CHARACTERIZATION OF SOME SOLITONS WITHIN THE FRAMEWORK OF VARIOUS DIFFERENTIABLE MANIFOLDS

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that thesis entitled **"CHARACTERIZATION OF SOME SOLITONS WITHIN THE FRAMEWORK OF VARIOUS DIFFERENTIABLE MANIFOLDS"** submitted by Sri **Dipen Ganguly** who got his name registered on 13th September, 2018 (INDEX NO : 170/18/Maths./26) for the award of Doctor of Philosophy (Science) degree of Jadavpur University, is absolutely based upon his own work under the supervision of **Prof. Arindam Bhattacharyya** and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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This Thesis is Dedicated to

My Beloved Parents

Sri Dilip Ganguly and Smt. Rama Ganguly

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Preface

The present thesis entitled, “Characterization of some solitons within the framework of various differentiable manifolds” consists of five chapters. The first chapter provides an introduction to different types of smooth manifolds and solitons.

In chapter two, first we study conformal Ricci soliton on generalized Sasakian space form and characterize the soliton in terms of shrinking steady and expanding. We consider the cases when the potential vector field is, pointwise collinear with the Reeb vector field and when it is of gradient type.

Next, we characterize almost coKähler manifolds admitting conformal Ricci soliton. We also investigate conformal Ricci soliton on a (k, μ)-almost coKähler manifold and prove the non-existence of conformal gradient Ricci soliton in this setup.

Then, we give some characterization of conformal Ricci soliton on $(LCS)_n$-manifolds and also consider the cases when the manifold is ξ-projectively flat, ξ-conharmonically flat and ξ-concircularly flat. After that, conformal Ricci soliton on $(LCS)_n$-manifolds satisfying certain curvature conditions are studied.

Next, we prove that if an warped product of two Riemannian manifolds admits a conformal Ricci soliton, then the base and the fiber both admit conformal Ricci soliton. Then, the converse of this result is discussed when the potential function is of gradient type. Also, we show that an warped product admitting conformal Ricci soliton is an Einstein one provided the potential vector field is Killing or concurrent. Finally, some applications of conformal Ricci soliton on generalized Robertson Walker spacetime are discussed.

In chapter three, we consider ϵ-Kenmotsu manifold admitting conformal η-Ricci soliton. Then, conformal η-Ricci solitons are characterized on ϵ-Kenmotsu manifold with Codazzi type, cyclic parallel and cyclic η-recurrent Ricci tensor and satisfying certain curvature conditions. After that, we study ϵ-Kenmotsu manifold admitting conformal η-Ricci soliton with torse-forming and gradient type potential vector field.

Next, we study conformal η-Ricci soliton on almost pseudo symmetric Kählerian spacetime manifold and characterize the nature of the soliton when the manifold is projectively flat and conharmonically flat. Finally, we study gradient conformal η-Ricci soliton on Kählerian spacetime manifold.

In chapter four, first we study some curvature properties of 3-dimensional quasi-Sasakian manifold with respect to Zamkovoy connection and then, the nature of Ricci soliton on 3-dimensional quasi-Sasakian manifold with respect to Zamkovoy connection is characterized.

Next, we study the nature of η-Ricci-Yamabe soliton on almost pseudo symmetric Kählerian spacetime manifold. Finally, it is shown that on a generalized Sasakian space form, a quasi-Yamabe soliton, with potential vector field pointwise collinear to the Reeb vector field, reduces to a Yamabe soliton.

In chapter five, we characterize η-Einstein soliton on a 3-dimensional trans-Sasakian manifold. Then, η-Einstein solitons are studied on 3-dimensional trans-Sasakian manifold with Codazzi type and cyclic parallel Ricci tensor and satisfying some curvature conditions. Finally, η-Einstein solitons with torse forming vector field are characterized on 3-dimensional trans-Sasakian manifold.

The thesis contains the subject matter of the following papers whose titles, journal information and chapterwise distribution are given below:

Authors	Title of the paper and journal information	Chapter
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Dipen Ganguly and Arindam Bhattacharyya	"A study on conformal Ricci solitons in the framework of $(LCS)_n$-manifolds", **GANITA,** Vol.70(2), 2020, 201-216.	2
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Dipen Ganguly, Santu Dey and Akram Ali	"Certain metrics on Kählerian spacetime manifolds", Communicated	3 and 4
Dipen Ganguly	"A note on quasi-Sasakian manifold with respect to the Zamkovoy connection" Communicated.	4
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Contents

1 Introduction

1.1 Introduction to manifolds

In the study of geometry, a manifold is a topological space that mimics Euclidean space locally around each point. Each point on a n-dimensional manifold has a neighbourhood that is homeomorphic to the n-dimensional Euclidean space. Lines and circles are examples of one-dimensional manifolds. Surfaces are another name for two-dimensional manifolds. The plane, sphere, and torus are all examples of objects that can be embedded (made without self-intersections) in three-dimensional real space. While a manifold may resemble Euclidean space locally, it may not do so globally. The sphere's surface, for example, is not an Euclidean space. As solution sets of systems of equations and graphs of functions, manifolds emerge spontaneously. Since it permits more sophisticated structures to be represented and understood in terms of the relatively well-known features of Euclidean space, the concept of a manifold is important to many sections of geometry and current mathematical physics.

Additional features may be present in manifolds. Differentiable manifolds are an important subclass of manifolds. Calculus on manifolds is possible thanks to this differentiable structure. Distances and angles can be measured using a Riemannian metric on a manifold. In the Hamiltonian formalism of classical mechanics, symplectic manifolds serve as phase spaces, whereas four-dimensional Lorentzian manifolds model spacetime in general relativity.

A differentiable manifold is a type of manifold that is comparable enough to an Euclidean space to allow calculus to be performed on it. A collection of charts, usually called as an atlas, can be used to describe any manifold. Since each chart sits within an Euclidean space to which the standard principles of calculus apply, ideas from calculus can therefore be applied while working within the various charts. Calculations performed in one chart are valid in any other differentiable chart if the charts are adequately compatible (that is, the transition from one chart to another is differentiable or smooth).

A topological manifold with a globally defined differential structure is called a differentiable manifold. Any topological manifold can be equipped with a differential structure locally with the help of the homeomorphisms in its atlas and the standard differentiable structure on an Euclidean space. From the local coordinate systems induced by the homeomorphisms, a global differentiable structure can be induced by taking their composition on chart intersections in the atlas as differentiable functions on the corresponding Euclidean space.

We start with, some curvature tensors which play an important role in the study of differential geometry and are used frequently throughout this thesis.

The projective curvature has an one-to-one correspondence between each coordinate neighbourhood of an n-dimensional Riemannian manifold and a domain of Euclidean space such that there is a one-to-one correspondence between geodesics of the Riemannian manifold with the straight lines in the Euclidean space. A transformation of a Riemannian manifold M of dimension n, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation.

Definition 1.1.1. *On an n-dimensional Riemannian (or, pseudo-Riemannian) manifold (M, g) the projective curvature tensor P [100], the concircular curvature tensor C [97], the conharmonic curvature tensor H [99], the W_2-curvature tensor [74], are defined by*

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[g(QY,Z)X - g(QX,Z)Y], \tag{1.1.1}$$

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.1.2}$$

$$H(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y] \tag{1.1.3}$$

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX], \tag{1.1.4}$$

where r is the scalar curvature, Q is the Ricci operator given by $g(QX,Y) = S(X,Y)$,

S denotes the Ricci tensor and R is the Riemannian curvature tensor defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.1.5}$$

for all vector fields $X, Y, Z \in \chi(M)$, where $\chi(M)$ denotes the set of all smooth vector fields on the manifold M.

In differential geometry and mathematical physics, an Einstein manifold is a Riemannian (or pseudo-Riemannian) differentiable manifold whose Ricci tensor is proportional to the metric. M. C. Chaki and R. K. Maity [23] introduced the concept of η-Einstein (or, quasi-Einstein) manifold as a natural generalization of Einstein manifolds.

Definition 1.1.2. *A Riemannian (or, pseudo-Riemannian) manifold (M,g) of dimension n is said to be an η-Einstein manifold if its Ricci tensor S satisfies*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{1.1.6}$$

for all $X, Y \in \chi(M)$ and smooth functions a, b on the manifold (M,g) and η is a 1-form given by $g(X,\xi) = \eta(X)$, where ξ is a unit vector field.

Definition 1.1.3. *A Riemannian (or, pseudo-Riemannian) manifold (M,g) of dimension n with non-zero Ricci tensor S is said to have,*

i) Codazzi type [44] Ricci tensor if,

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z), \tag{1.1.7}$$

ii) cyclic parallel [44] Ricci tensor if,

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0, \tag{1.1.8}$$

for all vector fields $X, Y, Z \in \chi(M)$.

Definition 1.1.4. *A smooth vector field V on a Riemannian (or, pseudo-Riemannian) manifold (M,g) of dimension n is said to be a conformal vector field [99, 100] if*

$$\mathcal{L}_V g = 2\rho g, \tag{1.1.9}$$

for some smooth function ρ on M and $\mathcal{L}_V g$ denotes the Lie derivative of the metric g along the direction of the vector field V.

In particular, if the smooth function ρ vanishes, i.e., if, $\mathcal{L}_V g = 0$, then the vector field V is called a Killing vector field. In this case, V is also called an infinitesimal isometry, as the local 1-parameter group of transformations generated by V in a neighbourhood of each point of M consists of local isometries.

K. Yano [97] investigated concircular vector fields to study concircular mappings, which are basically conformal mappings that preserve geodesic circles. In Mathematical Physics and General Relativity concircular vector fields have many applications. B.Y. Chen in [24] proved that a Lorentzian manifold is a generalised Robertson-Walker space-time if and only if it admits a timellike concircular vector field.

Definition 1.1.5. *[24] A vector field V on a Riemannian (or, pseudo-Riemannian) manifold M is called a concircular vector field, if the vector field V satisfies*

$$\nabla_X V = \alpha X, \tag{1.1.10}$$

for all vector fields $X \in \chi(M)$ and where α is a non-trivial smooth function on M.

Furthermore, the vector field V is called a concurrent vector field [25], if the smooth function α is constant function one, i.e., if

$$\nabla_X V = X. \tag{1.1.11}$$

for all vector fields $X \in \chi(M)$.

Definition 1.1.6. *A smooth vector field V on a Riemannian (or, pseudo-Riemannian) manifold (M, g) of dimension n is said to be a torse-forming vector field [99] if*

$$\nabla_X V = fX + \gamma(X)V, \tag{1.1.12}$$

where f is a smooth function and γ is a 1-form.

We start with the definition of contact manifold. D. E. Blair [15] introduced the notion of an almost contact manifold, given by the following definition:

Definition 1.1.7. *Let M be a $(2n+1)$-dimensional diffferentiable manifold and ϕ, ξ, η be a field of endomorphisms of the tangent spaces TM as a (1, 1)-tensor field, a vector field and a 1-form on M respectively. If this triplet (ϕ, ξ, η) satisfies*

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1.1.13}$$

for all vector fields $X, Y \in \chi(M)$ and I is the identity endomorphism, then (ϕ, ξ, η) is called an almost contact structure and M is called an almost contact manifold. Furthermore, if the manifold M is equipped with a Riemmanian metric g, then the almost contact manifold is called an almost contact metric manifold or an almost contact Riemannian manifold with almost contact metric structure (ϕ, ξ, η, g).

Theorem 1.1.1. *For an almost contact metric manifold the following relations hold:*

$$\eta(X) = g(X, \xi), \quad \phi(\xi) = 0, \quad \eta(\phi X) = 0, \tag{1.1.14}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.1.15}$$

$$g(X, \phi Y) + g(Y, \phi X) = 0, \tag{1.1.16}$$

for all vector fields $X, Y \in \chi(M)$.

Definition 1.1.8. *A smooth differentiable manifold M^{2n+1} with $1-$form η is said to be a contact manifold if $\eta \wedge (d\eta)^n \neq 0$. In particular, $\eta \wedge (d\eta)^n \neq 0$ is a volume element on M. The structure (ϕ, ξ, η, g) on M^{2n+1} is called contact metric structure or contact Riemannian structure and the manifold M^{2n+1} with a contact metric structure (ϕ, ξ, η, g) is said to be a contact metric manifold or contact Riemannian manifold.*

The fundamental 2-form Φ on an almost contact metric manifold is defined as:

$$\Phi(X, Y) = g(X, \phi Y) = d\eta(X, Y), \tag{1.1.17}$$

for all vector fields $X, Y \in \chi(M)$.

Definition 1.1.9. *A smooth vector field V on a contact metric manifold is said to be an infinitesimal contact transformation [90] if $\mathcal{L}_V \eta = h\eta$ for some smooth function h. In particular, if $h = 0$, then V is said to be a strict infinitesimal contact transformation.*

Now we mention some of the important contact manifolds which forms the basis of this thesis. These manifolds have been used extensively throughout the thesis.

Definition 1.1.10. *Let M be a $(2n+1)$ dimensional contact metric manifold with contact metric structure (ϕ, ξ, η, g). If the contact metric structure of M is normal, then M is said to have Sasakian structure or normal contact metric structure. The manifold M is called the Sasakian manifold or normal contact metric manifold.*

We recall the following theorems on almost contact metric structure:

Theorem 1.1.2. *An almost contact metric structure* (ϕ, ξ, η, g) *in* M *is called a Sasakian structure if*

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X.$$

for all vector fields $X, Y \in \chi(M)$ *and* ∇ *is the Levi-Civita connection.*

K. Kenmotsu [55] introduced a special class of contact Riemannian manifolds, satisfying certain conditions, which was later named as Kenmotsu manifold. Kenmotsu proved that a locally Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kähler manifold N with warping function $f(t) = ke^t$, where k is a non-zero constant.

