(i) $M(S)$ is an additively regular seminearring if and only if $(S, +)$ is a regular semigroup.

(ii) $M(S)$ is an additively inverse seminearring if and only if $(S, +)$ is an inverse semigroup.

(iii) $(M(S), +, \circ)$ is always a multiplicatively regular seminearring.

Definition 1.5.17. [98] An additively inverse seminearring $(S, +, \cdot)$ is said to satisfy *property D* if $a(b + b^*) = (a + a^*)b$ for all $a, b \in S$ where a^*, b^* denote the unique additive inverse of a, b, respectively.

Definition 1.5.18. [98] Let $(S, +, \cdot)$ be a seminearring. An equivalence relation ρ is said to be a *right congruence (left congruence)* on S if it satisfies the following conditions:

(i) $a\rho b \Rightarrow (a + c)\rho(b + c)$, $(c + a)\rho(c + b)$ for all $c \in S$,

(ii) $a\rho b \Rightarrow (ac)\rho(bc)$ (respectively, $(ca)\rho(cb)$) for all $c \in S$.

An equivalence relation ρ is said to be a *congruence* on S if it is both right and left congruence on S.

Remark 1.5.19. The right (left) attribute in the above definition corresponds to right (left) compatibility with respect to multiplication.

Definition 1.5.20. [98] Let ρ be a congruence on a seminearring $(S, +, \cdot)$. Then

$$S/\rho := \{[s] : [s] \text{ is the congruence class of } s \in S \text{under } \rho\}$$

forms a seminearring with respect to '+', '·', defined by

$$[x] + [y] = [x + y] \text{ and } [x] \cdot [y] = [xy] \text{ for all } x, y \in S.$$

Definition 1.5.21. A congruence ρ on a seminearring $(S, +, \cdot)$ is said to be a *near-ring congruence* on S if the seminearring S/ρ becomes a near-ring.
A congruence ρ on a seminearring $(S, +, \cdot)$ is said to be a *zero-symmetric near-ring congruence* on S if the seminearring S/ρ becomes a zero-symmetric near-ring.

Remark 1.5.22. A congruence ρ on a seminearring $(S, +, \cdot)$ is a near-ring congruence on S if and only if ρ is a group congruence on $(S, +)$.

Definition 1.5.23. [83] A subseminearring N of a seminearring $(S, +, \cdot)$ is said to be a *normal subseminearring* if

(i) for $a, b, c, d \in S$, $a + b + c + d \in N$ implies $a + c + b + d \in N$,

(ii) N is a k-ideal,

(iii) for each $s \in S$ there exists $x \in S$ such that $s + x \in N$.

Theorem 1.5.24 (Theorem 3.6 [83]). *Let S be a zero-symmetric and distributively generated seminearring. Then there exists an inclusion preserving bijection between the set of all additively commutative near-ring congruences on S and the set of all normal subseminearrings of S via the map $\phi : \rho \mapsto N_\rho$, where $N_\rho = \{a \in S : (a, 0) \in \rho\}$.*

Definition 1.5.25. [44, 58, 95] Let S and S' be two seminearrings. Then a mapping f from S to S' is said to be a *homomorphism of seminearrings* or a *seminearring homomorphism* if it satisfies the following properties

(i) $f(x + y) = f(x) + f(y)$ and

(ii) $f(xy) = f(x)f(y)$

for all $x, y \in S$.

A seminearring homomorphism g of S is said to be an *isomorphism of seminearrings* or a *seminearring isomorphism* if g is both surjective and injective.

Remark 1.5.26. If S is a seminearring with zero and f is a seminearring homomorphism, then $f(S)$ is also a seminearring with $f(0)$ as zero. If S is a zero-symmetric seminearring then $f(S)$ is also a zero-symmetric seminearring. If S, S' both contain zero, then $f(0)$ may be different from the zero of S'.

Definition 1.5.27. [44] Let S, S' be two seminearrings with zero and $f : S \to S'$ be a seminearring homomorphism. Then *Kernel* of f is the set $\{s \in S : f(s) = f(0)\}$ and is denoted by $ker f$.

In a seminearring with zero, $I(\subseteq S)$ is said to be an *ideal* of S if I is a kernel of some seminearring homomorphism of S.

Definition 1.5.28. [44] Let S be a seminearring and $D \subseteq S$. Then $SD = \{sd \in S : s \in S \text{ and } d \in D\}$ and $DS = \{ds \in S : s \in S \text{ and } d \in D\}$. A subset D of S is said to be *left (right) invariant* in the seminearring S if SD $(DS) \subseteq D$.

CHAPTER 2

NEAR-RING CONGRUENCES ON ADDITIVELY REGULAR SEMINEARRINGS

2

Near-ring Congruences on Additively Regular Seminearrings

A natural example of a seminearring is the set of all self maps of any additive semigroup under point wise addition and composition *i.e.,* $(M(S), +, \circ)$ where $(S, +)$ is a semigroup and $M(S)$ is the set of all self maps on S. It is observed that for any additive semigroup $(S, +)$, $(M(S), +, \circ)$ is always a multiplicatively regular semigroup (*cf.* Proposition 1.5.16). But in view of Example 2.3 of [98], it is evident that $(M(S), +, \circ)$ is not always an additively regular seminearring. In fact, $(M(S), +, \circ)$ is an additively regular seminearring if and only if $(S, +)$ is a regular semigroup (*cf.* Proposition 1.5.16). This motivates us to investigate as to how the theory of semigroups, more precisely regular semigroups, can be made into work in the study of seminearrings and additively regular seminearrings. Group congruences on regular semigroups play an important role in the study of regular semigroups. In [67] LaTorre studied group congruences, least group congruence on regular semigroups and established inclusion preserving bijection between the set of all group congruences and the set of all full, self-conjugate and closed subsemigroups (*cf.* Theorem 1.1.22). It leads us to study near-ring congruences on additively regular seminearrings. We deduce that for a distributively generated additively regular seminearring S, there exists an inclusion preserving bijection between the set of all near-ring congruences on S and the set of all normal full k-ideals of S. This study not only gives rise to refinements of some important results *viz.* Propositions 3.16, 3.17, Theorems 3.20 of [99] and Theorem 3.22 of [98] (involving mainly near-ring congruences) but also answers partially a question raised in [98]. In [98, 99],

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Kamalika Chakraborty et al., ***Near-ring congruences on additively regular seminearrings, Semigroup Forum*****, 101 (2020) 285-302.**

while studying congruences on additively inverse seminearrings, Sardar and Mukherjee obtained for a distributively generated additively inverse seminearring S with property D (*cf.* Definition 1.5.17) (i) an inclusion preserving bijective correspondence (Theorem 3.20 of [98, 99]) between the set of all near-ring congruences *i.e.,* normal congruences[1] on S and the set of all normal full k-ideals of S and (ii) the least near-ring congruence (Theorem 3.22 of [98]) on S. In the Concluding remark of [98], the authors raised a question as to whether Theorem 3.20 [98, 99] and Theorem 3.22 [98] can be obtained by removing the restriction(s) *viz.,* (a) 'property D' or (and) (b) 'distributively generated'. Some of our results *viz.,* Corollary 2.2.12 and Theorem 2.2.17 establish that Theorem 3.20 [98, 99] and Theorem 3.22 [98] can be obtained by removing the restriction of 'property D'. To conclude the chapter, we study the lattice structures of near-ring congruences and normal full k-ideals in a distributively generated additively regular seminearrings.

In **Section 1**, we define 'normal full k-ideal' (*cf.* Definition 2.1.6) in an additively regular seminearring and study some of its properties (*cf.* Proposition 2.1.11). Then we define 'normal congruence' (*cf.* Definition 2.1.13) in an additively regular seminearring and find its connection with near-ring congruences (*cf.* Theorem 2.1.16).

In **Section 2**, we first obtain the right sided analogue (*cf.* Theorem 2.2.1) and then the two sided analogue (*cf.* Theorem 2.2.2) of Theorem 1.1.21 in the setting of additively regular seminearrings. Then we show that for a distributively generated additively regular seminearring S, a normal full k-ideal of S corresponds to a near-ring congruence on S (*cf.* Theorem 2.2.3) and conversely a near-ring congruence on S corresponds to a normal full k-ideal of S (*cf.* Theorem 2.2.7). Combining these two results we obtain in Theorem 2.2.10 an inclusion preserving bijective correspondence between the set of all normal full k-ideals and the set of all near-ring congruences in a distributively generated additively regular seminearring. To conclude this section, we study least near-ring congruences on distributively generated additively regular and additively inverse seminearrings (*cf.* Theorems 2.2.15 and 2.2.17, Corollary 2.2.16).

In **Section 3**, we check that whether this correspondence (stated in Theorem 2.2.10) turns out to be a lattice isomorphism or not. In Theorem 2.3.15, we establish that in a distributively generated additively regular seminearring, the set of all near-ring congruences and the set of all normal full k-ideals are lattice isomorphic. In Theorems 2.3.17 and 2.3.18, we show that both the lattices mentioned in Theorem

[1] In an additively regular seminearring, the notions of near-ring congruences and normal congruences coincide (*cf.* Theorem 2.1.16) and they have been used here interchangeably.

2.3.15 are modular. In the study of distributivity of these lattices, we find that in a distributively generated additively regular seminearring, the lattice of normal full k-ideals may not be a distributive lattice (*cf.* Example 2.3.19) but imposition of some conditions ensures the distributivity (*cf.* Theorem 2.3.22). To conclude this section, in Theorems 2.3.23 and 2.3.24, we show that in a distributively generated additively regular seminearring, these latices are complete.

2.1 Normal full k-ideals of additively regular seminearrings

We start this section with examples of distributively generated additively regular and additively inverse seminearrings (*cf.* Definition 1.5.15).

Example 2.1.1. Let $(S,+)$ be a band. Let $End(S)$ denote the set of all endomorphisms of S. Consider the seminearring $\langle End(S)\rangle$ generated by $End(S)$. Let $f \in \langle End(S)\rangle$. Since $(S,+)$ is a band, $f(m)$ is an idempotent of S for all $m \in S$. Therefore $(f + f + f)(m) = f(m) + f(m) + f(m) = f(m)$ for all $m \in S$ whence $f + f + f = f$. Hence $\langle End(S)\rangle$ is an additively regular seminearring. Now in view of Example 1.5.9, $\langle End(S)\rangle$ is a distributively generated additively regular seminearring.

Example 2.1.2. Let $(S,+)$ be an inverse semigroup. Let $T := \{f \in End(S) : f^* \in End(S)\}$ where for all $s \in S$, $f^*(s) := (f(s))^*$ and a^* denotes the unique inverse of $a \in S$. Clearly if $f \in T$ then $f^* \in T$ since $(f^*)^* = f$. Let $h_1, h_2 \in T$. Now for $m_1, m_2 \in S$,

$$\begin{aligned}
(h_1 \circ h_2)^*(m_1 + m_2) &= ((h_1 \circ h_2)(m_1 + m_2))^* \\
&= (h_1(h_2(m_1) + h_2(m_2)))^* \text{ (since } h_2 \text{ is an endomorphism of } S) \\
&= (h_1)^*(h_2(m_1) + h_2(m_2)) \\
&= h_1^*(h_2(m_1)) + h_1^*(h_2(m_2)) \text{ (since } h_1^* \in End(S) \text{ as } h_1 \in T) \\
&= (h_1(h_2(m_1)))^* + (h_1(h_2(m_2)))^* \\
&= (h_1 \circ h_2)^*(m_1) + (h_1 \circ h_2)^*(m_2)
\end{aligned}$$

Therefore T is closed under composition. Hence $(T, \circ)$ is a semigroup. Now consider the seminearring $\langle T\rangle$ generated by T. Let $f = \sum_{i=1}^{n} f_i$ where $f_i \in T$ for each i, $1 \le i \le n$.

Now let $g = \sum_{i=n}^{1}(f_i)^*$. Then $g \in \langle T \rangle$ as $(f_i)^* \in T$ for each i, $1 \leq i \leq n$. Now

$$\begin{aligned}
(f+g+f)(m) &= (\sum_{i=1}^{n} f_i)(m) + (\sum_{i=n}^{1}(f_i)^*)(m) + (\sum_{i=1}^{n} f_i)(m) \\
&= \sum_{i=1}^{n} f_i(m) + \sum_{i=n}^{1}(f_i)^*(m) + \sum_{i=1}^{n} f_i(m) \\
&= \sum_{i=1}^{n} f_i(m) + \sum_{i=n}^{1}(f_i(m))^* + \sum_{i=1}^{n} f_i(m) \\
&= \sum_{i=1}^{n} f_i(m) \\
&= f(m) \text{ for all } m \in S.
\end{aligned}$$

Again $(g+f+g)(m) = g(m)$ for all $m \in S$. Therefore for each $f \in \langle T \rangle$, there exists $g \in \langle T \rangle$ such that $f+g+f = f$ and $g = g+f+g$. Hence $\langle T \rangle$ is an additively regular seminearring. If $f, g \in E^+(\langle T \rangle)$, then it can be easily verified that $f+g = g+f$. Then in view of Theorem 1.1.18, $(\langle T \rangle, +)$ is an inverse semigroup. Therefore $\langle T \rangle$ is an additively inverse seminearring (*cf.* Definition 1.5.15). Now let $f \in T$ and $g, h \in \langle T \rangle$. Then $f(g+h)(m) = f(g(m)+h(m)) = f(g(m))+f(h(m))$ for all $m \in S$ since $f \in T$ and $T \subseteq End(S)$. Therefore f is a distributive element of the seminearring $\langle T \rangle$. Then in view Definition 1.5.7, the seminearring $\langle T \rangle$ generated by T is a distributively generated additively inverse seminearring which is not a semiring.

Remark 2.1.3. S can be so chosen that T becomes non-empty, *e.g.,* we can take S to be the direct product of a semilattice and $D_{2n}(n \geq 3)$.

Notation 2.1.4. Let $(S, +, \cdot)$ be an additively regular seminearring. Throughout this thesis for each element $a \in S$, $V^+(a)$ always stands for the set of all additive inverses of a *i.e.,* the set $\{x \in S : a = a+x+a, x = x+a+x\}$. For $a \in S$, if $a+x+a = a$ for some $x \in S$, then $x+a+x \in V^+(a)$. Therefore for an additively regular seminearring $V^+(a)$ is always non-empty for each $a \in S$. In additively inverse seminearrings, for each $a \in S$, $V^+(a)$ is singleton and a^* denotes the unique additive inverse of a.

Definitions 2.1.5. In a seminearring $(S, +, \cdot)$,

(*i*) [98] a (left, right) $\mathcal{S}$-ideal I of S is said to be a *full (left, right) ideal* if $(I, +)$ is a full subsemigroup of $(S, +)$, *i.e.,* $E^+(S) \subseteq I$,

(*ii*) a (left, right) $\mathcal{S}$-ideal I of S is said to be a *reflexive (left, right) ideal* if $(I, +)$ is a reflexive subsemigroup of $(S, +)$, *i.e.,* if $a+b \in I$ then $b+a \in I$,

(iii) if S is an additively regular seminearring then a *normal (left, right) ideal* I is defined to be a (left, right) $\mathcal{S}$-ideal such that $(I,+)$ is a self-conjugate subsemigroup (*cf.* Definition 1.1.20) of $(S,+)$, *i.e.,* $x+i+x^{'} \in I$ for all $i \in I$, for all $x \in S$ and for all $x^{'} \in V^{+}(x)$.

Definitions 2.1.6. In a seminearring $(S,+,\cdot)$,

(i) a (left,right) $\mathcal{S}$-ideal I is said to be a *full (left, right) k-ideal* [98] if I is a full (left, right) ideal as well as a (left, right) k-ideal,

(ii) a *reflexive full (left, right) k-ideal* is defined to be a full (left, right) k-ideal which is also a reflexive (left, right) ideal,

(iii) if S is an additively regular seminearring then a *normal full (left, right) ideal* is defined to be a full (left, right) ideal which is also a normal (left, right) ideal and a *normal full (left, right) k-ideal* is defined to be a full (left, right) k-ideal which is also a normal (left, right) ideal.

Remark 2.1.7. Our definition of normal full k-ideal in additively regular seminearring coincides with that given in [99] for additively inverse seminearring.

Remark 2.1.8. For a (left, right) k-ideal T of a seminearring $(S,+,\cdot)$, $(T,+)$ is always a closed subsemigroup of $(S,+)$. As regards the converse we have 'for a normal full (left, right) ideal T of an additively regular seminearring $(S,+,\cdot)$, $(T,+)$ is a closed subsemigroup of $(S,+)$ (*cf.* Definition 1.1.2) if and only if T is a (left, right) k-ideal'.

The following result shows that normal full ideals exist in any distributively generated additively inverse seminearring.

Proposition 2.1.9. *Let $(S,+,\cdot)$ be a distributively generated additively inverse seminearring. Then $E^{+}(S)$ is a normal full ideal of S.*

Proof. In view of Proposition 3.2 [98], $E^{+}(S)$ is a right $\mathcal{S}$-ideal. Let $e \in E^{+}(S)$ and $s \in S$. Now $s = \sum_{i=1}^{n} t_i$ where each t_i is a distributive element of S. Clearly $t_i e \in E^{+}(S)$ for all t_i where $1 \leq i \leq n$. Therefore se $(= (\sum_{i=1}^{n} t_i)e = \sum_{i=1}^{n}(t_i e)) \in E^{+}(S)$ whence $E^{+}(S)$ is a left $\mathcal{S}$-ideal. Since $x+e+x^{*} \in E^{+}(S)$ for all $x \in S$ and for all $e \in E^{+}(S)$, $E^{+}(S)$ is a normal full ideal of S. □

Remark 2.1.10. (i) In an additively regular seminearring $(S,+,\cdot)$, $E^{+}(S)$ is not closed with respect to addition and hence not an $\mathcal{S}$-ideal though $E^{+}(S)$ is right absorbing (*i.e.,* $es \in E^+(S)$ for all $e \in E^+(S)$ and $s \in S$) and left absorbing for distributive element (*i.e.,* if t is a distributive element of S, $te \in E^+(S)$).

(*ii*) Proposition 2.1.9 is actually a refinement of Proposition 3.2 [98] in the setting of distributively generated additively inverse seminearrings.

In the following result we obtain some properties of normal full k-ideals in additively regular seminearring for its use in the sequel.

Proposition 2.1.11. *Let S be an additively regular seminearring and H be a full (left, right) k-ideal of S. Then the following conditions are equivalent.*

(*i*) *H is a normal full (left,right) ideal.*

(*ii*) *H is a reflexive ideal ($(H,+)$ is reflexive i.e., $a+b \in H$ implies $b+a \in H$).*

(*iii*) *$a+h+b \in H$ for all $a+b$, $h \in H$.*

Proof. $(i) \Rightarrow (ii)$: Let $a+b \in H$. Then $b+(a+b)+b^{'} \in H$ for all $b^{'} \in V^+(b)$. As H is a full ideal, $b+b^{'} \in H$ (*cf.* Definitions 2.1.5 (i)). Hence H being a k-ideal, $b+a \in H$ (*cf.* Definition 1.5.12).

$(ii) \Rightarrow (iii)$: Let $a+b$, $h \in H$. Then $(b+a)+h \in H$. Again using (ii) we obtain $a+h+b \in H$.

$(iii) \Rightarrow (i)$: Let $a \in S$ and $h \in H$. Then $a+a^{'} \in H$ for all $a^{'} \in V^+(a)$ as $a+a^{'}$ is an additive idempotent and H is a full ideal. So by (iii), $a+h+a^{'} \in H$. So (i) follows. □

The following example shows that Proposition 2.1.11 may not hold in absence of k-property of ideals.

Example 2.1.12. Let $S = \{f_i : 1 \leq i \leq 6\}$ where $f_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $f_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $f_4 = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$, $f_5 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $f_6 = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$ are partial maps on the set $\{1,2,3\}$. Let $(f_i+f_j)(s) = f_i(f_j(s))$ for all $s \in f_j^{-1}[Domf_i \cap Imf_j]$ and $f_if_j = f_3$ where $1 \leq i,j \leq 6$. Then $(S,+,\cdot)$ is an additively regular semiring, in fact, an additively inverse semiring [109]. Let $(D,+,\cdot)$ be a distributively generated near-ring. Then $S_1 = S \times D$ is a distributively generated additively regular, in fact, an additively inverse seminearring with pointwise addition and multiplication. Now $E^+(S_1) = \{(f_i,0) : 1 \leq i \leq 4\}$ is a normal full ideal (*cf.* Proposition 2.1.9) but not a k-ideal as $(f_2, 0) + (f_5, 0)$ $(= (f_1, 0)) \in E^+(S_1)$ and $(f_2, 0) \in E^+(S_1)$ but $(f_5, 0) \notin E^+(S_1)$. Now $(f_2, 0)+(f_5, 0) \in E^+(S_1)$ but $(f_5, 0)+(f_2, 0)$ $(= (f_5, 0)) \notin E^+(S_1)$ whence $(E^+(S_1), +)$ is not reflexive.

Definition 2.1.13. Let $(S, +, \cdot)$ be an additively regular seminearring. A congruence (right congruence, left congruence) ρ on S is said to be a *normal congruence (right normal congruence, left normal congruence)* on S if any two additive idempotents are ρ-related *i.e.,* $e\rho f$ for all $e, f \in E^+(S)$.

Remark 2.1.14. The normal congruence defined above is the same as that defined in Definition 3.5 [98] for additively inverse seminearring. In an additively inverse seminearring S, the condition that two additive idempotents are ρ-related for a congruence ρ on S is equivalent to two more conditions *viz.,* (i) $a\ \rho\ (a+e)$ and (ii) $a\ \rho\ (e+a)$ for all $e \in E^+(S)$ and for all $a \in S$ (Lemma 3.4 [98]). But in an additively regular seminearring these three conditions may not be equivalent which is illustrated in Example 2.1.15. But in an additively regular seminearring S, if a congruence ρ on S satisfies (i) and (ii) simultaneously then two additive idempotents are ρ-related and hence by Definition 2.1.13, ρ becomes a normal congruence on S.

The following example illustrates that for an additively regular seminearring Lemma 3.4 [98] may not be true, to be specific defining condition of Definition 2.1.13 and (i) and (ii) of the above remark are not equivalent.

Example 2.1.15. Let $(S, +)$ be a left-zero semigroup *i.e.,* $a+b = a$ for all $a, b \in S$ with more than one element and $M(S)$ denote the set of all self-maps of $(S, +)$. Then $M(S)$ forms an additively regular seminearring with point wise addition '+' and composition of maps. If we consider the identity relation $I_{M(S)}$ on $M(S)$ then $f\ I_{M(S)}\ (f+e)$ for all $f \in M(S)$ and for all $e \in E^+(M(S))$. Since S has more than one element and $E^+(M(S)) = M(S)$, any two additive idempotents are not $I_{M(S)}$-related.

Theorem 2.1.16. *Let $(S, +, \cdot)$ be an additively regular seminearring. Then ρ is a near-ring congruence on S if and only if ρ is a normal congruence on S.*

Proof. We omit the proof as it is same as the proof of Theorem 3.8 [98]. □

2.2 Near-ring congruences on seminearrings

We obtain below firstly the right sided analogue (*cf.* Theorem 2.2.1) and then the two sided analogue (*cf.* Theorem 2.2.2) of Theorem 1.1.21 (*i.e.,* the first part[2] of Theorem 1 of [67]) in the setting of additively regular seminearrings.

Theorem 2.2.1. *Let H be a normal full right ideal of an additively regular seminearring $(S, +, \cdot)$. Then the relation*

$$\beta_H = \{(a, b) \in S \times S : x + a = b + y \text{ for some } x, y \in H\}$$

is a right normal congruence on S. Moreover H is contained in $\{a \in S : a\beta_H e$ for some $e \in E^+(S)\} = Hw$ and Hw is a normal full right k-ideal of S [3].

Proof. Since H is a normal full right ideal of S, $(H, +)$ is a full and self-conjugate subsemigroup of $(S, +)$ (*cf.* Definitions 2.1.5). In view of Theorem 1 of [67], β_H is a group congruence on $(S, +)$. Then $e\beta_H f$ for all $e, f \in E^+(S)$. Let $a\beta_H b$ and $s \in S$. Then there exist $x, y \in H$ such that $x + a = b + y$. Therefore $(x + a)s = (b + y)s$ *i.e.,* $xs + as = bs + ys$ where $xs, ys \in H$ as H is a right $\mathcal{S}$-ideal. Then $as\ \beta_H\ bs$. Hence β_H is a right normal congruence on S (*cf.* Definition 2.1.13).

In view of 1st paragraph of the discussion below Theorem 2 of [67], $Hw = \{a \in S : a\beta_H e$ for some $e \in E^+(S)\}$ and it contains H (*cf.* Remark 1.1.3). By definition of β_H, any two elements of H are β_H related. Again H is a full ideal. Therefore $Hw = [e]_{\beta_H}$ for any $e \in E^+(S) = [h]_{\beta_H}$ for any $h \in H$. Since H is a normal full right ideal and β_H is a congruence on S, we deduce that Hw is a normal full right ideal. Then in view of Remark 1.1.3, $(Hw, +)$ is a closed subsemigroup of $(S, +)$. Therefore Hw is a normal full right k-ideal (*cf.* Remark 2.1.8). □

Theorem 2.2.2. *Let H be a normal full ideal of an additively regular distributively generated seminearring $(S, +, \cdot)$. Then the relation*

$$\beta_H = \{(a, b) \in S \times S : x + a = b + y \text{ for some } x, y \in H\}$$

is a normal congruence on S and hence a near-ring congruence (cf. Theorem 2.1.16). Moreover, Hw=$\{a \in S : a\beta_H e$ for some $e \in E^+(S)\}$ and Hw is a normal full k-ideal of S.

[2] The analogue of the second part is proved in Theorem 2.2.15.

[3] For definition of Hw we refer to Definitions 1.1.2.

Proof. In view of Theorem 2.2.1, it remains to prove that $\beta_{_H}$ is a left congruence and Hw is a left $\mathcal{S}$-ideal. Let $a\beta_{_H}b$ and $s \in S$. Then there exist $x, y \in H$ such that $x + a = b + y$. Now $s = \sum_{i=1}^{n} t_i$ where each t_i is a distributive element of S. $t_i(x + a) = t_i(b + y)$ *i.e.*, $t_ix + t_ia = t_ib + t_iy$ where $t_ix, t_iy \in H$ for all i, $1 \leq i \leq n$ as H is an $\mathcal{S}$-ideal. Therefore $t_ia\ \beta_{_H}\ t_ib$ and $\sum_{i=1}^{n}(t_ia)\ \beta_{_H}\ \sum_{i=1}^{n}(t_ib)$ *i.e.*, $sa\ \beta_{_H}\ sb$. This shows that $\beta_{_H}$ is a normal congruence and hence a near-ring congruence (*cf.* Theorem 2.1.16).

In view of Theorem 2.2.1, Hw=$\{a \in S : a\beta_{_H}e$ for some $e \in E^+(S)\} = [h]_{\beta_{_H}}$ for any $h \in H$ and Hw is a normal full right k-ideal of S. Since $\beta_{_H}$ is a congruence on S and H is an $\mathcal{S}$-ideal, we deduce that Hw is a left $\mathcal{S}$-ideal. Hence Hw is a normal full k-ideal of S. □

The above result takes the following form if we consider k-ideal instead of $\mathcal{S}$-ideal.

Theorem 2.2.3. *Let $(S, +, \cdot)$ be a distributively generated additively regular seminearring. Let H be a normal full k-ideal of S. Then the relation $\sigma_{_H}$, defined by*

$$a\sigma_{_H}b \text{ if and only if } a + b^{'} \in H \text{ for some } b^{'} \in V^+(b)$$

is a normal congruence and hence a near-ring congruence on S. Moreover, $H = \{a \in S : a\sigma_{_H}e$ for some $e \in E^+(S)\}$.

Proof. Since H is a normal full k-ideal of S, $(H, +)$ is a full, self-conjugate and closed subsemigroup of $(S, +)$ (*cf.* Definitions 2.1.5 and Remark 2.1.8). So, $Hw = H$ (*cf.* Definition 1.1.2). Then by Theorem 2 [67], $\sigma_{_H} = \beta_{_H}$. Therefore in view of Theorem 2.2.2, $\sigma_{_H}$ is a near-ring congruence on S.

Since $\sigma_{_H} = \beta_{_H}$, $\{a \in S : a\sigma_{_H}e$ for some $e \in E^+(S)\} = Hw$ (*cf.* Theorem 2.2.2). This together with the fact that $Hw = H$ implies $H = \{a \in S : a\sigma_{_H}e$ for some $e \in E^+(S)\}$. □

As a simple consequence of the above result we obtain the following result which can be considered to be a refinement of Proposition 3.16 of [99].

Corollary 2.2.4. *Let $(S, +, \cdot)$ be a distributively generated additively inverse seminearring. Let H be a normal full k-ideal of S. Then the relation $\sigma_{_H}$, defined by*

$$a\sigma_{_H}b \text{ if and only if } a + b^* \in H$$

is a near-ring congruence on S. Moreover, $H = \{a \in S : a\sigma_{_H}e$ for some $e \in E^+(S)\}$.

Remark 2.2.5. The relation σ of Proposition 3.16 [99] is denoted by $\sigma_{_H}$ in Corollary 2.2.4.

The following example shows that Theorem 2.2.3 may not be true in absence of k-property of an $\mathcal{S}$-ideal.

Example 2.2.6. In Example 2.1.12, $E^+(S_1)$ is a normal full ideal but not a normal full k-ideal. Let $H = E^+(S_1)$. Now $(f_4, 0) + (f_1, 0)$ $(= (f_1, 0)) \in E^+(S_1)$ and $(f_1, 0)^* = (f_1, 0)$. Therefore in view of Definition of $\sigma_{_H}$, $(f_4, 0)\ \sigma_{_H}\ (f_1, 0)$. Now $(f_1, 0) + (f_5, 0)$ $(= (f_1, 0)) \in E^+(S_1)$ and $(f_6, 0)^* = (f_5, 0)$. In view of Definition of $\sigma_{_H}$, $(f_1, 0)\ \sigma_{_H}\ (f_6, 0)$. But $(f_4, 0) + (f_6, 0)^*$ $(= (f_5, 0)) \notin E^+(S_1)$ implies that $((f_4, 0), (f_6, 0)) \notin \sigma_{_H}$. This shows that $\sigma_{_H}$ is not an equivalence relation hence not a near-ring congruence on S_1.

The following result may be considered to be the converse of Theorem 2.2.3.

Theorem 2.2.7. *Let $(S, +, \cdot)$ be a distributively generated additively regular seminearring. Let ρ be a near-ring congruence on S. Then $H_\rho := \{a \in S : a\rho e$ for some $e \in E^+(S)\}$ is a normal full k-ideal of S. Moreover, the relation $\sigma_{_{H_\rho}}$ on S defined by*

$$a\sigma_{_{H_\rho}}b \text{ if and only if } a + b' \in H_\rho \text{ for some } b' \in V^+(b)$$

coincides with ρ.

Proof. Since ρ is a near-ring, ρ is a congruence on S, ρ is a group congruence on $(S, +)$. Then in view of Theorem 3 [67], $H_\rho := \{a \in S : a\rho e$ for some $e \in E^+(S)\}$ is a self-conjugate, full, closed subsemigroup of $(S, +)$ and $\rho = \{(a, b) \in S \times S : a + b' \in H_\rho$ for some $b' \in V^+(b)\}$. Therefore ρ coincides with $\sigma_{_{H_\rho}}$. Let $s \in S$ and $a \in H_\rho$. Then $a\rho e$ for some $e \in E^+(S)$ and $as\ \rho\ es$. Now $es \in E^+(S)$ (*cf.* Remark 2.1.10 (i)). Therefore $as \in H_\rho$ and H_ρ is a right $\mathcal{S}$-ideal. Again $s = \sum_{i=1}^n t_i$ where each t_i is a distributive element. Therefore $t_ia\ \rho\ t_ie$ and $t_ie \in E^+(S)$ (*cf.* Remark 2.1.10 (i)) for all i, $1 \leq i \leq n$. Then $t_ia \in H_\rho$ for all i, $1 \leq i \leq n$. Since $(H_\rho, +)$ is a subsemigroup of $(S, +)$, $\sum_{i=1}^n t_ia \in H_\rho$ *i.e.*, $sa \in H_\rho$. Then H_ρ is a left $\mathcal{S}$-ideal as well and hence a normal full k-ideal (*cf.* Definitions 2.1.6 and Remark 2.1.8). □

As a simple consequence of the above result we obtain the following result which can be considered to be a refinement of Propositions 3.17 of [99].

Corollary 2.2.8. *Let $(S, +, \cdot)$ be a distributively generated additively inverse seminearring. Let ρ be a near-ring congruence on S. Then $H_\rho := \{a \in S : a\rho e$ for some $e \in E^+(S)\}$ is a normal full k-ideal of S. Moreover, the relation $\sigma_{_{H_\rho}}$ on S defined by*

$$a\sigma_{H_\rho} b \textit{ if and only if } a + b^* \in H_\rho$$

coincides with ρ.

Remark 2.2.9. The relation σ of Proposition 3.17 [99] is denoted by σ_{H_ρ} in Corollary 2.2.8.

The following result is the analogue of Theorem 1.1.22 (*i.e.,* Theorem 3 of [67]) in the setting of distributively generated additively regular seminearrings.

Theorem 2.2.10. *Let* $(S, +, \cdot)$ *be an additively regular distributively generated seminearring. Let* $\mathcal{I}(S)$ *denote the poset of all normal full* k*-ideals of* S *and* $\mathcal{C}(S)$ *denote the poset of all near-ring congruences on* S *(both under set inclusion). Then there exists an order-isomorphism between the two posets* $\mathcal{I}(S)$ *and* $\mathcal{C}(S)$ *via the map* $H \mapsto \sigma_H$ *where*

$$a\sigma_H b \textit{ if and only if } a + b' \in H \textit{ for some } b' \in V^+(b).$$

Proof. Theorem 2.2.3 shows the mapping is injective and Theorem 2.2.7 shows the mapping is surjective. Let $H, K \in \mathcal{I}(S)$. Then by definition of σ_H, $H \subseteq K$ implies $\sigma_H \subseteq \sigma_K$. Conversely, let $\sigma_H \subseteq \sigma_K$. Since $H = \{a \in S : a\sigma_H e$ for some $e \in E^+(S)\}$ and $K = \{a \in S : a\sigma_K e$ for some $e \in E^+(S)\}$ (*cf.* Theorem 2.2.3), $H \subseteq K$. Hence in view of Definition 1.2.3 the result follows. □

Remark 2.2.11. Let ρ be a near-ring congruence on a distributively generated additively regular seminearring S. Then S/ρ becomes a distributively generated near-ring. In view of (i) of Remark 1.4.9, S/ρ becomes a zero-symmetric near-ring. Therefore a near-ring congruence on a distributively generated additively regular seminearring S becomes a zero-symmetric near-ring congruence on S. Then in a distributively generated additively regular seminearring S, we get an order-isomorphism between the poset of all normal full k-ideals of S and the poset of all zero-symmetric near-ring congruences on S (both under set inclusion).

As a simple consequence of the above result we obtain the following result which can be considered to be a refinement of Theorem 3.20 of [99].

Corollary 2.2.12. *Let* $(S, +, \cdot)$ *be a distributively generated additively inverse seminearring. Let* $\mathcal{I}(S)$ *denote the poset of all normal full* k*-ideals of* S *and* $\mathcal{C}(S)$ *denote the poset of all near-ring congruences on* S *(both under set inclusion). Then there exists an order-isomorphism between the two posets* $\mathcal{I}(S)$ *and* $\mathcal{C}(S)$ *via the map* $H \mapsto \sigma_H$ *where* $a\sigma_H b$ *if and only if* $a + b^* \in H$.

Remark 2.2.13. Example 3.21 of [98] shows that we cannot do away with the restriction that S is distributively generated for obtaining the order-isomorphism between $\mathcal{I}(S)$ and $\mathcal{C}(S)$ as stated in Theorem 2.2.10.

The following result is a rephrased seminearring version of Theorem 1.1.23 (*i.e.,* Theorem 4 [67]).

Theorem 2.2.14. *Let* $(S, +, \cdot)$ *be an additively regular distributively generated seminearring and* τ *be a near-ring congruence on* S*. Then there exists a normal full* k*-ideal* H *of* S *such that the following are equivalent.*

(i) $a \ \tau \ b$,

(ii) $a + x + b^{'} \in H$ *for some* $x \in H$ *and for some (all)* $b^{'} \in V^{+}(b)$,

(iii) $a^{'} + x + b \in H$ *for some* $x \in H$ *and for some (all)* $a^{'} \in V^{+}(a)$,

(iv) $b + x + a^{'} \in H$ *for some* $x \in H$ *and for some (all)* $a^{'} \in V^{+}(a)$,

(v) $b^{'} + x + a \in H$ *for some* $x \in H$ *and for some (all)* $b^{'} \in V^{+}(b)$,

(vi) $a + x = y + b$ *for some* $x, y \in H$,

(vii) $H + a + H \cap H + b + H$ *is non-empty,*

(viii) $a \ \beta_{_H} \ b$,

(ix) $a \ \sigma_{_H} \ b$.

Proof. In view of Theorem 2.2.10, for a near-ring congruence τ, there exists a normal full k-ideal H of S such that $\tau = \sigma_{_H}$. Since $(H, +)$ is a self-conjugate, full and closed semigroup, $\beta_{_H} = \sigma_{_H}$ (*cf.* Theorem 2[67]). Rest of the proof follows immediately from Theorem 1.1.23. ☐

In view of Theorem 2.2.2 and equivalence of (i) and ($viii$) of Theorem 2.2.14, we deduce the following result which is the analogue of the last part of Theorem 1.1.21 in our setting.

Theorem 2.2.15. *Let* S *be a distributively generated additively regular seminearring. Then the relation* $\beta_U = \{(a, b) \in S \times S : x + a = b + y \text{ for some } x, y \in U\}$ *is the least near-ring congruence on* S *where* U *is the least normal full ideal of* S *i.e., the intersection of all normal full ideals of* S.

The following result is a seminearring version of Corollary 1.1.24.

Corollary 2.2.16. *Let σ denote the least near-ring congruence on a distributively generated additively regular seminearring $(S, +, \cdot)$ and U is the least normal full ideal of S. Then the following are equivalent.*

(i) $a \ \sigma \ b$,

(ii) $a + x + b^{'} \in U$ *for some* $x \in U$ *and for some (all)* $b^{'} \in V^{+}(b)$,

(iii) $a^{'} + x + b \in U$ *for some* $x \in U$ *and for some (all)* $a^{'} \in V^{+}(a)$,

(iv) $b + x + a^{'} \in U$ *for some* $x \in U$ *and for some (all)* $a^{'} \in V^{+}(a)$,

(v) $b^{'} + x + a \in U$ *for some* $x \in U$ *and for some (all)* $b^{'} \in V^{+}(b)$,

(vi) $a + x = y + b$ *for some* $x, y \in U$,

(vii) $U + a + U \cap U + b + U$ *is non-empty,*

(viii) $a \ \beta_{U} \ b$.

Proof. In view of Theorem 2.2.15, (i) and $(viii)$ are equivalent. Again $(U, +)$ is a self-conjugate, full subsemigroup of semigroup $(S, +)$ (*cf.* Definitions 2.1.5). This together with Theorem 1.1.23 implies that (ii)-$(viii)$ are equivalent □

The following result gives another description for the least near-ring congruence on a distributively generated additively inverse seminearring which refines Theorem 3.22 of [98].

Theorem 2.2.17. *Let $(S, +, \cdot)$ be an additively inverse distributively generated seminearring. Then the relation σ, defined on S, by $a \ \sigma \ b$ if and only if $a + f = b + f$ for some $f \in E^{+}(S)$ is the least near-ring congruence on S.*

Proof. By Theorem 1.1.25, σ is the least group congruence on $(S, +)$. Let $a \ \sigma \ b$ and $s \in S$. Then there exists $f \in E^{+}(S)$ such that $a + f = b + f$. In view of Proposition 3.2 [98], it follows that $as \ \sigma \ bs$. Hence σ is a right congruence on $(S, +, \cdot)$. Since S is distributively generated, $s = \sum_{i=1}^{n} t_i$ where each t_i is a distributive element for $1 \leq i \leq n$. Now $t_i(a + f) = t_i(b + f)$ *i.e.,* $t_i a + t_i f = t_i b + t_i f$ for $1 \leq i \leq n$. This together with the fact that $t_i f \in E^{+}(S)$ (*cf.* Proposition 2.1.9) shows that $t_i a \ \sigma \ t_i b$ for $1 \leq i \leq n$. Again σ is a congruence on $(S, +)$. Therefore $\sum_{i=1}^{n} t_i a \ \sigma \ \sum_{i=1}^{n} t_i b$ *i.e.,* $sa \ \sigma \ sb$ whence σ is a congruence on $(S, +, \cdot)$. This completes the proof. □

The following example illustrates that the relation σ defined in Theorem 2.2.17 may not even be a congruence if we remove the property of being 'additively inverse' from the hypothesis.

Example 2.2.18. Let S be the set of all maps from X into X where $X = \{1, 2, 3\}$. Let $(a+b)(s) = a(b(s))$ for all $s \in X$ and $ab = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ where $a, b \in S$. Then $(S, +, \cdot)$ is an additively regular semiring. Let $(S_2, +, \cdot)$ be the seminearring direct product of $(S, +, \cdot)$ and $(D, +, \cdot)$ where $(D, +, \cdot)$ is a distributively generated near-ring. Then $(S_2, +, \cdot)$ is a distributively generated additively regular seminearring and $E^+(S_2) = \{(e, 0) \in S_2 : e$ is an additive idempotent of $S\}$. Now let $(a, 0)$, $(b, 0)$, $(c, 0) \in S_2$ where $a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $c = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$ and let $(e, 0)$, $(f, 0) \in E^+(S_2)$ where $e = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}$ and $f = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$. Clearly, $(a, 0)+(e, 0) = (b, 0)+(e, 0) = (f, 0)$ and $(b, 0)+(f, 0) = (c, 0)+(f, 0) = (e, 0)$. Now let $(g, 0) \in E^+(S_2)$ such that $g(1) = 1$. Then $a + g \neq c + g$ as $(a + g)(1) = 2$ and $(c + g)(1) = 3$. Similarly, if we consider $g(1) = 2$ or $g(1) = 3$, we can easily see that $a + g \neq c + g$. Therefore there does not exist any $(g, 0) \in E^+(S_2)$ such that $(a, 0)+(g, 0) = (c, 0)+(g, 0)$. This shows that the relation $\sigma = \{(x, y) \in S_2 \times S_2 : x + s = y + s$ for some $s \in E^+(S_2)\}$ on S_2 is not a congruence.

2.3 Lattice of near-ring congruences on seminearrings

Definition 2.3.1. [83] Let A, B be two subsets of a seminearring $(S, +, \cdot)$. Then we define the sum of A and B to be the set $\{\sum_{i=1}^{n} x_i : n \in \mathbb{N}$ and $x_i \in A$ or $B\}$ and we denote it by $A + B$.

The following result on the sum of two normal full (right) k-ideals will be used frequently in what follows.

Proposition 2.3.2. *Let A, B be two normal full (right) k-ideals of an additively regular seminearring $(S, +, \cdot)$. Then for each $x \in A + B$,*

(i) there exists $z_1 \in A$ such that $z_1 + x$, $x + z_1 \in B$.

(ii) there exists $z_2 \in B$ such that $z_2 + x$, $x + z_2 \in A$.

Proof. Let $x \in A+B$ where A, B are two normal full (right) k-ideals of an additively regular seminearring $(S,+,\cdot)$. Then $x = \sum_{i=1}^{n} x_i$ where $x_i \in A \cup B$. Then there exists a subset $\{r_1, r_2, ..., r_k\}$ of $\{1, 2, ..., n\}$ such that $r_1 \leq r_2 \leq .... \leq r_k$, $k \leq n$ and x_{r_j} for all j where $1 \leq j \leq k$ are the only members of A. Now $x'_{r_1} + (x_1 + x_2 + ... + x_{r_1 - 1}) + x_{r_1} \in B$ for all $x'_{r_1} \in V^+(x_{r_1})$ as B is a normal (right) ideal and $(x_1 + x_2 + ... + x_{r_1-1}) \in B$. In a similar way we get $x'_{r_2} + (x'_{r_1} + x_1 + x_2 + ... + x_{r_1-1} + x_{r_1} + x_{r_1+1} + ... + x_{r_2-1}) + x_{r_2} \in B$ for all $x'_{r_2} \in V^+(x_{r_2})$, $x'_{r_1} \in V^+(x_{r_1})$ as $(x_{r_1+1} + ... + x_{r_2-1}) \in B$. Following this manner we finally get $x'_{r_k} + x'_{r_{k-1}} + ... + x'_{r_1} + x_1 + ... + x_{r_k} \in B$ for all $x'_{r_i} \in V^+(x_{r_i})$ where $1 \leq i \leq k$. Again $x_{r_k+1} + ... + x_n \in B$. Let $z_1 = x'_{r_k} + x'_{r_{k-1}} + ... + x'_{r_1}$ where $x'_{r_i} \in V^+(x_{r_i})$ for $1 \leq i \leq k$. Therefore $z_1 + x \in B$. Again A being full, for $1 \leq j \leq k$, $x_{r_j} + x'_{r_j} \in A$ for all $x'_{r_j} \in V^+(x_{r_i})$. Since A is a (right) k-ideal and $x_{r_j} \in A$, $x'_{r_j} \in A$ for $1 \leq j \leq k$. Therefore $z_1 \in A$. So by Proposition 2.1.11, $x + z_1 \in B$. This proves (i).

(ii) follows similarly due to symmetry of A and B in the sum $A+B$. □

Remark 2.3.3. It is well known that [83] the sum of two right $\mathcal{S}$-ideals is a right $\mathcal{S}$-ideal in any seminearring S and the sum of two $\mathcal{S}$-ideals is an $\mathcal{S}$-ideal in a distributively generated seminearring S. But the sum of two k-ideals need not be a k-ideal even in a distributively generated seminearring (see Example 3 [83]). Also the sum of two normal subseminearrings (see Definition 1.5.23) in a distributively generated seminearring is not a normal subseminearring which is evident from Example 4 [83]. In order to obtain a positive result in this direction in [83], the authors took the help of closure (Definition 2.6 and Proposition 2.8 [83]). In our setting *i.e.,* for an additively regular seminearring or for a distributively generated additively regular seminearring we are interested to see what happens to the sum of two normal full k-ideals. In this regard we provide the following example illustrating that the sum of two normal full right k-ideals of an additively regular seminearring is not a normal full right k-ideal. In Lemma 2.3.10 and Theorem 2.3.11 we obtain some restricted result in the desired direction with the help of k-closure (*cf.* Definition 2.3.5 and Remark 2.3.6).

Example 2.3.4. Let $S = \{a, b, c, d, e, f\}$ where

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

$$d = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}, e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

are six partial maps on the set $\{1, 2, 3, 4, 5, 6\}$. Let $(x + y)(i) = x(y(i))$ for all $i \in y^{-1}[Domx \cap Imy]$ where $x, y \in S$. Then $(S, +)$ is a commutative regular semigroup in particular an inverse semigroup where

$+$	a	b	c	d	e	f
a	e	c	b	b	a	a
b	c	e	a	a	b	b
c	b	a	e	e	c	c
d	b	a	e	f	c	d
e	a	b	c	c	e	e
f	a	b	c	d	e	f

Let us define $xy = x$ for all $x, y \in S$. Then $(S, +, \cdot)$ is an additively regular seminearring in particular an additively inverse seminearring where $E^+(S) = \{e, f\}$. Now $A = \{a, e, f\}$ and $B = \{b, e, f\}$ are two normal full right k-ideals. Clearly, $A + B = \{a, b, c, e, f\}$. Now $a + d \ (= b)$ and $a \in A + B$ but $d \notin A + B$. This shows that $A + B$ is a right $\mathcal{S}$-ideal but not a right k-ideal. Therefore $A + B$ is not a normal full right k-ideal.

Definition 2.3.5. Let A be a (left, right) $\mathcal{S}$-ideal of a seminearring $(S, +, \cdot)$. The *closure* [83] of A is defined to be the set $\{s \in S :$ for some $x, y \in A,\ x + s + y \in A\}$ and is denoted by $\bar{A}$.

Remark 2.3.6. Let A be a (left, right) $\mathcal{S}$-ideal of a seminearring $(S, +, \cdot)$. Then it is easy to observe that $A \subseteq \bar{A}$ and A is a k-ideal if and only if $\bar{A} = A$. So the closure of A defined above can be called k-closure of A.

In this connection the following remark is in order.

Remark 2.3.7. If H is a normal full ideal in a distributively generated additively regular seminearring S then Hw is a normal full k-ideal of S and $H \subseteq Hw$ (*cf.* Theorem 2.2.2). Let $x \in \bar{H}$. Then there exist $h_1, h_2 \in H$ such that $h_1 + x + h_2 \in H \subseteq Hw$. Then in view of Proposition 2.1.11, $h_2 + h_1 + x \in Hw$ whence $x \in Hw$. Therefore $\bar{H} \subseteq Hw$. If $z \in Hw$, then there exists $h_3 \in H$ such that $h_3 + z \in H$ whence $h_3 + z + h \in H$ for all $h \in H$. Then $z \in \bar{H}$ whence $Hw \subseteq \bar{H}$. Therefore $\bar{H} = Hw$ and this is the smallest normal full k-ideal containing H.

Definition 2.3.8. Let $(S, +, \cdot)$ be a seminearring and A, B be two (right) $\mathcal{S}$-ideals of S. Then we define four subsets of S as follows :

(*i*) $X_{1,A+B} := \{s \in S : \text{ for some } x \in B,\, x+s \in A\}$,

(*ii*) $X_{2,A+B} := \{s \in S : \text{ for some } x \in B,\, s+x \in A\}$,

(*iii*) $X_{3,A+B} := \{s \in S : \text{ for some } x \in A,\, x+s \in B\}$,

(*iv*) $X_{4,A+B} := \{s \in S : \text{ for some } x \in A,\, s+x \in B\}$.

Proposition 2.3.9. *Let $(S,+,\cdot)$ be an additively regular seminearring and A, B be two normal full (right) k-ideals of S. Then $\overline{A+B} = X_{1,A+B} = X_{2,A+B} = X_{3,A+B} = X_{4,A+B}$.*

Proof. Let $x \in \overline{A+B}$. Then there exist $s, t \in A+B$ such that $s+x+t \in A+B$. In view of Proposition 2.3.2 there exist $s_1, t_1 \in A$ such that $s_1+s,\, t+t_1 \in B$. Therefore by Definition 2.3.1, $s_1+s+x+t+t_1 \in A+B$ *i.e.,* $b+x+b_1 \in A+B$ where $b\ (= s_1+s)$, $b_1\ (= t+t_1) \in B$. Hence $b+x+b_1+x^{'}+x \in A+B$ for all $x^{'} \in V^{+}(x)$ as $x^{'}+x \in A,\, B$ *i.e.,* $(b+b_2)+x \in A+B$ where $b_2\ (= x+b_1+x^{'}) \in B$ as B is a normal (right) ideal. Again using Proposition 2.3.2 we get $b_3 \in B$ such that $b_3+(b+b_2+x) \in A$. Hence $x \in X_{1,A+B}$.

Now let $s \in X_{1,A+B}$. There exists $b \in B$ such that $b+s \in A$. Then $b+s+a \in A+B$ for any $a \in A$. So by Definitions 2.3.1, 2.3.5, $s \in \overline{A+B}$ whence $X_{1,A+B} \subseteq \overline{A+B}$. Hence $X_{1,A+B} = \overline{A+B}$. Using similar type of arguments as above we deduce the other equalities. □

Lemma 2.3.10. *Let $(S,+,\cdot)$ be an additively regular seminearring and A, B be two normal full right k-ideals of S. Then $\overline{A+B}$ is the smallest normal full right k-ideal of S containing A and B.*

Proof. In view of Definition 2.3.1 and Remark 2.3.6, $A, B \subseteq \overline{A+B}$. Let $x, y \in \overline{A+B}$. Then in view of Definition 2.3.8 (*iii*), (*iv*) and Proposition 2.3.9, there exist $a, a_1 \in A$ such that $a+x,\, y+a_1 \in B$ ...(*i*). Therefore $(a+x)+(y+a_1) \in B \subseteq A+B$. Then in view of Definition 2.3.5, $x+y \in \overline{A+B}$. Now let $s \in S$. Then using (*i*) we get $(a+x)s \in B$ as B is a right $\mathcal{S}$-ideal. Again $(a+x)s = as+xs$ and $as \in A$. Therefore $xs \in \overline{A+B}$ (*cf.* Definition 2.3.8 (*iii*) and Proposition 2.3.9) and $\overline{A+B}$ is a right $\mathcal{S}$-ideal of S containing A and B.

Now let $s+t, s \in \overline{A+B}$. Then there exist $b, b_1 \in B$ such that $b+s,\ s+t+b_1 \in A$ (*cf.* Definition 2.3.8 (*i*), (*ii*) and Proposition 2.3.9). Therefore $b+(s+t+b_1) \in A+B$ where $b+s, b_1 \in A+B$. Then in view of Definition 2.3.5, $t \in \overline{A+B}$. Similarly we deduce that if $s+t, t \in \overline{A+B}$ then $s \in \overline{A+B}$. Therefore $\overline{A+B}$ is a right k-ideal. Since A, B are full, $\overline{A+B}$ is a full right k-ideal. Again let $s \in \overline{A+B}$. So there exists $v \in B$ such that $s+v \in A$ (*cf.* Definition 2.3.8 (*ii*) and Proposition 2.3.9). Then for any $x \in S$, $s+x^{'}+x+v \in A$ for all $x^{'} \in V^{+}(x)$ (*cf.* Proposition 2.1.11). Again $x+(s+x^{'}+x+v)+x^{'} \in A$ for all $x^{'} \in V^{+}(x)$ as A is a normal right ideal. Now $(x+s+x^{'})+(x+v+x^{'}) \in A$ where $x+v+x^{'} \in B$ for all $x^{'} \in V^{+}(x)$. Then $x+s+x^{'}$ $\in \overline{A+B}$ for all $x^{'} \in V^{+}(x)$ (*cf.* Definition 2.3.8 (*ii*) and Proposition 2.3.9). Therefore $\overline{A+B}$ is a normal full right k-ideal of S containing A and B. It is a routine verification to check that $\overline{A+B}$ is the smallest normal full right k-ideal of S containing A and B. □

Theorem 2.3.11. *Let $(S,+,\cdot)$ be an additively regular distributively generated seminearring. Let A, B be two normal full k-ideals of S. Then $\overline{A+B}$ is the smallest normal full k-ideal containing A and B.*

Proof. By Lemma 2.3.10, $\overline{A+B}$ is the smallest normal full right k-ideal of S containing A and B. Let $x \in \overline{A+B}$ and $s \in S$. Then $s=\sum_{i=1}^{n} t_i$ where t_i is a distributive element for each i, $1 \leq i \leq n$. In view of Definition 2.3.8 (*iii*) and Proposition 2.3.9, there exists $a \in A$ such that $a+x \in B$. Then $t_i(a+x)=t_ia+t_ix \in B$ and $t_ia \in A$ for each i where $1 \leq i \leq n$. Therefore by Definition 2.3.8 (*iii*) and Proposition 2.3.9, t_ix $\in \overline{A+B}$ for each i. So $sx \in \overline{A+B}$. Hence $\overline{A+B}$ is a left $\mathcal{S}$-ideal. This completes the proof. □

It is a matter of routine verification to prove the following result.

Proposition 2.3.12. *Let $(S,+,\cdot)$ be an additively regular seminearring. Then intersection of two normal full k-ideals is a normal full k-ideal.*

Theorem 2.3.11 and Proposition 2.3.12 together imply the following result.

Theorem 2.3.13. *Let $(S,+,\cdot)$ be a distributively generated additively regular seminearring. Then the set $\mathcal{I}(S)$ of all normal full k-ideals of S under set inclusion forms a lattice where $I \wedge J = I \cap J$ and $I \vee J = \overline{I+J}$ for all $I, J \in \mathcal{I}(S)$.*

Theorem 2.3.14. *Let $(S,+,\cdot)$ be an additively regular seminearring. Then the set $\mathcal{C}(S)$ of all near-ring congruences on S under set inclusion forms a lattice where $\rho \wedge \sigma$ $= \rho \cap \sigma$ and $\rho \vee \sigma = \rho \circ \sigma$ for all $\rho, \sigma \in \mathcal{C}(S)$.*

Proof. It is easy to observe that the intersection of two near-ring congruences on an additively regular seminearring S is a near-ring congruence on S.

Let $\rho, \sigma \in \mathcal{C}(S)$. Then since $\rho \cap \sigma \in \mathcal{C}(S)$, $\rho \cap \sigma$ is a group congruence on $(S, +)$. Hence by Corollary 3 [27], $\rho \circ \sigma$ is the least group congruence on the semigroup $(S, +)$ containing ρ and σ. Let $a\ \rho \circ \sigma\ b$. Then there exists $z \in S$ such that $a \rho z$ and $z \sigma b$. Now for all $s \in S$, $as\ \rho\ zs$ and $zs\ \sigma\ bs$. Therefore $as\ \rho \circ \sigma\ bs$ for all $s \in S$. Similarly we can show that $sa\ \rho \circ \sigma\ sb$ for all $s \in S$. Then in view of Definitions 1.5.18 and 1.5.21, $\rho \circ \sigma$ is a near-ring congruence on $(S, +, \cdot)$. Hence $\mathcal{C}(S)$ is a lattice with $\rho \wedge \sigma = \rho \cap \sigma$ and $\rho \vee \sigma = \rho \circ \sigma$ where $\rho, \sigma \in \mathcal{C}(S)$. □

Proposition 1.2.10 together with Theorems 2.2.10, 2.3.13 and 2.3.14 gives rise to the following result.

Theorem 2.3.15. *Let $(S, +, \cdot)$ be a distributively generated additively regular seminearring. Then the lattice $\mathcal{C}(S)$ of all near-ring congruences on S and the lattice $\mathcal{I}(S)$ of all normal full k-ideals of S are isomorphic.*

Remark 2.3.16. Since a near-ring congruence on a distributively generated seminearring becomes a zero-symmetric near-ring congruence, in a distributively generated additively regular seminearring S, the lattice of all normal full k-ideals of S and the lattice of all zero-symmetric near-ring congruences on S are isomorphic.

Theorem 2.3.17. *Let $(S, +, \cdot)$ be a distributively generated additively regular seminearring. Then the set $\mathcal{I}(S)$ of all normal full k-ideals of S is a modular lattice.*

Proof. By Theorem 2.3.13, $\mathcal{I}(S)$ is a lattice. Let $I, J, K \in \mathcal{I}(S)$ and $K \subseteq I$. Then $\overline{(I \cap J) + K} \subseteq I \cap \overline{(J + K)}$ (*cf.* Lemma 1.2.11). Let $x \in I \cap \overline{(J + K)}$. Then $x \in I$ and there exists $y \in J$ such that $(x + y) \in K$ (*cf.* Definition 2.3.8 (*iv*) and Proposition 2.3.9). Therefore $(x + y) \in I$ whence $y \in I$. Hence $y \in I \cap J$ whence in view of the fact that $(x + y) \in K$ we obtain $x \in \overline{(I \cap J) + K}$ (*cf.* Definition 2.3.8 (*iv*) and Proposition 2.3.9). This completes the proof. □

Combining Theorems 2.3.15 and 2.3.17 we obtain the following result.

Theorem 2.3.18. *Let $(S, +, \cdot)$ be a distributively generated additively regular seminearring. Then the set $\mathcal{C}(S)$ of all near-ring congruences on S is a modular lattice.*

The following example shows that the lattice of all normal full k-ideals of a distributively generated additively regular seminearring may not be distributive.

Example 2.3.19. Let $(\mathbb{K}_4, +)$ be the Klein-4-group. Let us define $xy = e$ for all $x, y \in \mathbb{K}_4$. Then $\mathbb{K}_4$ is a ring. Let $(S_1, +, \cdot)$ be the seminearring direct product of $\mathbb{K}_4$ and a distributively generated additively regular seminearring $(D, +, \cdot)$. Then $(S_1, +, \cdot)$ is a distributively generated additively regular seminearring. Let $I_1 = \{e, a\} \times D$, $I_2 = \{e, b\} \times D$ and $I_3 = \{e, c\} \times D$ be three normal full k-ideals of S_1. Then $(I_1 \wedge I_2) \vee (I_1 \wedge I_3)$ $= \{e\} \times D$ but $I_1 \wedge (I_2 \vee I_3) = I_1$.

Now we want to obtain some sufficient conditions imposition of which ensures the distributivity of the lattice $\mathcal{I}(S)$ (*cf.* Theorem 2.3.22) of all normal full k-ideals of a distributively generated additively regular seminearring S. It is relevant to mention here that this condition is a modified version of its counterpart for the lattice of ideals of ring [12].

Definition 2.3.20. [2] Let A, B be two subsets of a seminearring $(S, +, \cdot)$. Then we define the product of A and B to be the set $\{\sum_{i=1}^{n} a_i b_i : n \in \mathbb{N}, a_i \in A \text{ and } b_i \in B\}$ and we denote it by AB.

Remark 2.3.21. According to [2], the product of two right $\mathcal{S}$-ideals is a right $\mathcal{S}$-ideal but the product of two $\mathcal{S}$-ideals (not necessarily an $\mathcal{S}$-ideal) becomes an $\mathcal{S}$-ideal if the seminearring is distributively generated.

Theorem 2.3.22. *Let $(S, +, \cdot)$ be a distributively generated additively regular seminearring. If $\overline{IJ} = I \cap J$ for all normal full k-ideals I, J of S, then the set $(\mathcal{I}(S), \subseteq)$ of all normal full k-ideals of S becomes a distributive lattice.*

Proof. By Theorem 2.3.13, $\mathcal{I}(S)$ is a lattice. Let I, J, $K \in \mathcal{I}(S)$. Then $\overline{(J \cap I) + (K \cap I)}$ $\subseteq \overline{(J + K)} \cap I$ (*cf.* Lemma 1.2.11) *i.e.*, $\overline{\overline{JI} + \overline{KI}} \subseteq \overline{\overline{(J + K)}I}$ as $\overline{JI} = J \cap I$. Now let $a \in \overline{(J + K)}I$. Clearly, $a = \sum_{i=1}^{n} x_i y_i$ where for all $1 \leq i \leq n$, $x_i \in \overline{(J + K)}$ and $y_i \in I$. Then by Definition 2.3.8 (*iv*) and Proposition 2.3.9, for each x_i, $1 \leq i \leq n$ there exists $z_i \in J$ such that $x_i + z_i \in K$. Now $x_i y_i + z_i y_i \in KI \subseteq \overline{KI}$ for all $1 \leq i \leq n$. By hypothesise $\overline{KI} = K \cap I$. So $\overline{KI}$ is a normal full k-ideal. Then $x_{n-1}y_{n-1} + (x_n y_n + z_n y_n) + z_{n-1}y_{n-1} \in \overline{KI}$ (*cf.* Proposition 2.1.11). Proceeding in this way we obtain $\sum_{i=1}^{n} x_i y_i + \sum_{k=n}^{1} z_k y_k \in \overline{KI}$. Therefore $a + b \in \overline{KI}$ where $b = \sum_{k=n}^{1} z_k y_k \in JI \subseteq \overline{JI}$. Then in view of Definition 2.3.8 (*iv*) and Proposition 2.3.9, $a \in \overline{\overline{JI} + \overline{KI}}$ whence $(\overline{J + K})I \subseteq \overline{\overline{JI} + \overline{KI}}$. Thus $\overline{\overline{(J + K)}I} \subseteq \overline{\overline{JI} + \overline{KI}}$. Therefore $\overline{\overline{(J + K)}I} = \overline{\overline{JI} + \overline{KI}}$. As $\overline{JI} = J \cap I$, we deduce from the last relation that $\overline{(J + K)} \cap I = \overline{J \cap I + K \cap I}$. This completes the proof. ☐

Theorem 2.3.23. *Suppose $(S, +, \cdot)$ is a distributively generated additively regular seminearring. Then the set $\mathcal{I}(S)$ of all normal full k-ideals of S is a complete lattice.*

Proof. By Theorem 2.3.13, $\mathcal{I}(S)$ is a lattice. Let $A \subseteq \mathcal{I}(S)$. Then $\bigcap_{I \in A} I$ is a k-ideal. Again $E^+(S) \subseteq I$ (*cf.* Definitions 2.1.5 (i)) for all $I \in A$ whence $E^+(S) \subseteq \bigcap_{I \in A} I$. Therefore $\bigcap_{I \in A} I$ is a full k-ideal. Now for any $a \in \bigcap_{I \in A} I$, $s + a + s' \in I$ for all $I \in A$, for all $s \in S$ and for all $s' \in V^+(s)$. Then $s + a + s' \in \bigcap_{I \in A} I$ for all $a \in \bigcap_{I \in A} I$, for all $s \in S$ and for all $s' \in V^+(s)$. Hence $\bigcap_{I \in A} I$ is a normal full k-ideal *i.e.,* $\bigcap_{I \in A} I \in \mathcal{I}(S)$. Also S is the greatest normal full k-ideal of $\mathcal{I}(S)$. So in view of Theorem 1.2.16, $\mathcal{I}(S)$ becomes a complete lattice. □

Combination of Theorems 2.3.15 and 2.3.23 gives rise to the following result.

Theorem 2.3.24. *Suppose $(S, +, \cdot)$ is a distributively generated additively regular seminearring. Then the set $\mathcal{C}(S)$ of all near-ring congruences on S is a complete lattice.*

Remark 2.3.25. All of the results of this section are also true for distributively generated additively inverse seminearrings. The proofs will be similar except that $a' \in V^+(a)$ will be replaced by unique additive inverse a^*.

CHAPTER 3

NEAR-RING CONGRUENCES ON ARBITRARY SEMINEARRINGS

3

Near-ring Congruences on Arbitrary Seminearrings

In the study of near-ring congruences on a seminearring, results obtained in [83, 99] and our previous chapter mainly establish inclusion preserving bijective correspondences between near-ring congruences and various types of ideals of seminearrings. In [83], the authors showed that there exists an inclusion preserving bijection between the set of all additively commutative near-ring congruences and the set of all normal subseminearrings of a distributively generated zero-symmetric seminearring S (*cf.* Theorem 1.5.24). In [99], the authors established an inclusion preserving bijection between the set of all near-ring congruences and the set of all normal full k-ideals of a distributively generated additively inverse seminearring S with property D (*cf.* Theorem 3.20 [99]). In our previous chapter, we established that in a distributively generated additively regular seminearring, the set of all near-ring congruences and the set of all normal full k-ideals are in an inclusion preserving bijective correspondence (*cf.* Theorem 2.2.10). While establishing these bijections, we observe that different kinds of restrictions were imposed either on the seminearring under consideration or on the near-ring congruence. The main purpose of this chapter is to obtain similar result connecting near-ring congruences on a seminearring with some new structures of that seminearring where the seminearring is neither distributively generated nor additively

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This chapter is mainly based on the work published in the following paper:

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regular and the near-ring congruence is not additively commutative. Papers of Gomes [36] and Gigoń [31] have guided us to achieve our desired results in this chapter.