Definition 1.1.11. *If the Levi-Civita connection* ∇ *of an almost contact metric manifold* (M, g, ϕ, ξ, η) *satisfies*

$$(\nabla_X \phi)Y = g(\phi X, Y) - \eta(Y)\phi X,$$

for all $X, Y \in \chi(M)$*, then the manifold* (M, g, ϕ, ξ, η) *is said to be a Kenmotsu manifold.*

In differential geometry, the sectional curvature of a Riemannian manifold plays a very important role. A Sasakian manifold with constant ϕ-sectional curvature c is called a Sasakian space form. Similarly, a Kenmotsu space form is a Kenmotsu manifold with constant ϕ-sectional curvature c. As a natural generalization of these spaces, P. Alegre, D. E. Blair and A. Carriazo [3] introduced the concept of generalized Sasakian space form.

Definition 1.1.12. *An almost contact metric manifold* (M, g, ϕ, ξ, η) *is called a generalized Sasakian space form if there exist three smooth functions* $f_1,\ f_2,\ f_3$ *on* M *such that the curvature tensor* R *satisfies*

$$\begin{aligned} R(X,Y)Z &= f_1[g(Y,Z)X - g(X,Z)Y] \\ &\quad + f_2[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z] \\ &\quad + f_3[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &\quad + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X], \end{aligned} \tag{1.1.18}$$

for all vector fields $X, Y, Z \in \chi(M)$.
In particular, for $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$ M *becomes a Sasakian space form. Again, if* $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$ *then* M *is a Kenmotsu space form.* M *is a cosymplectic space form if* $f_1 = f_2 = f_3 = \frac{c}{4}$.

In [4] the authors constructed various examples of generalized Sasakian space forms and showed that any three dimensional trans-Sasakian manifold under certain conditions is a generalized Sasakian space form. Also in [35], it has been proved that a conformally flat generalized Sasakian space form is locally ϕ-symmetric if and only if f_1 is constant.

The notion of trans-Sasakian manifold was introduced by J. A. Oubina [70] as a generalization of Sasakian and Kenmotsu manifolds. Later J. C. Marrero [60] completely characterized the local structures of trans-Sasakian manifolds of dimension $n \geq 5$ and showed that, a trans-Sasakian manifold of dim ≥ 5 is either cosymplectic or α-Sasakian or β-Kenmotsu. So, proper trans-Sasakian manifold exists only for dimension 3. The geometry of the almost Hermitian manifold $(M \times \mathbb{R}, G, J)$ gives rise to the geometry of the almost contact metric manifold (M, g, ϕ, ξ, η), where G is product metric of the product manifold $M \times \mathbb{R}$ with the complex structure J defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),$$

for all $X \in \chi(M)$ and smooth function f on the product manifold $M \times \mathbb{R}$.

Definition 1.1.13. *An almost contact metric manifold (M, g, ϕ, ξ, η) is called a trans-Sasakian manifold if the product manifold $(M \times \mathbb{R}, G, J)$ belongs to the class W_4 [45]. Equivalently, the expression for which an almost contact metric manifold (M, g, ϕ, ξ, η) becomes a trans-Sasakian manifold is given by*

$$(\nabla_X \phi)(Y) = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{1.1.19}$$

for all $X, Y \in \chi(M)$ and for some smooth functions α, β on the manifold M. Then such kind of manifold is called a trans-Sasakian manifold of type (α, β). In particular trans-Sasakian manifolds of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are called cosymplectic, α-Sasakian and β-Kenmotsu manifolds respectively.

In 1967, D. E. Blair [14] introduced the notion of quasi-Sasakian manifold as a normal almost contact metric manifold with closed fundamental 2-form ϕ, with an intention to unify the concepts of Sasakian and cosymplectic structures. He also proved that a quasi-Sasakian structure is locally the product of a Sasakian manifold and a Kähler manifold. Later, S. Tanno [92], S. Kanemaki [54], J. A. Oubina [70] and many other mathematicians studied quasi-Sasakian manifolds and developed some very important properties of these structures.

Definition 1.1.14. *An almost contact metric manifold M is said to be a quasi-Sasakian manifold if the almost contact structure (ϕ, ξ, η) is normal and the fundamental 2-form Φ is closed (i.e., $d\Phi = 0$).*

Z. Olszak [67] proved that, in a conformally flat three-dimensional Quasi-Sasakian manifold if the structure function β is constant, then (a) the manifold is a cosymplectic manifold, which is locally a product of the real line $\mathbb{R}$ and a two-dimensional Kähler space of constant Gauss curvature, or, (b) the manifold is of constant positive curvature and its structure can be obtained by a homothetic deformation of a Sasakian structure. Lately, Quasi-Sasakian manifolds have become a subject of great interest not only to mathematicians but also to theoretical physicists as it has wide applications in super gravity and string theory [1, 42].

The notion of Zamkovoy connection was introduced by S. Zamkovoy [101] in 2009, as a canonical paracontact connection whose torsion is the obstruction of paracontact manifold to be a para-Sasakian manifold. He further showed that the torsion of this connection vanishes exactly when the structure is para-Sasakian and also computed the gauze transformation of its scalar curvature. Later, this connection was studied by various researchers, within the framework of para-Kenmotsu manifold [9], Sasakian manifold [58] and LP-Sasakian manifold [59] etc.

Definition 1.1.15. *On an n-dimensional almost contact metric manifold (M, g, ϕ, ξ, η) equipped with a $(1, 1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, the Zamkovoy connection ∇^* is given by*

$$\nabla^*_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y, \tag{1.1.20}$$

for all $X, Y \in \chi(M)$, where ∇ is the Levi-Civita connection on M.

The geometry of coKähler manifolds as a special case of almost contact manifolds was studied primarily as an odd-dimensional analogy of the Kähler manifolds in complex geometry. Now, it is known that the product manifold $M^{2n+1} \times \mathbb{R}$ equipped with the structure J, as defined earlier, becomes an almost complex structure and if this almost complex structure J is integrable we say that the almost contact structure $(M^{2n+1}, \phi, \xi, \eta)$ is normal. Now we are in a position to define the concept of coKähler manifold [15, 22] and almost coKähler manifold.

Definition 1.1.16. *An almost contact metric manifold is called an almost coKähler manifold if both the* 1*-form* η *and the fundamental* 2*-form* Φ *are closed.*

In particular, if the associated almost contact structure is normal or equivalently $\nabla\phi = 0$ or $\nabla\Phi = 0$: then the almost coKähler manifold is called a coKähler manifold. Also, it is to be noted that, examples [27, 70] of almost coKähler manifolds exist, which are not globally the product of a almost Kähler manifold and the real line.

Definition 1.1.17. *An almost coKähler manifold is said to be a* (k, μ)*-almost coKähler manifold if the characteristic vector field* ξ *belongs to the generalised* (k, μ)*-nullity distribution i.e; if the Riemannian curvature tensor* R *satisfies*

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{1.1.21}$$

for all $X, Y \in \chi(M)$ *and for some smooth functions* (k, μ)*.*

Here, we call a (k, μ)-almost coKähler manifold with $k < 0$, a proper (k, μ)-almost coKähler manifold. Proper almost coKähler manifolds with k and μ being constants were introduced by H. Endo [39] and later Dacko and Olszak [30] further studied it in generalised cases. According to Dacko and Olszak [30] a (k, μ, ν)-almost coKähler manifold with $k < 0$ becomes a $(-1, \frac{\mu}{\sqrt{-k}})$-almost coKähler manifold, under some D-homothetic deformation.

We now give the definition of Lorentzian manifolds.

Definition 1.1.18. *[69] A smooth connected paracompact Hausdorff* n *dimensional manifold* (M, g) *is said to be a Lorentzian manifold if the metric* g *is Lorentzian metric, i.e;* M *admits a smooth symmetric tensor field* g *of type* $(0, 2)$ *such that for each point* $p \in M$*, the tensor* $g_p : T_pM \times T_pM \to \mathbb{R}$ *is a non-degenerate inner product of signature* $(-, +, +, ..., +)$*, where* T_pM *denotes the tangent space of the manifold* M *at point* p *and* $\mathbb{R}$ *is the real line.*
A non-zero vector $v \in T_pM$ *is said to be timelike(respectively; non-spacelike, null, spacelike) if it satisfies* $g_p(v, v) < 0$ *(respectively;*$\leq 0, = 0, > 0$*).*

Definition 1.1.19. *[61] Let M be an n-dimensional differentiable manifold equipped with a triple (ϕ, ξ, η), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form on M such that,*

$$\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \tag{1.1.22}$$

where I denotes the identity map on T_pM. Then M admits a Loretzian metric g such that,

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{1.1.23}$$

and M is said to admit a Lorentzian almost paracontact structure (g, ϕ, ξ, η). The manifold M equipped with a Lorentzian almost paracontact structure (g, ϕ, ξ, η) is said to be a Lorentzian almost paracontact manifold.

Definition 1.1.20. *A Lorentzian almost paracontact manifold M equipped with the structure (g, ϕ, ξ, η) is said to be Lorentzian para Sasakian (in brief, LP-Sasakian) manifold, if for all vector fields $X, Y \in \chi(M)$, the following relation holds,*

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X.$$

In 2003, A. A. Shaikh [78] introduced the notion of Lorentzian concircular structure manifolds (or, briefly, $(LCS)_n$-manifolds) which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [61]. After that, a lot of study has been carried out on locally ϕ-symmetric $(LCS)_n$-manifolds [80] and applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology [79].

Next, we give the definition of a concircular vector field in a Lorentzian manifold, which is essential for the study of $(LCS)_n$-manifolds.

Definition 1.1.21. *Let (M, g) be a Lorentzian manifold and P is a vector field in M defined by $g(U, P) = B(U)$, for any vector field U in M. Then the vector field P is said to be a concircular vector field if*

$$(\nabla_U B)(Y) = \alpha[g(U, Y) + \omega(U)B(Y)],$$

where α is a non-zero scalar and ω is closed 1-form and ∇ denotes the covariant differentiation operator of the manifold M with respect to the Lorentzian metric g.

Let (M, g) be a Lorentzian manifold of dimension n and let M admits a unit timelike concircular vector field ξ satisfying $g(\xi, \xi) = -1$. The vector field ξ is called the characteristic vector field of the manifold (M, g). Then ξ being unit concircular vector field, there exists a non-zero 1-form η such that

$$g(X, \xi) = \eta(X) \quad and \quad (\nabla_X \eta)(Y) = \alpha[g(X, Y) + \eta(X)\eta(Y)], \quad \alpha \neq 0. \tag{1.1.24}$$

Also the non-zero scalar α satisfies the equation

$$(\nabla_X \alpha) = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{1.1.25}$$

where ρ is a scalar function given by $\rho = -(\xi\alpha)$ and ∇ denotes the covariant differentiation operator of the manifold M with respect to the Lorentzian metric g. Now we consider a $(1, 1)$ tensor field ϕ given by, $\phi X = \frac{1}{\alpha}\nabla_X \xi$. Note that the tensor field ϕ is a symmetric $(1, 1)$ tensor field, called the structure tensor of the manifold.

Definition 1.1.22. *Let (M, g) be an n-dimensional Lorentzian manifold. Then the manifold (M, g) together with the unit timelike concircular vector field ξ, associated 1-form η, an $(1, 1)$ tensor field ϕ and the non-zero scalar function α is said to be a Lorentzian concircular structure manifold $(M, g, \xi, \eta, \phi, \alpha)$(briefly, $(LCS)_n$-manifold).*

It is to be noted that, if we consider the scalar function $\alpha = 1$, then we can obtain the LP-Sasakian structure introduced by Matsumoto [61]. So, in that sense $(LCS)_n$-manifolds are a generalization of LP-Sasakian manifolds.

In 1915, Albert Einstein introduced the theory of general relativity which realizes the gravitational field as the spacetime curvature and views the energy-momentum tensor as its source. In fact the Einstein field equations forms the basis of astrophysics, plasma physics and many other branches of the modern physics. Differential geometry plays an important role to understand general relativity in the mathematical language with the help of relativistic fluid models. In the study of general relativity the spacetimes can be modeled as a special subclass of pseudo-Riemannian manifolds namely the 4-dimensional Lorentzian manifolds. Einstein's gravitational equation can describe the characteristics of a perfect fluid inside spherical object which in turn helps us to understand the evolution of the universe.

A perfect fluid behaves isotropic in its rest frame and has no viscosity, shear stress and because of that it is used to model an isotropic universe. The form of the energy-momentum tensor [68] in a perfect fluid is given by

$$T(X,Y) = \rho g(X,Y) + (\sigma + \rho)\eta(X)\eta(Y), \tag{1.1.26}$$

for all $X, Y \in \chi(M)$, where ρ denotes the isotropic pressure, g is the metric tensor of the Minkowski spacetime, σ denotes the energy density and η is the g-dual 1-form of g given by $\eta(X) = g(X, \xi$ with ξ being the velocity vector of the perfect fluid satisfying $g(\xi, \xi) = -1$. In particular, if $\sigma = 3\rho$, the medium is a radiation fluid and if $\sigma = -\rho$, it is the vacuum case where the energy-momentum tensor becomes Lorentz-invariant.

The Einstein's gravitational field equation [68] that governs the motion of the perfect fluid is

$$\kappa T(X,Y) = S(X,Y) + \left(\omega - \frac{r}{2}\right) g(X,Y), \tag{1.1.27}$$

where κ is the gravitational constant and ω is the cosmological constant.