The flow of the content of **Section 1** is as follows. In [36], G. M. S. Gomes established that there is an inclusion preserving bijection between the set of all full, dense, reflexive and unitary subsemigroups of an arbitrary semigroup S and the set of all group congruences on S (*cf.* Theorem 7 and Lemma 8 [36]). In Theorem 1.1.29 (*i.e.,* Theorem 2.4 [31]), R. S. Gigoń simplified the proof of this statement and established an inclusion preserving bijection between the set of all dense, reflexive and closed subsemigroups of a semigroup $(S, +)$ and the set of all group congruences on S via the map $I \mapsto \rho_I$ where for a dense, reflexive and closed subsemigroup I of S,

$$(a, b) \in \rho_I \text{ if and only if there exists } x \in S \text{ such that } a + x,\ b + x \in I.$$

Since a near-ring congruence on a seminearring $(S, +, \cdot)$ means a congruence on the additive reduct $(S, +)$ as well as on the multiplicative reduct $(S, \cdot)$ such that it is a group congruence on $(S, +)$, our task reduces to find conditions on dense, reflexive and closed subsemigroup $(I, +)$ of $(S, +)$ to make ρ_I (as defined by Gigoń [31]) a right as well as left congruence on $(S, \cdot)$. In this regard, in a seminearring S, we see that ρ_I, corresponding to a dense, reflexive subsemigroup $(I, +)$ of $(S, +)$ with the property $IS \subseteq I$, becomes a right congruence on $(S, \cdot)$ as well as a group congruence on $(S, +)$ (*cf.* Proposition 3.1.1). So to make this ρ_I a left congruence on $(S, \cdot)$, too, we then introduce the notion of strong additive subsemigroup in a seminearring (*cf.* Definition 3.1.2). Then in a seminearring S, we establish that ρ_I, corresponding to a strong, dense, reflexive and closed additive subsemigroup I of S with the property $IS \subseteq I$ becomes a near-ring congruence on S (*cf.* Theorem 3.1.6) and conversely a near-ring congruence σ on S corresponds a strong, dense, reflexive and closed additive subsemigroup H_σ of S such that $H_\sigma S \subseteq I$ (*cf.* Theorem 3.1.7). Combining these two results, in a seminearring S we obtain our desired inclusion preserving bijective correspondence between the set of all near-ring congruences on S and the set $\{I \subseteq S | I$ is a strong, dense, reflexive and closed additive subsemigroup of S with $IS \subseteq I\}$ (*cf.* Theorem 3.1.8). Then we establish in Theorem 3.1.10 that in a seminearring S, the set of all zero-symmetric near-ring congruences on S and the set $\{I \subseteq S | I$ is a strong, dense, reflexive and closed additive subsemigroup of S with SI, $IS \subseteq I\}$ are in an inclusion preserving bijective correspondence.

In **Section 2**, we mainly study near-ring congruences (hence zero-symmetric near-ring congruences) on distributively generated seminearrings. We first characterize strong, dense, reflexive and closed additive subsemigroups in a distributively generated seminearring (*cf.* Proposition 3.2.1). Then in a distributively generated seminearring S, we obtain an inclusion preserving bijective correspondence between the set of all near-ring congruences on S (hence the set of all zero-symmetric near-ring congruences on S) and the set $\{I \subseteq S | I$ is a dense, reflexive and closed additive subsemigroup of S with SI, $IS \subseteq I\}$ (*cf.* Theorem 3.2.3). To conclude this section, in a seminearring S, we relate the notions (*viz.,* (*i*) normal full k-ideal (*cf.* Definition 2.1.6), (*ii*) normal subseminearring (*cf.* Definition 1.5.23) and (*iii*) strong, dense, reflexive and closed additive subsemigroup I of S such that $IS \subseteq I$) which correspond near-ring congruences on different kinds of seminearrings (*cf.* Observation 3.2.4).

3.1 Near-ring and zero-symmetric near-ring congruences on seminearrings

Proposition 3.1.1. *Suppose S is a seminearring and $(I, +)$ is a dense and reflexive subsemigroup of $(S, +)$ such that $IS \subseteq I$. Then the relation ρ_I on S is a right congruence on $(S, \cdot)$ and a group congruence on $(S, +)$ where*

$$(a, b) \in \rho_I \text{ if and only if there exists } x \in S \text{ such that } a + x,\ b + x \in I.$$

Proof. In view of Theorem 1.1.29, ρ_I is a group congruence on $(S, +)$. Let $(a, b) \in \rho_I$ and $s \in S$. Then there exists $x \in S$ such that $a + x$, $b + x \in I$. Since $IS \subseteq I$, $(a + x)s$, $(b + x)s \in I$. Therefore $as + xs$, $bs + xs \in I$ whence $(as, bs) \in \rho_I$. Then ρ_I is a right congruence on the semigroup $(S, \cdot)$. □

Definition 3.1.2. Let S be a seminearring. A subsemigroup $(I, +)$ of $(S, +)$ is said to be *strong* in S if for given s, $a \in S$ and $w \in I$ there exist i_1, i_2, i_3, $i_4 \in I$ such that $s(a + w) + i_1 = i_2 + sa$ and $s(w + a) + i_3 = i_4 + sa$.

The following example shows that there are plenty of instances of strong additive subsemigroup of a seminearring (not necessarily with zero).

Example 3.1.3. Let $(M, +)$ be a semigroup and N be a dense, reflexive and closed subsemigroup of M. Let us define a relation ρ_N on M by $\rho_N := \{(a, b) \in M \times M$: there exists $x \in M$ such that $a + x, b + x \in N\ \}$. Suppose $S = \{f : M \to M : a\ \rho_N\ b \Rightarrow f(a)\ \rho_N\ f(b)\}$. Let $f, g \in S$. Then for $a\ \rho_N\ b$, $f(a)\ \rho_N\ f(b)$ and $g(a)\ \rho_N\ g(b)$. Now in view of Theorem 1.1.29, ρ_N is a congruence on $(M, +)$. Therefore $(f(a) +$

$g(a))\ \rho_{_N}\ (f(b)+g(b))$, *i.e.*, $(f+g)(a)\ \rho_{_N}\ (f+g)(b)$ whence $f+g \in S$. Again $g(a)\ \rho_{_N}\ g(b)$, $f \in S$ show that $f(g(a))\ \rho_{_N}\ f(g(b))$ whence $fg \in S$. In view of Example 1.5.6, $M(M)$ is a seminearring under point wise addition and composition and $S \subseteq M(M)$ where S is closed under addition and composition. Therefore S is a seminearring.
Now $I = \{f \in S : f(M) \subseteq N\}$. Let $f_1, f_2 \in I$. Then $(f_1 + f_2)(M) \subseteq N$. Therefore $(I, +)$ is a subsemigroup of $(S, +)$. Now let $f \in I$ and $g, h \in S$. Then for each $s \in M$, $h(g + f)(s) = h(g(s) + n)$ where $f(s) = n \in N$. Clearly, $(g(s) + n)\ \rho_{_N}\ g(s)$. Therefore for $h \in S$, $h(g(s) + n)\ \rho_{_N}\ h(g(s))$ whence there exists $z \in M$ such that $h(g(s)+n)+z$, $h(g(s))+z \in N$. Let $n_s = z+h(g(s))$ and $n_s^{''} = h(g(s)+n)+z$. Then $h(g(s)+n)+n_s = n_s^{''}+h(g(s))$ where n_s, $n_s^{''} \in N$. Let us define $f_1 : M \to M$ by $f_1(s) = n_s$ and $f_2 : M \to M$ by $f_2(s) = n_s^{''}$. Clearly $f_1, f_2 \in I$ and $h(g + f) + f_1 = f_2 + hg$. Similarly we can show that there exist $f_3, f_4 \in I$ such that $h(f + g) + f_3 = f_4 + hg$. Therefore I is strong in S.

In the following proposition we simplify the defining conditions (*cf.* Definition 3.1.2) of a strong additive subsemigroup in a seminearring with zero.

Proposition 3.1.4. *Let S be a seminearring with zero and $(I, +)$ be a dense, reflexive and closed subsemigroup of $(S, +)$. Then the following are equivalent.*

(i) I is strong in S.

(ii) For each a, b, $s \in S$ and $i \in I$ there exist i_1, $i_2 \in I$ satisfying $s(a + i + b) + i_1 = i_2 + s(a + b)$.

Proof. Let $(I, +)$ be a dense, reflexive and closed subsemigroup of $(S, +)$ such that I is strong in S. Let $s, a, b \in S$ and $\delta \in I$. Since $(I, +)$ is a dense subsemigroup of $(S, +)$, there exist $b', a' \in S$ such that $b + b', a + a' \in I$. Then $(a + b) + (b' + a')$, $(a+\delta+b)+(b'+a') \in I$ (since $(I, +)$ is a reflexive subsemigroup of $(S, +)$). Now there exist $i_1, i_2, i_3, i_4 \in I$ such that

$$s(a + b + z + (a + \delta + b)) + i_1 = i_2 + s(a + b) \text{ and}$$
$$s(a + b + z + (a + \delta + b)) + i_3 = i_4 + s(a + \delta + b)$$

where $z = b' + a'$. Let $w \in S$ such that $s(a + \delta + b) + w \in I$ as $(I, +)$ is a dense subsemigroup of $(S, +)$. Now $s(a + b + z + (a + \delta + b)) + i_3 + w = i_4 + s(a + \delta + b) + w$ shows that $w+s(a+b+z+(a+\delta+b)) \in I$ as $(I, +)$ is a reflexive and closed subsemigroup of $(S, +)$. This together with the fact $s(a+b+z+(a+\delta+b))+i_1+w = i_2+s(a+b)+w$ shows that $s(a + b) + w \in I$ (since $(I, +)$ is a closed subsemigroup of $(S, +)$). Hence

$s(a+\delta+b)+(w+s(a+b)) = (s(a+\delta+b)+w)+s(a+b)$ where $(w+s(a+b))$, $(s(a+\delta+b)+w) \in I$.

Let I satisfy condition (ii). Since S contains zero, we can easily obtain that I satisfies condition (i). □

The following example shows that the above proposition does not hold if we remove zero from the seminearring under consideration.

Example 3.1.5. Let $S = \mathbb{N} \times \mathbb{N}$ where $\mathbb{N}$ denotes the set of all natural numbers. Let us define '$\oplus$' and '$\odot$' on $S \times S$ as follows :

(i) $(a, b) \oplus (c, d) = (a + c, d)$ and

(ii) $(a, b) \odot (c, d) = (a(c + d), b(c + d))$ where (a, b), $(c, d) \in S$.

Now $(S, \oplus)$ is a semigroup as it is a direct product of two semigroups, *viz.*, $\mathbb{N}$ under usual addition and $\mathbb{N}$ as right zero semigroup. For $(a, b), (c, d), (m, n) \in S$, $((a, b) \odot (c, d)) \odot (m, n) = (a(c+d)(m+n), b(c+d)(m+n)) = (a, b) \odot ((c, d) \odot (m, n))$. Therefore $(S, \odot)$ is a semigroup. Now $((a, b) \oplus (c, d)) \odot (m, n) = ((a+c)(m+n), d(m+n))$. Again $((a, b) \odot (m, n)) \oplus ((c, d)) \odot (m, n)) = (a(m + n), b(m + n)) \oplus (c(m + n), d(m + n)) = ((a+c)(m+n), d(m+n))$. Therefore $((a, b) \oplus (c, d)) \odot (m, n) = ((a, b) \odot (m, n)) \oplus ((c, d) \odot (m, n))$ for all $(a, b), (c, d), (m, n) \in S$. Hence $(S, \oplus, \odot)$ is a seminearring. But S is not with zero as $(S, +)$ is not a monoid. S is not a semiring since $(1, 1) \odot ((2, 2) \oplus (3, 3)) = (8, 8) \neq ((1, 1) \odot (2, 2)) \oplus ((1, 1) \odot (3, 3))$.

Let $I = \{(2n, m) \in S : n, m \in \mathbb{N}\}$.

(a) For $(2n_1, m_1)$, $(2n_2, m_2) \in I$, $(2n_1, m_1) \oplus (2n_2, m_2) = (2(n_1 + n_2), m_2) \in I$. Therefore $(I, \oplus)$ is a subsemigroup of $(S, \oplus)$.

(b) Now let $(a, b) \in S$. Then $(a, b) \oplus (a, b) = (2a, b) \in I$. Therefore $(I, \oplus)$ is a dense subsemigroup of $(S, \oplus)$.

(c) Let $(a, b), (c, d) \in S$ such that $(a, b) \oplus (c, d) \in I$. Therefore $(a + c, d) \in I$ whence $a + c$ is a multiple of 2. Hence $(c, d) \oplus (a, b) = (c + a, b) \in I$ whence $(I, \oplus)$ is a reflexive subsemigroup of $(S, \oplus)$.

(d) Let $(a, b), (c, d) \in S$ such that $(a, b) \oplus (c, d) \in I$ and $(a, b) \in I$. Then $(a + c, d), (a, b) \in I$. Therefore $2|(a + c)$ and $2|a$ whence $2|c$ whence $(c, d) \in I$. Hence $(I, \oplus)$ is a closed subsemigroup of $(S, \oplus)$.

(e) Let $(s_1, s_2), (a_1, a_2), (b_1, b_2) \in S$ and $(2n, m) \in I$. Now $(s_1, s_2) \odot ((a_1, a_2) \oplus (b_1, b_2)) = (s_1, s_2) \odot (a_1 + b_1, b_2) = (s_1(a_1 + b_1 + b_2), s_2(a_1 + b_1 + b_2))$. Again $(s_1, s_2) \odot ((a_1, a_2) \oplus (2n, m) \oplus (b_1, b_2)) = (s_1, s_2) \odot (a_1 + 2n + b_1, b_2) = (s_1(a_1 + 2n + b_1 + b_2), s_2(a_1 + 2n + b_1 + b_2))$. Let $(2ns_1, s_2(a_1 + b_1 + b_2)), (4ns_1, s_2(a_1 + b_1 + b_2)) \in I$. Then

$$(s_1, s_2) \odot ((a_1, a_2) \oplus (2n, m) \oplus (b_1, b_2)) \oplus (2ns_1, s_2(a_1 + b_1 + b_2)) =$$
$$(s_1(a_1 + 2n + b_1 + b_2), s_2(a_1 + 2n + b_1 + b_2)) \oplus (2ns_1, s_2(a_1 + b_1 + b_2)) =$$
$$(s_1(a_1 + 4n + b_1 + b_2), s_2(a_1 + b_1 + b_2)) \text{ and}$$

$$(4ns_1, s_2(a_1 + b_1 + b_2)) \oplus ((s_1, s_2) \odot ((a_1, a_2) \oplus (b_1, b_2))) =$$
$$(4ns_1, s_2(a_1 + b_1 + b_2)) \oplus (s_1(a_1 + b_1 + b_2), s_2(a_1 + b_1 + b_2)) =$$
$$(s_1(a_1 + 4n + b_1 + b_2), s_2(a_1 + b_1 + b_2)).$$

Therefore $(I, \oplus)$ is a dense, reflexive, closed subsemigroup of $(S, \oplus)$ and I satisfies condition (ii) of Proposition 3.1.4. But for $(1, 1)$, $(1, 2) \in S$ and $(2, 1) \in I$ there do not exist any pair of elements i_1, i_2 in I so that $(1, 1) \odot ((1, 2) \oplus (2, 1)) \oplus i_1 = i_2 \oplus ((1, 1) \odot (1, 2))$.

Theorem 3.1.6. *Suppose S is a seminearring and I is a strong, dense, reflexive and closed additive subsemigroup of S such that $IS \subseteq I$. Then the relation*

$$\rho_I = \{(a, b) \in S \times S : \text{there exists } x \in S \text{ such that } a + x, b + x \in I\}$$

is a near-ring congruence on S and $I = \{x \in S : (x, x + x) \in \rho_I\}$.

Proof. Since $(I, +)$ is a dense, reflexive and closed subsemigroup of $(S, +)$, in view of Theorem 1.1.29, ρ_I is a group congruence on $(S, +)$ and $I = \{x \in S : (x, x + x) \in \rho_I\}$. Again ρ_I is a right congruence on $(S, \cdot)$ (*cf.* Proposition 3.1.1). Now let $(a, b) \in \rho_I$ and $s \in S$. Then there exists $x \in S$ such that $a + x, b + x \in I$. Since I is strong in S, $x + b$, $a + x \in I$ show that there exist $i_1, i_2, i_3, i_4 \in I$ such that $s(a + x + b) + i_1 = i_2 + sa$ and $s(a + x + b) + i_3 = i_4 + sb$ ....(1). Let $z \in S$ such that $s(a + x + b) + z \in I$. This together with (1) and the fact that I is a reflexive and closed subsemigroup of $(S, +)$ shows that $sa + z$, $sb + z \in I$. Therefore $(sa, sb) \in \rho_I$ whence ρ_I is a near-ring congruence on S. □

Theorem 3.1.7. *Let S be a seminearring and σ be a near-ring congruence on S. Then $H_\sigma = \{x \in S : (x, x + x) \in \sigma\}$ is a strong, dense, reflexive and closed additive subsemigroup of S satisfying $H_\sigma S \subseteq H_\sigma$. Moreover, the relation ρ_{H_σ} on S defined by*

$$a\ \rho_{H_\sigma}\ b \text{ if and only if there exists } x \in S \text{ such that } a + x,\ b + x \in H_\sigma$$

coincides with σ.

Proof. Since σ is a near-ring congruence on the seminearring S, σ is a group congruence on $(S, +)$. Now $H_\sigma = \{x \in S : (x, x + x) \in \sigma\}$ is the zero of the near-ring S/σ. In view of Theorem 1.1.29, $\rho_{H_\sigma} = \sigma$ and H_σ is a dense, reflexive and closed subsemigroup of $(S, +)$. Let $h \in H_\sigma$ and $s, a \in S$. Then $(h, h + h) \in \sigma$ whence $(hs, hs + hs) \in \sigma$ (since σ is a congruence on the seminearring S). Therefore $hs \in H_\sigma$ and so $H_\sigma S \subseteq H_\sigma$. Now $(a + h, a) \in \sigma$ (since $h \in H_\sigma$ and H_σ is the zero of the near-ring S/σ). Then $(s(a + h), sa) \in \sigma$ (as σ is a congruence on the seminearring S). This together with the fact $\rho_{H_\sigma} = \sigma$ shows that there exists $w \in S$ such that $s(a + h) + w$, $sa + w \in H_\sigma$. Therefore $s(a+h)+(w+sa) = (s(a+h)+w)+sa$ where $w+sa$, $s(a+h)+w \in H_\sigma$. Similarly we can show that there exist h_1, $h_2 \in H_\sigma$ such that $s(h + a) + h_1 = h_2 + sa$. Thus H_σ is a strong, dense, reflexive and closed additive subsemigroup of S satisfying $H_\sigma S \subseteq H_\sigma$. □

Theorem 3.1.8. *Suppose S is a seminearring. Then the set $\{I \subseteq S | I$ is a strong, dense, reflexive and closed additive subsemigroup of S with $IS \subseteq I\}$ and the set of all near-ring congruences on S are in an inclusion preserving bijective correspondence via $I \mapsto \rho_I$ where*

$$a\ \rho_I\ b \text{ if and only if there exists } x \in S \text{ such that } a + x, b + x \in I.$$

Proof. Theorem 3.1.6 shows that the mapping is injective and Theorem 3.1.7 shows that the map is surjective. Again in view of definition of ρ_I and Theorem 3.1.6, $I \subseteq J$ if and only if $\rho_I \subseteq \rho_J$. This completes our proof. □

The following example shows that there are plenty of instances of strong, dense, reflexive and closed additive subsemigroup I of a seminearring S satisfying $IS \subseteq I$.

Example 3.1.9. Consider the seminearring S of Example 3.1.3 and $I = \{f \in S : f(M) \subseteq N\}$. Then in view of Example 3.1.3, I is a strong additive subsemigroup of S.

(i) Let $f \in S$ and $h \in I$. Then $h(M) \subseteq N$. Now $(hf)(M) = h(f(M)) \subseteq h(M) \subseteq N$. Therefore $hf \in I$ whence $IS \subseteq I$.

(ii) Let $f_1 + f_2 \in I$. Then $f_1(m) + f_2(m) \in N$ for all $m \in M$. Since N is a reflexive subsemigroup of M, $f_2(m) + f_1(m) \in N$ for all $m \in M$. Therefore $f_2 + f_1 \in I$ whence I is a reflexive additive subsemigroup of S.

(iii) Let $f+g, f \in I$. Then $f(m)+g(m)$, $f(m) \in N$ for all $m \in M$. Since N is a closed subsemigroup of M, $g(m) \in N$ for all $m \in M$. Then $g \in I$. Therefore I is a closed additive subsemigroup of S.

(iv) Now let $M = \cup_{\lambda \in \Lambda}[w_\lambda]_{\rho_N}$ where $\{w_\lambda : \lambda \in \Lambda\}$ be the representatives of the distinct ρ_N-classes. We fix these representatives and for each w_λ, we fix $w'_\lambda \in M$ such that $w_\lambda + w'_\lambda \in N$. Let $f \in S$ and define $g : M \to M$ by $g(m) = w'_\lambda$ where $f(m) \in [w_\lambda]_{\rho_N}$. Let $a \; \rho_N \; b$. Then $f(a) \; \rho_N \; f(b)$. Let $[w_{\lambda_i}]_{\rho_N} = [f(a)]_{\rho_N} = [f(b)]_{\rho_N}$ for some $\lambda_i \in \Lambda$. Then by definition of g, $g(a) = g(b) = w'_{\lambda_i}$. Hence $g(a) \; \rho_N \; g(b)$, *i.e.,* $g \in S$. Let $m \in M$ and $f(m) \in [w_\lambda]_{\rho_N}$. Then there exists $z \in M$ such that $f(m)+z$, $w_\lambda + z \in N$. Again $w_\lambda + w'_\lambda \in N$. Since $(N,+)$ is a reflexive subsemigroup, $w'_\lambda + (f(m)+z) + w_\lambda \in N$. This together with the fact $(N,+)$ is a reflexive and closed subsemigroup and $z + w_\lambda \in N$ shows that $f(m) + w'_\lambda \in N$ whence $f(m)+g(m) \in N$. Therefore $f+g \in I$ whence I becomes a dense subsemigroup of $(S,+)$, too.

Therefore I is a strong, dense, reflexive and closed additive subsemigroup of S and $IS \subseteq I$.

Theorem 3.1.10. *Suppose S is a seminearring. Then the set $\{I \subseteq S | I$ is a strong, dense, reflexive and closed additive subsemigroup of S with IS, $SI \subseteq I\}$ and the set of all zero-symmetric near-ring congruences on S are in an inclusion preserving bijective correspondence via $I \mapsto \rho_I$ where*

$$a \; \rho_I \; b \text{ if and only if there exists } x \in S \text{ such that } a+x, b+x \in I.$$

Proof. Let I be a strong, dense, reflexive and closed additive subsemigroup of S and IS, $SI \subseteq I$. Then in view of Theorem 3.1.6, the relation ρ_I is a near-ring congruence on S and $I = \{x \in S : (x, x+x) \in \rho_I\} = [i]_{\rho_I}$ is the zero of the near-ring S/ρ_I for any $i \in I$. Let $s \in S$ and $i \in I$. Then $si \in I$ (as $SI \subseteq I$). Hence $[s]_{\rho_I}[i]_{\rho_I} = [i]_{\rho_I}$ and consequently, S/ρ_I becomes a zero-symmetric near-ring.

Let σ be a zero-symmetric near-ring congruence on S. Let $H_\sigma = \{x \in S : (x, x+x) \in \sigma\}$. Then in view of Theorem 3.1.7, H_σ is the zero of the zero-symmetric near-ring S/σ and H_σ is a strong, dense, reflexive and closed additive subsemigroup of S and $H_\sigma S \subseteq S$. Clearly, $[h_1]_\sigma = H_\sigma$ for any $h_1 \in H_\sigma$. Let $h \in H_\sigma$ and $s \in S$. Since S/σ is a zero-symmetric near-ring, $[s]_\sigma [h]_\sigma = H_\sigma$ whence $sh \in H_\sigma$ and thus $SH_\sigma \subseteq H_\sigma$. □

3.2 Near-ring congruences on distributively generated seminearrings

Proposition 3.2.1. *Let S be a distributively generated seminearring and I be a dense, reflexive and closed additive subsemigroup of S. Then I is strong in S if and only if $SI \subseteq I$.*

Proof. Let I be strong in S. Let t be a distributive element of S, $a \in S$ and $\delta \in I$. Then there exist $i_1, i_2 \in I$ such that $t(a+\delta) + i_1 = i_2 + ta$, *i.e.*, $ta + t\delta + i_1 = i_2 + ta$. Since $(I,+)$ is a dense subsemigroup of $(S,+)$, there exists $w \in S$ such that $ta + w \in I$. Therefore $ta + t\delta + i_1 + w = i_2 + ta + w \in I$. This together with the fact that I is a closed and reflexive subsemigroup of $(S,+)$, shows that $t\delta \in I$. Now let $s \in S$. Then $s = \sum_{i=1}^{n} t_i$ where for each i, $1 \leq i \leq n$, t_i is a distributive element of S. Therefore $t_i\delta \in I$ for each i with $1 \leq i \leq n$. So $s\delta = \sum_{i=1}^{n}(t_i\delta) \in I$. Thus $SI \subseteq I$.

Conversely, let $SI \subseteq I$. Now let $s, a \in S$ and $\delta \in I$. Then $s = \sum_{i=1}^{n} t_i$ where for each i, $1 \leq i \leq n$, t_i is a distributive element of S. Since $(I,+)$ is a dense and reflexive subsemigroup of $(S,+)$ and $t_i\delta \in I$, for each i, there exists $w_i \in S$ such that $w_i + t_ia$, $t_ia + w_i$, $t_i\delta + t_ia + w_i$, $w_i + t_i\delta + t_ia \in I$. Then using the fact that $(I,+)$ is reflexive, we obtain $w_2 + (w_1 + t_1a) + t_2a \in I$ and $(t_1\delta + t_1a) + (t_2\delta + t_2a + w_2) + w_1 \in I$. Consequently, $\sum_{i=n}^{1} w_i + \sum_{i=1}^{n} t_ia \in I$, *i.e*, $\sum_{i=n}^{1} w_i + sa \in I$ ....(1) and $\sum_{i=1}^{n}(t_i\delta + t_ia) + \sum_{i=n}^{1} w_i \in I$, *i.e.*, $s(\delta+a) + \sum_{i=n}^{1} w_i \in I$...........(2). Using (1) and (2) we get $s(\delta+a) + (\sum_{i=n}^{1} w_i + sa) = (s(\delta+a) + \sum_{i=n}^{1} w_i) + sa$ where $\sum_{i=n}^{1} w_i + sa$, $s(\delta+a) + \sum_{i=n}^{1} w_i \in I$. Similarly we can show that $s(a+\delta) + (\sum_{i=n}^{1} w_i + sa) = (s(a+\delta) + \sum_{i=n}^{1} w_i) + sa$ where $\sum_{i=n}^{1} w_i + sa$, $s(a+\delta) + \sum_{i=n}^{1} w_i \in I$. Therefore I becomes strong in S. □

The following example shows that if a seminearring is not distributively generated then the above proposition may not hold.

Example 3.2.2. Let $(M,+)$ be a commutative monoid satisfying the following conditions (i) M is a regular semigroup, (ii) $E(M)$ is a closed subsemigroup of M where $E(M)$ denotes the set of all idempotents of M and (iii) there exists at least one pair $(a,b) \in M \times M$ such that there exist no e_1, $e_2 \in E(M)$ for which $e_1 + a = b + e_2$, *i.e.*, $e_1 + a \neq b + e_2$ for all $e_1, e_2 \in E(M)$. Let $w \in M \setminus E(M)$, $e \in E(M)$ such that $w \neq w + e$ (see $(S,+)$ of Example 2.3.4 for such a commutative monoid). Let $S = \{f : M \to M : f(E(M)) \subseteq E(M)$ and $f(0) = 0\}$. Let $g_1, g_2 \in S$. Then $(g_1 + g_2)(e) = g_1(e) + g_2(e) \in E(M)$ for all $e \in E(M)$ as $E(M)$ is a subsemigroup of

M and $g_1(e), g_2(e) \in E(M)$ for all $e \in E(M)$. Again $(g_1 + g_2)(0) = g_1(0) + g_2(0) = 0$. Therefore $g_1 + g_2 \in S$. Now $g_1(g_2(e)) \in E(M)$ for all $e \in E(M)$ and $g_1(g_2(0)) = 0$. Therefore $g_1 g_2 \in S$. Now $\{f : M \to M\}$ is a seminearring under point wise addition and composition of functions and $S \subseteq \{f : M \to M\}$ is closed under addition and composition. Therefore S is a seminearring.

Now let $I = \{f \in S : f(M) \subseteq E(M)\}$.

(i) Since $E(M)$ is a closed subsemigroup of M, it can be easily verified that I is a closed additive subsemigroup of S.

(ii) Since M is a commutative semigroup, S is an additively commutative seminearring and every additive subsemigroup of S is reflexive. Therefore I is a reflexive additive subsemigroup of S.

(iii) Let $f \in S$. Let us define $g : M \to M$ such that $g(0) = 0$ and $g(m) = f(m)'$ for some $f(m)' \in V(f(m))$. Let $e \in E(M)$. Then $f(e) \in E(M)$. Since $E(M)$ is a closed subsemigroup of M, $V(f(e)) \subseteq E(M)$. Therefore $g(e) \in E(M)$. Hence $g \in S$. Now in view of Definition of g, $(f+g)(m) = f(m)+g(m) = f(m)+f(m)'$ (for some $f(m)' \in V(f(m))) \in E(M)$ for all $m \in M$. Therefore $f + g \in I$ and I is a dense additive subsemigroup of S.

(iv) Let $f \in S$ and $g \in I$. Then $f(g(M)) \subseteq f(E(M))$ (as $g(M) \subseteq E(M)) \subseteq E(M)$ (as $f(E(M) \subseteq E(M))$ and $g(f(M)) \subseteq g(M) \subseteq E(M)$ (as $g(M) \subseteq E(M)$). Therefore $fg, gf \in I$ whence SI, $IS \subseteq I$.

Thus $(I, +)$ is a dense, reflexive and closed subsemigroup of $(S, +)$ such that SI, $IS \subseteq I$.

Let $f \in I$ such that $f(0) = 0$ and $f(m) = e$ for all $m \in M \setminus \{0\}$. Again let $g \in S$ such that $g(w) = a$ and $g(w + e) = b$. Then there exist no $i, i' \in I$ such that $g \circ (id + f) + i = i' + g \circ id$ where $id(m) = m$ for all $m \in M$ (since for all e_1, $e_2 \in E(M)$, $g(id + f)(w) + e_1 = g(w + e) + e_1 = b + e_1 \neq e_2 + a = e_2 + g(w) = e_2 + g(id(w))$). Therefore in view of Definition 3.1.2, I fails to be strong in S.

Now we obtain the analogue of Theorem 3.1.8 in the setting of distributively generated seminearrings.

Theorem 3.2.3. *In a distributively generated seminearring S, the set $\{I \subseteq S|I$ is a dense, reflexive and closed additive subsemigroup of S with IS, $SI \subseteq I\}$ and the set of all near-ring congruences (and hence the set of all zero-symmetric near-ring* *congruences) on S are in an inclusion preserving bijective correspondence via $I \mapsto \rho_I$ where*

$$a\rho_I b \text{ if and only if there exists } x \in S \text{ such that } a + x, b + x \in I.$$

Proof. It follows from Theorem 3.1.8, Proposition 3.2.1 and the fact that a distributively generated near-ring is always zero-symmetric (*cf.* (i) of Remark 1.4.9). □

Before we conclude, we highlight (*cf.* Observation 3.2.4) the connection of our main result *viz.,* Theorem 3.1.8 with its counter parts Theorem 1.5.24 (*i.e.,* Theorem 3.6 [83]) and Theorem 2.2.10. In order to accomplish this we recall that there exist inclusion preserving bijective correspondences between

(i) the set of all normal subseminearrings and the set of all additively commutative near-ring congruences in a distributively generated zero-symmetric seminearring (*cf.* Theorem 1.5.24),

(ii) the set of all normal full k-ideals and the set of all near-ring congruences in a distributively generated additively regular seminearring (*cf.* Theorem 2.2.10),

(iii) the set $\{I \subseteq S | I$ is a strong, dense, reflexive and closed additive subsemigroup of S with $IS \subseteq I\}$ and the set of all near-ring congruences in a seminearring S (*cf.* Theorem 3.1.8).

So we make the following observation which relates the above notions in a seminearring S *viz.,* (i) normal subseminearring, (ii) normal full k-ideal, (iii) strong, dense, reflexive and closed additive subsemigroup I of S with $IS \subseteq I$.

Observation 3.2.4. (i) Let S be a distributively generated zero-symmetric seminearring and I be a normal subseminearring (see Definition 1.5.23) of S. Then in view of Definitions 1.1.2, 1.5.11 and 1.5.23, I is a dense, closed additive subsemigroup of S such that $IS, SI \subseteq I$. Since S is zero-symmetric, in view of Note 1 [83] I is a reflexive additive subsemigroup. Then in view of Proposition 3.2.1, I is a strong, dense, reflexive and closed additive subsemigroup of S with $IS \subseteq I$.

But the converse is not true which is evident from the fact that in a distributively generated zero-symmetric near-ring N whose addition is not abelian, $\{0\}$ is a strong, dense, reflexive and closed additive subsemigroup of N with $\{0\}N = \{0\}$ but not a normal subseminearring of N (*cf.* Theorem 2.3 [83]).

(*ii*) If S is not a distributively generated seminearring, then a normal subseminearring I of S may not be a strong, dense, reflexive and closed additive subsemigroup of S with $IS \subseteq I$. Consider I of the seminearring S of Example 3.2.2. Then I is a normal subseminearring (since $(S, +)$ is commutative, I satisfies condition (i) of Definition 1.5.23, (ii) and (iii) of Definition 1.5.23 follow from Definitions 1.1.2, 1.5.12 and Example 3.2.2). But I is not a strong additive subsemigroup whence I is not a strong, dense, reflexive and closed additive subsemigroup of S with $IS \subseteq I$.

(*iii*) If S is a distributively generated additively regular seminearring, then for any normal full k-ideal (see Definition 2.1.6) I of S there exists a near-ring congruence σ_I such that $I = \{a \in S : a\sigma_I e \text{ for some } e \in E^+(S)\}$ (*cf.* Theorem 2.2.3) where $E^+(S)$ denotes the set of all additive idempotents of S. As σ_I is a near-ring congruence, $\{a \in S : a\sigma_I e \text{ for some } e \in E^+(S)\} = \{a \in S : (a + a, a) \in \sigma_I\}$ and hence in view of Theorem 3.1.7, I is a strong, dense, reflexive and closed additive subsemigroup of S with $IS \subseteq I$. Following a similar kind of argument as previous and Theorem 3.1.6, we see that if I is a strong, dense, reflexive and closed additive subsemigroup of S with $IS \subseteq I$ then I must be a normal full k-ideal of S, too.

(*iv*) If S is an additively regular seminearring but not distributively generated, then a normal full k-ideal I may not be a strong, dense, reflexive and closed additive subsemigroup of S with $IS \subseteq I$. In Example 3.2.2, S is an additively regular seminearring (since for any $f \in S$, if we define $g : M \to M$ such that $g(m) = f(m)'$ for some $f(m)' \in V(f(m))$, $g(0) = 0$ then $f + g + f = f$ and $g \in S$). Let $I = \{f \in S : f(M) \subseteq E(M)\}$. Then in view of Example 3.2.2, I is a dense, reflexive and closed additive subsemigroup of S and $IS, SI \subseteq I$. Then in view of Definition 1.5.12, I is a k-ideal such that I is a reflexive additive subsemigroup of S. Again in view of Remark 1.1.28, I is a full additive subsemigroup of S too. Therefore I is a full k-ideal such that I is a reflexive additive subsemigroup of S whence in view of Proposition 2.1.11, I is a normal full k-ideal of S. Since I is not a strong additive subsemigroup, I is not a strong, dense, reflexive and closed additive subsemigroup of S with $IS \subseteq I$.

(*v*) If S is an additively regular seminearring but not distributively generated, then a strong, dense, reflexive and closed additive subsemigroup I of S with $IS \subseteq I$ may not be a normal full k-ideal. For example, consider $E^+(S')$ of the seminearring S' of Example 3.21 [98] and the near-ring congruence η on S' as mentioned in Example 3.21 [98]. Then $E^+(S') = \{x \in S' : (x, x+x) \in \eta\}$. Therefore in view of Theorem 3.1.7, $E^+(S')$ is a strong, dense, reflexive and closed additive subsemigroup of S' with $E^+(S')S' \subseteq E^+(S')$. But in Example 3.21 [98], it has been proved that $E^+(S')$ is not a normal full k-ideal of S'.

CHAPTER 4

LATTICE OF NEAR-RING CONGRUENCES ON SEMINEARRINGS

4

Lattice of Near-ring Congruences on Seminearrings

As a continuation of our study in Chapter 3, in this chapter, our goal is to extend the bijections established in Theorem 3.1.8 and Theorem 3.1.10 to lattice isomorphisms. We also study the lattice structures of the set of all near-ring congruences and the set of all zero-symmetric near-ring congruences of a seminearring, in detail. We organize this chapter as follows.

In the study of near-ring congruences on a seminearring, accomplished in [83, 99] and Chapter 2, inclusion preserving bijections between near-ring congruences and various types of ideals have been established in seminearrings with different kinds of restrictions. In order to upgrade these studies, in Chapter 3, we consider a seminearring S without any restriction and establish inclusion preserving bijective correspondences between (i) the set of all near-ring congruences on S and the set $\{I \subseteq S | I$ is a strong, dense, reflexive and closed additive subsemigroup of S with $IS \subseteq I\}$ and (ii) the set of all zero-symmetric near-ring congruences on S and the set $\{I \subseteq S | I$ is a strong, dense, reflexive and closed additive subsemigroup of S with IS, $SI \subseteq I\}$ (*cf.* Theorems 3.1.8 and 3.1.10). In **Section 1** of this chapter, we describe explicitly the motivation behind considering a strong, dense, reflexive and closed additive subsemigroup I of a seminearring S with $(IS \subseteq I)$ $IS, SI \subseteq I$ to be a suitable structure to get the bijections

This chapter is mainly based on the work of the following paper:
Kamalika Chakraborty et al., *Lattice of Near-ring Congruences on Seminearrings*, Communicated.

established in Theorem 3.1.8 and Theorem 3.1.10. In this direction we first adapt the notion of strong ideal[1] in the setting of seminearring with zero (which need not be zero-symmetric) and redefine it in Definition 4.1.8. Then we define strong dense ideal and characterize it in a seminearring with zero (*cf.* Definition 4.1.11, Theorem 4.1.12). Thereafter in view of Remark 4.1.14, to get a suitable substructure which corresponds to a near-ring congruence on a seminearring, we generalise the notion of strong dense ideal in the setting of a seminearring which need not contain zero and call it a 'generalised strong dense reflexive (right) k-ideal' (*cf.* Definition 4.1.15). We observe that in a seminearring S, I is a generalised strong dense reflexive (right) k-ideal if and only if I is a strong, dense, reflexive and closed additive subsemigroup of S with ($IS \subseteq I$) $IS, SI \subseteq I$ (*cf.* Remark 4.1.17). Then in Theorem 4.1.18, we rewrite Theorem 3.1.8 and Theorem 3.1.10 in terms of generalised strong dense reflexive (right) k-ideals.

In **Section 2**, we focuss on the study of various aspects of lattice structures of generalised strong dense reflexive k-ideals and zero-symmetric near-ring congruences on a seminearring. As a first step to that, in Example 4.2.3, we show that in an arbitrary seminearring, both the set of all generalised strong dense reflexive k-ideals and the set of all zero-symmetric near-ring congruences need not form lattices under set inclusion. This makes us quest after a sufficient condition to form these sets lattices. With the help of Proposition 4.2.6, Proposition 4.2.13 and Theorem 4.2.14, we establish that in a seminearring with left local units (*cf.* Definition 4.2.4) the set of all generalised strong dense reflexive k-ideals forms a lattice with respect to the set inclusion (*cf.* Theorem 4.2.15). Then in view of Proposition 4.2.16, we obtain that the set of all zero-symmetric near-ring congruences on a seminearring with left local units forms a lattice, too (*cf.* Theorem 4.2.17). Finally in Theorem 4.2.18, it has been made possible to extend the bijection between the set of all generalised strong dense reflexive k-ideals and that of all zero-symmetric near-ring congruences in a seminearring with left local units to a lattice isomorphism. Thereafter with the help of this lattice isomorphism, in Theorem 4.2.19, the modularity of both the lattices of Theorem 4.2.18 in a seminearring with left local units has been established. Then in Example 4.2.20, we exhibit that the lattice of all generalised strong dense reflexive k-ideals (and hence the lattice of zero-symmetric near-ring congruences) need not be distributive even if the seminearring under consideration is with left local units. To conclude this section, we obtain some sufficient conditions imposition of which ensures the distributivity of the lattice of

[1] The notion of strong ideal was defined in [44] in the setting of zero-symmetric left distributive seminearring.

generalised strong dense reflexive k-ideals (and hence the lattice of zero-symmetric near-ring congruences) of a seminearring (*cf.* Theorems 4.2.22 and 4.2.26).

In **Section 3**, in order to check the extendibility of the bijection connecting near-ring congruences and generalised strong dense reflexive right k-ideals (stated in (i) of Theorem 4.1.18) to a lattice isomorphism, we first see that in a seminearring, even with left local units, both the set of all generalised strong dense reflexive right k-ideals and the set of all near-ring congruences need not form lattices under set inclusion (*cf.* Example 4.3.1). But in an E^+-inversive seminearring, each of them becomes lattice (*cf.* Theorem 4.3.7). It is also evident from Theorem 4.3.7 that Theorems 4.2.15 and 4.2.17 are also true in the setting of E^+-inversive seminearrings (which need not contain left local units). Then in Theorem 4.3.9, we are able to extend the bijective correspondences, stated in Theorem 4.1.18, to lattice isomorphisms. We conclude this section with the study of modularity, distributivity and completeness of the above-mentioned lattices in an E^+-inversive seminearring (*cf.* Theorem 4.3.10, Example 4.3.11 and Theorem 4.3.12).

4.1 Generalised strong dense reflexive (right) k-ideals

In this section we mainly describe explicitly the motivation behind considering a strong, dense, reflexive and closed additive subsemigroup I of a seminearring S with $IS \subseteq I$ ($IS, SI \subseteq I$) as a suitable structure such that ρ_I (see Theorems 3.1.8 and 3.1.10) becomes a near-ring (zero-symmetric near-ring) congruence. To accomplish this, we first recall some preliminary notions, results and deduce some relevant results of semigroup and seminearring theories for their use in the sequel.

Definition 4.1.1. [68] A non-empty subset I of a semigroup $(S, +)$ is said to be a *normal subsemigroup*[2] if for any x and y which are elements of S or empty symbols and for any k and k_1 lying in I or being empty symbols (given only that x, k_1, y are not all empty symbols),

$$x + k + y \in I \text{ always implies } x + k_1 + y \in I.$$

Remark 4.1.2. (i) A normal subsemigroup of a monoid always contains 0, the identity of the monoid (consider $x = y = 0$, an empty symbol k_1 and k being an element of I).

[2] This is different from [31].

(*ii*) In view of Theorem 4.11 of chapter VII [68], a subset I of a monoid $(M,+)$ is a kernel of some morphism of M if and only if I is a normal subsemigroup.

Remark 4.1.3. [68] Now let I be a normal subsemigroup of a monoid $(M,+)$. Let us define the relation 'r_I' by,

$$a\ r_I\ b \text{ if and only if } a,b \in x+I+y \text{ for some } x,y \in M.$$

Now $a\ r_I\ a$ as $a \in a+I+0$ and $0 \in I$ (*cf.* Remark 4.1.2). Therefore r_I is reflexive. In view of Definition of r_I, it is symmetric and compatible with respect to addition. Let $r_I^{''}$ be the transitive closure (*cf.* Definition 1.1.9) of r_I. Since r_I is compatible with respect to addition, $r_I^{''}$ is also compatible with respect to addition. Therefore $r_I^{''}$ is a semigroup congruence on $(M,+)$.

If $x,y \in I$ then $x,y \in 0+I+0$. Therefore any two elements of I are r_I-related and $I \subseteq [0]_{r_I^{''}}$. Let $x\ r_I\ y$ and $x \in I$. Then in view of Definition of r_I, $x = a+i_1+b$, $y = a+i_2+b$ for some $a,b \in M$ and $i_1,i_2 \in I$ and $a+i_1+b \in I$. Again I is a normal subsemigroup. Then y (*i.e.,* $a+i_2+b$) $\in I$. Therefore

$$\text{if } x\ r_I\ y \text{ and } x \in I \text{ then } y \in I. \text{ .... } (1)$$

Now let $x \in [0]_{r_I^{''}}$. Then (1) shows that $x \in I$. Therefore $I = [0]_{r_I^{''}}$ and I is the identity of the monoid $(M/r_I^{''},+)$.

Let λ_I be the corresponding semigroup homomorphism, *i.e.,*

$$\lambda_I : (M,+) \to (M/r_I^{''},+), s \mapsto [s]_{r_I^{''}}.$$

Clearly, I is the kernel of λ_I. Let ϕ be a semigroup morphism of M such that kernel of ϕ is I and r_ϕ be the corresponding congruence (*i.e.,* $a\ r_\phi\ b$ if and only if $\phi(a) = \phi(b)$). Let $a\ r_I\ b$. Then $a = x+\xi_1+y$ and $b = x+\xi_2+y$ for some $x,y \in M$ and $\xi_1,\xi_2 \in I$. Therefore $\phi(a) = \phi(b) = \phi(x)+\phi(y)$. Then $r_I \subseteq r_\phi$ whence $r_I^{''} \subseteq r_\phi$. Thus for a congruence ρ on $(M,+)$ such that $[0]_\rho = I$, $r_I^{''} \subseteq \rho$.

Notation 4.1.4. Let $(I,+)$ be a normal subsemigroup of a monoid $(M,+)$. Throughout this thesis, unless mentioned otherwise, r_I , $r_I^{''}$ and λ_I stand for what we have stated in Remark 4.1.3.

Remark 4.1.5. Let S be a seminearring with zero and I be an ideal of S. Then in view of Definition 1.5.27 and Remark 4.1.2, $(I,+)$ is a normal subsemigroup of $(S,+)$ and I is a right invariant subset (*cf.* Definition 1.5.28) of S.

Now we rewrite the definitions of property Q and strong ideal (*cf.* [44]) in the setting of seminearring with zero (which need not be zero-symmetric always).

Definition 4.1.6. Let S be a seminearring with zero.

(i) A normal subsemigroup $(I,+)$ of $(S,+)$ is said to *have property* Q if the condition

$$Q(I): \text{ for all } a,b,s \in S \text{ and } \delta \in I, \quad s(a+\delta+b)\; r_I^{''}\; s(a+b)$$

holds.

(ii) An ideal (*cf.* Definition 1.5.27) I of S is said to be a *strong ideal* if I satisfies property Q.

Remark 4.1.7. Let $(S,+,\cdot)$ be a seminearring with zero.

(i) Let $(I,+)$ be a normal subsemigroup of $(S,+)$. Then from Theorem 3 and Theorem 5 of [44] we obtain that I has property Q if and only if $r_I^{''}$ is a left congruence on S and I is right invariant if and only if $r_I^{''}$ is a right congruence on S.

(ii) In view of Remark 4.1.5, a strong ideal I of S satisfies the following conditions: (1) $(I,+)$ is a normal subsemigroup of $(S,+)$, (2) I is a right invariant subset of the seminearring S, (3) I satisfies property Q. On the other hand let us consider a subset J of S which satisfies the above-mentioned three properties. Then in view of (i) of this remark, $r_J^{''}$ is a congruence on S. Then J becomes an ideal of S since $ker\lambda_J = [0]_{r_J^{''}} = J$, where λ_J is the corresponding homomorphism (*cf.* Remark 4.1.3). Hence J is a strong ideal. So we can rewrite the notion of strong ideal (*cf.* Definition 4.1.6) as follows.

Definition 4.1.8. In a seminearring $(S,+,\cdot)$ with zero, a subset I of S is said to be a *strong ideal* if I satisfies the following conditions:

(1) $(I,+)$ is a normal subsemigroup of $(S,+)$,

(2) I is a right invariant subset of the seminearring S, *i.e.,* $IS \subseteq I$ and

(3) I satisfies property Q.

The following result *viz.,* Proposition 4.1.9 related with a monoid plays an important role in the sequel.

Proposition 4.1.9. *Let* $(M,+)$ *be a monoid. Then* *(i) a dense normal subsemigroup of M is same as a dense, reflexive and closed subsemigroup of M,*

(ii) if I is a dense normal subsemigroup of M,

$$r_I^{''} = \{(a, b) \in M \times M\colon \textit{ there exist } x, y \in I \textit{ such that } a + x = y + b \}.$$

Proof. (i) Let $(I, +)$ be a dense normal subsemigroup ($i.e.$, a subsemigroup which is normal as well as dense) of $(M, +)$. $(I, +)$ is a closed subsemigroup of M since if $x+y$, $x \in I$ then $y \in I$. Let $a + b \in I$. There exists $a^{'} \in M$ such that $a^{'} + a \in I$ since I is a dense subsemigroup of M. Now $a^{'} + 0 + a \in I$ and $a + b \in I$. Then in view of Definition 4.1.1, $a^{'} + (a + b) + a \in I$. Therefore $(a^{'} + a) + (b + a)$, $a^{'} + a \in I$ and I is a closed subsemigroup. Then $b + a \in I$ whence I is a reflexive subsemigroup. Thus I is a dense, reflexive and closed subsemigroup of M. On the other hand, suppose $(I, +)$ is a dense, reflexive and closed subsemigroup of a monoid $(M, +)$. Let $x + i + y \in I$ where $x, y \in M$ and $i \in I$. Let $i_1 \in I$. Now $y + x + i \in I$ since I is a reflexive subsemigroup. Then $y + x \in I$ since I is a closed subsemigroup. Therefore $y + x + i_1 \in I$ as I is a subsemigroup of M whence $x + i_1 + y \in I$ ($\because$ $(I, +)$ is reflexive). Thus I is a dense normal subsemigroup of M.

(ii) Let I be a dense normal subsemigroup of $(M, +)$ and $\pi_I = \{(a, b) \in M \times M\colon$ there exist $x, y \in I$ such that $a + x = y + b \}$. Clearly, $(I, +)$ is a dense, reflexive and closed subsemigroup of M and π_I is a group congruence on M (*cf.* Theorem 1.1.29). Let $(a, b) \in r_I$. Then there exist $i_1, i_2 \in I$ and $x, y \in M$ such that $a = x + i_1 + y$ and $b = x + i_2 + y$. Since I is dense and reflexive, there exist y' and x' such that $y' + y$, $y + y'$, $x + x'$ and $x' + x \in I$. Let $w_1 = y' + ((x' + x) + i_2) + y$ and $w_2 = x + (i_1 + (y + y')) + x'$. Then $w_1, w_2 \in I$ as I is a normal subsemigroup of $(M, +)$. Then $a + w_1 = w_2 + b$ whence $r_I \subseteq \pi_I$. Therefore $r_I^{''} \subseteq \pi_I$ since π_I is a congruence and $r_I^{''}$ is the transitive closure of r_I. Conversely, suppose $(a, b) \in \pi_I$. Then there exist $i_1, i_2 \in I$ such that $a + i_1 = i_2 + b$. Clearly, $(a, a + i_1)$, $(i_2 + b, b) \in r_I$. Therefore $(a, b) \in r_I''$. Hence $\pi_I \subseteq r_I''$. □

The following example shows that a normal subsemigroup may not be a reflexive subsemigroup.

Example 4.1.10. Let $M(\mathbb{N})$ be the set of all functions from the set of all natural numbers $\mathbb{N}$ to $\mathbb{N}$. Then $(M(\mathbb{N}), \circ)$ is a monoid where $\circ$ denotes the composition of functions. Clearly $id_{\mathbb{N}}$, the identity mapping of $\mathbb{N}$ ($i.e.,$ $id_{\mathbb{N}}(x) = x$ for all $x \in \mathbb{N}$) is the identity of the monoid $M(\mathbb{N})$ and $\{id_{\mathbb{N}}\}$ is a normal subsemigroup of $M(\mathbb{N})$. Let us define $f, g : \mathbb{N} \to \mathbb{N}$ by the following ways:

$$f(x) = x \text{ if } x \text{ is odd and } f(x) = x/2 \text{ if } x \text{ is even and}$$
$$g(x) = 2x \text{ for all } x \in \mathbb{N}.$$

Then $f \circ g \in \{id_{\mathbb{N}}\}$ but $g \circ f \notin \{id_{\mathbb{N}}\}$.

Definition 4.1.11. A subset I of a seminearring S with zero is said to be a *strong dense ideal* if I is a strong ideal of S and $(I, +)$ is a dense subsemigroup of $(S, +)$.

In view of Definitions 4.1.8 and 4.1.11 and Proposition 4.1.9, we obtain the following characterization of strong dense ideals of a seminearring with zero.

Theorem 4.1.12. *Let S be a seminearring with zero and $I \subseteq S$. Then I is a strong dense ideal of S if and only if I satisfies the following three conditions :*

(1) $(I, +)$ is a dense, reflexive and closed subsemigroup of $(S, +)$,

(2) I is a right invariant subset of the seminearring S (i.e., $IS \subseteq I$),

(3) for any $a, b, s \in S$ and $i \in I$ there exist $i_1, i_2 \in I$ such that

$$s(a + i + b) + i_1 = i_2 + s(a + b).$$

Now we rewrite Proposition 3.1.4 in the following manner.

Proposition 4.1.13. *Let S be a seminearring with zero and I be a dense, reflexive and closed additive subsemigroup of S. Then the following conditions are equivalent.*

(i) For any $a, b, s \in S$ and $i \in I$ there exist $i_1, i_2 \in I$ such that $s(a + i + b) + i_1 = i_2 + s(a + b)$.

(ii) For any $s, a \in S$ and $w \in I$ there exist i_1, i_2, i_3, $i_4 \in I$ such that $s(a + w) + i_1 = i_2 + sa$ and $s(w + a) + i_3 = i_4 + sa$.

But if the seminearring is without zero, then the above Proposition may not be true which is evident from Example 3.1.5.

Remark 4.1.14. Let I be a strong dense ideal in a seminearring S with zero. Then r_I'' is a congruence on the seminearring S (*cf.* Remark 4.1.7 and Definition 4.1.8). Since I is a dense, reflexive and closed additive subsemigroup of S (*cf.* Theorem 4.1.12), Theorem 1.1.29 together with Proposition 4.1.9 shows that r_I'' is a group congruence on the semigroup $(S, +)$. Then in view of Remark 1.5.22, r_I'' becomes a near-ring congruence on the seminearring S. Therefore in a seminearring with zero, a strong dense ideal corresponds a near-ring congruence.

Now in order to get a substructure which corresponds a near-ring congruence in a seminearring (which need not contain zero), we aim to generalize the structure of 'strong dense ideal' in the setting of seminearrings which need not contain zero. But in a seminearring without zero, definitions of ideal, strong ideal and strong dense ideal (*cf.* Definitions 1.5.27, 4.1.6 and 4.1.11), in terms of kernels of some morphisms, are no longer tenable. So, in what follows, in view of Theorem 4.1.12 and Proposition 4.1.13, we choose right invariant subset whose additive reduct is a dense, reflexive, closed subsemigroup satisfying condition (ii) of Proposition 4.1.13 as a suitable one (*cf.* Definition 4.1.15) for generalizing the notion of strong dense ideal.

Definition 4.1.15. A non-empty subset I of a seminearring S is said to be a *generalised strong dense reflexive (left, right) k-ideal* if I satisfies the following conditions :

(i) $(I, +)$ is a dense, reflexive subsemigroup of $(S, +)$,

(ii) I is a (left, right) k-ideal,

(iii) for $s, a \in S$ and $w \in I$ there exist $i_1, i_2, i_3, i_4 \in I$ such that $s(a+w)+i_1 = i_2+sa$ and $s(w + a) + i_3 = i_4 + sa$.

Remark 4.1.16. In a seminearring S with zero, I is a strong dense ideal of S if and only if I is a generalised strong dense reflexive right k-ideal.

Remark 4.1.17. In a seminearring S,

(i) I is a generalised strong dense reflexive right k-ideal if and only if I is a strong, dense, reflexive and closed additive subsemigroup of S with $IS \subseteq I$,

(ii) I is a generalised strong dense reflexive k-ideal if and only if I is a strong, dense, reflexive and closed additive subsemigroup of S with $IS, SI \subseteq I$.

We now rewrite Theorem 3.1.8 and Theorem 3.1.10 in the following way.

Theorem 4.1.18. *Suppose S is a seminearring. Then there exist inclusion preserving bijective correspondences between*

(i) the set of all generalised strong dense reflexive right k-ideals of S and the set of all near-ring congruences on S,

(ii) the set of all generalised strong dense reflexive k-ideals of S and the set of all zero-symmetric near-ring congruences on S

via $I \mapsto \rho_I$ where $a \; \rho_I \; b$ if and only if there exists $x \in S$ such that $a + x, b + x \in I$.

Remark 4.1.19. In a seminearring S, the inverse of the bijection mentioned in Theorem 4.1.18 is $\rho \mapsto ker\rho$ which is evident from its proof (*cf.* Proof of Theorems 3.1.8 and 3.1.10) where $ker\rho = \{x \in S : (x+x, x) \in \rho\}$. Clearly, $ker\rho$ is a generalised strong dense reflexive right k-ideal and $ker\rho = 0_{S/\rho}$. It is obvious that $(a, b) \in \rho$ if and only if there exists $x \in S$ such that $a + x$, $b + x \in ker\rho$ where ρ is a near-ring congruence on the seminearring S.

4.2 Lattice of zero-symmetric near-ring congruences

Now want to investigate whether the bijection between the set of all zero-symmetric near-ring congruences and the set of all generalised strong dense reflexive k-ideals in an arbitrary seminearring (stated in (*ii*) of Theorem 4.1.18) can be extended to a lattice isomorphism. In order to accomplish this we first obtain following two propositions.

Proposition 4.2.1. *Let I be a reflexive closed subsemigroup of a semigroup $(S, +)$. If for $a, b \in S$, there exists $x \in S$ such that $a + x, b + x \in I$ then for $y \in S$, $a + y \in I$ implies $b + y \in I$.*

Proof. Since $b+x, a+y \in I$, $(b+x)+(a+y) \in I$. Therefore $y+b+x+a \in I$ as $(I, +)$ is a reflexive subsemigroup of $(S, +)$. This together with the facts that $x + a \in I$ and I is a closed subsemigroup of $(S, +)$ shows that $y + b \in I$. Hence $b + y \in I$. □

Proposition 4.2.2. *Let S be a seminearring and I be a (right) k-ideal of S such that I is a dense and reflexive additive subsemigroup of S. Then I is a generalised strong dense reflexive (right) k-ideal if and only if for $s, a \in S$ and $i \in I$ there exists $z \in S$ such that $s(a + i) + z$, $s(i + a) + z$, $sa + z \in I$.*

Proof. Let I be a generalised strong dense reflexive (right) k-ideal of S and $s, a \in S$, $i \in I$. Then in view of Definition 4.1.15, there exist $i_1, i_2, i_3, i_4 \in I$ such that $s(a+i)+i_1 = i_2+sa$ and $s(i+a)+i_3 = i_4+sa$. Since $(I,+)$ is a dense subsemigroup of $(S,+)$, there exists $z \in S$ such that $sa+z \in I$. This together with the facts $s(i+a)+i_3+z = i_4+sa+z$ and $s(a+i)+i_1+z = i_2+sa+z$ shows that $s(i+a)+z$, $s(a+i)+z \in I$. Conversely, suppose for $s, a \in S$ and $i \in I$ there exists $z \in S$ such that $s(a+i)+z$, $s(i+a)+z$, $sa+z \in I$. Since $(I,+)$ is a reflexive subsemigroup of $(S,+)$, $z+sa \in I$. Now $s(a+i)+(z+sa) = (s(a+i)+z)+sa$ and $s(i+a)+(z+sa) = (s(i+a)+z)+sa$ where $z+sa$, $s(a+i)+z$, $s(i+a)+z \in I$. Therefore I is a generalised strong dense reflexive (right) k-ideal of S. □

In an arbitrary seminearring, the set of all generalised strong dense reflexive k-ideals need not form a lattice which is evident from the following example. So as a first step towards achieving our desired lattice isomorphism (as stated at the very beginning of this section), we would like to search for a sufficient condition to make the set of all generalised strong dense reflexive k-ideals form a lattice.

Example 4.2.3. Consider the semigroup $(\mathbb{R}_0^+, +)$ where $\mathbb{R}_0^+$ denotes the set of all non-negative real numbers and '+' is the usual addition of real numbers. Let $I = \{2n : n \in \mathbb{N}\} \cup \{0\}$ and $J = \{\sqrt{2}n : n \in \mathbb{N}\} \cup \{0\}$. Clearly, $(I,+)$ and $(J,+)$ are dense, closed and reflexive subsemigroups of $(\mathbb{R}_0^+, +)$. Then in view of Definition 1.1.2, $(I+J)w = \{2a + b\sqrt{2} \in \mathbb{R}_0^+ | a, b \in \mathbb{Z}\}$. Clearly, $(I+J)w$ is a closed subsemigroup of $(\mathbb{R}_0^+, +)$ containing both I and J. Let $\Delta = \mathbb{R}_0^+ \setminus (I+J)w$. Let

$$a \cdot_\Delta b = \begin{cases} a & \text{if } b \in \Delta \\ 0 & \text{if } b \notin \Delta. \end{cases}$$

Then in view of Example 1.4 (b)[91], $(\mathbb{R}_0^+, +, \cdot_\Delta)$ becomes a seminearring. Using Propositions 4.2.1 and 4.2.2, it can be checked that both I and J are generalised strong dense reflexive k-ideals but $I \cap J = \{0\}$ fails to be a generalised strong dense reflexive k-ideal.

Again in view of Theorem 4.1.18 (ii), ρ_I, ρ_J are zero-symmetric near-ring congruences on the seminearring $\mathbb{R}_0^+$. Let $a \in \mathbb{R}_0^+$. Now there does not exist any $x \in \mathbb{R}_0^+$ such that $a+x \in I \cap J$. Again in view of Theorem 3.1.6, $I = \{x \in \mathbb{R}_0^+ : (x+x, x) \in \rho_I\}$ and $J = \{x \in \mathbb{R}_0^+ : (x+x, x) \in \rho_J\}$. Therefore there does not exist any $x \in \mathbb{R}_0^+$ such that $(a+x, a+x+a+x) \in \rho_I \cap \rho_J$. This shows that $\rho_I \cap \rho_J$ is not a group congruence on the semigroup $(\mathbb{R}_0^+, +)$. Hence $\rho_I \cap \rho_J$ is not a zero-symmetric near-ring congruence on the seminearring $\mathbb{R}_0^+$.

It is evident from the above example that the intersection of two generalised strong dene reflexive k-ideals is not necessarily a generalised strong dene reflexive k-ideal. To overcome this we take impetus from [83] since normal subseminearring is a generalised strong dense reflexive k-ideal in a distributively generated zero-symmetric seminearring (*cf.* Remark 4.1.17 and Observation 3.2.4). In [83], the authors considered seminearrings with multiplicative identity to make the intersection of two normal subseminearrings a normal subseminearring again (*cf.* Proposition 2.4 [83]). This motivates us to consider seminearring with left local units (*cf.* Definition 4.2.4) as the notion of left local units generalizes the notion of multiplicative identity.

Definition 4.2.4. A seminearring $(S, +, \cdot)$ is said to be a *seminearring with left (right) local units* if for each $a \in S$, there exists $l_a \in S$ (resp., $r_a \in S$) such that $l_a a = a$ (resp., $ar_a = a$). A seminearring $(S, +, \cdot)$ is said to be a *seminearring with local units* if for each $a \in S$, there exists $e_a \in S$ such that $e_a a = a = ae_a$.

Example 4.2.5. Let $(S, +)$ be a semigroup. Let us define $\odot$ on S by $a \odot b = a$ for all $a, b \in S$. Then $(S, +, \odot)$ is a seminearring with left local units since $a \odot a = a$ for all $a \in S$.

Proposition 4.2.6. *Let S be a seminearring with left local units. Then intersection of any two generalised strong dense reflexive k-ideals is a generalised strong dense reflexive k-ideal.*

Proof. Let I, J be two generalised strong dense reflexive k-ideals of S. Clearly $I \cap J$ is a k-ideal of S such that $(I \cap J, +)$ is a reflexive subsemigroup of $(S, +)$. Let $s \in S$. Then there exists $x \in S$ such that $s + x \in I$. Since S is with left local units, for $s + x \in S$, there exists $l_{s+x} \in S$ such that $l_{s+x}(s + x) = s + x$. Again there exists $z \in S$ such that $l_{s+x} + z \in J$ as $(J, +)$ is a dense subsemigroup of $(S, +)$. Hence $(l_{s+x} + z)(s + x) \in I \cap J$ *i.e.,* $s + (x + z(s + x)) \in I \cap J$. Therefore $I \cap J$ is a dense subsemigroup of $(S, +)$.

Let $s, a \in S$ and $w \in I \cap J$. Since I, J are generalised strong dense reflexive k-ideals, in view of Proposition 4.2.2, there exist $x, y \in S$ such that $s(a + w) + x$, $s(w + a) + x$, $sa + x \in I$ and $s(a + w) + y$, $s(w + a) + y$, $sa + y \in J$. Now there exists $z \in S$ such that $sa + z \in I \cap J$ as $(I \cap J, +)$ is dense in $(S, +)$. Proposition 4.2.1 together with $sa + z, s(a + w) + x$, $s(w + a) + x$, $sa + x \in I$ shows that $s(a + w) + z$, $s(w + a) + z \in I$. Similarly it can be shown that $s(a + w) + z$, $s(w + a) + z \in J$. Therefore $s(a + w) + z$, $s(w + a) + z$, $sa + z \in I \cap J$. So in view of Proposition 4.2.2, $I \cap J$ is a generalised strong dense reflexive k-ideal. □

It is evident from the above Proposition that in a seminearring S with left local units, for any two members I, J of the set of all generalised strong dense reflexive k-ideals of S, $I \cap J$ becomes the meet of I and J. Now we are going to construct the join of I and J. The following result on the sum (see Definition 2.3.1) of two generalised strong dense reflexive k-ideals will be used frequently in what follows.

Proposition 4.2.7. *Let A, B be two generalised strong dense reflexive k-ideals of a seminearring $(S, +, \cdot)$ with left local units. Then for each $x \in A + B$,*

(i) there exists $z_1 \in A$ such that $z_1 + x$, $x + z_1 \in B$,

(ii) there exists $z_2 \in B$ such that $z_2 + x$, $x + z_2 \in A$.

Proof. Let $x \in A + B$. Then $x = \sum_{i=1}^{n} x_i$ where $x_i \in A \cup B$. Now there exists a subset $\{r_1, r_2, ..., r_k\}$ of $\{1, 2, ..., n\}$ such that $r_1 \leq r_2 \leq .... \leq r_k$, $k \leq n$ and $\{x_{r_j} : 1 \leq j \leq k\} \subseteq A$. Since $A \cap B$ is a generalised strong dense reflexive k-ideal (*cf.* Proposition 4.2.6), for each j where $1 \leq j \leq k$, there exists $y_{r_j} \in S$ such that $y_{r_j} + x_{r_j} \in A \cap B$. Now $y_{r_1} + (x_1 + x_2 + ... + x_{r_1 - 1}) + x_{r_1} \in B$ (as B is a generalised strong dense reflexive k-ideal and $(x_1 + x_2 + ... + x_{r_1 - 1})$, $y_{r_1} + x_{r_1} \in B$). In a similar way we get $y_{r_2} + (y_{r_1} + x_1 + x_2 + ... + x_{r_1 - 1} + x_{r_1} + x_{r_1 + 1} + ... + x_{r_2 - 1}) + x_{r_2} \in B$ as $(x_{r_1 + 1} + ... + x_{r_2 - 1}) \in B$. Following this manner we finally get $y_{r_k} + y_{r_{k-1}} + ... + y_{r_1} + x_1 + ... + x_{r_k} \in B$. Again $x_{r_k + 1} + ... + x_n \in B$. Let $z_1 = y_{r_k} + y_{r_{k-1}} + ... + y_{r_1}$. Therefore $z_1 + x \in B$. Now for each j where $1 \leq j \leq k$, $y_{r_j} + x_{r_j}, x_{r_j} \in A$. This together with the fact A is a generalised strong dense reflexive k-ideal shows that $y_{r_j} \in A$ for each j, $1 \leq j \leq k$. Therefore $z_1 \in A$. This proves (i).