Definition 1.1.23. *Let (M, g) be a four-dimensional smooth Riemannian manifold having the structure of a general relativistic perfect fluid spacetime. Then (M, g) is said to be a Kählerian spacetime manifold if it admits a $(1,1)$ tensor field J satisfying*

$$\begin{aligned} J^2(X) &= -X, & (1.1.28)\\ g(JX, JY) &= g(X,Y), & (1.1.29)\\ (\nabla_X J)(Y) &= 0, & (1.1.30) \end{aligned}$$

for all smooth vector fields $X, Y \in \chi(M)$.

Now, we define warped product of two Riemannian manifolds. The concept of warped product was introduced by Bishop and O'Neill [8] to construct examples of complete Riemannian manifolds of negative sectional curvature. Also note that, the notion of warped product generalizes the concept of surface of revolution.

Definition 1.1.24. *Let (B, g_B) and (F, g_F) be two Riemannian manifolds and $f > 0$ be a smooth function on B. The warped product $B \times_f F$ is the product manifold $M = B \times F$ with the Riemannian structure such that $||X||^2 = ||\pi^*(X)||^2 + f^2(\pi(p))||\sigma^*(X)||^2$, for any* *vector field X on M and $\pi : B \times F \to B$ and $\sigma : B \times F \to F$ are natural projections on M. Thus we have that*

$$g_M = g_B + f^2 g_F, \tag{1.1.31}$$

holds on M. Here B is called the base of M and F is called the fiber. The function f is called the warping function of the warped product.

An example of a very well-known warped product space is a generalized Robertson-Walker spacetime which is an extension of the classical Robertson-Walker spacetime. It is to be noted that, generalized Robertson-Walker spacetime also obeys the Weyl hypothesis, i.e; the world lines should be everywhere orthogonal to a family of spacelike slices. M. Sánchez [76] characterized generalized Robertson-Walker spacetimes in terms of timelike and spatially conformal conformal vector fields. Also, a characterization of generalized Robertson-Walker spacetimes in terms of timelike concircular vector field has been studied by B.Y. Chen [24].

Definition 1.1.25. *A generalized Robertson-Walker spacetime is a warped product manifold $M = I \times_f F$ endowed with the Lorentzian metric*

$$g = -dt^2 \oplus f^2 g_F, \tag{1.1.32}$$

where the base is an open interval I of $\mathbb{R}$ with its usual metric reversed $(I, -dt^2)$, the fiber is an n-dimensional Riemannian manifold (F, g_F) and the warping function is any positive function $f > 0$ on I.

Next, we discuss about indefinite structures on manifolds which are used in our study. A. Bejancu et.al. [7] in 1993, introduced the concept of an idefinite manifold namely ϵ-Sasakian manifold and after that, X. Xufeng et.al. [96] established that the class of ϵ-Sasakian manifolds are real hypersurfaces of indefinite Kaehlerian manifolds. On the other hand K. Kenmotsu [55] introduced a special class of contact Riemannian manifolds, satisfying certain conditions, which was later named as Kenmotsu manifold. U. C. De et.al. [32] introduced the concept of ϵ-Kenmotsu manifolds and further proved that the existence of the new indefinite structure on the manifold influences the curvatures of the manifold. After that several authors [50, 51, 87] studied ϵ-Kenmotsu manifolds and many interesting results have been obtained on this indefinite structure.

An n-dimensional smooth manifold (M, g) is said to be an ϵ-almost contact metric manifold [7] if it admits a $(1,1)$ tensor field ϕ, a characteristic vector field ξ, a global 1-form η and an indefinite metric g on M satisfying the following relations

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1.1.33}$$

$$\eta(X) = \epsilon g(X, \xi), \quad g(\xi, \xi) = \epsilon, \tag{1.1.34}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{1.1.35}$$

for all $X, Y \in \chi(M)$. Here the value of the quantity ϵ is either $+1$ or -1 according as the characteristic vector field ξ is spacelike or timelike vector field. Also it can be easily seen that rank of ϕ is $(n-1)$ and $\phi(\xi) = 0$, $\eta \circ \phi = 0$. Now if we define

$$d\eta(X, Y) = g(X, \phi Y), \tag{1.1.36}$$

for all $X, Y \in \chi(M)$, then the manifold (M, g) is called an ϵ-contact metric manifold.

If the Levi-Civita connection ∇ of an ϵ-contact metric manifold satisfies

$$(\nabla_X \phi)(Y) = -g(X, \phi Y) - \epsilon\eta(Y)\phi X, \tag{1.1.37}$$

for all $X, Y \in \chi(M)$, then the manifold (M, g) is called an ϵ-Kenmotsu manifold [32].

Again an ϵ-almost contact metric manifold is an ϵ-Kenmotsu manifold if and only if, for all $X \in \chi(M)$ the manofold satisfies

$$\nabla_X \xi = \epsilon(X - \eta(X)\xi). \tag{1.1.38}$$

1.2 Introduction to solitons

In 1982, R. S. Hamilton [47] introduced the Ricci soliton as a self similar solution to the Ricci flow equation given by:

$$\frac{\partial}{\partial t}(g(t)) = -2S(g(t)),$$

where S denotes the Ricci tensor and $g(t)$ is an one parameter family of metrics on the manifold.

Definition 1.2.1. *A Riemannian (or, pseudo-Riemannian) metric g defined on a smooth manifold M, of dimension n, is said to be a Ricci soliton, if for some real constant λ there exists a smooth vector field V on M satisfying the equation*

$$S + \frac{1}{2}\mathcal{L}_V g = \lambda g, \tag{1.2.1}$$

where $\mathcal{L}_V$ denotes the Lie derivative along the direction of V.

The Ricci soliton is called shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$. Ricci solitons can also be viewed as natural generalizations of Einstein metrics which moves only by an one-parameter group of diffeomorphisms and scaling [49].

A. E. Fischer [41] in 2005, introduced conformal Ricci flow equation which is a modified version of the Hamilton's Ricci flow equation that modifies the volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow equations on a smooth closed connected oriented n-dimensional manifold M are given by

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg, \tag{1.2.2}$$

$$r(g) = -1,$$

where p is a non-dynamical (time dependent) scalar field and $r(g)$ is the scalar curvature of the manifold. The term $-pg$ acts as the constraint force to maintain the scalar curvature constraint. Thus these evolution equations are analogous to famous Navier-Stokes equations in fluid mechanics where the constraint is divergence free. That is why sometimes p is also called the conformal pressure.

Recently, in 2015, N. Basu et.al. [6] introduced the concept of conformal Ricci soliton as a generalization of the classical Ricci soliton.

Definition 1.2.2. *A Riemannian (or, pseudo-Riemannian) metric g on an n-dimensional smooth manifold M, is called a conformal Ricci soliton, if there exists a real constant λ and a vector field V such that*

$$\mathcal{L}_V g + 2S = [2\lambda - (p + \frac{2}{n})]g, \tag{1.2.3}$$

where p is the conformal pressure.

It can be easily checked that the foregoing soliton equation satisfies the conformal Ricci flow equation (1.2.2). Conformal Ricci solitons have been studied by many authors on Lorentzian α-Sasakian manifold [38], f-Kenmotsu manifold [52], Lagrangian submanifold in a complex space form [84]. M. D. Siddiqi et al. [85], established that a conformal Ricci soliton, on a perfect fluid spacetime with torse-forming vector field and without cosmological constant, is expanding.

Furthermore, if the soliton vector field is gradient of some smooth function on the manifold, that is if, $V = grad\ f = Df$, for some smooth function f, then the soliton is called conformal gradient Ricci soliton. In that case the soliton equation (1.2.2) becomes

$$S + \nabla\nabla f = [\lambda - (\frac{p}{2} + \frac{1}{n})]g, \tag{1.2.4}$$

where ∇ is the Riemannian connection on the manifold.

J. T. Cho and M. Kimura [28] introduced the concept of η-Ricci soliton and later C. Calin and M. Crasmareanu [18] studied it on Hopf hypersufaces in complex space forms. A Riemannian manifold (M, g) is said to admit an η-Ricci soliton if for a smooth vector field V, the metric g satisfies the following equation

$$\mathcal{L}_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0,$$

where $\lambda, \mu \in \mathbb{R}$. Note that, for $\mu = 0$ the η-Ricci soliton becomes a Ricci soliton.

Very recently M. D. Siddiqi [83] introduced the notion of conformal η-Ricci soliton.

Definition 1.2.3. *A Riemannian (or, pseudo-Riemannian) metric g on an n-dimensional smooth manifold M, is called a conformal η-Ricci soliton, if there exists a smooth vector field V such that*

$$\mathcal{L}_V g + 2S + [2\lambda - (p + \frac{2}{n})]g + 2\mu\eta \otimes \eta = 0, \tag{1.2.5}$$

where n is the dimension of the manifold, p is the non-dynamical scalar field, η is the g-dual of V and λ, μ are real constants.

It is easy to see that, for $\mu = 0$ the conformal η-Ricci soliton (g, V, λ, μ) reduces to the conformal Ricci soliton (g, V, λ).

The notion of Yamabe flow was introduced by R. S. Hamilton [49] as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian metrics on a Riemannian manifold of dimension greater than or equal to three. The Yamabe flow on a smooth Riemannian manifold (M, g) is defined as the evolution equation of the Riemannian metric $g = g(t)$ as follows

$$\frac{\partial}{\partial t}(g(t)) = -r(g(t)), \tag{1.2.6}$$

where r denotes the scalar curvature of the manifold. It is to be noted that, in dimension two the Yamabe flow is equivalent to the Ricci flow, but in dimension greater than two the Yamabe flow and the Ricci flow do not agree in general, as the Yamabe flow preserves the conformal class of the metric whereas the Ricci flow does not. In mathematical physics, the Yamabe flow corresponds to the fast diffusion case of the plasma equation.

Definition 1.2.4. *[49] Let (M, g) be an n-dimensional complete Riemannian manifold. If the Riemannian metric g satisfies*

$$\frac{1}{2}\mathcal{L}_V g = (r - \sigma)g, \tag{1.2.7}$$

for some smooth vector field V and some $\sigma \in \mathbb{R}$, then it is known as a Yamabe soliton.

The Yamabe soliton is said to be shrinking, steady or expanding according to $\sigma < 0$, $\sigma = 0$ or $\sigma > 0$ respectively. Just like Ricci solitons are self similar solutions of the Ricci flow, Yamabe solitons are also self similar solutions to the Yamabe flow which moves by an one parameter family of diffeomorphisms and scaling. Over the years many authors have studied Yamabe solitons [20, 26, 74].

As a generalization of Yamabe soliton, recently, B. Y. Chen and S. Deshmukh [26] introduced the notion of quasi-Yamabe soliton.

Definition 1.2.5. *A Riemannian metric (M, g) is said to be a quasi-yamabe soliton if*

$$\frac{1}{2}\mathcal{L}_V g = (r - \sigma)g + \mu V^* \otimes V^*, \tag{1.2.8}$$

for some smooth function μ, real constant σ and V^* is the dual 1-form of V.

In particular, if $\mu = 0$ then the quasi-Yamabe soliton (g, V, σ, μ) reduces to the Yamabe soliton (g, V, σ). Quasi-yamabe solitons are studied on contact metric manifolds in [37] and on warped product manifolds in [15].

In 2016, G. Catino and L. Mazzieri [22] introduced the notion of Einstein soliton which can be viewed as a self-similar solution to the Einstein flow

$$\frac{\partial g}{\partial t} = -2(S - \frac{r}{2}g), \tag{1.2.9}$$

where g denotes the Riemannian metric. It is evident that, the Einstein soliton is analogous to the Ricci soliton. Since the study of Ricci soliton has tremendous contribution in solving the longstanding Thurston's geometric conjecture, so, it is interesting to study the Einstein soliton from various directions to solve many physical and geometrical problems. Here, we consider a slight perturbation of the Einstein soliton by the term $\eta \otimes \eta$, called the η-Einstein soliton [10].

Definition 1.2.6. *A Riemannian (or, pseudo-Riemannian) metric g on an n-dimensional smooth manifold M, is said to admit an η-Einstein soliton, if there exists a smooth vector field V such that*

$$\mathcal{L}_\xi g + 2S + (2\lambda - r)g + 2\mu\eta \otimes \eta = 0, \tag{1.2.10}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative along direction of ξ and $\lambda, \mu \in \mathbb{R}$.

The η-Einstein soliton is called shrinking if $\lambda < 0$, steady if $\lambda = 0$ and expanding if $\lambda > 0$. In particular, if $\mu = 0$, the η-Einstein soliton reduces to the Einstein soliton (g, ξ, λ).

Very recently, in 2019, S. Güler and M. Crasmareanu [46] introduced the geometric flow Ricci-Yamabe map, which is a scalar combination of Ricci flow and Yamabe flow. The Ricci-Yamabe map is also called the Ricci-Yamabe flow of type (α, β) and on a Riemannian or semi-Riemannian manifold (M,g) it is defined by

$$\frac{\partial}{\partial t}g(t) = -2\alpha S(t) + \beta r(t)g(t),$$

where S is the Ricci tensor, r is the scalar curvature and α, $\beta \in \mathbb{R}$. According to the choice of the scalars α, β the Ricci-Yamabe flow can be a Riemannian or semi-Riemannian or singular Riemannian flow and for this reason it is very useful in various geometrical and general relativistic models. A self-similar solution to this Ricci-Yamabe flow of type (α, β) is called a Ricci-Yamabe soliton of type (α, β).