(ii) follows similarly due to symmetry of A and B in the sum $A + B$. □

Remark 4.2.8. It is well known that the sum of two right $\mathcal{S}$-ideals is a right $\mathcal{S}$-ideal in any seminearring S and the sum of two $\mathcal{S}$-ideals is an $\mathcal{S}$-ideal in a distributively generated seminearring S [83]. But the sum of two k-ideals need not be a k-ideal even in a distributively generated seminearring (see Example 3 [83]). It is here noteworthy that since the sum of two normal subseminearrings (see Definition 1.5.23) in a distributively generated seminearring need not be a normal subseminearring (*cf.* Example 4 [83]), the authors took the help of the notion of closure (Definition 2.6 and Proposition 2.8 [83]) in order to obtain a positive result in this direction in [83]. As a consequence, in our setting, *i.e.,* for a seminearring with left local units, we are also interested to see what happens to the sum of two generalised strong dense reflexive k-ideals. In this regard we provide the following example illustrating that the sum of two generalised strong dense reflexive k-ideals of a seminearring with left local units is not always a generalised strong dense reflexive k-ideal. In Theorem 4.2.14 we obtain the join of two generalised strong dense reflexive k-ideals in the form of k-closure (*cf.* Definition 2.3.5) of the sum of two generalised strong dense reflexive k-ideals.

Example 4.2.9. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then $(\mathbb{N}_0, +, \cdot)$ is a semiring under usual addition and multiplication. $\mathbb{N}_0$ is also a seminearring with left local units (here $l_a = 1$ for each $a \in \mathbb{N}_0$). Now $2\mathbb{N}_0 = \{2n : n \in \mathbb{N}_0\}$ and $3\mathbb{N}_0 = \{3n : n \in \mathbb{N}_0\}$ are generalised strong dense reflexive k-ideals of $\mathbb{N}_0$. But $2\mathbb{N}_0 + 3\mathbb{N}_0$ is not a k-ideal of $\mathbb{N}$ as $1+2, 2 \in 2\mathbb{N}_0 + 3\mathbb{N}_0$ but $1 \notin 2\mathbb{N}_0 + 3\mathbb{N}_0$. Therefore $2\mathbb{N}_0 + 3\mathbb{N}_0$ is not a generalised strong dense reflexive k-ideal of $\mathbb{N}_0$.

For our ready references, we recall Definitions 2.3.5 and 2.3.8 of Chapter 2.

Definition 4.2.10. Let A be a (left, right) $\mathcal{S}$-ideal of a seminearring $(S, +, \cdot)$. The *closure* [83] of A is defined to be the set $\{s \in S : \text{ for some } x, y \in A,\ x + s + y \in A\}$ and is denoted by $\bar{A}$.

Definition 4.2.11. Let $(S, +, \cdot)$ be a seminearring and A, B be two (right) $\mathcal{S}$-ideals of S. Then we define four subsets of S as follows :

(*i*) $X_{1,A+B} := \{s \in S : \text{ for some } b \in B,\ b + s \in A\}$,

(*ii*) $X_{2,A+B} := \{s \in S : \text{ for some } b \in B,\ s + b \in A\}$,

(*iii*) $X_{3,A+B} := \{s \in S : \text{ for some } a \in A,\ a + s \in B\}$,

(*iv*) $X_{4,A+B} := \{s \in S : \text{ for some } a \in A,\ s + a \in B\}$.

Observation 4.2.12. (1) In a seminearring S, if A is a reflexive subsemigroup of $(S, +)$, then $X_{1,A+B} = X_{2,A+B}$ and if B is a reflexive subsemigroup of $(S, +)$, then $X_{3,A+B} = X_{4,A+B}$.

(2) In view of Proposition 4.2.7, in a seminearring S with left local units, if A, B are generalised strong dense reflexive k-ideals of S, $A + B \subseteq X_{i,A+B}$ for $1 \leq i \leq 4$.

Proposition 4.2.13. *Let $(S, +, \cdot)$ be a seminearring with left local units and A, B be two generalised strong dense reflexive k-ideals of S. Then* $\overline{A+B} = X_{1,A+B} = X_{2,A+B} = X_{3,A+B} = X_{4,A+B}$.

Proof. Let $x \in \overline{A+B}$. Then there exist $s, t \in A+B$ such that $s+x+t \in A+B$. In view of Proposition 4.2.7, there exist $s_1, t_1 \in A$ such that s_1+s, $t+t_1 \in B$. Therefore $s_1+s+x+t+t_1 \in A+B$, *i.e.,* $b+x+b_1 \in A+B$ where $b(=s_1+s)$, $b_1(=t+t_1)$ $\in B$. Again there exists $y \in S$ such that $y+x, x+y \in A \cap B$ as $A \cap B$ is a generalised strong dense reflexive k-ideal of S (*cf.* Proposition 4.2.6). Hence $b+x+b_1+y+x \in A+B$, *i.e.,* $(b+b_2)+x \in A+B$ where $b_2(=x+b_1+y) \in B$ as B is a generalised strong dense reflexive k-ideal. Again using Proposition 4.2.7 we get $b_3 \in B$ such that $b_3+(b+b_2+x) \in A$. Hence $x \in X_{1,A+B}$.

Now let $s \in X_{1,A+B}$. There exists $b \in B$ such that $b+s \in A$. Then $b+s+a \in A+B$ for any $a \in A$. So $s \in \overline{A+B}$ whence $X_{1,A+B} \subseteq \overline{A+B}$. Hence $X_{1,A+B} = \overline{A+B}$. Using similar type of arguments as above and in view of (i) of Observation 4.2.12, we deduce the other equalities. □

Theorem 4.2.14. *Let $(S,+,\cdot)$ be a seminearring with left local units and A, B be two generalised strong dense reflexive k-ideals of S. Then $\overline{A+B}$ is the smallest generalised strong dense reflexive k-ideal containing A and B.*

Proof. It follows from the definition of closure that A, $B \subseteq \overline{A+B}$. Let $x, y \in \overline{A+B}$. Then in view of Definition 4.2.11 (iii), (iv) and Proposition 4.2.13, there exist a, a_1 $\in A$ such that $a+x$, $y+a_1 \in B$. Therefore $(a+x)+(y+a_1) \in B \subseteq A+B$. Then $x+y \in \overline{A+B}$. Now let $s+t, s \in \overline{A+B}$. Then there exist $b, b_1 \in B$ such that $b+s$, $s+t+b_1 \in A$ (*cf.* Definition 4.2.11 (i), (ii) and Proposition 4.2.13). Therefore $b+(s+t+b_1) \in A+B$ where $b+s, b_1 \in A+B$. Then $t \in \overline{A+B}$ whence $(\overline{A+B},+)$ is a closed subsemigroup of $(S,+)$. Now for $s+t \in \overline{A+B}$, there exists $b_1 \in B$ such that $s+t+b_1 \in A$ (*cf.* Definition 4.2.11 (ii) and Proposition 4.2.13). Now for $t \in S$, there exists $w \in S$ such that $t+w \in A \cap B$ since $A \cap B$ is a generalised strong dense reflexive k-ideal (*cf.* Proposition 4.2.6). Then $t+(s+t+b_1)+w \in A$ and $t+b_1+w \in B$. Therefore $(t+s)+z \in A$ where $z = t+b_1+w \in B$ whence $t+s \in \overline{A+B}$ (*cf.* Definition 4.2.11 (ii) and Proposition 4.2.13). Since $\overline{A+B}$ contains both A, B and $(A,+), (B,+)$ are dense subsemigroups of $(S,+)$, $(\overline{A+B},+)$ is also a dense subsemigroup of $(S,+)$ whence $\overline{A+B}$ becomes a dense, reflexive and closed subsemigroup of $(S,+)$.

Now let $x \in \overline{A+B}$ and $s \in S$. Then there exists $a \in A$ such that $x+a \in B$ (*cf.* Definition 4.2.11 (iv) and Proposition 4.2.13). Therefore $xs+as \in B$ where $as \in A$. Hence $xs \in \overline{A+B}$ and $\overline{A+B}$ becomes a dense reflexive right k-ideal of S.

Let $s, x \in S$ and $w \in \overline{A+B}$. Then there exists $a \in A$ such that $a+w$, $w+a \in B$ (*cf.* Definition 4.2.11 (iii) and Proposition 4.2.13). Since $A \cap B$ is a generalised strong dense reflexive k-ideal, there exists $z \in S$ such that $s(x+w+a)+z \in A \cap B$. Now for $w+a \in B$, there exists $z_1 \in S$ such that $s(x+w+a)+z_1$, $sx+z_1 \in B$ and for $a \in A$, there exists $z_2 \in S$ such that $s(x+w+a)+z_2$, $s(x+w)+z_2 \in A$ (*cf.* Proposition 4.2.2). Proposition 4.2.1 together with $s(x+w+a)+z$, $s(x+w+a)+z_1$, $sx+z_1 \in B$ shows that $sx+z \in B \subseteq \overline{A+B}$. Similarly, Proposition 4.2.1 together with the fact $s(x+w+a)+z$, $s(x+w+a)+z_2$, $s(x+w)+z_2 \in A$ shows that $s(x+w)+z \in A \subseteq \overline{A+B}$. Therefore $s(x+w)+z, sx+z \in \overline{A+B}$. In a similar way we can show that there exists $y \in S$ such that $s(w+x)+y$, $sx+y \in \overline{A+B}$. Therefore $(s(w+x)+y)+(sx+z) \in \overline{A+B}$. This together with the facts that $(\overline{A+B},+)$ is a reflexive, closed subsemigroup of $(S,+)$ and $sx+y \in \overline{A+B}$ shows that $s(w+x)+z \in \overline{A+B}$. Hence in view of Proposition 4.2.2, $\overline{A+B}$ is a generalised strong dense reflexive right k-ideal.

Let $w \in \overline{A+B}$ and $s \in S$. Then there exists $a \in A$ such that $w+a \in B$. Since A is a generalised strong dense reflexive k-ideal, there exist $a_1, a_2 \in A$ such that $s(w+a)+a_1 = a_2+sw$. Since B is a generalised strong dense reflexive k-ideal, $s(w+a) \in B \subseteq A+B \subseteq \overline{A+B}$. Therefore $s(w+a)+a_1 \in \overline{A+B}$ *i.e.*, $a_2+sw \in \overline{A+B}$ and $a_2 \in A \subseteq A+B \subseteq \overline{A+B}$. This together with the fact $\overline{A+B}$ is a generalised strong dense reflexive right k-ideal shows that $sw \in \overline{A+B}$. Therefore $\overline{A+B}$ is a generalised strong dense reflexive k-ideal. □

Now Proposition 4.2.6 and Theorem 4.2.14 together imply the following result.

Theorem 4.2.15. *Let $(S,+,\cdot)$ be a seminearring with left local units. Then the set of all generalised strong dense reflexive k-ideals of S under set inclusion forms a lattice where for any two generalised strong dense reflexive k-ideals I, J of S, $I \wedge J = I \cap J$ and $I \vee J = \overline{I+J}$.*

Now we are going to study the lattice structure of the set of all zero-symmetric near-ring congruences on a seminearring with left local units, in which (ii) of Theorem 4.1.18 plays an important role.

Proposition 4.2.16. *In a seminearring S with left local units, for any two zero-symmetric near-ring congruences ρ, γ on S*

(i) $ker(\rho \cap \gamma) = ker\rho \cap ker\gamma$,

(ii) $ker(\rho \circ \gamma) = \overline{ker\rho + ker\gamma}$,

(iii) $\rho \cap \gamma = \{(x,y) \in S \times S : \text{there exists } z \in S \text{ such that } x+z, y+z \in ker(\rho \cap \gamma)\}$,

(iv) $\rho \circ \gamma = \{(x, y) \in S \times S : \textit{there exists } z \in S \textit{ such that } x + z, y + z \in \overline{ker\rho + ker\gamma}\}$ *where* $ker\rho = \{x \in S : (x + x, x) \in \rho\}$.

Proof. (i) We omit the proof of (i) as it is a matter of routine verification.

(ii) Let $x \in ker(\rho \circ \gamma)$. Then there exists $z \in S$ such that $(x + x, z) \in \rho$ and $(z, x) \in \gamma$. Since $ker\rho \cap ker\gamma$ is a generalised strong dense reflexive k-ideal (*cf.* Proposition 4.2.6), there exists $t \in S$ such that $z + t \in ker\rho \cap ker\gamma$. Now $(z + t, x + t) \in \gamma$ and $(z + t) \in ker\gamma$ show that $[(x + t)]_\gamma = [(z + t)]_\gamma = 0_{S/\gamma}$. Again $(x + (x + t), (z + t)) \in \rho$ and $(z + t) \in \ker\rho$ show that $x + (x + t) \in ker\rho$ where $(x + t) \in ker\gamma$. Then in view of Proposition 4.2.13, $x \in \overline{ker\rho + ker\gamma}$. Now let $w \in \overline{ker\rho + ker\gamma}$. Then in view of Proposition 4.2.13, there exists $y \in ker\rho$ such that $w + y \in ker\gamma$. Since $y \in ker\rho$ and ρ is a zero-symmetric near-ring congruence on S, $(w + w, w + y + w) \in \rho$. Again since $w + y \in ker\gamma$ and γ is a zero-symmetric near-ring congruence on S, $(w + y + w, w) \in \gamma$. Therefore $(w + w, w) \in \rho \circ \gamma$ whence $w \in ker(\rho \circ \gamma)$.

(iii) Let $(x, y) \in \rho \cap \gamma$. Then there exist $z_1, z_2 \in S$ such that $x + z_1, y + z_1 \in ker\rho$ and $x + z_2, y + z_2 \in ker\gamma$ (*cf.* Remark 4.1.19). Again in view of Proposition 4.2.6, there exists $w \in S$ such that $x + w \in ker\rho \cap ker\gamma$. This together with Proposition 4.2.1 shows that $x + w, y + w \in ker(\rho \cap \gamma)$ since $ker(\rho \cap \gamma)$ is a reflexive closed subsemigroup of $(S, +)$ (*cf.* Proposition 4.2.6). Again if there exists $z \in S$ such that $a + z, b + z \in ker(\rho \cap \gamma)$=$ker\rho \cap ker\gamma$, then in view of Remark 4.1.19, $(a, b) \in (\rho \cap \gamma)$.

(iv) Let $(x, y) \in \rho \circ \gamma$. Then there exists $z \in S$ such that $(x, z) \in \rho$ and $(z, y) \in \gamma$. Now in view of Remark 4.1.19, there exist $x_1, x_2 \in S$ such that $x + x_1, z + x_1 \in ker\rho$ and $z + x_2, y + x_2 \in ker\gamma$. Again in view of Proposition 4.2.6, there exists $w \in S$ such that $z + w \in ker\rho \cap ker\gamma$. This together with Proposition 4.2.1 shows that $x + w \in ker\rho \subseteq \overline{ker\rho + ker\gamma}$ and $y + w \in ker\gamma \subseteq \overline{ker\rho + ker\gamma}$ since $ker\rho$, $ker\gamma$ are reflexive closed subsemigroups of $(S, +)$. Therefore $x + w$, $y + w$ $\in \overline{ker\rho + ker\gamma}$. Again let there exist $z \in S$ such that $a + z, b + z \in \overline{ker\rho + ker\gamma}$. Then there exist $w_1 \in ker\rho$, $w_2 \in ker\gamma$ such that $w_1 + a + z \in ker\gamma$ and $b + z + w_2 \in ker\rho$ (*cf.* Proposition 4.2.13). Therefore $(a, b + z + w_2 + w_1 + a) \in \rho$. Again $w_1 + a + z, w_2 \in ker\gamma$ shows that $z + w_2 + w_1 + a \in ker\gamma$ (as $ker\gamma$ is a reflexive subsemigroup of $(S, +)$). Hence $(b + z + w_2 + w_1 + a, b) \in \gamma$. Thus $(a, b) \in \rho \circ \gamma$.

□

Theorem 4.2.17. *Suppose $(S, +, \cdot)$ is a seminearring with left local units. Then the set of all zero-symmetric near-ring congruences on S is a lattice under set inclusion where for any two zero-symmetric near-ring congruences σ, γ on S, $\sigma \wedge \gamma = \sigma \cap \gamma$ and $\sigma \vee \gamma = \sigma \circ \gamma$.*

Proof. Let S be a seminearring with left local units and σ, γ be two zero-symmetric near-ring congruences S. Then in view of Remark 4.1.19, $ker\sigma$, $ker\gamma$ are generalised strong dense reflexive k-ideals of S. Again in view of Theorem 4.2.15, $ker\sigma \cap ker\gamma$, $\overline{ker\sigma + ker\gamma}$ are generalised strong dense reflexive k-ideals of S. This together with (ii) of Theorem 4.1.18 shows that $\rho_{ker\sigma \cap ker\gamma}$, $\rho_{\overline{ker\sigma + ker\gamma}}$ are zero-symmetric near-ring congruences on S. Now in view of (iii) and (iv) of Proposition 4.2.16, $\rho_{ker\sigma \cap ker\gamma} = \sigma \cap \gamma$ and $\rho_{\overline{ker\sigma + ker\gamma}} = \sigma \circ \gamma$. This completes our proof. □

In view of (ii) of Theorem 4.1.18, combination of Theorem 4.2.15 and Theorem 4.2.17 gives our desired lattice isomorphism which is stated below.

Theorem 4.2.18. *Suppose $(S, +, \cdot)$ is a seminearring with left local units. Then the lattice of all generalised strong dense reflexive k-ideals of S and the lattice of all zero-symmetric near-ring congruences on S are isomorphic.*

Theorem 4.2.19. *In a seminearring S with left local units, (i) the set of all generalised strong dense reflexive k-ideals of S and (ii) the set of all zero-symmetric near-ring congruences on S are modular lattices.*

Proof. The proof can be done in a similar manner to the proof of Theorem 2.3.17. □

In a seminearring S, the lattice of all generalised strong dense reflexive k-ideals of S may not be distributive even if S satisfies the hypothesis of Theorem 4.2.18 which is evident from the following example[3].

Example 4.2.20. Let $\mathbf{Q}[x, y]$ be the polynomial ring in two variables over the rational numbers $\mathbf{Q}$. Then $(\mathbf{Q}[x, y], +, \cdot)$ is a seminearring with left local units under usual addition and multiplication of polynomials. It can be easily shown that every ideal of $\mathbf{Q}[x, y]$ is a generalised strong dense reflexive k-ideal. Now let $I_1 = \langle x \rangle$, $I_2 = \langle x^2, x+y \rangle$ and $I_3 = \langle x^2, x - y \rangle$. Now $I_1 \cap (\overline{I_2 + I_3}) = \langle x \rangle$ but $\overline{(I_1 \cap I_2) + (I_1 \cap I_3)} = \langle x^2, xy \rangle$.

[3] This example is inspired from [28].

It is well known that the set of all ideals of a ring R becomes a distributive lattice if $IJ = I \cap J$ for all ideals I, J of R [12]. Now inspired by this concept, we want to obtain some sufficient conditions imposition of which on the lattice of all generalised strong dense reflexive k-ideals ensures the distributivity of the same. To accomplish this, our first step is to investigate whether the product (see Definition 2.3.20) of two generalised strong dense reflexive k-ideals becomes a generalised strong dense reflexive k-ideal or not.

Example 4.2.21. Let S be the seminearring of Example 3.18 [98]. Then $S' = S \times \mathbb{Z}_2$ is clearly a seminearring where $(\mathbb{Z}_2, +, \cdot)$ has the usual meaning. It is easy to verify that the set of all additive idempotents $E^+(S') = \{(0,0), (a,0), (b,0), (c,0), (d,0)\}$ is a generalised strong dense reflexive k-ideal of S'. Now

$$E^+(S')E^+(S') = \{(0,0), (a,0), (b,0), (d,0)\}.$$

Now $(c,0)+(a,0) = (d,0)$ and $(a,0), (d,0) \in E^+(S')E^+(S')$ but $(c,0) \notin E^+(S')E^+(S')$. Therefore $E^+(S')E^+(S')$ is not a closed subsemigroup of $(S',+)$ whence $E^+(S')E^+(S')$ is not a generalised strong dense reflexive k-ideal of the seminearring S'.

As the above example exhibits that the product of two generalised strong dense reflexive k-ideals in a seminearring S usually lacks the property of being a closed subsemigroup of $(S,+)$, we focus on dealing with the closure of the product of two generalised strong dense reflexive k-ideals in what follows.

Theorem 4.2.22. *Let $(S,+,\cdot)$ be a seminearring with left local units. If $\overline{IJ} = I \cap J$ for all generalised strong dense reflexive k-ideals I, J of S, then (i) the set of all generalised strong dense reflexive k-ideals of S and (ii) the set of all zero-symmetric near-ring congruences on S become distributive lattices.*

Proof. (i) Let I, J, K be three generalised strong dense reflexive k-ideals. Then $\overline{(J \cap I) + (K \cap I)} \subseteq \overline{(J+K)} \cap I$ (*cf.* Lemma 1.2.11). Now let $a \in \overline{(J+K)}I$. Clearly, $a = \sum_{i=1}^{n} x_i y_i$ where for all $1 \leq i \leq n$, $x_i \in \overline{(J+K)}$ and $y_i \in I$. Then by Definition 4.2.11 (iv) and Proposition 4.2.13, for each x_i, $1 \leq i \leq n$ there exists $z_i \in J$ such that $x_i + z_i \in K$. Now $x_i y_i + z_i y_i \in KI \subseteq \overline{KI}$ for all $1 \leq i \leq n$. By hypothesise $\overline{KI} = K \cap I$. So $\overline{KI}$ is a generalised strong dense reflexive k-ideal. Then $x_{n-1}y_{n-1} + (x_n y_n + z_n y_n) + z_{n-1}y_{n-1} \in \overline{KI}$ since $(\overline{KI}, +)$ is a reflexive subsemigroup of $(S,+)$ and $x_{n-1}y_{n-1} + z_{n-1}y_{n-1}$, $x_n y_n + z_n y_n \in \overline{KI}$. Proceeding in this way we obtain $\sum_{i=1}^{n} x_i y_i + \sum_{k=n}^{1} z_k y_k \in \overline{KI}$. Therefore $a + b \in \overline{KI}$ where $b = \sum_{k=n}^{1} z_k y_k \in JI \subseteq \overline{JI}$. Then in view of Definition 4.2.11 (iv)

and Proposition 4.2.13, $a \in \overline{\overline{JI} + \overline{KI}}$ whence $(\overline{J+K})I \subseteq \overline{\overline{JI} + \overline{KI}}$. Thus $\overline{\overline{(J+K)}I}$ $\subseteq \overline{\overline{JI} + \overline{KI}}$, *i.e.*, $\overline{(J+K)} \cap I \subseteq \overline{(J \cap I) + (K \cap I)}$. Therefore $\overline{\overline{(J+K)}I} = \overline{\overline{JI} + \overline{KI}}$. As $\overline{JI} = J \cap I$, we deduce from the last relation that $\overline{(J+K)} \cap I = \overline{J \cap I + K \cap I}$. This completes the proof of (i).

Consequently (ii) holds in view of Theorem 4.2.18. □

Now we are going to find a suitable class of seminearrings where $\overline{IJ} = I \cap J$ for all generalised strong dense reflexive k-ideals I, J.

Definition 4.2.23. A seminearring $(S, +, \cdot)$ is said to be a *multiplicatively regular seminearring* [42] if $(S, \cdot)$ is a regular semigroup *i.e.,* for each $a \in S$ there exists $b \in S$ such that $a = aba$.

A seminearring S is said to be a *k-regular seminearring* [83] if for each $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$.

Remark 4.2.24. Though a multiplicatively regular seminearring is not necessarily a k-regular seminearring (see Example 9 [83]), a zero-symmetric multiplicatively regular seminearring is always a k-regular seminearring, whereas a k-regular seminearring need not be either multiplicatively regular or additively regular seminearring which is evident from Example 10 [83].

Proposition 4.2.25. *If S is a multiplicatively regular seminearring or a k-regular seminearring, then $\overline{IJ} = I \cap J$ for any two generalised strong dense reflexive k-ideals I, J of S.*

Proof. Let I, J be two generalised strong dense reflexive k-ideals of a seminearring S. Then $IJ \subseteq I \cap J$ whence $\overline{IJ} \subseteq I \cap J$ as $I \cap J$ is a k-ideal. Now it is a matter of routine verification to show that if S is multiplicatively regular then $\overline{IJ} = I \cap J$. Again let S be a k-regular seminearring and $a \in I \cap J$. Then there exist $x, y \in S$ such that $a + axa = aya$. Now $(ax)a, (ay)a \in IJ$. Then in view of Definition 2.3.5, $a \in \overline{IJ}$. □

Consequently, we obtain the following theorem in view of Theorem 4.2.22.

Theorem 4.2.26. *Let S be a k-regular seminearring with left local units or a multiplicatively regular seminearring. Then (i) the set of all generalised strong dense reflexive k-ideals of S and (ii) the set of all zero-symmetric near-ring congruences on S are distributive lattices.*

4.3 Lattice of near-ring congruences

In (i) of Theorem 4.1.18, we have seen that the set of all near-ring congruences and the set of all generalised strong dense reflexive right k-ideals are in a bijective correspondence in an arbitrary seminearring. In this connection, we now want to investigate whether this bijection can be extended to a lattice isomorphism as in the previous section we have managed to extend the bijection between the set of all zero-symmetric near-ring congruences and the set of all generalised strong dense reflexive k-ideals (as stated in (ii) of Theorem 4.1.18) to a lattice isomorphism (*cf.* Theorem 4.2.18) in a seminearring with left local units.

In order to accomplish this, we now exhibit in the following example that the set of all generalised strong dense reflexive right k-ideals need not form a lattice even in a seminearring containing left local units and zero.

Example 4.3.1. Consider the seminearring $(\mathbb{R}_0^+, +, \odot)$ where $\mathbb{R}_0^+$ denotes the set of all non-negative real numbers, '+' is the usual addition on real numbers and $a \odot b = a$ for all $a, b \in \mathbb{R}_0^+$. Clearly, this is a seminearring with left local units. Let $I = \{2n : n \in \mathbb{N}\} \cup \{0\}$ and $J = \{\sqrt{2}n : n \in \mathbb{N}\} \cup \{0\}$ where $\mathbb{N}$ denotes the set of all natural numbers. It can be easily verified that $(I, +)$, $(J, +)$ are dense, reflexive and closed subsemigroups of $(\mathbb{R}_0^+, +)$ and I, J are right invariant subsets of the seminearring $\mathbb{R}_0^+$. Since $s \odot (a+x) = s \odot (x+a) = s \odot a = s$ for all $s, a, x \in \mathbb{R}_0^+$, $s \odot (x+i)+i = i+s \odot x$, $s \odot (i+x)+i = i+s \odot x$, $s \odot (x+j)+j = j+s \odot x$, $s \odot (j+x)+j = j+s \odot x$ for all $s, x \in \mathbb{R}_0^+$, for all $i \in I$ and for all $j \in J$. Then in view of Definition 4.1.15, I and J are generalised strong dense reflexive right k-ideals. Now $I \cap J = \{0\}$ which is not a dense subsemigroup of $(\mathbb{R}_0^+, +)$. Therefore $I \cap J$ is not a generalised strong dense reflexive right k-ideal of $\mathbb{R}_0^+$.

Again in view of Theorem 4.1.18 (i), ρ_I, ρ_J are near-ring congruences on the seminearring $(\mathbb{R}_0^+, +, \odot)$. Let $a \in \mathbb{R}_0^+$. Now there does not exist any $x \in \mathbb{R}_0^+$ such that $a + x \in I \cap J$. Again in view of Theorem 3.1.6, $I = \{x \in S : (x + x, x) \in \rho_I\}$ and $J = \{x \in S : (x + x, x) \in \rho_J\}$. Therefore there does not exist any $x \in \mathbb{R}_0^+$ such that $(a + x, a + x + a + x) \in \rho_I \cap \rho_J$. This shows that $\rho_I \cap \rho_J$ is not a group congruence on $(\mathbb{R}_0^+, +)$. Hence $\rho_I \cap \rho_J$ is not a near-ring congruence on the seminearring $\mathbb{R}_0^+$.

It is evident from the above example that the intersection of two generalised strong dene reflexive right k-ideals may not be a generalised strong dene reflexive right k-ideal. To overcome this we take impetus from Chapter 2 since in a distributively generated additively regular seminearring, a normal full k-ideal coincides with a generalised strong dense reflexive right k-ideal (*cf.* Remark 4.1.16 and Observation 3.2.4). In Proposition 2.3.12, we showed that in a distributively generated additively regular seminearring, intersection of two normal full k-ideals is a normal full k-ideal. This motivates us to consider E^+-inversive seminearrings[4] as the notion of E^+-inversive seminearring generalizes the notion of additively regular seminearring.

Definition 4.3.2. A seminearring $(S,+,\cdot)$ is said to be an E^+*-inversive* or *additively E-inversive seminearring* if $(S,+)$ is an E-inversive semigroup *i.e.,* for any $s \in S$ there exists $x \in S$ such that $s+x \in E^+(S)$ where $E^+(S)$ denotes the set of all additive idempotents of S.

Notation 4.3.3. Let $(S,+,\cdot)$ be a seminearring. Throughout the thesis, for each element $a \in S$, $W^+(a)$ always stands for the set of all additive weak inverses of a *i.e.,* the set $\{x \in S : x = x+a+x\}$.

Remark 4.3.4. It is easy to see that a seminearring $(S,+,\cdot)$ is E^+-inversive if and only if $W^+(a)$ (*cf.* Notation 4.3.3) is non-empty for all $a \in S$ (*cf.* [31]).

Proposition 4.3.5. *For a semigroup $(S,+)$, the seminearring $M(S)$ of all self maps under point-wise addition and composition is an E^+-inversive seminearring if and only if $(S,+)$ is an E-inversive semigroup.*

Proof. Let $(S,+)$ be an E-inversive semigroup and $f \in M(S)$. As $(S,+)$ is an E-inversive semigroup, for each $s \in S$, there exists $x_s \in S$ such that $f(s)+x_s$ is an idempotent of S, *i.e.,* $f(s)+x_s+f(s)+x_s = f(s)+x_s$. By invoking axiom of choice we define a function $g : S \longrightarrow S$ by $g(s) = x_s$ for all $s \in S$. Then for all $s \in S$, $(f+g)(s)+(f+g)(s) = f(s)+x_s+f(s)+x_s = f(s)+x_s = f(s)+g(s) = (f+g)(s)$. whence $f+g$ is an additive idempotent of the seminearring $M(S)$. So $M(S)$ is an E^+-inversive seminearring.

Conversely, suppose $M(S)$ is an E^+-inversive seminearring. Then for each $s \in S$, let us define $f_s : S \longrightarrow S$ by $f_s(x) = s$ for all $x \in S$. Clearly $f_s \in M(S)$. So there exists $f'_s \in M(S)$ such that $f_s + f'_s$ is an additive idempotent of the seminearring $M(S)$

[4] We have defined E^+-inversive seminearring *i.e.,* additively E-inversive seminearring adapting the technique applied in [80, 81, 82, 98] where the authors have replaced regular, inverse, completely regular, Clifford semigroups respectively by additively regular, additively inverse, additively completely regular, additively Clifford seminearrings.

(as $M(S)$ is an E^+-inversive seminearring). Let $x \in S$ and $f'_s(x) = z$. Then

$$\begin{aligned}(f_s + f'_s)(x) + (f_s + f'_s)(x) &= (f_s + f'_s)(x) \\ (s + f'_s(x)) + (s + f'_s(x)) &= (s + f'_s(x)) \\ (s + z) + (s + z) &= (s + z)\end{aligned}$$

Therefore $s + z$ is an idempotent of the semigroup $(S, +)$. Consequently, $(S, +)$ is an E-inversive semigroup. □

The following result related with an E-inversive semigroup plays an important role in the sequel.

Proposition 4.3.6. *In an E-inverse semigroup S, a closed subsemigroup N is a full subsemigroup if and only if it is a dense subsemigroup.*

Proof. Let N be a full, closed subsemigroup of an E-inversive semigroup $(S, +)$. Then for each $s \in S$, there exists $t \in S$ such that $s + t \in E(S)$, the set of all idempotents of S. Again $E(S) \subseteq N$ (since N is a full subsemigroup of S (*cf.* Definition 1.1.20)). Therefore N is a dense subsemigroup of S. Let N be a dense, closed subsemigroup of S and $e \in E(S)$. Then there exists $x \in S$ such that $e + x \in N$. Now $e + (e + x)$, $e + x \in N$ and N is a closed subsemigroup of S show that $e \in N$ whence N is a full subsemigroup of S. □

Now we are going to establish in Theorem 4.3.7 that in the setting of an E^+-inversive seminearring, Propositions 4.2.6, 4.2.7, 4.2.13 and Theorems 4.2.14, 4.2.15 hold for both generalised strong dense reflexive right k-ideals and generalised strong dense reflexive k-ideals and Proposition 4.2.16 and Theorem 4.2.17 hold for both near-ring and zero-symmetric near-ring congruences even if we remove the existence of left local units from the seminearring under consideration.

Theorem 4.3.7. *Let $(S, +, \cdot)$ be an E^+-inversive seminearring. Then the following are true.*

(i) Intersection of any two generalised strong dense reflexive (right) k-ideals is a generalised strong dense reflexive (right) k-ideal.

(ii) If A, B are two generalised strong dense reflexive (right) k-ideals of S, then for each $x \in A + B$, there exist $z_1 \in A$ and $z_2 \in B$ such that $z_1 + x$, $x + z_1 \in B$ and $z_2 + x$, $x + z_2 \in A$.

(iii) *If* A, B *are two generalised strong dense reflexive (right)* k*-ideals of* S, $\overline{A+B} = X_{1,A+B} = X_{2,A+B} = X_{3,A+B} = X_{4,A+B}$.

(iv) *For two generalised strong dense reflexive (right)* k*-ideals* A, B *of* S, $\overline{A+B}$ *is the smallest generalised strong dense reflexive (right)* k*-ideal containing* A *and* B.

(v) *The set of all generalised strong dense reflexive (right)* k*-ideals of* S *under set inclusion forms a lattice where for any two generalised strong dense reflexive (right)* k*-ideals* I, J *of* S, $I \wedge J = I \cap J$ *and* $I \vee J = \overline{I+J}$.

(vi) *For any two (zero-symmetric) near-ring congruences* ρ, γ *on* S, (1) $ker(\rho \cap \gamma) = ker\rho \cap ker\gamma$, (2) $ker(\rho \circ \gamma) = \overline{ker\rho + ker\gamma}$, (3) $\rho \cap \gamma = \{(x, y) \in S \times S :$ *there exists* $z \in S$ *such that* $x + z, y + z \in ker(\rho \cap \gamma)\}$, *and* (4) $\rho \circ \gamma = \{(x, y) \in S \times S :$ *there exists* $z \in S$ *such that* $x + z, y + z \in \overline{ker\rho + ker\gamma}\}$.

(vii) *The set of all (zero-symmetric) near-ring congruences on* S *becomes a lattice under set inclusion where for any two (zero-symmetric) near-ring congruences* ρ, γ *on* S, $\rho \wedge \gamma = \rho \cap \gamma$ *and* $\rho \vee \gamma = \rho \circ \gamma$.

Proof. In an E^+-inversive seminearring S, for a generalised strong dense reflexive (right) k-ideal I of S, $(I, +)$ is always a full subsemigroup of $(S, +)$ (*cf.* Proposition 4.3.6). Since intersection of two generalised strong dense reflexive (right) k-ideals is a closed, full subsemigroup of $(S, +)$, it becomes a dense subsemigroup of $(S, +)$, too. Then following the manner of the proof of Proposition 4.2.6, it can be proved that (i) holds.

Using result (i) of this Theorem instead of Proposition 4.2.6 in the proof of Proposition 4.2.7, (ii) can be proved.

If we use results (i) and (ii) of this Theorem instead of Proposition 4.2.6 and Proposition 4.2.7 in the proof of Proposition 4.2.13, (iii) is obtained.

If we use results (i) and (iii) of this Theorem instead of Proposition 4.2.6 and Proposition 4.2.13 in the proof of Theorem 4.2.14, (iv) is obtained.

(i) and (iv) of this Theorem together imply (v).

In the proof of Proposition 4.2.16, if we use result (i) of this Theorem instead of Proposition 4.2.6, result (iii) of this Theorem instead of Proposition 4.2.13, (vi) can be proved.

Proof of (vii) follows in view of results (v) and (vi) of this Theorem, Theorem 4.1.18 and Remark 4.1.19. □

Remark 4.3.8. If a seminearring S is with left local units, the set of all generalised strong dense reflexive k-ideals of S (and hence the set of all zero-symmetric near-ring congruences on S) forms a lattice. (*cf.* Theorems 4.2.15 and 4.2.17). But in a seminearring with left local units, neither the set of all generalised strong dense reflexive right k-ideals nor the set of all near-ring congruences need to form lattice which is evident from Example 4.3.1. Again if we consider an E^+-inversive seminearring S (which need not be with left local units), then both the set of all generalised strong dense reflexive right k-ideals of S (and hence the set of all near-ring congruences on S) and the set of all generalised strong dense reflexive k-ideals of S (and hence the set of all zero-symmetric near-ring congruences on S) become lattices (*cf.* Theorem 4.3.7).

In view of Theorem 4.1.18, combination of (v) and (vii) of Theorem 4.3.7 gives our desired lattice isomorphism which is stated below.

Theorem 4.3.9. *Suppose* $(S, +, \cdot)$ *is an* E^+*-inversive seminearring. Then*

(i) the lattice of all generalised strong dense reflexive right k*-ideals of* S *and the lattice of all near-ring congruences on* S *are isomorphic,*

(ii) the lattice of all generalised strong dense reflexive k*-ideals of* S *and the lattice of all zero-symmetric near-ring congruences on* S *are isomorphic.*

In the following result we study the modularity, distributivity of the lattices of near-ring congruences and zero-symmetric near-ring congruences on an E^+-inversive seminearring.

Theorem 4.3.10. *Suppose* $(S, +, \cdot)$ *is an* E^+*-inversive seminearring. Then the following are true.*

(i) The set of all generalised strong dense reflexive right k*-ideals of* S *and the set of all near-ring congruences on* S *become modular lattices.*

(ii) The set of all generalised strong dense reflexive k*-ideals of* S *and the set of all zero-symmetric near-ring congruences on* S *become modular lattices.*

(iii) If $\overline{IJ} = I \cap J$ *for all generalised strong dense reflexive right* k*-ideals* I, J *of* S*, then the set of all generalised strong dense reflexive right* k*-ideals and the set of all near-ring congruences on* S *become distributive lattices.*

(iv) *If* $\overline{IJ} = I \cap J$ *for all generalised strong dense reflexive* k*-ideals* I, J *of* S*, then the set of all generalised strong dense reflexive* k*-ideals and the set of all zero-symmetric near-ring congruences on* S *become distributive lattices.*

Proof. We omit the proof since (*i*) and (*ii*) follow in a similar manner to the proof of Theorem 2.3.17 and (*iii*), (*iv*) hold in a similar manner to the proof of Theorem 2.3.22. □

To conclude this section we establish (*cf.* Theorem 4.3.12) the completeness of four lattices obtained above for E^+-inversive seminearrings. But two of these lattices that occur in the setting of seminearrings with left local units are not necessarily complete which is evident from the following example.

Example 4.3.11. $\mathbb{N}_0$, the set of all non-negative integers, is a semiring with 1 under usual addition and multiplication whence a seminearring with left local units. Now for each $n \in \mathbb{N}$, $I_n = \{nx : x \in \mathbb{N}_0\}$ is a generalised strong dense reflexive k-ideal of $\mathbb{N}_0$. But $\bigcap_{n\in\mathbb{N}} I_n = \{0\}$ is not a generalised strong dense reflexive k-ideal.

Theorem 4.3.12. *In an* E^+*-inversive seminearring* S*,*

(i) *the set of all generalised strong dense reflexive right* k*-ideals of* S *and the set of all near-ring congruences on* S *are complete lattices,*

(ii) *the set of all generalised strong dense reflexive* k*-ideals of* S *and the set of all zero-symmetric near-ring congruences on* S *are complete lattices.*

Proof. (i) By (*v*) of Theorem 4.3.7, the set of all generalised strong dense reflexive right k-ideals of S is a lattice. Let A be a subset of the set of all generalised strong dense reflexive right k-ideals. Then $(\bigcap_{I\in A} I, +)$ is a closed, reflexive subsemigroup of $(S, +)$. Since I is a dense, closed subsemigroup, $E^+(S) \subseteq I$ for all $I \in A$ where $E^+(S)$ denotes the set of all additive idempotents of S. Therefore $E^+(S) \subseteq \bigcap_{I\in A} I$. Then in view of Proposition 4.3.6, $(\bigcap_{I\in A} I, +)$ is a dense subsemigroup of $(S, +)$. Therefore $(\bigcap_{I\in A} I, +)$ is a dense, reflexive and closed subsemigroup of $(S, +)$. It can be easily verified that $\bigcap_{I\in A} I$ is a right invariant subset of the seminearring S. Now let $s, a \in S$ and $w \in \bigcap_{I\in A} I$. Then for each $I \in A$, there exists $u_I \in S$ such that $s(a + w) + u_I, sa + u_I, s(w + a) + u_I \in I$ (*cf.* Proposition 4.2.2). Again there exists $z \in S$ such that $sa + z \in E^+(S) \subseteq I$ for each $I \in A$ as $(S, +)$ is an E-inversive semigroup. Since $s(a + w) + u_I, sa + u_I, s(w + a) + u_I, sa + z \in I$ and $(I, +)$ is a reflexive, closed subsemigroup of $(S, +)$, in view of

Proposition 4.2.1 $s(w + a) + z, sa + z, s(a + w) + z \in I$ for each $I \in A$. Therefore $s(w + a) + z, sa + z, s(a + w) + z \in \bigcap_{I \in A} I$. Then in view of Proposition 4.2.2, $\bigcap_{I \in A} I$ is a generalised strong dense reflexive right k-ideal whence in view of Theorem 1.2.16, the set of all generalised strong dense reflexive right k-ideals of S becomes a complete lattice. Then in view of Theorem 4.3.9 (i), the set of all near-ring congruences on S becomes a complete lattice.

(ii) follows similarly. □

CHAPTER 5

FULL SUBDIRECT PRODUCTS OF A BI-SEMILATTICE AND A ZERO-SYMMETRIC NEAR-RING

5

Full Subdirect products of a Bi-semilattice and a Zero-symmetric Near-ring

Seminearrings find their natural home in the set of all self maps of any additive semigroup (not necessarily commutative). So it is natural to investigate as to how the theory of semigroups can be made into work in the study of seminearrings. This study has been accomplished in a good many number of papers such as [80, 81, 82, 98]. Major part of this study is devoted to obtain analogues of some structure theorems of semigroups such as "a semigroup is completely regular if and only if it is a semilattice of completely simple semigroups if and only if it is a union of groups", "a semigroup is Clifford if and only if it is a semilattice of groups if and only if it is a strong semilattice of groups". This trend of development of seminearring theory together with a structure theorem of semigroups, *viz.*, "a semigroup is a full subdirect product of a semilattice and a group if and only if it is an E-inversive sturdy semilattice of cancellative monoids" obtained by Mitsch in Theorem 14 [78] (*cf.* Theorem 1.1.33), has motivated us to characterize full subdirect products of a bi-semilattice and a (zero-symmetric) near-ring and subdirect products of a distributive lattice and a (zero-symmetric) near-ring in the class of 'E^+-inversive seminearring'.

In **Section 1**, we first recall the notions of 'bi-semilattice' (*cf.* Definition 5.1.1) and 'strong bi-semilattice of seminearrings' (*cf.* Definition 5.1.9). Then with the help of

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This chapter is mainly based on the work of the following paper:

Rajlaxmi Mukherjee et al., ***On full subdirect products of a bi-semilattice and a zero-symmetric near-ring*****,Communicated.**

Proposition 5.1.11, in Theorem 5.1.12 and Theorem 5.1.16, we characterize the seminearrings which are full subdirect products of a bi-semilattice and a (zero-symmetric) near-ring as the E^+-inversive seminearrings which are strong bi-semilattice of additively cancellative (zero-symmetric) seminearrings. Then we obtain some variants of the above results *viz.*, Theorem 5.1.17 and Theorem 5.1.18 which characterize subdirect products of a distributive lattice and a (zero-symmetric) near-ring as the E^+-inversive seminearrings which are strong distributive lattice of additively cancellative (zero-symmetric) seminearrings. Each of these four theorems is not only an analogue of Mitsch's Theorem 14 [78] (*cf.* Theorem 1.1.33) in our setting, but also an analogue of Ghosh's Theorem 2.3 [29] (*cf.* Theorem 1.3.14) on semirings. In Theorem 2.3 [29] (*cf.* Theorem 1.3.14), Ghosh characterized the class of semirings which are subdirect products of a distributive lattice and a ring. In Theorem 2.10 [29] (*cf.* Theorem 1.3.15), Ghosh obtained a different element-wise characterization of this class of semirings. This motivates us to make an attempt to obtain an analogue of Ghosh's Theorem 2.10 [29] (*cf.* Theorem 1.3.15) in our setting. In this direction, with the help of Proposition 5.1.25 and Proposition 5.1.26, we obtain Theorem 5.1.27 and Theorem 5.1.29 which respectively provide different characterizations of the classes of seminearrings which are full subdirect products of a bi-semilattice and a zero-symmetric near-ring and which are subdirect products of a distributive lattice and a zero-symmetric near-ring. In fact Ghosh's [29] Theorem 2.3 and Theorem 2.10 (*cf.* Theorems 1.3.14 and 1.3.15) become particular cases of Theorem 5.1.18 and Theorem 5.1.29, respectively. The present study has also answered (*cf.* Theorem 5.1.29) the question[1] "how to characterize a seminearring which is a subdirect product of a distributive lattice and a zero-symmetric near-ring but not necessarily additively regular with 0 and 1?"

In [81, 82, 98], the authors mainly characterized the classes of seminearrings which are (*i*) bi-semilattice (distributive lattice) of zero-symmetric near-rings (*cf.* Corollary 3.12 and Theorem 3.14 [82]), (*ii*) strong bi-semilattice (distributive lattice) of zero-symmetric near-rings (*cf.* Theorem 2.28 and Theorem 2.35 [81]), (*iii*) additively regular zero-symmetric seminearring with 1 and subdirect product of a distributive lattice and a zero-symmetric near-ring (*cf.* Theorem 4.14 [98]). All of these classes belong to the class of additively regular seminearrings. In **Section 2**, we mainly explore the

[1] This question is a natural one based on the characterization of "a subdirect product of a distributive lattice and a zero-symmetric near-ring in the class of additively regular seminearrings containing 0 and 1" obtained in Theorem 4.14 [98]. It should be mentioned here that the bi-semilattice version of this result (*cf.* Theorem 4.14 [98]) was not addressed in [98].

relationships of these classes of additively regular (and hence E^+-inversive) seminearrings with the classes of seminearrings which are characterized in Theorem 5.1.16 and Theorem 5.1.18 (*cf.* Theorem 5.2.1). To conclude this section we obtain the analogue of Theorem 5.2.1 replacing zero-symmetric near-ring by near-ring (*cf.* Theorem 5.2.6).

In **Section 3**, we mainly discuss the validity of some results obtained in Chapter 4 for the classes of seminearrings obtained in this chapter. In this direction, we obtain lattice isomorphisms (*cf.* Theorem 5.3.5) between the lattice of near-ring (zero-symmetric near-ring) congruences and the lattice of generalised strong full reflexive right k-ideals (k-ideals) in the classes of seminearrings obtained in Theorems 5.1.12, 5.1.16, 5.1.17 and 5.1.18. Then we study the modularity, distributivity and completeness of theses lattices and obtain the least near-ring (zero-symmetric near-ring) congruence and the smallest generalised strong full reflexive right k-ideal (k-ideal) of a full subdirect product of a bi-semilattice and a (zero-symmetric) near-ring and of a subdirect product of a distributive lattice and a (zero-symmetric) near-ring (*cf.* Theorems 5.3.6, 5.3.7 and 5.3.8).

5.1 Characterization

Definition 5.1.1. [93] A non-empty set S with two semilattice operations '$\cdot$', '$+$' is called a *bi-semilattice*. One regards $(S,\cdot)$ as a *meet-semilattice* and $(S,+)$ as a *join-semilattice*.

A bi-semilattice is called a *meet-distributive bi-semilattice* if the meet operation '$\cdot$' distributes over the join operation '$+$'.

Remark 5.1.2. Throughout this thesis '*bi-semilattice*' stands for '*meet-distributive bi-semilattice*'

Example 5.1.3. [93] For a given semilattice $(V,\cdot)$ let us consider the set of all finite non-empty subsemilattices of $(V,\cdot)$, denoted by $S(V,\cdot)$. If we define the following

$$ST = \{st|s \in S, t \in T\}$$

and

$$S + T = S \cup T \cup S \cdot T$$

then $S(V,\cdot)$ forms a bi-semilattice.

Example 5.1.4. [50] $B = \{0, \alpha, 1\}$ is a bi-semilattice with respect to the following operations :

$+$	0	α	1
0	0	α	1
α	α	α	α
1	1	α	1

$\cdot$	0	α	1
0	0	α	0
α	α	α	α
1	0	α	1

Clearly, the join semilattice $(B, +)$ is the chain $0 < 1 < \alpha$ and the meet semilattice $(B, \cdot)$ is the chain $\alpha < 0 < 1$. This bi-semilattice is not a lattice since $1 + 1\alpha = \alpha$.

Observation 5.1.5. Every bi-semilattice is a semiring with join as addition and meet as multiplication. If in a semiring $(S, +, \cdot)$, $(S, +)$ and $(S, \cdot)$ are commutative bands then $(S, +, \cdot)$ is a bi-semilattice.

Remark 5.1.6. Let $(B, +, \cdot)$ be a bi-semilattice.
(i) Though there are two partial orders in the bi-semilattice B (see Definition 5.1.1), one with respect to the join semilattice $(B, +)$ and another with respect to the meet semilattice $(B, \cdot)$, following [81], throughout this thesis, unless mentioned otherwise, $\alpha \leq \beta$ in B stands for $\alpha \leq \beta$ in $(B, +)$ *i.e.,* $\alpha + \beta = \beta$.
(ii) For any α and $\beta \in B$, (1) if $\alpha \leq \beta$ then $\alpha\beta \leq \beta$; (2) $\alpha + \beta + \alpha\beta = \alpha + \beta$, *i.e.,* $\alpha\beta \leq \alpha + \beta$ [81].

Definition 5.1.7. [82] A congruence σ on a seminearring $(S, +, \cdot)$ is said to be a *bi-semilattice congruence* on S if the seminearring S/σ becomes a bi-semilattice.

Definition 5.1.8. [82] A seminearring $(S, +, \cdot)$ is called a *bi-semilattice I of seminearrings $S_i (i \in I)$* if S admits of a bi-semilattice congruence β such that $I = S/\beta$ with each S_i a β-class.

Definition 5.1.9. [81] Let B be a bi-semilattice (distributive lattice) and $\{S_\alpha : \alpha \in B\}$ be a family of seminearrings which are indexed by the elements of B. For each $(\alpha \leq \beta)$ in B, we now define a seminearring monomorphism $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ satisfying the following conditions:

(1) $\phi_{\alpha,\alpha} = I_{S_\alpha}$, where I_{S_α} denotes the identity mapping on S_α,

(2) $\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$, if $\alpha \leq \beta \leq \gamma$,

(3) $\phi_{\alpha,\gamma}(S_\alpha)\phi_{\beta,\gamma}(S_\beta) \subseteq \phi_{\alpha\beta,\gamma}(S_{\alpha\beta})$, if $\alpha + \beta \leq \gamma$.

On $S = \cup S_\alpha$ (the disjoint union S_α's) we define $\oplus$ and $\odot$ as follows:

(4) $a \oplus b = \phi_{\alpha,\alpha+\beta}(a) + \phi_{\beta,\alpha+\beta}(b)$ and

(5) $a \odot b = c \in S_{\alpha\beta}$ such that $\phi_{\alpha\beta,\alpha+\beta}(c) = \phi_{\alpha,\alpha+\beta}(a)\ \phi_{\beta,\alpha+\beta}(b)$ where $a \in S_\alpha$, $b \in S_\beta$.

We denote the above system by $\langle B, S_\alpha, \phi_{\alpha,\beta}\rangle$. This is a seminearring and we call it a *strong bi-semilattice (distributive lattice) B of seminearrings S_α, $\alpha \in B$.*

Definition 5.1.10. A seminearring isomorphic with a subseminearring H of the direct product of two seminearrings S and T is called a *subdirect product* of S and T if the two projections $\pi_1 : H \to S$, $\pi_1(s,t) = s$ and $\pi_2 : H \to T$, $\pi_2(s,t) = t$ are surjective.

A subdirect product S of two seminearrings S_1 and S_2 is said to be a *full subdirect product* if $E^+(S)$=$E^+(S_1) \times E^+(S_2)$.

For its immediate use in Theorem 5.1.12, an analogue of Mitsch's result, we obtain the following result which is also an analogue of Proposition 1.6 [29] in our setting.

Proposition 5.1.11. *Let $S = \langle B, S_\alpha, \phi_{\alpha,\beta}\rangle$ be a strong bi-semilattice (distributive lattice) B of seminearrings $\{S_\alpha : \alpha \in B\}$. Let θ be a binary relation on S defined by*

$$a\ \theta\ b \Leftrightarrow \phi_{\alpha,\alpha+\beta}(a) = \phi_{\beta,\alpha+\beta}(b)$$

where $a \in S_\alpha$ and $b \in S_\beta$. Then θ is a congruence on S and S is a subdirect product of B and S/θ.

Proof. If $a \in S_\alpha$, $\phi_{\alpha,\alpha}(a) = I_{S_\alpha}(a) = a$. Therefore $a\ \theta\ a$ for all $a \in S$ whence θ is reflexive. θ is symmetric by its construction. Let $a\ \theta\ b$ and $b\ \theta\ c$ where $a \in S_\alpha$, $b \in S_\beta$ and $c \in S_\gamma$. Then

$$\begin{gathered}\phi_{\alpha,\alpha+\beta}(a) = \phi_{\beta,\alpha+\beta}(b) \text{ and } \phi_{\beta,\beta+\gamma}(b) = \phi_{\gamma,\beta+\gamma}(c).\\ \text{Therefore } \phi_{\alpha+\beta,\alpha+\beta+\gamma}(\phi_{\alpha,\alpha+\beta}(a)) = \phi_{\alpha+\beta,\alpha+\beta+\gamma}(\phi_{\beta,\alpha+\beta}(b)) \text{ and}\\ \phi_{\beta+\gamma,\alpha+\beta+\gamma}(\phi_{\beta,\beta+\gamma}(b)) = \phi_{\beta+\gamma,\alpha+\beta+\gamma}(\phi_{\gamma,\beta+\gamma}(c)).\\ \text{Hence } \phi_{\alpha,\alpha+\beta+\gamma}(a) = \phi_{\beta,\alpha+\beta+\gamma}(b) = \phi_{\gamma,\alpha+\beta+\gamma}(c).\\ \text{Then } \phi_{\alpha+\gamma,\alpha+\beta+\gamma}(\phi_{\alpha,\alpha+\gamma}(a)) = \phi_{\alpha+\gamma,\alpha+\beta+\gamma}(\phi_{\gamma,\alpha+\gamma}(c)).\\ \text{Since } \phi_{\alpha+\gamma,\alpha+\beta+\gamma} \text{ is one-one, } \phi_{\alpha,\alpha+\gamma}(a) = \phi_{\gamma,\alpha+\gamma}(c).\end{gathered}$$

Therefore $a\ \theta\ c$ whence θ is transitive. Hence θ is an equivalence relation.

Let $a\ \theta\ b$ and $c \in S$ where $a \in S_\alpha$, $b \in S_\beta$ and $c \in S_\gamma$. Then $\phi_{\alpha,\alpha+\beta}(a) = \phi_{\beta,\alpha+\beta}(b)$. Then operating both sides by $\phi_{\alpha+\beta,\alpha+\beta+\gamma}$, we get

$$\phi_{\alpha+\beta,\alpha+\beta+\gamma}(\phi_{\alpha,\alpha+\beta}(a)) = \phi_{\alpha+\beta,\alpha+\beta+\gamma}(\phi_{\beta,\alpha+\beta}(b)), \; i.e.,$$
$$\phi_{\alpha,\alpha+\beta+\gamma}(a) = \phi_{\beta,\alpha+\beta+\gamma}(b).$$

Now $\phi_{\gamma,\alpha+\beta+\gamma}(c) + \phi_{\alpha,\alpha+\beta+\gamma}(a) = \phi_{\gamma,\alpha+\beta+\gamma}(c) + \phi_{\beta,\alpha+\beta+\gamma}(b)$.

Therefore $\phi_{\alpha+\gamma,\alpha+\beta+\gamma}(c \oplus a) = \phi_{\beta+\gamma,\alpha+\beta+\gamma}(c \oplus b)$ where $c \oplus a \in S_{\alpha+\gamma}$ and $c \oplus b \in S_{\beta+\gamma}$.

Then $(c \oplus a)\ \theta\ (c \oplus b)$. Similarly we can show that $(a \oplus c)\ \theta\ (b \oplus c)$. Again from $\phi_{\alpha,\alpha+\beta}(a) = \phi_{\beta,\alpha+\beta}(b)$ we can write

$$\phi_{\gamma,\alpha+\beta+\gamma}(c)\ \phi_{\alpha,\alpha+\beta+\gamma}(a) = \phi_{\gamma,\alpha+\beta+\gamma}(c)\ \phi_{\beta,\alpha+\beta+\gamma}(b).$$

Let $s = c \odot a$ and $t = c \odot b$. Then

$$\phi_{\gamma(\alpha+\beta),\alpha+\beta+\gamma}(\phi_{\gamma\alpha,\gamma(\alpha+\beta)}(s)) = \phi_{\gamma\alpha,\alpha+\beta+\gamma}(s) = \phi_{\alpha+\gamma,\alpha+\beta+\gamma}(\phi_{\gamma\alpha,\alpha+\gamma}(s)) =$$
$$\phi_{\alpha+\gamma,\alpha+\beta+\gamma}(\phi_{\gamma,\alpha+\gamma}(c)\phi_{\alpha,\alpha+\gamma}(a)) = (\phi_{\gamma,\alpha+\beta+\gamma}(c)\phi_{\alpha,\alpha+\beta+\gamma}(a)).$$

Similarly, we can show that

$$\phi_{\gamma(\alpha+\beta),\alpha+\beta+\gamma}(\phi_{\gamma\beta,\gamma(\alpha+\beta)}(t)) = (\phi_{\gamma,\alpha+\beta+\gamma}(c)\phi_{\beta,\alpha+\beta+\gamma}(b)).$$

Since $\phi_{\gamma,\alpha+\beta+\gamma}(c)\ \phi_{\alpha,\alpha+\beta+\gamma}(a) = \phi_{\gamma,\alpha+\beta+\gamma}(c)\ \phi_{\beta,\alpha+\beta+\gamma}(b)$ holds we can write

$$\phi_{\gamma(\alpha+\beta),\alpha+\beta+\gamma}(\phi_{\gamma\alpha,\gamma(\alpha+\beta)}(s)) = \phi_{\gamma(\alpha+\beta),\alpha+\beta+\gamma}(\phi_{\gamma\beta,\gamma(\alpha+\beta)}(t))$$
$$\text{whence } \phi_{\gamma\alpha,\gamma(\alpha+\beta)}(s) = \phi_{\gamma\beta,\gamma(\alpha+\beta)}(t).$$

Thus $(c \odot a)\ \theta\ (c \odot b)$. Similarly, we can show that $(a \odot c)\ \theta\ (b \odot c)$. Therefore θ is a congruence on the seminearring S. Hence S/θ becomes a seminearring.

Now following Proposition 1.6 of [29], we define a mapping $\psi : S \longrightarrow B \times S/\theta$ by $a \longmapsto (\alpha, [a]_\theta)$ where $a \in S_\alpha$. It is a matter of routine verification to check that ψ is a monomorphism and $\psi(S)$ is a subdirect product of B and S/θ. □

Theorem 5.1.12. *The following conditions on a seminearring $(S, +, \cdot)$ are equivalent.*

(1) $(S, +, \cdot)$ *is a full subdirect product of a bi-semilattice and a near-ring.*

(2) S *is an E^+-inversive strong bi-semilattice of additively cancellative seminearrings with zero.*

(3) S *is an E^+-inversive strong bi-semilattice of seminearrings with zero, each of which contains a single additive idempotent.*

Proof. (1) ⇒ (2) Let S be a full subdirect product of a bi-semilattice B and a near-ring N. Then $(S, +)$ is a full subdirect product of the semilattice $(B, +)$ and the group $(N, +)$. Hence by Theorem 1.1.33, $(S, +)$ is an E-inversive strong semilattice

B of cancellative monoids $(S_\alpha, +)$ $(\alpha \in B)$ with injective linking homomorphisms $\phi_{\alpha,\beta}$: $(S_\alpha, +) \to (S_\beta, +)$ $(\alpha \leq \beta, \alpha, \beta \in B)$ where $S_\alpha = (\{\alpha\} \times N) \cap S$ and $\phi_{\alpha,\beta}(\alpha, n) = (\alpha, n) + (\beta, 0_N) = (\beta, n)$, $n \in N$ (*cf.* Remark 5.1.6(i)). Let (α, a), $(\alpha, b) \in S_\alpha$. Then $(\alpha, a)(\alpha, b) = (\alpha, ab) \in S_\alpha$ whence $(S_\alpha, +, \cdot)$ is an additively cancellative seminearring with zero where $0_{S_\alpha} = (\alpha, 0_N)$. Let $\alpha \leq \beta$. Then $\phi_{\alpha,\beta}((\alpha, a)(\alpha, b)) = \phi_{\alpha,\beta}((\alpha, ab)) = (\beta, ab) = (\phi_{\alpha,\beta}(\alpha, a))(\phi_{\alpha,\beta}(\alpha, b))$. Therefore $\phi_{\alpha,\beta}$ $(\alpha \leq \beta, \alpha, \beta \in B)$ is a seminearring monomorphism. Thus it is sufficient to verify the conditions (3) and (5) of Definition 5.1.9. Let $\alpha + \beta \leq \gamma$, $(\alpha, a) \in S_\alpha$ and $(\beta, b) \in S_\beta$ where $\alpha, \beta, \gamma \in B$ and $a, b \in N$. Then in view of Remark 5.1.6 (ii), $\alpha\beta \leq \alpha + \beta \leq \gamma$. Now

$$\phi_{\alpha,\gamma}(\alpha, a)\ \phi_{\beta,\gamma}(\beta, b) = (\gamma, a)(\gamma, b) = (\gamma, ab) = \phi_{\alpha\beta,\gamma}(\alpha\beta, ab)\ .....\ (i).$$

Therefore $\phi_{\alpha,\gamma}(S_\alpha)\phi_{\beta,\gamma}(S_\beta) \subseteq \phi_{\alpha\beta,\gamma}(S_{\alpha\beta})$, if $\alpha + \beta \leq \gamma$. Now $(\alpha, a) \odot (\beta, b) = (\alpha\beta, ab) \in S_{\alpha\beta}$. Let $c = (\alpha\beta, ab)$. Then in view of (i),

$$\phi_{\alpha\beta,\alpha+\beta}(c) = \phi_{\alpha,\alpha+\beta}(\alpha, a)\ \phi_{\beta,\alpha+\beta}(\beta, b).$$

Hence S is an E^+-inversive strong bi-semilattice B of additively cancellative seminearrings S_α $(\alpha \in B)$ with zero.

(2) $\Rightarrow$ (3) It follows from the additive cancellative property of each S_α and the fact that each S_α contains zero.

(3) $\Rightarrow$ (1) Let S be an E^+-inversive strong bi-semilattice B of seminearrings S_α $(\alpha \in B)$ with zero element 0_α such that each S_α contains a single additive idempotent *i.e.,* $E^+(S_\alpha) = \{0_\alpha\}$. Now in view of Proposition 5.1.11, S is a subdirect product of the bi-semilattice B and the seminearring S/θ via the seminearring monomorphism ψ: $S \longrightarrow B \times S/\theta$, $a \longmapsto (\alpha, [a]_\theta)$, where $a \in S_\alpha$ and θ is defined on S by $b\ \theta\ c$ if and only if $\phi_{\alpha,\alpha+\beta}(b) = \phi_{\beta,\alpha+\beta}(c)$ for $b \in S_\alpha$ and $c \in S_\beta$. Since, each S_α contains a single additive idempotent 0_α, $\phi_{\alpha,\alpha+\beta}(0_\alpha) = \phi_{\beta,\alpha+\beta}(0_\beta) = 0_{\alpha+\beta}$ for all $\alpha, \beta \in B$. Hence $0_\alpha\ \theta\ 0_\beta$ for all $\alpha, \beta \in B$. Again $a + 0_\alpha = a$ whence $[a]_\theta + [0_\alpha]_\theta = [a]_\theta$ for all $a \in S_\alpha$ and for all $\alpha \in B$. Therefore $(S/\theta, +)$ is a monoid where $0_{S/\theta} = [0_\alpha]_\theta$ for any $\alpha \in B$. Let $a \in S_\alpha$. Since S is an E^+-inversive strong bi-semilattice B of seminearrings S_α $(\alpha \in B)$, there exists $b \in S_\beta$ for some $\beta \in B$ such that $a \oplus b$ is an additive idempotent of S. Therefore $a \oplus b = \phi_{\alpha,\alpha+\beta}(a) + \phi_{\beta,\alpha+\beta}(b) = 0_{\alpha+\beta}$, since $S_{\alpha+\beta}$ contains a single additive idempotent. Then $[a]_\theta + [b]_\theta = [0_{\alpha+\beta}]_\theta = 0_{S/\theta}$. Hence $(S/\theta, +)$ is a group where $0_{S/\theta} = [0_\alpha]_\theta$ for any $\alpha \in B$. Therefore $(S/\theta, +, \cdot)$ is a near-ring. Now for each $\alpha \in B$, $\psi(0_\alpha) = (\alpha, [0_\alpha]_\theta)$. Hence S is a full subdirect product of the bi-semilattice B and the near-ring S/θ. $\square$

The following example illustrates that the inclusion (that comes from (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) of Theorem 5.1.12) of the class of seminearrings which are full subdirect products of a bi-semilattice and a near-ring in the class of E^+-inversive seminearrings is strict.

Example 5.1.13. Let $(T, +)$ be a semilattice containing at least two distinct elements a and b. Hence $(T, +)$ is an E-inversive semigroup. Then by Proposition 4.3.5, $(M(T), +, \cdot)$ is an E^+-inversive seminearring. Now let $f \in E^+(M(T)) = M(T)$ such that $f(a) = b$ and $f(b) = a$. Then clearly $f^2 \neq f$. Therefore $(M(T), +, \cdot)$ is not a full subdirect product of a bi-semilattice and a near-ring since in a full subdirect product of a bi-semilattice and a near-ring, each additive idempotent is a multiplicative idempotent.

The following example shows that not every subdirect product of a bi-semilattice and a near-ring is full.

Example 5.1.14. Let $B = \{0, \alpha, 1\}$ be the bi-semilattice of Example 5.1.4. Then the join semilattice $(B, +)$ is the chain $0 < 1 < \alpha$ and the meet semilattice $(B, \cdot)$ is the chain $\alpha < 0 < 1$ (see Definition 5.1.1 and Remark 5.1.6). Let $(\mathbb{Z}, +, \cdot)$ be a near-ring where '+' is the usual addition of integers and $ab = a|b|$ for all $a, b \in \mathbb{Z}$. Let $S = \{(0, -2n)|0 \in B,\ n \in \mathbb{N}\} \cup \{(1, -n)|1 \in B,\ n \in \mathbb{N}\} \cup \{(\alpha, a)|\alpha \in B,\ a \in \mathbb{Z}\} \subseteq B \times \mathbb{Z}$. Since for all $m, n \in \mathbb{N}$ and for all $a, b \in \mathbb{Z}$,

$$(0, -2n)+(0, -2m) = (0, -2(n+m)), (0, -2n)(0, -2m) = (0, -2n|-2m|) = (0, -4nm),$$

$$(1, -n) + (1, -m) = (1, -(n + m)), (1, -n)(1, -m) = (1, -n| - m|) = (1, -nm),$$

$$(\alpha, a) + (\alpha, b) = (\alpha, a + b), (\alpha, a)(\alpha, b) = (\alpha, ab),$$

$\{(0, -2n)|0 \in B,\ n \in \mathbb{N}\}$, $\{(1, -n)|1 \in B,\ n \in \mathbb{N}\}$, $\{(\alpha, a)|\alpha \in B,\ a \in \mathbb{Z}\}$ are seminearrings. Again for all $z \in \mathbb{Z}$ and $m, n \in \mathbb{N}$

$$(0, -2n)+(1, -m)(= (1, -2n-m)), (1, -m)+(0, -2n) \in \{(1, -n)|1 \in B,\ n \in \mathbb{N}\} \subseteq S,$$

$$(0, -2n) + (\alpha, z)(= (\alpha, -2n + z)), (\alpha, z) + (0, -2n) \in \{(\alpha, a)|\alpha \in B,\ a \in \mathbb{Z}\} \subseteq S,$$

$$(1, -n) + (\alpha, z)(= (\alpha, -n + z)), (\alpha, z) + (1, -n) \in \{(\alpha, a)|\alpha \in B,\ a \in \mathbb{Z}\} \subseteq S,$$

$$(0, -2n)(1, -m)(= (0, -2nm)), (1, -m)(0, -2n) \in \{(0, -2n)|0 \in B,\ n \in \mathbb{N}\} \subseteq S,$$

$$(0, -2n)(\alpha, z)(= (\alpha, -2n|z|)), (\alpha, z)(0, -2n) \in \{(\alpha, a)|\alpha \in B,\ a \in \mathbb{Z}\} \subseteq S,$$

$$(1, -n)(\alpha, z)(= (\alpha, -n|z|)), (\alpha, z)(1, -n)(= (\alpha, z| - n|)) \in \{(\alpha, a)|\alpha \in B,\ a \in \mathbb{Z}\} \subseteq S.$$

Therefore S is closed under '+' and '$\cdot$' whence S is a seminearring. Since for any $z \in \mathbb{Z}$, $(\alpha, z) \in S$ and $(0, -2)$, $(1, -1), (\alpha, 0) \in S$, S is a subdirect product of the bi-semilattice B and the near-ring $(\mathbb{Z}, +, \cdot)$, but it is not a full subdirect product of B and $\mathbb{Z}$ as $(0, 0)$ is not in S.