Definition 1.2.7. *A Riemannian (or, pseudo-Riemannian) manifold (M,g) is said to admit a Ricci-Yamabe soliton of type* (α, β)*, if for some smooth vector field* V*, the metric* g *satisfies*

$$\mathcal{L}_V g + 2\alpha S + [2\lambda - \beta r]g = 0, \tag{1.2.11}$$

where $\lambda, \alpha, \beta \in \mathbb{R}$.

In particular, a Ricci-Yamabe soliton of type

- $(1, 0)$ is called a Ricci soliton.
- $(0, 1)$ is called a Yamabe soliton.
- $(1, -1)$ is called an Einstein soliton.

M. D. Siddiqi [86] extended this notion of Ricci-Yamabe soliton to η-Ricci-Yamabe soliton. On a Riemannian manifold (M,g) the data $(g, V, \lambda, \mu, \alpha, \beta)$ is said to be an η-Ricci-Yamabe soliton of type (α, β) (or, simply an η-Ricci-Yamabe soliton) if the metric g satisfies

$$\mathcal{L}_V g + 2\alpha S + [2\lambda - \beta r]g + 2\mu\eta \otimes \eta = 0, \tag{1.2.12}$$

where λ, μ, α, β are real constants.

Among the five chapters of this thesis, this first chapter consists an introduction to different types of smooth manifolds and solitons.

In chapter two, first we study conformal Ricci soliton on generalized Sasakian space forms and characterize the soliton in terms of shrinking steady and expanding. It is proved that a generalized Sasakian space form admitting a conformal Ricci soliton having, potential vector field pointwise collinear with the Reeb vector field, is an Einstein one. Then we find the condition which makes the potential function of a conformal gradient Ricci soliton constant.

Next, we investigate almost coKähler manifolds admitting conformal Ricci soliton and we show that conformal Ricci soliton on a (k, μ)-almost coKähler manifold the soliton becomes expanding, depending on the conformal pressure p. After that, it is proved that a (k, μ)-almost coKähler manifold, with the potential vector field V pointwise collinear with the Reeb vector field ξ, does not admit conformal gradient Ricci soliton.

Then, we give some characterization of conformal Ricci soliton on $(LCS)_n$-manifolds and we show that the manifold becomes an η-Einstein manifold. Also it is proved that, an $(LCS)_n$-manifold admitting a conformal Ricci soliton is ξ-projectively flat. Next, we find necessary and sufficient conditions for an $(LCS)_n$-manifold admitting a conformal Ricci soliton, to be ξ-conharmonically flat and ξ-concircularly flat. After that, conformal Ricci soliton on $(LCS)_n$-manifolds satisfying certain curvature conditions like $R(\xi, X) \cdot \tilde{P} = 0$ and $R(\xi, X) \cdot \tilde{M} = 0$; are studied.

Next, we prove that if an warped product of two Riemannian manifolds admits a conformal Ricci soliton, then the base and the fiber both admit conformal Ricci soliton. Then, the converse of this result is discussed when the potential function is of gradient type. After that, we show that, an warped product admitting conformal Ricci soliton is an Einstein one provided the potential vector field is Killing or concurrent. Finally, some applications of conformal Ricci soliton on generalized Robertson Walker spacetime are discussed.

In chapter three, first we study ϵ-Kenmotsu manifolds admitting conformal η-Ricci solitons and establish the relation between the soliton constants λ and μ. Then, conformal η-Ricci solitons are characterized on ϵ-Kenmotsu manifolds with Codazzi type, cyclic parallel and cyclic η-recurrent Ricci tensor. Moving further, we investigate conformal η-Ricci solitons on ϵ-Kenmotsu manifolds satisfying curvature conditions $R \cdot S = 0$, $C \cdot S = 0$, $Q \cdot C = 0$. After that, we consider torse-forming vector field on ϵ-Kenmotsu manifolds admitting conformal η-Ricci solitons. Then we characterize gradient conformal η-Ricci soliton on ϵ-Kenmotsu manifold.

Next, we consider conformal η-Ricci soliton on almost pseudo symmetric Kählerian spacetime manifolds and characterize the nature of the soliton when the manifolds are projectively flat and conharmonically flat. Finally, we study gradient conformal η-Ricci soliton on Kählerian spacetime manifolds.

In chapter four, first we study some curvature properties of 3-dimensional quasi-Sasakian manifolds with respect to Zamkovoy connection. Then, the nature of Ricci soliton on 3-dimensional quasi-Sasakian manifolds with respect to Zamkovoy connection is characterized. It is proved that, if a quasi-Sasakian 3-manifold with respect to Zamkovoy connection admits a Ricci soliton, with the Reeb vector field as the potential vector field, then it is a steady Ricci soliton.

Next, we characterize the nature of η-Ricci-Yamabe soliton on almost pseudo symmetric Kählerian spacetime manifolds. Finally, it is shown that, if a generalized Sasakian spaceform admits a quasi-Yamabe soliton, with potential vector field pointwise collinear to the Reeb vector field, then it becomes a manifold of constant scalar curvature and the soliton reduces to a Yamabe soliton.

In chapter five, we prove that, a 3-dimensional trans-Sasakian manifold admitting an η-Einstein soliton becomes an eta-Einstein manifold of constant scalar curvature and we also characterize the soliton in terms of shrinking, steady and expanding. Then, η-Einstein solitons are characterized on 3-dimensional trans-Sasakian manifolds with Codazzi type and cyclic parallel Ricci tensor. Moving further, we study some curvature conditions $R \cdot S = 0$, $W_2 \cdot S = 0$, $R \cdot E = 0$, $B \cdot S = 0$, $S \cdot R = 0$ admitting η-Einstein solitons on 3-dimensional trans-Sasakian manifold. Finally, η-Einstein solitons with torse forming vector field are characterized on 3-dimensional trans-Sasakian manifolds.

2

On conformal Ricci solitons

2.1 Introduction

In this chapter we study conformal Ricci soliton on generalized Sasakian space form, almost coKähler manifold, $(LCS)_n$-manifold and warped product space. This chapter is divided into thirteen sections. The first and second section contain introduction and preliminaries respectively.

In section three, we study conformal Ricci soliton and conformal gradient Ricci soliton on generalized Sasakian space forms and we characterize the soliton in terms of shrinking steady and expanding. Then in section four, some illustrative examples of generalized Sasakian space forms are given and some of our results for conformal Ricci soliton are verified within this framework.

In section five, we investigate almost coKähler manifolds admitting the conformal Ricci soliton. Section six is devoted to the study of conformal Ricci soliton on a (k, μ)-almost coKähler manifold and we show that in this case, depending on the conformal pressure p, the soliton becomes expanding. After that in section seven, it is proved that a (k, μ)-almost coKähler manifold, with the potential vector field V pointwise collinear with the Reeb vector field ξ, does not admit conformal gradient Ricci soliton.

In section eight, we study $(LCS)_n$-manifolds admitting conformal Ricci soliton and we give some characterization of the soliton. After that, conformal Ricci soliton on ξ-projectively flat, ξ-conharmonically flat and ξ-concircularly flat $(LCS)_n$-manifolds are taken into consideration. Section nine deals with conformal Ricci soliton on $(LCS)_n$-

manifolds satisfying curvature conditions like $R(\xi, X) \cdot \tilde{P} = 0$ and $R(\xi, X) \cdot \tilde{M} = 0$; where R is the Riemann curvature tensor, $\tilde{P}$ is the pseudo-projective curvature tensor and $\tilde{M}$ is the M-projective curvature tensor.

In section ten, we study conformal Ricci soliton on warped product spaces and try to see its impact on the base and the fiber spaces. Sections eleven and twelve deal with warped product spaces admitting conformal Ricci soliton whose potential vector fields are Killing and concurrent respectively. Finally in section thirteen, we present some applications of conformal Ricci soliton on generalized Robertson Walker spacetime.

2.2 Preliminaries

Definitions of generalized Sasakian spaceforms, almost coKähler manifolds, $(LCS)_n$-manifolds and warped product spaces are given in the introductory chapter one. Now we discuss some basic results on these spaces.

From hereon, the notation $M(f_1, f_2, f_3)$ denotes a $(2n+1)$-dimensional generalized Sasakian space form with $f_1 \neq f_3$ in general. According to [3], in a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$, the following relations hold:

$$\nabla_X \xi = (f_3 - f_1)\phi(X), \tag{2.2.1}$$

$$(\nabla_X \eta)(Y) = (f_3 - f_1) g(\phi(X), Y), \tag{2.2.2}$$

$$(\nabla_X \phi)(Y) = (f_3 - f_1)[\eta(Y)X - g(X, Y)\xi], \tag{2.2.3}$$

$$R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.2.4}$$

$$R(\xi, X)Y = (f_3 - f_1)[\eta(Y)X - g(X, Y)\xi], \tag{2.2.5}$$

$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.2.6}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.2.7}$$

$$S(X, \xi) = 2n(f_1 - f_3)\eta(X), \tag{2.2.8}$$

$$Q\xi = 2n(f_1 - f_3)\xi, \tag{2.2.9}$$

for all vector fields $X, Y \in \chi(M)$ and where R is the curvature tensor, S is the Ricci tensor and Q is the Ricci operator respectively.

Next, we set two symmetric operators h and h' given by, $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' = h \circ \phi$ on the almost coKähler manifold $(M^{2n+1}, g, \phi, \xi, \eta)$. Then setting $l := R(\,.\,, \xi)\xi$, the following relations can be obtained on an almost coKähler manifold [see [70],[30]]

$$h\xi = 0, \quad h\phi + \phi h = 0, \quad tr(h) = tr(h') = 0, \tag{2.2.10}$$

$$\nabla_\xi \phi = 0, \quad \nabla \xi = h', \quad div\xi = 0, \tag{2.2.11}$$

$$S(\xi, \xi) + \|h\|^2 = 0, \tag{2.2.12}$$

$$\phi l \phi - l = 2h^2, \tag{2.2.13}$$

$$\nabla_\xi h = -h^2\phi - \phi l. \tag{2.2.14}$$

Again, on an n-dimensional $(LCS)_n$-manifold the following relations hold [78, 79, 80]

$$\phi^2 X = X + \eta(X)\xi \ , \quad \eta(\xi) = -1, \tag{2.2.15}$$

$$\phi(\xi) = 0 \ , \quad \eta \circ \phi = 0, \tag{2.2.16}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.2.17}$$

$$R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.2.18}$$

$$R(\xi, X)Y = (\alpha^2 - \rho)[g(X, Y)\xi - \eta(Y)X], \tag{2.2.19}$$

$$\eta(R(X, Y)Z) = (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.2.20}$$

$$S(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{2.2.21}$$

for all vector fields $X, Y, Z \in \chi(M)$.

Next, we recall a very important result (for details see [8]) on warped products, which is used in later sections of this chapter.

Lemma 2.2.1. *Let $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$ be an warped product of two Riemannian manifolds (B, g_B) and (F, g_F) with $dimB = m$ and $dimF = n$. Then for all $X, Y \in \mathfrak{X}(B)$ and $U, V \in \mathfrak{X}(F)$*

i) $D_X U = D_U X = \frac{X(f)}{f} U$,

ii) $S(X, U) = 0$,

iii) $S(X, Y) = S^B(X, Y) - \frac{n}{f} H^f(X, Y)$,

iv) $S(U, V) = S^F(U, V) - (\frac{\Delta f}{f} + (n-1)\frac{\|\nabla f\|^2}{f^2}) g(U, V)$,

where $D_X Y$ is the lift of $\nabla_X Y$ on B and S^B, S^F are the lifts of the Ricci tensors on the base B and the fiber F respectively.

2.3 Conformal Ricci soliton on generalized Sasakian space form

In this section we characterize generalized Sasakian space form admitting conformal Ricci soliton with various conditions on the potential vector field. Then we study conformal gradient Ricci soliton on generalized Sasakian space form. First we prove the following:

Theorem 2.3.1. *If a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a conformal Ricci soliton (g, V, λ) then the soliton is*

i) shrinking if $p < [4n(f_3 - f_1) - \frac{2}{2n+1}]$,

ii) steady if $p = [4n(f_3 - f_1) - \frac{2}{2n+1}]$ and

iii) expanding if $p > [4n(f_3 - f_1) - \frac{2}{2n+1}]$.