The above situation differs as follows when the bi-semilattice is replaced by a distributive lattice.

Proposition 5.1.15. *A seminearring S is a subdirect product of a distributive lattice D and a near-ring N if and only if S is a full subdirect product of D and N.*

Proof. Let S be a subdirect product of a distributive lattice D and a near-ring N. Clearly, $E^+(S) = (D \times \{0_N\}) \cap S$ where 0_N is the zero of the near-ring N. Let $\alpha \in D$. Then $(\alpha, n) \in S$ for some $n \in N$. Since S is a subdirect product of D and N, $(\beta, -n) \in S$ for some $\beta \in D$. Therefore $(\alpha, n) + (\beta, -n) = (\alpha + \beta, 0_N) \in S$. Hence $(\alpha + \beta, 0_N)(\alpha, n) \in S$, *i.e.,* $((\alpha + \beta)\alpha, 0_N) \in S$. Since D is a distributive lattice, $(\alpha + \beta)\alpha = \alpha$. Therefore $(\alpha, 0_N) \in S$ whence $(D \times \{0_N\}) \subseteq S$. Then in view of Definition 5.1.10, S is a full subdirect product of the distributive lattice D and the near-ring N. $\square$

While obtaining the analogues of the results of semigroups in the setting of seminearrings, the semilattice is replaced either by bi-semilattice or distributive lattice and the group is replaced by near-ring or zero symmetric near-ring. So Mitsch's analogue obtained in Theorem 5.1.12 has the following three variants viz., Theorems 5.1.16, 5.1.17 and 5.1.18.

The zero-symmetric version of Theorem 5.1.12 is the following.

Theorem 5.1.16. *The following conditions on a seminearring $(S, +, \cdot)$ are equivalent.*

1. *$(S, +, \cdot)$ is a full subdirect product of a bi-semilattice and a zero-symmetric near-ring.*
2. *S is an E^+-inversive strong bi-semilattice of additively cancellative zero-symmetric seminearrings.*
3. *S is an E^+-inversive strong bi-semilattice of zero-symmetric seminearrings, each of which contains a single additive idempotent.*

Proof. The proof follows by a slight modification of the argument of the proof of Theorem 5.1.12. □

The following is the distributive lattice version of Theorem 5.1.12 which follows in view of Theorem 5.1.12 and Proposition 5.1.15.

Theorem 5.1.17. *The following conditions on a seminearring $(S, +, \cdot)$ are equivalent.*

(1) $(S, +, \cdot)$ *is a subdirect product of a distributive lattice and a near-ring.*

(2) S *is an* E^+*-inversive strong distributive lattice of additively cancellative seminearrings with zero.*

(3) S *is an* E^+*-inversive strong distributive lattice of seminearrings with zero, each of which contains a single additive idempotent.*

The zero-symmetric version of Theorem 5.1.17 is the following.

Theorem 5.1.18. *The following conditions on a seminearring $(S, +, \cdot)$ are equivalent.*

(1) $(S, +, \cdot)$ *is a subdirect product of a distributive lattice and a zero-symmetric near-ring.*

(2) S *is an* E^+*-inversive strong distributive lattice of additively cancellative zero-symmetric seminearrings.*

(3) S *is an* E^+*-inversive strong distributive lattice of zero-symmetric seminearrings, each of which contains a single additive idempotent.*

Having obtained four analogues, viz., Theorems 5.1.12, 5.1.16, 5.1.17 and 5.1.18, of Mitsch's result (*i.e.,* Theorem 1.1.33) in our setting our task now reduces (see the Introduction) to obtain the analogue of Ghosh's Theorem 1.3.15 in our setting. We accomplish this in Theorem 5.1.27 and Theorem 5.1.29, which respectively provide different characterizations of the class of E^+-inversive seminearrings obtained in Theorems 5.1.16 and 5.1.18. In the deduction of these results Proposition 5.1.25 and Proposition 5.1.26 play the key role. The necessary prerequisites for these propositions are built up in Propositions 5.1.19, 5.1.20, 5.1.21 and 5.1.22.

Proposition 5.1.19. *Let S be a seminearring with non-empty $E^+(S)$ and $ef = fe$, $e^2 = e$ for all $e, f \in E^+(S)$. Then (i) $e + f = f + e$, for all $e, f \in E^+(S)$ implies $E^+(S)$ is a bi-semilattice and (ii) for $a \in S$, $D_a := \{e \in E^+(S) : a + e = a$ and $ag = eg$ for all $g \in E^+(S)\}$ is non-empty implies it is singleton.*

Proof. (i) It follows by the Definition 5.1.1.
(ii) Let $a \in S$ and $e, f \in E^+(S)$ such that $e, f \in D_a$. Then $a + e = a + f = a$ and $ag = eg = fg$ for all $g \in E^+(S)$. Now for $g = e$, $ae = e = fe$ and for $g = f$, $af = ef = f$. Again $ef = fe$. Therefore $e = f$ whence D_a is a singleton set. □

Proposition 5.1.20. *Let S be an E^+-inversive seminearring in which for all $e \in E^+(S)$ and for all $a \in S$*

(1) $e + a = a + e$,

(2) $ea = ae$ and

(3) $e^2 = e$.

Then $\mathfrak{S} = \{a \in S : D_a$ is non-empty$\}$ forms an E^+-inversive, full right ideal of S. Moreover, if $s(a+e) = sa + se$ for all $e \in E^+(S)$ and for all $s, a \in S$, then $\mathfrak{S}$ becomes an E^+-inversive, full ideal of S, too.

Proof. Since S is E^+-inversive, $E^+(S)$ is non-empty. Let $e \in E^+(S)$. Then by Proposition 5.1.19 (ii), $D_e = \{e\}$. So $e \in \mathfrak{S}$ whence $\mathfrak{S} \neq \emptyset$ as well as $E^+(S) \subseteq \mathfrak{S}$. Let $a, b \in \mathfrak{S}$ and $s \in S$. Then in view of Proposition 5.1.19 (ii), there exist $e, f \in E^+(S)$ such that $D_a = \{e\}$ and $D_b = \{f\}$. Now $(a + b) + (e + f) = (a + e) + (b + f)$ (using condition (1)) $= a + b$ (since $e \in D_a$ and $f \in D_b$), $(a+b)g = ag + bg = eg + fg$ (since $e \in D_a$ and $f \in D_b$) for all $g \in E^+(S)$. Therefore $D_{a+b} = \{e+f\}$. Again $(as)+(es) = (a+e)s = as$ (since $e \in D_a$) and $(as)g = (ag)s$ (using condition (2)) $= (eg)s$ (since $e \in D_a$) $= (es)g$ (using condition (2)) for all $g \in E^+(S)$. Therefore $D_{as} = \{es\}$ whence $a + b, as \in \mathfrak{S}$. Now for $a \in \mathfrak{S}$, there exist $x, y \in S$ for which $a + x$, $x + y \in E^+(S)$. Let $z = x + x + y$. Then in view of Proposition 5.1.19 (i) we obtain $a + z = (a + x) + (x + y) \in E^+(S)$. Again Proposition 5.1.19 (ii) together with condition (2) shows that $D_z = \{x + y\}$ whence $z \in \mathfrak{S}$. Therefore $\mathfrak{S}$ is an E^+-inversive, full right ideal of S.
Moreover, if $s(x + f) = sx + sf$ for all $f \in E^+(S)$ and for all $s, x \in S$, then for $a \in \mathfrak{S}$ and $t \in S$, in view of Proposition 5.1.19 (ii) we deduce that $D_{ta} = \{te\}$ where $D_a = \{e\}$, whence $\mathfrak{S}$ becomes an E^+-inversive, full ideal of S. □

Proposition 5.1.21. *Suppose S is an E^+-inversive seminearring such that $e^2 = e$ for all $e \in E^+(S)$ and $D_a := \{e \in E^+(S) : a + e = a$ and $ag = eg$ for all $g \in E^+(S)\}$ is non-empty for all $a \in S$. Then $ef = fe$ if and only if $ea = ae$ for all $e, f \in E^+(S)$ and for all $a \in S$.*

Proof. Let $ef = fe$ for all $e, f \in E^+(S)$. Let $a \in S$ and $e \in E^+(S)$. Then by hypothesis that D_a is non-empty and Proposition 5.1.19 (ii), $D_a = \{f\}$ for some $f \in E^+(S)$ and $ag = fg$ for all $g \in E^+(S)$. Now $fg(= (f+f)g) \in E^+(S)$ for all $g \in E^+(S)$ whence $ag \in E^+(S)$ for all $g \in E^+(S)$. Also $ga\ (= (g+g)a) \in E^+(S)$ for all $g \in E^+(S)$. These together with the repeated use of the hypothesis that each additive idempotent is a multiplicative idempotent and they commute with each other, lead to the following deduction: $ea = e(ea) = (ea)e = e(ae) = (ae)e = ae$.
The other implication is obvious. □

In view of Proposition 5.1.19 (i) and (ii), the following result easily follows from the proof of Proposition 5.1.21.

Proposition 5.1.22. *Suppose S is an E^+-inversive seminearring satisfying the following conditions: $e+f = f+e$ for all $e, f \in E^+(S)$, $ef = fe$ for all $e, f \in E^+(S)$, $e^2 = e$ for all $e \in E^+(S)$ and D_a is non-empty for all $a \in S$. Then $E^+(S)$ is an $\mathcal{S}$-ideal of S.*

Notations 5.1.23. In many places, that follow, we will refer to the following conditions on an E^+-inversive seminearring S without their explicit descriptions. (1) $e+a = a+e$ for all $a \in S$ and for all $e \in E^+(S)$, (2) $ef = fe$ for all $e, f \in E^+(S)$, (3) $e^2 = e$ for all $e \in E^+(S)$ and (4) D_a is non-empty for all $a \in S$.

Remark 5.1.24. Under the hypothesis of Proposition 5.1.20, an E^+-inversive seminearring has an E^+-inversive subseminearring (an E^+-inversive, full right ideal, to be precise) such that for each of its element a, D_a is non-empty. So in view of Proposition 5.1.21, the restriction on an E^+-inversive seminearring S given by (4) is not quite unjustified when S satisfies (1), (2) and (3) (*cf.* Notations 5.1.23).

Proposition 5.1.25. *Let S be an E^+-inversive seminearring satisfying the conditions* (1) *to* (4) *(cf. Notations 5.1.23). Then the relation ρ on S, defined by*

$$a\rho b \text{ if and only if } D_a = D_b,$$

is a congruence on S such that S/ρ is isomorphic to $E^+(S)$. Hence ρ is a bi-semilattice congruence on S.

Proof. Clearly, ρ is an equivalence relation. Let $a\rho b$ and $c \in S$. Then in view of Proposition 5.1.19 (ii), there exist $f, h \in E^+(S)$ such that $D_a = D_b = \{f\}$........(i) and $D_c = \{h\}$ ....(ii). This together with the condition (1) and Proposition 5.1.19 (ii)

shows that $D_{a+c} = D_{b+c} = \{f + h\}$ and $D_{c+a} = D_{c+b} = \{h + f\}$. Therefore $(a + c)$ ρ $(b + c)$ and $(c + a)$ ρ $(c + b)$. Now in view of (ii), $hf = cf$. Also by condition (2) and Proposition 5.1.21, $cg = gc$ for all $g \in E^+(S)$. Hence $fh = fc$. This together with (i) and (ii) shows that $D_{ac} = D_{bc} = \{fh\}$ whence $ac\ \rho\ bc$. In a similar manner we can show that $hf = ha = hb$ and $D_{ca} = D_{cb} = \{hf\}$ whence $ca\ \rho\ cb$. Hence ρ is a congruence on S. Now it is easy to see that $\phi : S/\rho \to E^+(S)$, defined by $\phi([a]_\rho) = e$ where $D_a = \{e\}$ for all $[a]_\rho \in S/\rho$, is a seminearring isomorphism. Hence $(S/\rho, +, \cdot)$ is isomorphic to $(E^+(S), +, \cdot)$ which is a bi-semilattice (*cf.* Proposition 5.1.19(i)) and so in view of Definition 5.1.7, ρ becomes a bi-semilattice congruence on S. □

Proposition 5.1.26. *Let S be an E^+-inversive seminearring satisfying* (1) *to* (4) *(cf. Notations 5.1.23) and the following condition:* (5) *if $D_a = D_b$ and $I(a) \cap I(b)$ is non-empty then $a = b$ where $I(a) := \{x \in S : a + x \in E^+(S)\}$. Then $E^+(S)$ is a reflexive full k-ideal of S and the following are equivalent.*

(i) *The relation σ on S defined by,*

$$a\ \sigma\ b \text{ if and only if for some } x \in S,\ a + x,\ b + x \in E^+(S),$$

is a near-ring congruence on S.

(ii) *$c(a + e) = ca + ce$ for all $e \in E^+(S)$ and for all $c, a \in S$.*

Proof. By Proposition 5.1.22, $E^+(S)$ is an $\mathcal{S}$-ideal. Let $e, x \in S$ such that $x + e, e \in E^+(S)$. Then in view of Proposition 5.1.19 (ii), $D_x = \{f\}$ for some $f \in E^+(S)$. Again in view of Proposition 5.1.19 (ii), $D_f = \{f\}$. Therefore $D_x = D_f$ and $x + e, f + e \in E^+(S)$ whence $e \in I(x) \cap I(f)$. Hence by condition (5), $x = f$ whence $x \in E^+(S)$. In view of condition (1), the above argument shows that if $e + x, e \in E^+(S)$ then $x \in E^+(S)$. Thus $E^+(S)$ is a full k-ideal. Let $a + b \in E^+(S)$. Then in view of condition (1), $b + (a + b) + b' \in E^+(S)$ for all $b' \in W^+(b)$ (*cf.* Notation 4.3.3). Again $b + b' \in E^+(S)$. Hence $E^+(S)$ being a k-ideal, $b + a \in E^+(S)$. Hence by Definition 2.1.6, $E^+(S)$ is a reflexive full k-ideal.
$(i) \Rightarrow (ii)$ Let c, $a \in S$ and $e \in E^+(S)$. Then $ce \in E^+(S)$ as $E^+(S)$ is an $\mathcal{S}$-ideal. Since σ is a near-ring congruence on S, it is a group congruence on $(S, +)$. Hence $(a + e)\ \sigma\ a$ whence $c(a + e)\ \sigma\ ca$. So there exists $x \in S$ such that $c(a + e) + x$, $ca + x \in E^+(S)$. Since $ce \in E^+(S)$, $ce + ca + x \in E^+(S)$ whence by condition (1), $ca + ce + x \in E^+(S)$. This together with $c(a + e) + x \in E^+(S)$ implies that

$x \in I(c(a+e)) \cap I(ca+ce)$. By invoking Proposition 5.1.19 (ii), let $D_c = \{c_0\}$ and $D_a = \{a_0\}$ where $c_0, a_0 \in E^+(S)$. Then $c\ \rho\ c_0$, $e\ \rho\ e$ and $a\ \rho\ a_0$ where ρ is as defined in Proposition 5.1.25. Since ρ is a bi-semilattice congruence on S, we deduce that $c(a+e)$ ρ $c_0(a_0+e)$ and $(ca+ce)$ ρ $(c_0a_0+c_0e)$. Again in view of Proposition 5.1.19 (i), $c_0(a_0+e) = c_0a_0+c_0e$. Therefore $c(a+e)$ ρ $(ca+ce)$. Hence by definition of ρ, $D_{c(a+e)}$ $= D_{(ca+ce)}$. This together with the fact $x \in I(c(a+e)) \cap I(ca+ce)$ and condition (5) implies that $c(a+e) = ca+ce$.
$(ii) \Rightarrow (i)$ By condition (1), the additive idempotents are additively central in the E^+-inversive seminearring S. So $(S,+)$ is an E-inversive as well as an E-semigroup (*cf.* Definition 1.1.16 (vii)). Then in view of Theorem 4.5 of [31], σ is a group congruence on $(S,+)$. Let $a\sigma b$ and $s \in S$. Then there exists $x \in S$ such that $a+x$, $b+x \in E^+(S)$. Hence $E^+(S)$ being an $\mathcal{S}$-ideal, $(a+x)s$, $(b+x)s \in E^+(S)$, *i.e.*, $as+xs, bs+xs \in E^+(S)$. Therefore (as) σ (bs). Since $E^+(S)$ is reflexive, $x+a, x+b \in E^+(S)$. Since S is E^+-inversive, there exists $s' \in S$ such that $s+s' \in E^+(S)$. Hence $E^+(S)$ being an $\mathcal{S}$-ideal, $(s+s')(a+x+b)$, $(s+s')(b+a+x) \in E^+(S)$. This together with the fact $a+x, x+b$ $\in E^+(S)$ and the hypothesis given by (ii) implies that $sa+s(x+b)+s'(a+x+b)$, $sb+s(a+x)+s'(b+a+x) \in E^+(S)$. Now by condition (1), $b+(a+x) = (a+x)+b$. Hence using the fact $s(x+b)$, $s(a+x) \in E^+(S)$ ($\because E^+(S)$ is an $\mathcal{S}$-ideal) and condition (1), we deduce that $sa+(s(x+b)+s'(a+x+b)+s(a+x))$, $sb+(s(x+b)+s'(a+x+b)+s(a+x))$ $\in E^+(S)$. Consequently, (sa) σ (sb). Hence σ is a near-ring congruence on S. □

Theorem 5.1.27. *A seminearring S is a full subdirect product of a bi-semilattice and a zero-symmetric near-ring if and only if S is an E^+-inversive seminearring satisfying the following :*

(1) $e+a = a+e$ *for all* $a \in S$ *and for all* $e \in E^+(S)$,

(2) $ef = fe$ *for all* $e, f \in E^+(S)$,

(3) $e^2 = e$ *for all* $e \in E^+(S)$,

(4) $D_a := \{e \in E^+(S) : a+e = a$ *and* $ag = eg$ *for all* $g \in E^+(S)\}$ *is non-empty for each* $a \in S$,

(5) *if* $D_a = D_b$ *and* $I(a) \cap I(b)$ *is non-empty then* $a = b$ *where* $I(a) := \{x \in S : a+x \in E^+(S)\}$,

(6) $s(a+e) = sa+se$ *for all* $s, a \in S$ *and for all* $e \in E^+(S)$.

Proof. Let S be an E^+-inversive seminearring satisfying (1)-(6). Then in view of Propositions 5.1.25 and 5.1.26, ρ and σ are two congruences on S such that S/ρ is a bi-semilattice and S/σ is a near-ring where $a\rho b$ if and only if $D_a = D_b$ and $a\sigma b$ if and only if $I(a) \cap I(b)$ is non-empty. We define $\psi : S \to S/\rho \times S/\sigma$ by $a \mapsto ([a]_\rho, [a]_\sigma)$ where $a \in S$. It is easy to verify that ψ is a seminearring morphism. Now $\psi(a) = \psi(b)$ implies that $([a]_\rho, [a]_\sigma) = ([b]_\rho, [b]_\sigma)$ whence $a\rho b$ and $a\sigma b$. Therefore $D_a = D_b$ and $I(a) \cap I(b)$ is non-empty whence by condition (5), $a = b$. Hence ψ is a monomorphism. Clearly, S is a subdirect product of the bi-semilattice S/ρ and the near-ring S/σ via the monomorphism ψ. Let $x \in E^+(S/\rho) \times E^+(S/\sigma)$. Then $x = ([a]_\rho, 0_{S/\sigma})$ where $a \in S$. Now let $D_a = \{e\}$ for some $e \in E^+(S)$ (*cf.* Proposition 5.1.19 (*ii*)). Then $a\ \rho\ e$. Since S/σ is a near-ring, $[e]_\sigma = 0_{S/\sigma}$. So $\psi(e) = ([e]_\rho, [e]_\sigma) = ([a]_\rho, 0_{S/\sigma})= x$. Hence $x \in \psi(E^+(S))$. Therefore $\psi(E^+(S)) = E^+(S/\rho) \times E^+(S/\sigma)$. Hence S is a full subdirect product of the bi-semilattice S/ρ and the near-ring S/σ. Now in view of Proposition 5.1.21, $af = fa$ for all $f \in E^+(S)$ and for all $a \in S$. This together with the fact that $[f]_\sigma = 0_{S/\sigma}$ for all $f \in E^+(S)$ shows that the near-ring S/σ is zero-symmetric.

Conversely, let S be a full subdirect product of a bi-semilattice B and a zero-symmetric near-ring N. Then by Theorem 5.1.16, S is an E^+-inversive seminearring. Also by the fullness of the subdirect product we obtain $E^+(S) = E^+(B) \times E^+(N) = B \times \{0_N\}$. Let $a \in S$ and $e, f \in E^+(S)$. Then $a = (\alpha, n)$, $e = (\beta, 0_N)$, $f = (\gamma, 0_N)$ for some $\alpha, \beta, \gamma \in B$ and $n \in N$. Therefore $a + e = (\alpha, n) + (\beta, 0_N) = (\alpha + \beta, n) = e + a$. Now $e^2 = (\beta, 0_N)(\beta, 0_N) = (\beta, 0_N) = e$. Again $ef = (\beta, 0_N)(\gamma, 0_N) = (\beta\gamma, 0_N)$ $= (\gamma\beta, 0_N) = fe$. Therefore S satisfies conditions $(1) - (3)$. Now $a + (\alpha, 0_N) = (\alpha, n) + (\alpha, 0_N) = (\alpha, n) = a$ and $ae = (\alpha, n)(\beta, 0_N) = (\alpha\beta, 0_N) = (\alpha, 0_N)(\beta, 0_N)$ (since N is zero-symmetric). Therefore $D_a = D_{(\alpha,n)} = \{(\alpha, 0_N)\}$. Let $b \in S$ such that $D_a = D_b$ and $I(a) \cap I(b)$ is non-empty. Since $D_a = D_b$, $b = (\alpha, m)$ for some $m \in N$. Again since $I(a) \cap I(b)$ is non-empty, there exists $z(= (\delta, t)$ for some $\delta \in B$ and $t \in N) \in S$ such that $a + z$, $b + z \in E^+(S)$. Therefore $a + z = (\alpha + \delta, n + t) = (\alpha + \delta, 0_N)$ and $b + z = (\alpha + \delta, m + t) = (\alpha + \delta, 0_N)$. Then $m + t = 0_N = n + t$ whence $m = n$. Hence $a = b$. Now for $s(= (\lambda, w)$ for some $\lambda \in B$ and $w \in N) \in S$ and $e(= (\beta, 0_N)$ for some $\beta \in B)$, $s(a + e) = (\lambda, w)(\alpha + \beta, n) =(\lambda(\alpha + \beta), wn) = (\lambda\alpha + \lambda\beta, wn) = (\lambda\alpha, wn) + (\lambda\beta, 0_N) = sa + se$. Therefore S satisfies conditions (5) and (6). □

In order to obtain our final result *viz.,* Theorem 5.1.29, we need, among others, the following result.

Proposition 5.1.28. *Let S be an E^+-inversive seminearring satisfying the conditions*

(2), (3), (6) *(Notations are the same as those in Theorem 5.1.27) and the following condition:* (4′) $a = a{+}ae$ *for all* $e \in E^+(S)$ *and for all* $a \in S$. *Then for any* $a \in S$, D_a *is singleton and* $D_a = \{a(a+x)\}$ *for any* $x \in I(a) := \{s \in S : a + s \in E^+(S)\}$.

Proof. Let $a \in S$ and $e \in E^+(S)$. Then $ea(= (e+e)a) \in E^+(S)$. Again by condition (6), $ae(= a(e+e)) \in E^+(S)$. Hence $ae = (ae)(ae)$ (by condition (3)) $= a(ae)e$ (by condition (2)) $= a^2e$.......(i). Since S is E^+-inversive, $I(a)$ is non-empty. Let $x \in I(a)$. Then $(a+x) \in E^+(S)$. So by condition (4′), $a + a(a+x) = a$. This implies, for each $g \in E^+(S)$, $ag = (a+a(a+x))g = ag+a(a+x)g = ag+a(ag+xg) = ag+(a^2g+axg)$ (using condition (6) as $xg \in E^+(S)$) $= a^2g+axg$ (using (i) and $ag \in E^+(S)$) $= a(a+x)g$ (using condition (6) and right distributive property of S). Therefore by definition of D_a (*cf.* Proposition 5.1.19(ii)), $a(a+x) \in D_a$. So D_a is non-empty. Hence by Proposition 5.1.19(ii), D_a is singleton. Hence for any $a \in S$, $D_a = \{a(a+x)\}$ where $x \in I(a)$. □

Theorem 5.1.29. *A seminearring* S *is a subdirect product of a distributive lattice and a zero-symmetric near-ring if and only if* S *is an* E^+*-inversive seminearring satisfying the following conditions :*

(1) $e + a = a + e$ *for all* $e \in E^+(S)$ *and for all* $a \in S$,

(2) $ef = fe$ *for all* $e, f \in E^+(S)$,

(3) $e^2 = e$ *for all* $e \in E^+(S)$,

(4′) $a = a + ae$ *for all* $e \in E^+(S)$ *and for all* $a \in S$ *(Same as* (4′) *of Proposition 5.1.28),*

(5) *if* $D_a = D_b$ *and* $I(a) \cap I(b)$ *is non-empty then* $a = b$ *where* $I(a) := \{x \in S : a + x \in E^+(S)\}$,

(6) $s(a+e) = sa + se$ *for all* $e \in E^+(S)$ *and for all* $a, s \in S$.

Proof. Let S be an E^+-inversive seminearring satisfying conditions (1)-(3), (4′), (5) and (6). Then in view of Proposition 5.1.28, S satisfies condition (4) of Theorem 5.1.27 and hence all the conditions of Theorem 5.1.27. So S is a full subdirect product of the bi-semilattice S/ρ and the zero-symmetric near-ring S/σ where $a\rho b$ if and only if $D_a = D_b$ and $a\sigma b$ if and only if there exists $x \in S$ such that $a+x$, $b+x \in E^+(S)$. Again in view of Proposition 5.1.25 and Proposition 5.1.28, S/ρ is isomorphic to $E^+(S)$ and in view of condition (4′), $E^+(S)$ is a distributive lattice. Hence in view of Proposition 5.1.15,

S becomes a subdirect product of the distributive lattice S/ρ and the zero-symmetric near-ring S/σ.
The other implication follows easily. □

The following remark is in order.

Remark 5.1.30. Theorem 5.1.29 is not only an analogue of, but also includes, Ghosh's Theorem 2.10 [29] (*cf.* Theorem 1.3.15). Because if the seminearring is replaced by a semiring then Theorem 5.1.29 reduces to Ghosh's Theorem 1.3.15. In fact, in that case the conditions (1) and (6) of Theorem 5.1.29 respectively come from commutativity of addition (In [29], the author has considered additively commutative semiring) and left distributive property of multiplication over addition and the other conditions viz., (2), (3), (4′), (5) are respectively the same[2] as the conditions (2.1), (2.2), (2.3) and the combination of (2.4) and (2.5) of Theorem 1.3.15.
Theorem 5.1.27, being the bi-semilattice version of Theorem 5.1.29, is also an analogue of Theorem 1.3.15. Since commutativity of addition and left distributive property of semiring are absent in a seminearring (*cf.* Definition 1.5.1), in the comparison of our results with Ghosh's Theorem 1.3.15, we see that there are two additional conditions viz., (1) and (6) in Theorems 5.1.27 and 5.1.29. That these conditions are essential as well as independent of the other conditions is evident from Example 5.1.31, Example 5.1.32(i) and Example 5.1.32(ii). In this connection, it is relevant to examine the independence of the other conditions of our results viz., Theorems 5.1.27 and 5.1.29, which we accomplish in Example 5.1.33.

The following example illustrates that without the condition (1) the conclusions of the Theorems 5.1.27, 5.1.29 need not follow. This example also illustrates the independence of (1) from the other conditions.

Example 5.1.31. Let $(S, \cdot)$ be a semilattice containing at least two elements. We define $a + b := a$ for all $a, b \in S$. Then $(S, +)$ is a band but not a semilattice and $(S, +, \cdot)$ is a semiring. Let $(\mathbb{Z}, +, \cdot)$ be the zero-symmetric near-ring of Example 5.1.14. Let T be the direct product of S and $\mathbb{Z}$. Then $E^+(T) = S \times \{0\}$. Clearly, T satisfies (2) and (3) of Theorems 5.1.27 and 5.1.29.

Let $a \in T$ and $w \in E^+(T)$. Then $a = (f, x)$ for some $f \in S$ and $x \in \mathbb{Z}$ and $w = (e, 0)$ for some $e \in S$. Then $a + (f, 0) = (f, x) + (f, 0) = (f, x) = a$ and $aw = (f, x)(e, 0) = (fe, 0) = (f, 0)w$. Therefore $(f, 0) \in D_a = D_{(f,x)}$. Then T satisfies

[2] The detailed explanation is not included in order to avoid possible digression.

(4) of Theorem 5.1.27. In view of Proposition 5.1.19 (*ii*), $D_a = D_{(f,x)} = \{(f,0)\}$. Let $b \in I(a)$ and so $a+b \in E^+(T)$. Clearly, $b = (g,-x)$ for some $g \in S$. Then $a(a+b) = (f,x)((f,x)+(g,-x)) = (f,x)(f+g,0) = (f,x)(f,0) = (f,0)$. Therefore $D_a = \{a(a+b) : b \in I(a)\}$. Now for $a \in T$ and $w \in E^+(T)$, where $a = (f,x)$ for some $f \in S$ and $x \in \mathbb{Z}$ and $w = (e,0)$ for some $e \in S$, $a + aw = (f,x)+(f,x)(e,0) = (f,x)+(fe,0) = (f+fe,x) = (f,x) = a$. Therefore T satisfies (4′) of Theorem 5.1.29.

Let $a, b \in T$ such that $D_a = D_b$ and $I(a) \cap I(b)$ is non-empty where $a = (f,x), b = (g,y)$ for some $f, g \in S$ and $x, y \in \mathbb{Z}$. Now we have already shown that $D_a = \{(f,0)\}$ and $D_b = \{(g,0)\}$. Since $D_a = D_b$, $f = g$. Let $c = (h,z) \in I(a) \cap I(b)$. Then $(f,x)+(h,z) = (f+h,x+z) = (f,x+z) \in E^+(T)$ and hence $x = -z$. Similarly, we can show that $y = -z$. Therefore $x = y$. Hence $a = b$ whence T satisfies condition (5) of Theorems 5.1.27 and 5.1.29. It can be easily shown that T satisfies condition (6) of Theorems 5.1.27 and 5.1.29. But T does not satisfy condition (1) of Theorems 5.1.27 and 5.1.29.

Now we provide below two examples to illustrate that the condition (6) in both the theorems (Theorem 5.1.27 and Theorem 5.1.29) is essential as well as independent of other conditions.

Example 5.1.32. Let $(S,+)$ be the semigroup described in the Exercise 21 of pp-43, [45] where $S = \{e,a,f,b\}$ and '+' is defined as follows.

$+$	e	a	f	b
e	e	a	f	b
a	a	e	b	f
f	f	b	f	b
b	b	f	b	f

(*i*) Suppose $T = \{f_0, f_1, f_2, f_3\} \subseteq M(S)$ where

$$e \xmapsto{f_0} e,\ a \xmapsto{f_0} e,\ f \xmapsto{f_0} f,\ b \xmapsto{f_0} f;$$
$$e \xmapsto{f_1} f,\ a \xmapsto{f_1} f,\ f \xmapsto{f_1} f,\ b \xmapsto{f_1} f;$$
$$e \xmapsto{f_2} e,\ a \xmapsto{f_2} a,\ f \xmapsto{f_2} f,\ b \xmapsto{f_2} f;$$
$$e \xmapsto{f_3} f,\ a \xmapsto{f_3} b,\ f \xmapsto{f_3} f,\ b \xmapsto{f_3} f.$$

Then $(T,+,\circ)$ is an E^+-inversive seminearring (under point wise addition and composition) with $E^+(T) = \{f_0, f_1\}$ where $+$ and $\circ$ are as follows.

$+$	f_0	f_1	f_2	f_3
f_0	f_0	f_1	f_2	f_3
f_1	f_1	f_1	f_3	f_3
f_2	f_2	f_3	f_0	f_1
f_3	f_3	f_3	f_1	f_1

$\circ$	f_0	f_1	f_2	f_3
f_0	f_0	f_1	f_0	f_1
f_1	f_1	f_1	f_1	f_1
f_2	f_0	f_1	f_2	f_1
f_3	f_1	f_1	f_3	f_1

It is a matter of routine verification to show that T satisfies conditions (1), (2) and (3) of Theorem 5.1.27. T also satisfies condition (4) of Theorem 5.1.27 as $D_{f_0} = D_{f_2} = \{f_0\}$ and $D_{f_1} = D_{f_3} = \{f_1\}$. This together with the facts that $I(f_0) = I(f_1) = \{f_0, f_1\}$, $I(f_2) = I(f_3) = \{f_2, f_3\}$ shows that T satisfies condition (5) of Theorem 5.1.27. But T does not satisfy condition (6) of Theorem 5.1.27 as $f_3 \circ (f_2 + f_1) \neq f_3 \circ f_2 + f_3 \circ f_1$. Consequently, in view of the proof of Theorem 5.1.27 T is not a full subdirect product of a bi-semilattice and a zero-symmetric near-ring.

(ii) Let $P = \{g_0, g_1, g_2, g_3\} \subseteq M(S)$ where

$$e \xmapsto{g_0} e,\ a \xmapsto{g_0} e,\ f \xmapsto{g_0} e,\ b \xmapsto{g_0} e;$$
$$e \xmapsto{g_1} e,\ a \xmapsto{g_1} e,\ f \xmapsto{g_1} f,\ b \xmapsto{g_1} f;$$
$$e \xmapsto{g_2} e,\ a \xmapsto{g_2} e,\ f \xmapsto{g_2} e,\ b \xmapsto{g_2} a;$$
$$e \xmapsto{g_3} e,\ a \xmapsto{g_3} e,\ f \xmapsto{g_3} f,\ b \xmapsto{g_3} b.$$

Then $(P, +, \circ)$ is an E^+-inversive seminearring (under point wise addition and composition) with $E^+(P) = \{g_0, g_1\}$ where $+$ and $\circ$ are as follows.

$+$	g_0	g_1	g_2	g_3
g_0	g_0	g_1	g_2	g_3
g_1	g_1	g_1	g_3	g_3
g_2	g_2	g_3	g_0	g_1
g_3	g_3	g_3	g_1	g_1

$\circ$	g_0	g_1	g_2	g_3
g_0	g_0	g_0	g_0	g_0
g_1	g_0	g_1	g_0	g_1
g_2	g_0	g_0	g_0	g_2
g_3	g_0	g_1	g_0	g_3

It is a matter of routine verification to show that P satisfies conditions (1), (2), (3) and (4′) of Theorem 5.1.29. Now $I(g_0) = \{g_0, g_1\} = I(g_1)$ and $I(g_2) = \{g_2, g_3\} = I(g_3)$. This together with $D_{g_0} = D_{g_2} = \{g_0\}$ and $D_{g_1} = D_{g_3} = \{g_1\}$ shows that P satisfies condition (5). But $g_2 \circ (g_2 + g_1) \neq g_2 \circ g_2 + g_2 \circ g_1$ shows that P does not satisfy condition (6) of Theorem 5.1.29. Consequently, in view of the proof of Theorem 5.1.29 T is not a subdirect product of a distributive lattice and a zero-symmetric near-ring.

The following examples respectively prove the independence of each of the remaining conditions of both the Theorems 5.1.27 and 5.1.29.

Example 5.1.33. (i) All but the condition (2) of Theorems 5.1.27 and 5.1.29 are satisfied by the E^+-inversive seminearring $(U, +, \cdot)$, where $(U, +)$ is a join semilattice and $(U, \cdot)$ is left zero semigroup.

(ii) All except condition (3) of Theorems 5.1.27 and 5.1.29 are satisfied by the E^+-inversive seminearring $(L, +, \cdot)$, where $L = \{1, 2\}$ and $+$ is the supremum with respect to the usual ordering of natural numbers and $ab = 1$ for all $a, b \in L$.

(iii) All the conditions of Theorem 5.1.27 except (4) are satisfied by the E^+-inversive subseminearring $S = \{(0, n)|0 \in B,\ n \in \mathbb{N}\} \cup \{(\alpha, a)|\alpha \in B,\ a \in \mathbb{Z}\}$ of the E^+-inversive seminearring $B \times \mathbb{Z}$ where B is the bi-semilattice and $(\mathbb{Z}, +, \cdot)$ is the zero-symmetric near-ring as defined in Example 5.1.14.

(iv) All the conditions of Theorem 5.1.29 except ($4'$) are satisfied by the E^+-inversive seminearring $(B, +, \cdot)$, which is a bi-semilattice but not a distributive lattice.

(v) All except condition (5) of Theorems 5.1.27 and 5.1.29 are satisfied by the E^+-inversive seminearring $S = \{0, a, b\}$ (see Example 1 [10]) with zero 0 where $a^2 = 2a = ab = 0$ and $b^2 = 2b = a + b = b$ (we note that it is a distributive lattice of rings but not a strong distributive lattice of rings (*cf.* [29])).

5.2 Relationships among different classes of seminearrings

Any additively regular (*cf.* Definition 1.5.15) seminearring is clearly additively E-inversive, i.e., E^+-inversive (*cf.* Definition 4.3.2). Hence any seminearring belonging to each of the classes characterized in [81, 82, 98] is E^+-inversive. The relationships, of these classes of additively regular (and hence E^+-inversive) seminearrings with the classes of E^+-inversive seminearrings characterized in Theorems 5.1.16, 5.1.27 and Theorems 5.1.18, 5.1.29, are explored in the following result.

Theorem 5.2.1. (i) *The class of strong bi-semilattices of zero-symmetric near-rings (abbreviated as* ***SBSLZSNR****) [81] is a subclass of the class of full subdirect products of a bi-semilattice and a zero-symmetric near-ring (****FSDPBSLZSNR****) (cf. Theorems 5.1.16, 5.1.27) i.e.,* ***SBSLZSNR*** $\subseteq$ ***FSDPBSLZSNR****.*

(ii) *The class of strong distributive lattices of zero-symmetric near-rings (****SDLZSNR****) [81] is a subclass of the class of subdirect products of a distributive lattice and a* *zero-symmetric near-ring (**SDPDLZSNR**) (cf. Theorems 5.1.18, 5.1.29) i.e., **SDLZSNR** $\subseteq$ **SDPDLZSNR**.*

(iii) ***SBSLZSNR** = **FSDPBSLZSNR** $\cap$ **AR** = **FSDPBSLZSNR** $\cap$ **BSLZSNR** where the class of bi-semilattices of zero-symmetric near-rings is abbreviated as **BSLZSNR** [82] and the class of additively regular seminearrings is abbreviated as **AR**.*

(iv) ***SDLZSNR** = **SDPDLZSNR** $\cap$ **AR** = **SDPDLZSNR** $\cap$ **DLZSNR** = **SDPDLZSNR** $\cap$ **SBSLZSNR** where the class of distributive lattices of zero-symmetric near-rings is abbreviated as **DLZSNR** [82].*

(v) ***SDPDLZSNR1** $\subseteq$ **SDLZSNR** where the class of additively regular zero-symmetric seminearrings with* 1 *which are subdirect product of a distributive lattice and a zero-symmetric near-ring is abbreviated as **SDPDLZSNR1** (cf. Theorem 4.14 [98]).*

Proof. (i) follows from Theorem 2.28 [81] and Theorem 5.1.16.

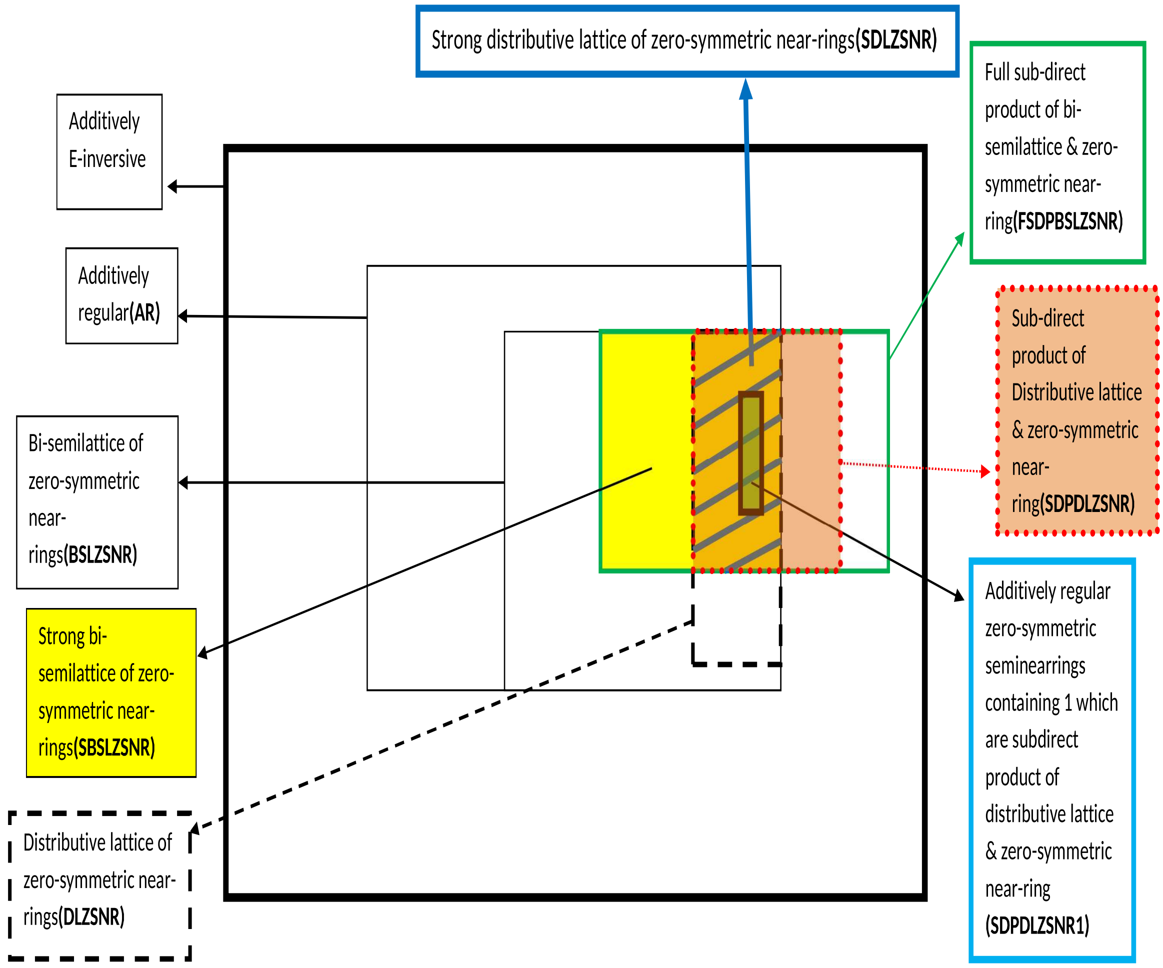
Theorem 2.35 [81] together with Theorem 5.1.18 implies (ii).

Now a strong bi-semilattice of zero-symmetric near-rings is additively regular (*cf.* Theorem 2.28 [81]) as well as an E^+-inversive strong bi-semilattice of additively cancellative zero-symmetric seminearrings. So in view of Theorem 5.1.16 we obtain **SBSLZSNR** $\subseteq$ **FSDPBSLZSNR** $\cap$ **AR**. Again in presence of additive regularity, an E^+-inversive strong bi-semilattice of additively cancellative zero-symmetric seminearrings becomes a strong bi-semilattice of zero-symmetric near-rings whence the 1^{st} equality of (iii) follows. By Corollary 3.12 [82], **BSLZSNR** $\subseteq$ **AR**. Clearly, **SBSLZSNR** $\subseteq$ **BSLZSNR** (*cf.* Corollary 3.12 [82] and Theorem 2.28 [81]). From (i) of this theorem, **SBSLZSNR** $\subseteq$ **FSDPBSLZSNR**. These relations together with the 1^{st} equality of (iii) imply the 2^{nd} equality of (iii).

The 1^{st} equality and the 2^{nd} equality of (iv) can be proved in an analogous way to those of (iii) by using Theorem 5.1.18 instead of Theorem 5.1.16, property (ii) instead of property (i) of this theorem, Theorem 3.14 [82] instead of Corollary 3.12 [82] and Theorem 2.35 [81] instead of Theorem 2.28 [81]. The 3^{rd} equality of (iv) follows from the 1^{st} equality of (iv) and the fact that **SDLZSNR** $\subseteq$ **SBSLZSNR** $\subseteq$ **AR** (*cf.* Theorem 2.28 [81]).

(v) is an easy consequence of (iv). □

What Theorem 5.2.1 says can be expressed with the help of the following (possibly not so nice looking) diagram .



Remark 5.2.2. The result given by the 1^{st} equality in (iii) of Theorem 5.2.1 is the analogue of a result implicit in Theorem 2.7 [101].

The following example illustrates that the inclusion in (i) of Theorem 5.2.1 is strict.

Example 5.2.3. Let $B = \{0, \alpha, 1\}$ be the bi-semilattice and let $(\mathbb{Z}, +, \cdot)$ be the zero-symmetric near-ring as described in Example 5.1.14. Let $T = T_0 \cup T_1 \cup T_\alpha$ where $T_0 = \{(0, -2n) \in B \times \mathbb{Z} \mid 0 \in B,\ n \in \mathbb{N} \cup \{0\}\}$, $T_1 = \{(1, -n) \in B \times \mathbb{Z} \mid 1 \in B,\ n \in \mathbb{N} \cup \{0\}\}$ and $T_\alpha = \{(\alpha, a) \in B \times \mathbb{Z} \mid \alpha \in B,\ a \in \mathbb{Z}\}$. Then T is a full subdirect product of the bi-semilattice B and the zero-symmetric near-ring $(\mathbb{Z}, +, \cdot)$ with $E^+(T) = \{(0, 0), (1, 0), (\alpha, 0)\}$. But T is not additively regular as $(0, -2n)$ where $n \in \mathbb{N}$ is not a regular element in $(T, +)$. So in view of Theorem 5.2.1 (iii), T is not a strong bi-semilattice of zero-symmetric near-rings.

The following example illustrates that the inclusion in (ii) of Theorem 5.2.1 is strict.

Example 5.2.4. In view of Example 1.4 (b) (pp. 8, [91]), $(\mathbb{Z}, +, \cdot_\triangle)$ is a zero-symmetric near-ring with '+' as the usual addition of integers and $a \cdot_\triangle b \ := \ a$ if $b \in \triangle$ and $a \cdot_\triangle b \ := \ 0$ if $b \notin \triangle$ where $I = \{2n : n \in \mathbb{N} \cup \{0\}\}$ and $\triangle = \{z \in \mathbb{Z} \ : \ z \notin I\}$. Let D be the Boolean algebra $\{0, 1\}$. Let $S = \{(0, a) \ | 0 \in D, \ a \in I\} \cup \{(1, a) \ | 1 \in D, \ a \in \mathbb{Z}\}$. Then S is a subdirect product of the distributive lattice D and the zero-symmetric near-ring $(\mathbb{Z}, +, \cdot_\triangle)$. But it is not additively regular as $(0, 2n) \in S$ where $n \in \mathbb{N}$ is not a regular element in $(S, +)$. Hence in view of Theorem 5.2.1 (iv), S is not a strong distributive lattice of zero-symmetric near-rings.

Remark 5.2.5. Existence of zero-symmetric near-ring without multiplicative identity 1 proves that the inclusion in (v) of Theorem 5.2.1 is strict.

Theorem 5.2.6. (i) *The class of strong bi-semilattices of near-rings (abbreviated as **SBSLNR**) [81] is a subclass of the class of full subdirect products of a bi-semilattice and a near-ring (**FSDPBSLNR**) (cf. Theorem 5.1.12) i.e., **SBSLNR** $\subseteq$ **FSDPBSLNR**.*

(ii) *The class of strong distributive lattices of near-rings (**SDLNR**) [81] is a subclass of the class of subdirect products of a distributive lattice and a near-ring (**SDPDLNR**) (cf. Theorem 5.1.17) i.e., **SDLNR** $\subseteq$ **SDPDLNR**.*

(iii) ***SBSLNR** = **FSDPBSLNR** $\cap$ **AR** = **FSDPBSLNR** $\cap$ **BSLNR** where the class of bi-semilattices of near-rings is abbreviated as **BSLNR** [82] and the class of additively regular seminearrings is abbreviated as **AR**.*

(iv) ***SDLNR** = **SDPDLNR** $\cap$ **AR** = **SDPDLNR** $\cap$ **DLNR** = **SDPDLNR** $\cap$ **SBSLNR** where the class of distributive lattices of near-rings is abbreviated as **DLNR** [82].*

Proof. (i) follows from Theorem 2.23 [81] and Theorem 5.1.12.

Theorem 2.35 [81] together with Theorem 5.1.17 implies (ii).

Now a strong bi-semilattice of near-rings is additively regular (*cf.* Theorem 2.23 [81]) as well as an E^+-inversive strong bi-semilattice of additively cancellative seminearrings. So in view of Theorem 5.1.12 we obtain **SBSLNR** $\subseteq$ **FSDPBSLNR** $\cap$ **AR**.

Again in presence of additive regularity, an E^+-inversive strong bi-semilattice of additively cancellative seminearrings becomes a strong bi-semilattice of near-rings whence the 1^{st} equality of (*iii*) follows. By Corollary 3.10 [82], **BSLNR** $\subseteq$ **AR**. Clearly, **SBSLNR** $\subseteq$ **BSLNR** (*cf.* Corollary 3.10 [82] and Theorem 2.23 [81]). From (*i*) of this theorem, **SBSLNR** $\subseteq$ **FSDPBSLNR**. These relations together with the 1^{st} equality of (*iii*) imply the 2^{nd} equality of (*iii*).

The 1^{st} equality and the 2^{nd} equality of (*iv*) can be proved in an analogous way to those of (*iii*) by using Theorem 5.1.17 instead of Theorem 5.1.12, property (*ii*) instead of property (*i*) of this theorem, Theorem 3.13 [82] instead of Corollary 3.10 [82] and Theorem 2.35 [81] instead of Theorem 2.23 [81]. The 3^{rd} equality of (*iv*) follows from the 1^{st} equality of (*iv*) and the fact that **SDLNR** $\subseteq$ **SBSLNR** $\subseteq$ **AR** (*cf.* Theorem 2.23 [81]). □

5.3 Near-ring congruences

Definition 5.3.1. A non-empty subset I of a seminearring S is said to be a *generalised strong full reflexive (left, right) k-ideal* if I satisfies the following conditions :

(*i*) $(I, +)$ is a full, reflexive subsemigroup of $(S, +)$,

(*ii*) I is a (left, right) k-ideal,

(*iii*) for $s, a \in S$ and $w \in I$ there exist i_1, i_2, i_3, $i_4 \in I$ such that $s(a+w)+i_1 = i_2+sa$ and $s(w+a)+i_3 = i_4+sa$.

Remark 5.3.2. Let S be a full subdirect product of a bi-semilattice and (zero-symmetric) near-ring or a subdirect product of a distributive lattice and a (zero-symmetric) near-ring. Then in view of Theorems 5.1.12, 5.1.16, 5.1.17 and 5.1.18, S is an E^+-inversive seminearring. Let $I \subseteq S$. Then in view of Definition 4.1.15, Definition 5.3.1 and Proposition 4.3.6, I is a generalised strong full reflexive (left, right) k-ideal if and only if I is a generalised strong dense reflexive (left, right) k-ideal.

Theorem 5.3.3. *Suppose S is a full subdirect product of a bi-semilattice and a (zero-symmetric) near-ring or a subdirect product of a distributive lattice and a (zero-symmetric) near-ring. Then there exist inclusion preserving bijective correspondences between*

(*i*) *the set of all generalised strong full reflexive right k-ideals of S and the set of all near-ring congruences on S,* *(ii) the set of all generalised strong full reflexive k-ideals of S and the set of all zero-symmetric near-ring congruences on S*

via $I \mapsto \rho_I$ where $a\ \rho_I\ b$ if and only if there exists $x \in S$ such that $a + x, b + x \in I$.

Proof. It follows from Theorem 4.1.18 and Remark 5.3.2. □

Remark 5.3.4. Let S be a full subdirect product of a bi-semilattice and a zero-symmetric near-ring or a subdirect product of a distributive lattice and a zero-symmetric near-ring. Now let ρ be a near-ring congruence on S. Then S/ρ becomes a zero-symmetric near-ring. Therefore a near-ring congruence on S is a zero-symmetric near-ring congruence on S. Again if I is a generalised strong full reflexive right k-ideal of S, then ρ_I is a near-ring congruence on S whence a zero-symmetric near-ring congruence on S. Then $I = \{x \in S : (x, x + x) \in \rho_I\}$ becomes a generalised strong full reflexive k-ideal of S. Therefore I is a generalised strong full reflexive right k-ideal of S if and only if I is a generalised strong full reflexive k-ideal of S.

The following result is the counter part of Theorem 4.3.9 for the seminearrings which are full subdirect products of a bi-semilattice and a (zero-symmetric) near-ring or subdirect products of a distributive lattice and a (zero-symmetric) near-ring.

Theorem 5.3.5. *Suppose $(S, +, \cdot)$ is a full subdirect product of a bi-semilattice and a (zero-symmetric) near-ring or a subdirect product of a distributive lattice and a (zero-symmetric) near-ring. Then*

(i) the lattice of all generalised strong full reflexive right k-ideals of S and the lattice of all near-ring congruences on S are isomorphic,

(ii) the lattice of all generalised strong full reflexive k-ideals of S and the lattice of all zero-symmetric near-ring congruences on S are isomorphic.

Proof. In view of (v) of Theorem 4.3.7 and Remark 5.3.2, the set of all generalised strong full reflexive (right) k-ideals of S under set inclusion forms a lattice where for any two generalised strong full reflexive (right) k-ideals I, J of S, $I \wedge J = I \cap J$ and $I \vee J = \overline{I + J}$. Since S is an E^+-inversive seminearring, (vii) of Theorem 4.3.7 shows that the set of all (zero-symmetric) near-ring congruences on S becomes a lattice under set inclusion where for any two (zero-symmetric) near-ring congruences ρ, γ on S, $\rho \wedge \gamma = \rho \cap \gamma$ and $\rho \vee \gamma = \rho \circ \gamma$. The rest of the proof follows in view of Proposition 1.2.10 and Theorem 5.3.3. □

The following result is the counter part of Theorems 4.3.10 and 4.3.12 for the seminearrings which are full subdirect products of a bi-semilattice and a (zero-symmetric) near-ring or subdirect products of a distributive lattice and a (zero-symmetric) near-ring. In this result we mainly study the modularity, distributivity and completeness of the lattices obtained in Theorem 5.3.5.

Theorem 5.3.6. *Suppose* $(S, +, \cdot)$ *is a full subdirect product of a bi-semilattice and a (zero-symmetric) near-ring or a subdirect product of a distributive lattice and a (zero-symmetric) near-ring. Then the following are true.*

(i) The set of all generalised strong full reflexive right k*-ideals of* S *and the set of all near-ring congruences on* S *become modular lattices.*

(ii) The set of all generalised strong full reflexive k*-ideals of* S *and the set of all zero-symmetric near-ring congruences on* S *become modular lattices.*

(iii) If $\overline{IJ} = I \cap J$ *for all generalised strong full reflexive right* k*-ideals* I, J *of* S*, then the set of all generalised strong full reflexive right* k*-ideals of* S *and the set of all near-ring congruences on* S *become distributive lattices.*

(iv) If $\overline{IJ} = I \cap J$ *for all generalised strong full reflexive* k*-ideals* I, J *of* S*, then the set of all generalised strong full reflexive* k*-ideals of* S *and the set of all zero-symmetric near-ring congruences on* S *become distributive lattices.*

(v) The lattice of all generalised strong full reflexive right k*-ideals of* S *and the lattice of all near-ring congruences on* S *are complete.*

(vi) The lattice of all generalised strong full reflexive k*-ideals of* S *and the lattice of all zero-symmetric near-ring congruences on* S *are complete.*

Proof. We omit the proof since (i) and (ii) follow in a similar manner to the proof of Theorem 2.3.17, (iii), (iv) hold in a similar manner to the proof of Theorem 2.3.22 and (v), (vi) hold in a similar manner to the proof of Theorem 4.3.12. □

Theorem 5.3.7. *Suppose* $(S, +, \cdot)$ *is a full subdirect product of a bi-semilattice and a near-ring or a subdirect product of a distributive lattice and a near-ring. Then*

(i) $E^+(S)$ *is the smallest generalised strong full reflexive right* k*-ideal of* S *and*

(ii) $\sigma = \{(a, b) \in S \times S : a + b' \in E^+(S)$ *for some (all)* $b' \in W^+(b)\}$ *is the least near-ring congruence on* S*.*

Proof. (i) Let S be a full subdirect product of a bi-semilattice B and a near-ring N. Then $E^+(S) = \{(e, o) \in B \times N : e \in B\}$. Clearly, $E^+(S)$ is a full subsemigroup of $(S, +)$. Let $x, y \in S$ such that $x + y \in E^+(S)$. Then $x = (e_1, a)$, $y = (e_2, -a)$ for some $e_1, e_2 \in B$ and $a \in N$. Now $y + x = x + y = (e_1 + e_2, 0)$. Therefore $y + x \in E^+(S)$ whence $E^+(S)$ is a reflexive subsemigroup of $(S, +)$.

Let $w \in E^+(S)$ and $s \in S$. Then $w = (e, 0)$ and $s = (f, c)$ for some $e, f \in B$ and $c \in N$. Now $ws = (e, 0)(f, c) = (ef, 0)$. Therefore $ws \in E^+(S)$ whence $E^+(S)$ is a right $\mathcal{S}$-ideal.

Let $g \in E^+(S)$ and $t \in S$ such that $g + t \in E^+(S)$. Then $g = (e, 0)$ and $t = (f, n)$ for some $e, f \in B$ and $n \in N$. Now $g + t = (e + f, n)$. Since $g + t \in E^+(S)$, $n = 0$. Therefore $t \in E^+(S)$ whence $E^+(S)$ is a right k-ideal.

Now let $g \in E^+(S)$ and $s, x \in S$. Then $g = (e, 0)$, $s = (f_1, b)$ and $x = (f_2, z)$ for some $e, f_1, f_2 \in B$ and $b, z \in N$. Now

$$s(g + x) = s(x + g) = (f_1, b)(e + f_2, z) = (f_1(e + f_2), bz)$$

and $sx = (f_1 f_2, bz)$. Then

$$s(g + x) + (f_1 e, 0) = (f_1 e, 0) + sx = s(x + g) + (f_1 e, 0)$$

where $(f_1 e, 0) \in E^+(S)$. Therefore $E^+(S)$ is the smallest generalised strong full reflexive right k-ideal of S.

(ii) Since $E^+(S)$ is a generalised strong full reflexive right k-ideal of S, in view of (i) of Theorem 5.3.3, $\rho_{E^+(S)}$ is a near-ring congruence on S where $(a, b) \in \rho_{E^+(S)}$ if and only if there exists $x \in S$ such that $a + x$, $b + x \in E^+(S)$. Again $E^+(S)$ a reflexive and closed subsemigroup of $(S, +)$. Then in view of Definitions 1.1.16 and 1.1.2, $(S, +)$ is an E-unitary E-inversive semigroup. Now in view of Proposition 5.3 [31], $\sigma = \{(a, b) \in S \times S : a + b' \in E^+(S)$ for some (all) $b' \in W^+(b)\}$ is the least group congruence on $(S, +)$ and $\sigma = \rho_{E^+(S)}$. Therefore σ is the least near-ring congruence on the seminearring S.

Now a subdirect product of a distributive lattice D and a near-ring N is again a full subdirect product of the bi-semilattice D and the near-ring N. Then for a subdirect product of a distributive lattice and a near-ring, (i) and (ii) follow similarly. □

Theorem 5.3.8. *Suppose $(S, +, \cdot)$ is a full subdirect product of a bi-semilattice and a zero-symmetric near-ring or a subdirect product of a distributive lattice and a zero-symmetric near-ring. Then* *(i)* $E^+(S)$ *is the smallest generalised strong full reflexive* k*-ideal of* S *and*

(ii) $\sigma = \{(a, b) \in S \times S : a + b' \in E^+(S)$ *for some (all)* $b' \in W^+(b)\}$ *is the least zero-symmetric near-ring congruence on* S.

Proof. We omit the proof since it follows from Remark 5.3.4 and Theorem 5.3.7. □

Some Remarks and Scope of Further Study

We list below some remarks and observations which are mainly related with some possible extension of the research work undertaken in this thesis.

1. In almost all results of **Chapter 4**, we have considered some restricted type of seminearrings, *viz.,* 'seminearring with left local units', 'E^+-inversive seminearring' so that the set of all generalised strong dense reflexive right k-ideals (k-ideals) and the set of all near-ring (zero-symmetric near-ring) congruences form lattices (*cf. Remark 4.3.8*). It will be nice if one can find some wider class of seminearrings where the above sets always form lattices and consequently the correspondences of Theorem 4.1.18 become lattice isomorphisms.

2. In **Chapter 5**, as analogues of Mitsch's Theorem 14 [78] (*cf.* Theorem 1.1.33) as well as of Ghosh's Theorem 2.3 [29] (*cf.* Theorem 1.3.14), Theorems 5.1.12, 5.1.16, 5.1.17 and 5.1.18 characterize four classes of seminearrings. But, only for the two classes of seminearrings, *i.e.,* full subdirect products of a bi-semilattice and a zero-symmetric near-ring (obtained in Theorems 5.1.16) and subdirect products of a distributive lattice and a zero-symmetric near-ring (obtained in 5.1.18), we have been able to obtain analogue of Ghosh's Theorem 2.10 [29] (*cf.* Theorem 1.3.15) viz., Theorems 5.1.27, 5.1.29. A possible future work is to obtain analogue of Ghosh's Theorem 1.3.15 for the classes of seminearrings which are full subdirect products of a bi-semilattice and a near-ring (characterized in Theorem 5.1.12) and which are subdirect products of a distributive lattice a near-ring (characterized in Theorem 5.1.17). It would also be nice to investigate about the forms of the Theorems 5.1.12, 5.1.16, 5.1.17, 5.1.18, 5.1.27, 5.1.29 if the seminearrings are replaced by distributively generated seminearrings.

3. In view of Observation 5.1.5, a bi-semilattice $(B, +, \cdot)$ is a semiring where $(B, +)$ and $(B, \cdot)$ are semilattices. So in **Chapter 5**, we mainly characterize different classes of seminearrings which are subdirect products of a semiring and a seminearring. Replacing a bi-semilattice by an idempotent seminearring I (*i.e.,* a seminearring $(I, +, \cdot)$ where $(I, +)$ is a semilattice and $(I, \cdot)$ is a band) or by a band seminearring B (*i.e.,* a seminearring $(B, +, \cdot)$ where $(B, +)$ and $(B, \cdot)$ are both bands), one can characterize the classes of seminearrings which are

   (i) full subdirect product of an idempotent seminearring and a (zero-symmetric) near-ring and

   (ii) full subdirect product of a band seminearring and a (zero-symmetric) near-ring.

4. It is well known that for two topological spaces X and G, $T(X, G)$, the family of all continuous functions from X into G can be given an algebraic structure by defining point-wise operations, provided G has an algebraic structure compatible with the topological structure. However, even in the absence of any algebraic structure on G one can, in a natural way, equip $T(X, G)$ with an algebraic structure as follows. For each continuous function α from G into X there corresponds an associative binary operation on $T(X, G)$ defined by $fg = f \circ \alpha \circ g$ for any $f, g \in T(X, G)$ where $f \circ \alpha \circ g$ denotes the usual composition of functions. The resulting semigroup is denoted by $T(X, G, \alpha)$ [70]. Moreover, if G is an additive topological group and $X = G$ then the semigroup $T(X, G, \alpha)$ becomes the near-ring of all continuous self-maps of G, denoted by $N(G, G, \alpha)$, where the addition is defined point-wise [70]. Analogue of Magill's [70] problem can be formulated in our setting by replacing the additive topological group G by an additive topological semigroup S which in turn makes $N(S, S, \alpha)$ a seminearring. The study of algebraic as well as of topological aspects of this type of seminearrings may be a possible future work.

5. Following the formulation of minimum group congruence on inverse semigroup and regular semigroup, we have obtained least near-ring congruence (*cf.* Theorem 2.2.15, Corollary 2.2.16 and Theorem 2.2.17) on additively regular seminearrings and additively inverse seminearrings. In a similar manner adopting the formulation of maximum idempotent separating congruence on inverse semigroup one can formulate maximum additive idempotent separating congruence in a class of

additively regular seminearrings.

6. We have studied the lattice of near-ring congruences and that of zero-symmetric near-ring congruences on E^+-inversive seminearrings (*cf.* Theorems 4.3.9, 4.3.10). A possible future work is to study additively regular congruences and additively completely simple congruences on E^+-inversive seminearrings in order to obtain analogues of some results connecting regular and completely simple congruences with its kernel and trace on E-inversive semigroups (*cf.* [30, 69]).

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List of Publications Based on the Thesis

A list of publications resulted from the work of this thesis has been appended below.

(1) **Kamalika Chakraborty, Pavel Pal, Sujit Kumar Sardar, Near-ring congruences on additively regular seminearrings, Semigroup Forum, 101, 285-302 (2020)**.

(2) **Kamalika Chakraborty, Rajlaxmi Mukherjee, Sujit Kumar Sardar, Near-ring congruences on seminearrings, Semigroup Forum, 104, 584-593 (2022)**.

(3) **Kamalika Chakraborty, Rajlaxmi Mukherjee, Lattice of near-ring congruences on seminearrings, Bull. Calcutta Math. Soc., (accepted).**

(4) **Rajlaxmi Mukherjee, Kamalika Chakraborty, Sujit Kumar Sardar, On full subdirect products of a bi-semilattice and a (zero-symmetric) near-ring, (communicated).**

Index of Symbols

$End(S)$ The set of all endomorphisms of an additively written semigroup S, page 27

$M(S)$ The seminearring of all self maps of an additively written semigroup S, page 26

$M_0(S)$ The seminearring of zero fixing self maps of an additively written semigroup S, page 26

$M_c(S)$ The seminearring of all constant self maps of an additively written semigroup S, page 27

r_I $a\ r_I\ b$ if and only if $a, b \in x + I + y$ for some $x, y \in M$, page 71

r_I'' Transitive closure of the relation r_I, page 71

$V(a)$ The set of all inverses of an element a of a semigroup, page 14

$V^+(a)$ The set of all additive inverses of an element a of a seminearring S *i.e.,* the set $\{x \in S : a = a + x + a, x = x + a + x\}$, page 35

$W^+(a)$ The set of all additive weak inverses of an element a of a seminearring S *i.e.,* the set $\{x \in S : x = x + a + x\}$, page 88

Subject Index