Proof. Let (g, V, λ) be a conformal Ricci soliton on a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$. Then for all vector fields $X, Y \in \chi(M)$, from (1.2.3) we have

$$(\mathcal{L}_V g)(X,Y) + 2S(X,Y) = [2\lambda - (p + \frac{2}{2n+1})]g(X,Y). \tag{2.3.1}$$

Now consider a $(0,2)$ tensor field defined by

$$\mathfrak{T} = \mathcal{L}_V g + 2S. \tag{2.3.2}$$

It can be easily seen that the tensor field $\mathfrak{T}$ is a symmetric tensor field. Again since g is a metric connection, we have $\nabla g = 0$ and hence from (2.3.1) note that $\mathcal{L}_V g + 2S$ is parallel with the Levi-Civita connection. Therefore (2.3.2) implies that $\mathfrak{T}$ is a parallel, symmetric $(0,2)$ tensor field. Thus we have $\nabla \mathfrak{T} = 0$, which can be written as

$$\mathfrak{T}(R(X,Y)Z, W) + \mathfrak{T}(Z, R(X,Y)W) = 0.$$

Putting $X = W = Z = \xi$ in the above equation and using (2.2.5) we get

$$\mathfrak{T}(Y,\xi) = \mathfrak{T}(\xi,\xi)\eta(Y). \tag{2.3.3}$$

Taking covariant differentiation of (2.3.3) along arbitrary vector field X, then recalling (2.2.1) and (2.2.2) we obtain

$$\mathfrak{T}(\nabla_X Y, \xi) + (f_3 - f_1)\mathfrak{T}(Y, \phi X) = \mathfrak{T}(\xi,\xi)[(f_3 - f_1)g(\phi X, Y) + \eta(\nabla_X Y)].$$

In view of (2.3.3) the above equation reduces to

$$\mathfrak{T}(Y,\phi X)=\mathfrak{T}(\xi,\xi)g(\phi X,Y).$$

Replacing X by ϕX in the foregoing equation and then using (2.3.3) we arrive at

$$\mathfrak{T}(X,Y)=\mathfrak{T}(\xi,\xi)g(X,Y). \tag{2.3.4}$$

Again fron (2.3.2) we can write

$$\mathfrak{T}(X,Y)=(\mathcal{L}_V g)(X,Y)+2S(X,Y).$$

Taking $X=Y=\xi$ in above, the using (2.2.2) and (2.2.8) yields

$$\mathfrak{T}(\xi,\xi)=4n(f_1-f_3).$$

Using the above value in (2.3.4) and then recalling (2.3.2) we get

$$(\mathcal{L}_V g)(X,Y)+2S(X,Y)=4n(f_1-f_3)g(X,Y). \tag{2.3.5}$$

Finally equating (2.3.1) and (2.3.5) we obtain

$$\lambda=2n(f_1-f_3)+(\frac{p}{2}+\frac{1}{2n+1}). \tag{2.3.6}$$

Hence the soliton is shrinking if $\lambda>0$, steady if $\lambda=0$ and expanding if $\lambda<0$. This completes the proof. □

Again for a Sasakian space form $f_1=\frac{c+3}{4}$ and $f_3=\frac{c-1}{4}$, then from (2.3.6) we get $\lambda=2n+(\frac{p}{2}+\frac{1}{2n+1})$. Thus we have the following

Corollary 2.3.1. *A conformal Ricci soliton in a* $(2n+1)$*-dimensional Sasakian space form is shrinking if* $(p+4n+\frac{2}{2n+1})<0$*, steady if* $(p+4n+\frac{2}{2n+1})=0$ *and expanding if* $(p+4n+\frac{2}{2n+1})>0$.

Similarly in a Kenmotsu space form $f_1=\frac{c-3}{4}$ and $f_3=\frac{c+1}{4}$, then from (3.6) we deduce $\lambda=-2n+(\frac{p}{2}+\frac{1}{2n+1})$. Thus we have the following

Corollary 2.3.2. *A conformal Ricci soliton in a* $(2n+1)$*-dimensional Kenmotsu space form is shrinking if* $(p-4n+\frac{2}{2n+1})<0$*, steady if* $(p-4n+\frac{2}{2n+1})=0$ *and expanding if* $(p-4n+\frac{2}{2n+1})>0$.

Now we consider a conformal Ricci soliton (g, V, λ) with V pointwise collinear with the Reeb vector field ξ. In this regard our next theorem is

Theorem 2.3.2. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian space form admiting a conformal Ricci soliton (g, V, λ), whose potential vector field V is pointwise collinear with the Reeb vector field ξ. Then V is a constant multiple of ξ and the manifold M is an Einstein manifold of scalar curvature $r = 2n(2n+1)(f_1 - f_3)$.*

Proof. Let us assume that $V = b\xi$ for some smooth function b, then from the definition of the conformal Ricci soliton equation (1.2.3) we can write

$$\begin{aligned} & bg(\nabla_X \xi, Y) + X(b)\eta(Y) + bg(X, \nabla_Y \xi) + Y(b)\eta(X) \\ & = [2\lambda - (p + \frac{2}{2n+1})]g(X, Y) - 2S(X, Y). \end{aligned}$$

Using (2.2.1) the foregoing equation reduces to

$$X(b)\eta(Y) + Y(b)\eta(X) + 2S(X, Y) = [2\lambda - (p + \frac{2}{2n+1})]g(X, Y). \tag{2.3.7}$$

Replacing Y by ξ in (2.3.7) and recalling (2.2.8) we have

$$X(b) = [2\lambda - (p + \frac{2}{2n+1}) - 4n(f_1 - f_3) - \xi(b)]\eta(X). \tag{2.3.8}$$

Taking $X = \xi$ in the previous equation yields

$$\xi(b) = [\lambda - (\frac{p}{2} + \frac{1}{2n+1}) - 2n(f_1 - f_3)]. \tag{2.3.9}$$

In view of (2.3.9), the equation (2.3.8) becomes

$$db = [\lambda - (\frac{p}{2} + \frac{1}{2n+1}) - 2n(f_1 - f_3)]\eta. \tag{2.3.10}$$

Operating d on both sides of (2.3.10) and using Poincare lemma $d^2 = 0$ we get

$$[\lambda - (\frac{p}{2} + \frac{1}{2n+1}) - 2n(f_1 - f_3)]d\eta = 0.$$

But as $d\eta \neq 0$, the foregoing equation gives us

$$\lambda = 2n(f_1 - f_3) + (\frac{p}{2} + \frac{1}{2n+1}). \tag{2.3.11}$$

Now using (2.3.11) in (2.3.10) we obtain $db = 0$, which eventually implies that

$$b = constant. \tag{2.3.12}$$

Therefore V is a constant multiple of ξ. This proves first part of the theorem.

Again considering an orthonormal basis $\{e_i : 1 \le i \le (2n+1)\}$ of the tangent space at each point of the manifold and then putting $X = Y = e_i$ in (2.3.7) and summing over $1 \le i \le (2n+1)$ we get

$$\xi(b) + r = (2n+1)[\lambda - (\frac{p}{2} + \frac{1}{2n+1})].$$

Using (2.3.12) in the previous equation infers that

$$r = (2n+1)[\lambda - (\frac{p}{2} + \frac{1}{2n+1})]. \tag{2.3.13}$$

Combining (2.3.11) and (2.3.13) we obtain

$$r = 2n(2n+1)(f_1 - f_3). \tag{2.3.14}$$

Also recalling (2.3.7) and then using (2.3.12) we get

$$S(X,Y) = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]g(X,Y). \tag{2.3.15}$$

Thus in view of (2.3.14) and (2.3.15) we can conclude that the manifold M is an Einstein manifold of scalar curvature $r = 2n(2n+1)(f_1 - f_3)$, which proves the second part of the theorem. Hence completes the proof. □

Corollary 2.3.3. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional Sasakian space form admiting a conformal Ricci soliton (g, V, λ), whose potential vector field V is pointwise collinear with the Reeb vector field ξ. Then V is a constant multiple of ξ and the manifold M is an Einstein manifold of constant scalar curvature $r = 2n(2n+1)$.*

Corollary 2.3.4. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional Kenmotsu space form admiting a conformal Ricci soliton (g, V, λ), whose potential vector field V is pointwise collinear with the Reeb vector field ξ. Then V is a constant multiple of ξ and the manifold M is an Einstein manifold of constant scalar curvature $r = -2n(2n+1)$.*

Next we characterize the potential vector field V of a conformal Ricci soliton (g, V, λ) on a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ which satisfies the curvature condition Ricci semi-symmetry. Regarding this we prove the following:

Theorem 2.3.3. *If a $(2n+1)$-dimensional Ricci semi-symmetric generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a conformal Ricci soliton (g, V, λ), then M is an Einstein manifold and the potential vector field V is a conformal vector field.*

Proof. Let us assume that $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian space form admitting a conformal Ricci soliton (g, V, λ), and the manifold is Ricci semi-symmetric. Then we have $R(X,Y) \cdot S = 0$, which can be written as

$$S(R(X,Y)Z, U) + S(Z, R(X,Y)U) = 0.$$

Replacing U by ξ in above yields

$$S(R(X,Y)Z, \xi) + S(Z, R(X,Y)\xi) = 0.$$

Using (2.2.4) and (2.2.9) in the previous equation we obtain

$$2n(f_1 - f_3)\eta(R(X,Y)Z) + (f_1 - f_3)\eta(Y)S(X,Z) - (f_1 - f_3)\eta(X)S(Y,Z) = 0.$$

Taking $X = \xi$ in the foregoing equation, then recalling (2.2.5) and (2.2.8) infers that

$$S(Y,Z) = 2n(f_1 - f_3)g(Y,Z), \tag{2.3.16}$$

which implies that the manifold is an Einstein manifold and this proves the first part of the theorem.

Again, as (g, V, λ) is a conformal Ricci soliton on the $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$, equation (1.2.3) holds and using (2.3.16) in it we have

$$(\mathcal{L}_V g)(X,Y) = [2\lambda - 4n(f_1 - f_3) - (p + \frac{2}{2n+1})]g(X,Y), \tag{2.3.17}$$

for all vector fields $X, Y \in \chi(M)$. Thus from it can be written that

$$\mathcal{L}_V g = 2\rho g,$$

where $\rho = [\lambda - 2n(f_1 - f_3) - (\frac{p}{2} + \frac{1}{2n+1})]$. Thus in view of the equation (1.1.9), we can conclude that V is a conformal vector field. This completes the proof. □

Finally, we concentrate on the generalized Sasakian space form $M(f_1, f_2, f_3)$ admitting conformal Ricci soliton whose potential vector field V is gradient of some smooth function f, i.e; we characterize conformal gradient Ricci soliton on $M(f_1, f_2, f_3)$. But before proving our main theorem, let us first prove the following:

Lemma 2.3.1. *If a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a conformal gradient Ricci soliton (g, Df, λ), then the curvature tensor R satisfies*

$$\begin{aligned} R(X,Y)Df &= (2ndf_1 + 3df_2 - df_3)(Y)X - (3df_2 + (2n-1)df_3)(Y)\eta(X)\xi \\ &\quad +(3df_2 + (2n-1)df_3)(X)\eta(Y)\xi - (2ndf_1 + 3df_2 - df_3)(X)Y \\ &\quad +(f_1 - f_3)(3f_2 + (2n-1)f_3)[g(X,\phi Y)\xi - g(\phi X, Y)\xi \\ &\quad +\eta(X)\phi Y - \eta(Y)\phi X], \end{aligned} \tag{2.3.18}$$

for all vector fields $X, Y \in \chi(M)$.

Proof. Let us assume that (g, Df, λ) be a conformal gradient Ricci soliton on $M(f_1, f_2, f_3)$. Then from the conformal gradient Ricci soliton equation (1.2.4) we can write

$$\nabla_X Df = [\lambda - 2n(f_1 - f_3) - (\frac{p}{2} + \frac{1}{2n+1})]X - QX, \tag{2.3.19}$$

for any vector field X on M and Q is the Ricci operator.
Taking covariant differentiation of (2.3.19) along an arbitrary vector field Y we obtain

$$\nabla_Y \nabla_X Df = [\lambda - 2n(f_1 - f_3) - (\frac{p}{2} + \frac{1}{2n+1})]\nabla_Y X - \nabla_Y QX. \tag{2.3.20}$$

Interchanging X and Y in the foregoing equation infers that

$$\nabla_X \nabla_Y Df = [\lambda - 2n(f_1 - f_3) - (\frac{p}{2} + \frac{1}{2n+1})]\nabla_X Y - \nabla_X QY. \tag{2.3.21}$$

Also from (2.3.19) it can be written that

$$\nabla_{[X,Y]} Df = [\lambda - 2n(f_1 - f_3) - (\frac{p}{2} + \frac{1}{2n+1})](\nabla_X Y - \nabla_Y X) - Q(\nabla_X Y - \nabla_Y X). \tag{2.3.22}$$

Using (2.3.20)-(2.3.22) in Riemannian curvature tensor expression (1.1.5) we obtain

$$R(X,Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y. \tag{2.3.23}$$

Again recalling (2.2.7) and covariantly differentiating it along Y yields

$$\begin{aligned} \nabla_Y QX &= (2nf_1 + 3f_2 - f_3)\nabla_Y X + (2ndf_1 + 3df_2 - df_3)(Y)X \\ &\quad -(3f_2 + (2n-1)f_3)[\nabla_Y \eta(X)\xi + \eta(X)\nabla_Y \xi] \\ &\quad -(3df_2 + (2n-1)df_3)(Y)\eta(X)\xi. \end{aligned} \tag{2.3.24}$$

Also from (2.2.7) we can write

$$Q(\nabla_Y X) = (2nf_1 + 3f_2 - f_3)\nabla_Y X - (3f_2 + (2n-1)f_3)\eta(\nabla_Y X)\xi. \tag{2.3.25}$$

Using (2.3.24) and (2.3.25) in $(\nabla_Y Q)X = \nabla_Y QX - Q(\nabla_Y X)$, then recalling (2.2.1) and (2.2.2) we obtain

$$\begin{aligned}(\nabla_Y Q)X &= (2ndf_1 + 3df_2 - df_3)(Y)X \\ &\quad -(3df_2 + (2n-1)df_3)(Y)\eta(X)\xi \\ &\quad +(f_1 - f_3)(3f_2 + (2n-1)f_3)[g(X, \phi Y)\xi + \eta(X)\phi Y].\end{aligned} \tag{2.3.26}$$

Interchanging X and Y in (2.3.26) yields

$$\begin{aligned}(\nabla_X Q)Y &= (2ndf_1 + 3df_2 - df_3)(X)Y \\ &\quad -(3df_2 + (2n-1)df_3)(X)\eta(Y)\xi \\ &\quad +(f_1 - f_3)(3f_2 + (2n-1)f_3)[g(\phi X, Y)\xi + \eta(Y)\phi X].\end{aligned} \tag{2.3.27}$$

Finally making use of (2.3.26) and (2.3.27) in (2.3.23) completes the proof. □

Now we conclude this section with our main result on conformal gradient Ricci soliton which is the following:

Theorem 2.3.4. *If a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a conformal gradient Ricci soliton (g, Df, λ) then the potential function f is constant, provided f_1 and f_3 are both constants. Furthermore, the soliton is shrinking if $p < [2f_3 - 6f_2 - 2nf_1 - \frac{2}{2n+1}]$, steady if $p = [2f_3 - 6f_2 - 2nf_1 - \frac{2}{2n+1}]$ or, expanding if $p > [2f_3 - 6f_2 - 2nf_1 - \frac{2}{2n+1}]$.*

Proof. Let us assume that (g, Df, λ) is a conformal gradient Ricci soliton on the generalized Sasakian space form $M(f_1, f_2, f_3)$. Now putting $X = \xi$ in (2.3.18) and then taking inner product with arbitrary vector field Z we obtain

$$\begin{aligned}g(R(\xi, Y)Df, Z) &= 2n(df_1 - df_3)(Y)\eta(Z) \\ &\quad -2n(df_1 - df_3)(\xi)[g(Y, Z) - \eta(Y)\eta(Z)] \\ &\quad +(f_1 - f_3)(3f_2 + (2n-1)f_3)g(\phi Y, Z),\end{aligned} \tag{2.3.28}$$

for all vector fields Y, Z on the manifold.

Again in view of (2.2.5) and making use of the curvature property $g(R(\xi, Y)Df, Z) = -g(R(\xi, Y)Z, Df)$, we can write

$$g(R(\xi, Y)Df, Z) = (f_3 - f_1)[g(Y, Z)(\xi f) - \eta(Z)(Yf)]. \tag{2.3.29}$$

Equating (2.3.28) and (2.3.29) we deduce

$$\begin{aligned}(f_3 - f_1)[g(Y, Z)(\xi f) - \eta(Z)(Yf)] &= 2n(df_1 - df_3)(Y)\eta(Z) \\ &\quad -2n(df_1 - df_3)(\xi)[g(Y, Z) - \eta(Y)\eta(Z)] \\ &\quad +(f_1 - f_3)(3f_2 + (2n-1)f_3)g(\phi Y, Z).\end{aligned}$$

Now replacing Z by ξ, the foregoing equation infers that

$$2n(df_1 - df_3)(Y) = (f_3 - f_1)[\eta(Y)(\xi f) - (Yf)],$$

which reduces to

$$[\eta(Y)(\xi f) - (Yf)] = 0,$$

provided f_1 and f_3 are both constants. Also this can be rewritten as

$$g(Y, (\xi f)\xi) = g(Y, Df).$$

Since the above holds for all vector field Y on the manifold, we obtain

$$Df = (\xi f)\xi. \tag{2.3.30}$$

Differentiating (2.3.30) covariantly along arbitrary vector field X and then using 2.6 yields

$$\nabla_X Df = (X(\xi f))\xi + (f_3 - f_1)(\xi f)\phi X. \tag{2.3.31}$$

Equating (2.3.31) with (2.3.19) we deduce

$$QX = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]X - (X(\xi f))\xi - (f_3 - f_1)(\xi f)\phi X. \tag{2.3.32}$$

Comparing the coefficients of ϕX from (2.3.32) and (2.2.7) we get $(\xi f) = 0$. Using this in (2.3.30) infers that $Df = grad\ f = 0$, which eventually implies that f is constant. This proves first part of the theorem.

Again comparing the coefficients of X from (2.3.32) and (2.2.7) we obtain

$$\lambda = [(\frac{p}{2} + \frac{1}{2n+1}) + (2nf_1 - 3f_2 - f_3)]. \tag{2.3.33}$$

Hence the soliton is shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$. This completes the proof. □

2.4 Examples of generalized Sasakian space form admitting conformal Ricci soliton

In this section we discuss examples of generalized Sasakian space form admitting conformal Ricci soliton and Yamabe soliton.

P. Alegre, D. E. Blair and A. Carriazo in their seminal work [3] constructed an example of generalized Sasakian space form as follows:

Example 2.4.1. *Consider the manifold* $\mathbb{R} \times \mathbb{C}^m$ *endowed with three smooth functions given by*

$$f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f'}, \tag{2.4.1}$$

for some smooth real valued function $f = f(t)$ *and* f' *denotes the derivative of* f *with respect to* t.

Now if we consider $f(t) = e^{\alpha t}$, *for some real number* α, *then from* (2.3.6) *we can compute* $\lambda = -m\alpha + \frac{p}{2} + \frac{1}{2m+1}$. *Therefore we can comment that the generalized Sasakian space form* $(\mathbb{R} \times \mathbb{C}^m, f_1, f_2, f_3)$ *admits a conformal Ricci soliton with the soliton constant* λ *as computed above. Furthermore the conformal Ricci soliton is shrinking if* $p < [2m\alpha - \frac{2}{m+1}]$, *steady if* $p = [2m\alpha - \frac{2}{m+1}]$ *and expanding if* $p > [2m\alpha - \frac{2}{m+1}]$.

Next, we give a non-trivial example of a conformal Ricci soliton on a three dimensional generalized Sasakian space form as constructed in [77].

Example 2.4.2. *Let us consider the* 3*-dimensional manifold* $M = \{(u, v, w) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}\}$. *Define a linearly independent set of vector fields* $\{E_i : 1 \leq i \leq 3\}$ *on the manifold* M *given by*

$$E_1 = \frac{\partial}{\partial u} - v\frac{\partial}{\partial w}, \qquad E_2 = \frac{\partial}{\partial v}, \qquad E_3 = \frac{\partial}{\partial w}.$$

Let us define the Riemannian metric g *on* M *by*

$$g(E_i, E_j) = \begin{cases} 1, & \text{for} \quad i = j \\ 0, & \text{for} \quad i \neq j \end{cases}$$

for all $i, j = 1, 2, 3$. *Now considering* $E_3 = \xi$, *let us take the* 1*-form* η, *on the manifold* M, *defined by*

$$\eta(U) = g(U, E_3), \quad \forall U \in \chi(M).$$

Then it can be observed that $\eta(\xi) = 1$. *Let us define the* $(1,1)$ *tensor field* ϕ *on* M *as*

$$\phi(E_1) = -E_2, \quad \phi(E_2) = E_1, \quad \phi(E_3) = 0.$$

Using the linearity of g *and* ϕ *it can be easily checked that*

$$\phi^2(U) = -U + \eta(U)\xi, \quad g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V), \quad \forall U, V \in \chi(M).$$

Hence the structure (g, ϕ, ξ, η) *defines an almost contact metric structure on the manifold* M. *Now, using the definitions of Lie bracket, after some direct computations we get* $[E_1, E_2] = E_3$ *and* $[E_1, E_3] = [E_2, E_3] = 0$. *Again the Riemannian connection* ∇ *of the metric* g *is defined by the well-known Koszul's formula which is given by*

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]). \end{aligned}$$

Using the above formula one can easily calculate that

$$\begin{gathered} \nabla_{E_1} E_1 = 0, \quad \nabla_{E_1} E_2 = \frac{1}{2} E_3, \quad \nabla_{E_1} E_3 = -\frac{1}{2} E_2, \\ \nabla_{E_2} E_1 = -\frac{1}{2} E_3, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_2} E_3 = \frac{1}{2} E_1, \\ \nabla_{E_3} E_1 = -\frac{1}{2} E_2, \quad \nabla_{E_3} E_2 = \frac{1}{2} E_1, \quad \nabla_{E_3} E_3 = 0. \end{gathered}$$

Thus from the above relations and using (1.1.5), *the non-vanishing components of the Riemannian curvature tensor* R *can easily be computed as*

$$\begin{gathered} R(E_1, E_2)E_1 = \frac{3}{4} E_2, \quad R(E_1, E_3)E_1 = -\frac{1}{4} E_3, \quad R(E_2, E_2)E_3 = \frac{1}{4} E_2, \\ R(E_1, E_2)E_2 = -\frac{3}{4} E_1, \quad R(E_2, E_3)E_2 = -\frac{1}{4} E_3, \quad R(E_1, E_3)E_3 = \frac{1}{4} E_1. \end{gathered}$$

Hence we can calculate the non-vanishing components of the Ricci tensor as follows

$$S(E_1, E_1) = -\frac{1}{2}, \quad S(E_2, E_2) = -\frac{1}{2}, \quad S(E_3, E_3) = \frac{1}{2}.$$

Therefore in view of the above values of the Ricci tensor, we can say that the manifold M *is a generalized Sasakian space form with the functions* $f_1 = -\frac{1}{4}$, $f_2 = 0$ *and* $f_3 = -\frac{1}{3}$.

Now if we take the soliton vector field $V = \xi = E_3$, *then from the equation* (1.2.3) *we obtain* $\lambda = (\frac{p}{2} - \frac{1}{6})$. *Hence for this value of* λ *the data* (g, ξ, λ) *defines a conformal*

Ricci soliton on the generalized Sasakian space form $M(f_1, f_2, f_3)$. Moreover we can see that (M, g) is a manifold of constant scalar curvature $r = -\frac{1}{2} = 2 \times 3 \times (f_1 - f_3)$ and hence the theorem (2.3.4) is verified.

Again on this generalized Sasakian space form $M(f_1, f_2, f_3)$, considering $V = \xi$ in the equation (1.2.8), we compute that $\sigma = -\frac{1}{2}$ and $\mu = 0$. Therefore for this values of σ and μ the data (g, ξ, σ, μ) defines a quasi-Yamabe soliton, which eventually reduces to a Yamabe soliton as $\mu = 0$ and hence the theorem (4.6.1) is verified.

2.5 Conformal Ricci soliton on almost coKähler manifold

Let us consider $(M^{2n+1}, g, \phi, \xi, \eta)$ be a $(2n+1)$-dimensional almost coKähler manifold that admits a conformal Ricci soliton (g, V, λ), then equation (1.2.3) holds. In view of the definition of Lie derivative, it follows from (1.2.3) that, for all $Y, Z \in \chi(M)$

$$g(\nabla_Y \beta\xi, Z) + g(Y, \nabla_Z \beta\xi) + 2S(Y, Z) = [2\lambda - (p + \frac{2}{2n+1})]g(Y, Z). \tag{2.5.1}$$

Now, let the vector field V be pointwise collinear with the Reeb vector field ξ, i.e; $V = \beta\xi$, where $\beta$is a non-zero smooth function on the corresponding manifold. Then taking covariant differentiation of both sides of $V = \beta\xi$, along the direction of X we get

$$\nabla_X V = X(\beta)\xi + \beta\nabla_X \xi,$$

and using $\nabla\xi = h'$ from equation (2.2.11) the above equation eventually becomes

$$\nabla_X V = X(\beta)\xi + \beta h' X. \tag{2.5.2}$$

Then using (2.5.2) in the equation (2.5.1) we get

$$g(Y\beta\xi + \beta h'Y, Z) + g(Y, Z\beta\xi + \beta h'Z) + 2S(Y, Z) = [2\lambda - (p + \frac{2}{2n+1})]g(Y, Z).$$

Again using from the fact that h' is symmetric and after simplification the above equation finally becomes

$$Y(\beta)\eta(Z) + Z(\beta)\eta(Y) + 2\beta g(h'Y, Z) + 2S(Y, Z) = [2\lambda - (p + \frac{2}{2n+1})]g(Y, Z). \tag{2.5.3}$$

Next, we consider a local ϕ-basis $\{e_j : 1 \leq j \leq 2n+1\}$ on the tangent space T_pM for each point $p \in M^{2n+1}$. Then putting $Y = Z = e_j$ in (2.5.3) and taking summation over $1 \leq j \leq 2n+1$ and also using $tr(h') = 0$ from (2.2.10) we get

$$\xi(\beta) + r = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})](2n+1). \tag{2.5.4}$$

Again putting $Z = \xi$ in the equation (2.5.3) and using symmetry of h' we have

$$Y(\beta) + \xi(\beta)\eta(Y) + 2S(Y, \xi) = [2\lambda - (p + \frac{2}{2n+1})]\eta(Y). \tag{2.5.5}$$

Now, combining equations (2.5.4) and (2.5.5) and after some calculations we get

$$Y(\beta) + 2S(Y, \xi) = [[\lambda - (\frac{p}{2} + \frac{1}{2n+1})](1 - 2n) + r]\eta(Y).$$

Thus, from the above it is easily seen that

$$\xi(\beta) + 2S(\xi, \xi) = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})](1 - 2n) + r. \tag{2.5.6}$$

Eliminating $\xi(\beta)$ from equations (2.5.4) and (2.5.6) and after simplification we arrive at

$$2n[\lambda - (\frac{p}{2} + \frac{1}{2n+1})] - r + S(\xi, \xi) = 0.$$

Using equation (2.2.12) in the above equation and using the fact that for conformal Ricci flow the scalar curvature $r = -1$, and then simplifying we get the value of the soliton constant as

$$\lambda = \frac{\|h\|^2 - 1}{2n} + (\frac{p}{2} + \frac{1}{2n+1}). \tag{2.5.7}$$

Therefore in view of the fact that the soliton is shrinking, steady or expanding according as $\lambda > 0$, $\lambda = 0$ or, $\lambda < 0$; from the above equation (3.8) we can state the following theorem

Theorem 2.5.1. *Let $(M^{2n+1}, g, \phi, \xi, \eta)$ be an almost coKähler manifold such that the metric g is a conformal Ricci soliton. If the potential vector field V be non-zero pointwise collinear with the Reeb vector field ξ, then*

i) the soliton is shrinking if, $p > \frac{1-(2n+1)\|h\|^2}{(2n^2+n)}$,

ii) the soliton is steady if, $p = \frac{1-(2n+1)\|h\|^2}{(2n^2+n)}$,

iii) the soliton is expanding if, $p < \frac{1-(2n+1)\|h\|^2}{(2n^2+n)}$.

Again, if we have $S = [\frac{\|h\|^2-1}{2n}]g$, then from conformal Ricci soliton equation (1.2.3) and using value of the soliton constant λ from (2.5.7) we have $\mathcal{L}_V g = 0$. Therefore we can see that $V = \beta\xi$ is a Killing vector field and hence the soliton becomes trivial. Hence we can state the following corollary.

Corollary 2.5.1. *Let $(M^{2n+1}, g, \phi, \xi, \eta)$ be an almost coKähler manifold such that the metric g is a conformal Ricci soliton. If the potential vector field V be non-zero pointwise collinear with the Reeb vector field ξ and the Ricci tensor S be a constant multiple of the metric g, with the constant $\frac{\|h\|^2-1}{2n}$, (i.e; if $S = [\frac{\|h\|^2-1}{2n}]g$), then the soliton is trivial.*

2.6 Conformal Ricci soliton on (k, μ)-almost coKähler manifold

This section is devoted to the study of (k, μ)-almost coKähler manifold which admits a conformal Ricci soliton. Then equation (1.1.21) holds.

Now, putting $Y = \xi$ in (1.1.21) we get

$$R(X, \xi)\xi = k[X - \eta(X)\xi] + \mu[hX - \eta(X)h\xi].$$

Then using the definition of $l := R(., \xi)\xi$ and from equation (2.2.10) using the fact that $h\xi = 0$, we can write

$$l = -k\phi^2 + \mu h.$$

Combining the previous equation with (2.2.13) and after a brief calculation we get $h^2 = k\phi^2$. Thus, it is clear that the manifold M^{2n+1} is K-almost coKähler if and ony if, $k = 0$.

Now, we state a lemma (for proof see lemma 4.1 of [95]) which is used later in this section and in the next section of this chapter.

Lemma 2.6.1. *Let $(M^{2n+1}, g, \phi, \xi, \eta)$ be a (k, μ)-almost coKähler manifold of dimension greater than 3 with $k < 0$. Then the Ricci operator is given by*

$$Q = \mu h + 2nk\eta \otimes \xi, \tag{2.6.1}$$

where k is a non-zero constant and μ is a smooth function satisfying $d\mu \wedge \eta = 0$.

Now let us consider the metric g of the (k, μ)-almost coKähler manifold admits a conformal Ricci soliton. Then from the soliton equation (1.2.3) and using the definition of the Lie derivative we can write

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) + 2S(X, Y) = [2\lambda - (p + \frac{2}{2n+1})]g(X, Y). \tag{2.6.2}$$

Then, substituting $V = \xi$ in the above equation (2.6.2) and using the result $\nabla\xi = h'$ from (2.2.11) we get

$$g(h'X, Y) + g(X, h'Y) + 2S(X, Y) = [2\lambda - (p + \frac{2}{2n+1})]g(X, Y).$$

Again as h' is symmetric the above equation implies

$$g(h'X, Y) + g(QX, Y) = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]g(X, Y). \tag{2.6.3}$$

Now, in view of the lemma(4.1) putting value of the Ricci operator Q, from equation (2.6.1), in the above equation (2.6.3) we get

$$g(h'X, Y) + g(\mu hX, Y) + 2nk\eta(X)\eta(Y) = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]g(X, Y). \tag{2.6.4}$$

Thus putting $Y = \xi$ in the above (2.6.4) and using $h\phi + \phi h = 0$ from (2.2.10) we finally get

$$2nk = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]. \tag{2.6.5}$$

Now, as it is mentioned in the lemma(4.1) that $k < 0$, so from the above relation (2.6.5) we can conclude that $[\lambda - (\frac{p}{2} + \frac{1}{2n+1})] < 0$ that is; $\lambda < (\frac{p}{2} + \frac{1}{2n+1})$. Thus if $(\frac{p}{2} + \frac{1}{2n+1}) \leq 0$, i.e; if, $p \leq \frac{-2}{2n+1}$ then $\lambda < 0$ and therefore the soliton is expanding. So, in view of the above we have the following theorem.

Theorem 2.6.1. *Let $(M^{2n+1}, g, \phi, \xi, \eta)$ be a (k, μ)-almost coKähler manifold of dimension greater than 3 with $k < 0$ and the metric g admits a conformal Ricci soliton. Then the soliton is expanding if the conformal pressure p satisfy the inequality $p \leq \frac{-2}{2n+1}$.*

2.7 Conformal gradient Ricci soliton on (k, μ)-almost coKähler manifold

This section is devoted to the study of conformal gradient Ricci soliton on (k, μ)-almost coKähler manifold. So, let us first give the statement of our main theorem of this section.

Theorem 2.7.1. *Let $(M^{2n+1}, g, \phi, \xi, \eta)$ be a (k, μ)-almost coKähler manifold of dimension greater than 3 with $k < 0$. Then there exist no conformal gradient Ricci soliton on the manifold, with the potential vector field V pointwise collinear with the Reeb vector field ξ.*

Proof. Let us assume that the manifold admits a conformal gradient Ricci soliton (g, V, λ). Then equation (1.2.4) holds. Now as the soliton is of gradient type, i.e; $V = Df$, for some smooth function f and D is the gradient operator. Thus for any vector field $X \in \chi(M)$, the equation (1.2.4) can be rewritten as

$$\nabla_X Df + QX = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]X. \tag{2.7.1}$$

Replacing X by Y in the above equation (2.7.1) we get

$$\nabla_Y Df + QY = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]Y. \tag{2.7.2}$$

Similarly replacing X by $[X, Y]$ in (2.7.1) we get

$$\nabla_{[X,Y]} Df + Q[X, Y] = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})][X, Y]. \tag{2.7.3}$$

Using equations (2.7.1)-(2.7.3) in the Riemannian curvature formula (1.1.5) and after some simple calculations we get

$$R(X, Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y. \tag{2.7.4}$$

Again for any vector fields $X, Y \in \chi(M)$, using equation (2.6.1) we obtain

$$\begin{aligned}(\nabla_Y Q)X - (\nabla_X Q)Y = \mu((\nabla_Y h)X - (\nabla_X h)Y)& \\ + 2nk(\eta(X)h'Y - \eta(Y)h'X) + Y(\mu)hX - X(\mu)hY.&\end{aligned} \tag{2.7.5}$$

Now we recall an equation from Proposition 9 of the paper [71]. The result is, for any vector fields $X, Y \in \chi(M)$,

$$(\nabla_X h)Y - (\nabla_Y h)X = k(\eta(Y)\phi X - \eta(X)\phi Y + 2g(\phi X, Y)\xi) + \mu(\eta(X)h'Y - \eta(Y)h'X). \tag{2.7.6}$$

Then using (2.7.5) in (2.7.4) and then using (2.7.6), a simple computation gives that

$$\begin{aligned}R(X, Y)Df = k\mu(\eta(X)\phi Y - \eta(Y)\phi X + 2g(X, \phi Y)\xi) - \mu^2(\eta(X)h'Y - \eta(Y)h'X)& \\ + Y(\mu)hX - X(\mu)hY + 2nk(\eta(X)h'Y - \eta(Y)h'X),&\end{aligned} \tag{2.7.7}$$

for any vector fields $X, Y \in \chi(M)$. Putting $X = \xi$ in the above equation (2.7.7) we get

$$R(\xi, Y)Df = k\mu(\phi Y) - \xi(\mu)hY - \mu^2(h'Y) + 2nk(h'Y).$$

Replacing Y by X in the above equation and then taking inner product with respect to arbitrary vector Y gives us

$$g(R(\xi, X)Df, Y) = k\mu g(\phi X, Y) - \xi(\mu)g(hX, Y) - \mu^2 g(h'X, Y) + 2nkg(h'X, Y). \quad (2.7.8)$$

Again for a (k, μ)-almost coKähler manifold, using equation (1.1.21) we can write

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX].$$

Taking inner-product of the equation with respect to the vector field Df and using the fact that $g(X, Df) = (Xf)$ we get

$$g(R(\xi, X)Y, Df) = k[g(X, Y)(\xi f) - \eta(Y)(Xf)] + \mu[g(hX, Y)(\xi f) - \eta(Y)((hX)f)]. \quad (2.7.9)$$

Now combining (2.7.8) and (2.7.9) and using $g(R(X, Y)Z, U) = -g(R(X, Y)U, Z)$, for any vector fields $X, Y, Z, U \in \chi(M)$, yields

$$\begin{aligned} &k\mu g(\phi X, Y) - \xi(\mu)g(hX, Y) - \mu^2 g(h'X, Y) + 2nkg(h'X, Y) \\ &= k\eta(Y)(Xf) - kg(X, Y)(\xi f) - \mu g(hX, Y)(\xi f) + \mu\eta(Y)((hX)f). \end{aligned} \quad (2.7.10)$$

Antisymmetrizing the above equation we get

$$\begin{aligned} k\mu[g(\phi X, Y) - g(X, \phi Y)] &= k[\eta(Y)(Xf) - \eta(X)(Yf)] \\ &\quad + \mu[\eta(Y)((hX)f)] - \eta(X)((hY)f). \end{aligned} \quad (2.7.11)$$

Now as per our assumption $V = b\xi$, it is easy to see that $h'(Df) = 0$. This again implies, $(h'X)f = g(h'X, Df) = g(X, h'(Df)) = 0$. Similarly $(h'Y)f = 0$. Thus

$$(h(\phi X))f = 0, \quad (h(\phi Y))f = 0. \quad (2.7.12)$$

Using antisymmetry of ϕ and then putting $X = \phi X$ in equation (2.7.10) and using (2.7.11) we get

$$-2\mu g(X, Y) + \mu\eta(X)\eta(Y) = \eta(Y)((\phi X)f). \quad (2.7.13)$$

Putting $Y = \xi$ in the above (2.7.12) yields

$$-\mu g(X, \xi) = g(\phi X, Df). \quad (2.7.14)$$

Then again using $X = \phi X$ in the above equation (2.7.13) we get $g(X, Df) = g(X, \xi(\xi f))$. This gives us

$$Df = (\xi f)\xi. \tag{2.7.15}$$

Covariant differentiation of equation (2.7.14) along the direction of X we get

$$\nabla_X Df = (X(\xi f))\xi + (\xi f)(h'X). \tag{2.7.16}$$

Again from the equation (2.7.1) we have

$$\nabla_X Df = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]X - QX. \tag{2.7.17}$$

Thus combining equations (2.7.15) and (2.7.16) we get

$$QX = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]X - (X(\xi f))\xi - (\xi f)(h'X). \tag{2.7.18}$$

Again, the value of Q from lemma(4.1) gives us

$$QX = \mu hX + 2nk\eta(X)\xi. \tag{2.7.19}$$

Now, comparing right hand sides of (2.7.17) and (2.7.18) we get, $d^2 f = -2nk$, where d is the exterior derivative of f. Again from the well-known Poincare lemma of exterior differentiation we know that, $d^2 = 0$ and this implies, $-2nk = 0$, which is a contradiction to our assumption that $k < 0$. This completes the proof. $\square$

2.8 Conformal Ricci soliton on $(LCS)_n$-manifolds

Let us consider $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$-manifold. Again we know that, for all vector fields $X, Y \in \chi(M)$, the 1-form η satisfies the equation

$$(\nabla_X \eta)(Y) = \nabla_X \eta(Y) - \eta(\nabla_X Y). \tag{2.8.1}$$

Using the equation (1.1.24) in the above equation (2.8.1), after a simple calculation, we get

$$(\mathcal{L}_\xi g)(X, Y) = 2\alpha[g(X, Y) + \eta(X)\eta(Y)]. \tag{2.8.2}$$

Now applying the conformal Ricci soliton equation (1.2.3) in the above equation (2.8.2) we have

$$S(X, Y) = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]g(X, Y) - \alpha\eta(X)\eta(Y). \tag{2.8.3}$$

Let us take $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$. Then we can rewrite the above equation (2.8.3) as

$$S(X,Y) = \sigma g(X,Y) - \alpha\eta(X)\eta(Y). \tag{2.8.4}$$

which shows that the manifold is an η-Einstein manifold.

Now since the above is true for all vector fields X and Y, using the relation $S(X,Y) = g(QX,Y)$ in the above equation (2.8.4) we have

$$QX = \sigma X - \alpha\eta(X)\xi. \tag{2.8.5}$$

Again taking $Y = \xi$ in the equation (2.8.4) we get

$$S(X,\xi) = (\sigma + \alpha)\eta(X). \tag{2.8.6}$$

Let us consider an orthonormal basis $\{e_i : 1 \leq i \leq n\}$ of the manifold (M,g). Then putting $X = Y = e_i$ in the equation (2.8.4) and summing over $1 \leq i \leq n$, we have $r(g) = n\sigma + \alpha$. But we know that for conformal Ricci flow, $r(g) = -1$, which leads us to get $\sigma = -(\frac{\alpha+1}{n})$. Again we have $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$, using this in the previous result we get

$$\lambda = \frac{p}{2} + (1 - \frac{1}{n})\alpha. \tag{2.8.7}$$

So, from the above discussions, using equations (2.8.4) and (2.8.7), we can state the following theorem

Theorem 2.8.1. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton. Then*

i) The manifold becomes an η-Einstein manifold.

ii) The value of the soliton scalar λ is equal to $\lambda = \frac{p}{2} + (1 - \frac{1}{n})\alpha$.

iii) The soliton is shrinking, steady or expanding according as the conformal pressure $p < 2(\frac{1-n}{n})\alpha$, $p = 2(\frac{1-n}{n})\alpha$ or $p > 2(\frac{1-n}{n})\alpha$.

Next, we present some results regarding various curvature tensors. Let us first consider the projective curvature tensor on an n-dimensional $(LCS)_n$-manifold. Now for an $(LCS)_n$-manifold of dimension n, putting $Z = \xi$ in (1.1.1) we get

$$P(X,Y)\xi = R(X,Y)\xi - \frac{1}{(n-1)}[S(Y,\xi)X - S(X,\xi)Y].$$

Using (2.2.18) and (2.8.6) the above equation becomes

$$P(X,Y)\xi = [(\alpha^2-\rho) - \frac{\sigma+\alpha}{(n-1)}][\eta(Y)X - \eta(X)Y]. \tag{2.8.8}$$

Again combining equations (2.2.21) and (2.8.6) we have

$$[(\alpha^2-\rho)(n-1) - \sigma - \alpha]\eta(X) = 0, \tag{2.8.9}$$

which essentially gives us

$$[(\alpha^2-\rho)(n-1)] = (\sigma+\alpha). \tag{2.8.10}$$

Now in view of (2.8.10), the equation (2.8.8) yields us $P(X,Y)\xi = 0$ for any vector fields $X, Y \in \chi(M)$. Thus we have the following

Theorem 2.8.2. *If $(M, g, \xi, \eta, \phi, \alpha)$ is an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton, then the manifold becomes ξ-projectively flat, ξ being the characteristic vector field of the manifold.*

Now we consider the concircular curvature tensor on an n-dimensional $(LCS)_n$-manifold. So, for an $(LCS)_n$-manifold of dimension n, putting $Z = \xi$ in (1.1.2) we get

$$C(X,Y)\xi = R(X,Y)\xi - \frac{r}{n(n-1)}[\eta(Y)X - \eta(X)Y].$$

Using (2.2.18) the above equation becomes

$$C(X,Y)\xi = [(\alpha^2-\rho) - \frac{r}{n(n-1)}][\eta(Y)X - \eta(X)Y]. \tag{2.8.11}$$

Again in view of equation (2.8.10), the above equation (2.8.11) becomes

$$C(X,Y)\xi = [\frac{(\sigma+\alpha)}{(n-1)} - \frac{r}{n(n-1)}][\eta(Y)X - \eta(X)Y]. \tag{2.8.12}$$

Now in view of equation (2.8.12), we can say that $C(X,Y)\xi = 0$ iff $r = n(\sigma+\alpha)$. Again using the fact that for conformal Ricci flow $r = -1$ and using $\sigma = [(\lambda-\alpha) - (\frac{p}{2} + \frac{1}{n})]$ we eventually get $C(X,Y)\xi = 0$ iff $\lambda = \frac{p}{2}$. This leads to the following theorem

Theorem 2.8.3. *If $(M, g, \xi, \eta, \phi, \alpha)$ is an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton, then the manifold becomes ξ-concircularly flat iff $\lambda = \frac{p}{2}$, ξ being the characteristic vector field of the manifold and p is the conformal pressure.*

Let us now consider the conharmonic curvature tensor on an $(LCS)_n$-manifold of dimension n. Then, putting $Z = \xi$ in (1.1.3) we have

$$H(X,Y)\xi = R(X,Y)\xi - \frac{1}{(n-2)}[\eta(Y)QX - \eta(X)QY + S(Y,\xi)X - S(X,\xi)Y].$$

Using (2.2.18), (2.8.5) and (2.8.6) the above equation yields

$$H(X,Y)\xi = [(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{(n-2)}][\eta(Y)X - \eta(X)Y]. \tag{2.8.13}$$

Again in view of equation (2.8.10), the above equation (2.8.13) becomes

$$H(X,Y)\xi = [\frac{(-n\sigma - \alpha)}{(n-1)(n-2)}][\eta(Y)X - \eta(X)Y]. \tag{2.8.14}$$

Thus from the above (2.8.14) we can conclude that $H(X,Y)\xi = 0$ iff $n\sigma = -\alpha$. Moreover, using the value $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$ and after few steps of calculations we have $H(X,Y)\xi = 0$ iff $\lambda = \frac{p}{2} + \frac{1}{n} + (1 - \frac{1}{n})\alpha$. Thus we can state the following:

Theorem 2.8.4. *If $(M, g, \xi, \eta, \phi, \alpha)$ is an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton, then the manifold becomes ξ-conharmonically flat iff $\lambda = \frac{p}{2} + \frac{1}{n} + (1 - \frac{1}{n})\alpha$, ξ being the characteristic vector field of the manifold and p is the conformal pressure.*

Next, let us consider a conformal Ricci soliton (g, V, λ) on an n-dimensional $(LCS)_n$-manifold M and hence equation (1.2.3) holds. Now assume that, the potential vector field V is pointwise collinear with the Reeb vector field ξ, that is, $V = b\xi$, where b is a smooth function on M. Then for any vector fields $X, Y \in \chi(M)$, from equation (1.2.3) we can write

$$\mathcal{L}_{b\xi}g(X,Y) + 2S(X,Y) = [2\lambda - (p + \frac{2}{n})]g(X,Y). \tag{2.8.15}$$

Again from the property of the Lie derivative of the Levi-Civita connection we know that $\mathcal{L}_Z g(X,Y) = g(\nabla_X Z, Y) + g(\nabla_Y Z, X)$. Applying this formula in the above equation (2.8.15) and then using $\phi X = \frac{1}{\alpha}\nabla_X \xi$ we get

$$b\alpha g(\phi X, Y) + (Xb)\eta(Y) + b\alpha g(\phi Y, X) + (Yb)\eta(X) + 2S(X,Y) = [2\lambda - (p + \frac{2}{n})]g(X,Y). \tag{2.8.16}$$

Putting $Y = \xi$ in (2.8.16) and using the equations (2.2.16) we obtain

$$2S(X,\xi) - (Xb) + (\xi b)\eta(X) = [2\lambda - (p + \frac{2}{n})]\eta(X). \tag{2.8.17}$$

Using equation (2.8.6) in the above (2.8.17) and then putting the value $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$ gives us

$$(Xb) = (\xi b)\eta X. \tag{2.8.18}$$

Again putting $X = \xi$ in the equation (2.8.17) we have

$$S(\xi, \xi) - (\xi b) + [\lambda - (\frac{p}{2} + \frac{1}{n})] = 0. \tag{2.8.19}$$

Now, in view of equation (2.8.6) and $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$, the above equation (2.8.19) yields $(\xi b) = 0$. Furthermore, using $(\xi b) = 0$ in equation (2.8.18) we can conclude that (Xb)=0, for any vector field $X \in \chi(M)$. And this implies that the function b is constant and hence V is a constant multiple of ξ. Therefore we have the following theorem

Theorem 2.8.5. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$-manifold which admits a conformal Ricci soliton (g, V, λ), V being the potential vector field of the manifold. If the potential vector field V is pointwise collinear with the characteristic vector field ξ, i.e; if $V = b\xi$, then b is constant, i.e; V becomes constant multiple of ξ.*

Next, we study an important curvature property called ξ-Ricci semi symmetry.

Let $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$-manifold. Then we say that the manifold M is ξ-Ricci semi symmetric if, $R(\xi, X) \cdot S = 0$ in M, where ξ is the characteristic vector field, R is the Riemannian curvature tensor, S is the Ricci tensor.

Let us start with the known formula that for any vector fields X, Y, Z on M,

$$R(\xi, X) \cdot S = S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z). \tag{2.8.20}$$

Now, using (2.2.19) the above equation (2.8.20) yields

$$R(\xi, X) \cdot S = (\alpha^2 - \rho)[g(X, Y)S(\xi, Z) - \eta(Y)S(X, Z) + S(Y, \xi)g(X, Z) - \eta(Z)S(Y, X)].$$

Using (2.2.21) in the above equation and after few steps we get

$$R(\xi, X) \cdot S = \alpha(\alpha^2 - \rho)[g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)]. \tag{2.8.21}$$

Now note that $(\alpha^2 - \rho) = 0$ implies $\lambda = \frac{p}{2} + \frac{1}{n}$, which is the trivial case. Thus for non-triviality we assume $(\alpha^2 - \rho) \neq 0$. Again as α is a non-zero scalar, from (2.8.21) we can state the following:

Theorem 2.8.6. *If $(M, g, \xi, \eta, \phi, \alpha)$ is an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton, then the manifold becomes ξ-Ricci semi symmetric, i.e; $R(\xi, X) \cdot S = 0$ iff the Lorentzian metric g satisfies the relation*

$$g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z) = 0,$$

for any vector fields $X, Y, Z \in \chi(M)$, ξ being the characteristic vector field.

2.9 Conformal Ricci soliton on $(LCS)_n$-manifolds satisfying certain curvature conditions

In this section we characterize conformal Ricci soliton on $(LCS)_n$-manifolds satisfying certain types of curvature conditions.

First let (M, g) be an n-dimensional $(LCS)_n$-manifold and then the conharmonic curvature tensor on M is given by equation (1.1.3). Interchanging Z and X and then putting $Z = \xi$ in (1.1.3) we get

$$H(\xi, X)Y = R(\xi, X)Y - \frac{1}{n-2}[S(X, Y)\xi - S(\xi, Y)X + g(X, Y)Q\xi - g(\xi, Y)QX].$$

Using (2.2.19), (2.8.4), (2.8.5) and (2.8.6) in the previous equation yields

$$H(\xi, X)Y = [(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{(n-2)}][g(X, Y)\xi - \eta(Y)X]. \tag{2.9.1}$$

Also from (2.9.1) we can write

$$\eta(H(\xi, X)Y) = -[(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{(n-2)}][g(X, Y) + \eta(X)\eta(Y)]. \tag{2.9.2}$$

Now we assume that $H(\xi, X) \cdot S = 0$ holds. Then we have

$$S(H(\xi, X)Y, Z) + S(Y, H(\xi, X)Z) = 0. \tag{2.9.3}$$

In view of (2.8.4) the above (2.9.3) yields

$$\sigma[g(H(\xi, X)Y, Z) + g(Y, H(\xi, X)Z)] - \alpha[\eta(H(\xi, X)Z)\eta(Y) + \eta(H(\xi, X)Y)\eta(Z)] = 0.$$

Using (2.9.1) and (2.9.2) in the above equation we get

$$\alpha[(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{(n-2)}][g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)] = 0. \tag{2.9.4}$$

Finally taking $Z = \xi$ in equation (2.9.4) and then using (2.2.17) we arrive at

$$\alpha[(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{(n-2)}]g(\phi X, \phi Y) = 0. \tag{2.9.5}$$

Since α is non-zero and $g(\phi X, \phi Y) \neq 0$ always; then $[(\alpha^2 - \rho) - \frac{(2\sigma+\alpha)}{(n-2)}] = 0$ i.e; $\lambda = \frac{p}{2} + \frac{1}{n} + (1 - \frac{1}{n})\alpha$. Therefore we can state the following theorem:

Theorem 2.9.1. *If $(M, g, \xi, \eta, \phi, \alpha)$ is an n-dimensional $(LCS)_n$-manifold which admits a conformal Ricci soliton, and satisfies the condition $H(\xi, X) \cdot S = 0$ i.e; the manifold is ξ-Ricci conharomnically symmetric. Then the soliton constant is given by $\lambda = \frac{p}{2} + \frac{1}{n} + (1 - \frac{1}{n})\alpha$; where H is the conharmonic curvature tensor and S is the Ricci tensor of the manifold and ξ is the characteristic vector field.*

Next we study another important curvature tensor called $\tilde{M}$-projective curvature tensor [5]. The $\tilde{M}$-projective curvature tensor on an $(LCS)_n$-manifold is given by

$$\tilde{M}(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{2.9.6}$$

Taking inner product with respect to the vector field ξ, the above (2.9.5) yields

$$\begin{aligned} \eta(\tilde{M}(X,Y)Z) &= \eta(R(X,Y)Z) - \frac{1}{2(n-1)}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y) \quad (2.9.7) \\ &\quad + g(Y,Z)S(X,\xi) - g(X,Z)S(Y,\xi)]. \end{aligned}$$

Using (2.2.20), (2.8.4) and (2.8.5) in the above equation we get

$$\eta(\tilde{M}(X,Y)Z) = [(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{2(n-1)}][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{2.9.8}$$

Now we assume the condition that $R(\xi, X) \cdot \tilde{M} = 0$. Then we have

$$\begin{aligned} &R(\xi,X)\tilde{M}(Y,Z)W - \tilde{M}(R(\xi,X)Y,Z)W \\ &\qquad - \tilde{M}(Y,R(\xi,X)Z)W - \tilde{M}(Y,Z)R(\xi,X)W = 0. \quad (2.9.9) \end{aligned}$$

Using (2.2.19) in (2.9.8) and then taking an inner product with respect to ξ we get

$$\begin{aligned} &- g(X,\tilde{M}(Y,Z)W) - \eta(X)\eta(\tilde{M}(Y,Z)W) - g(X,Y)\eta(\tilde{M}(\xi,Z)W) \\ &+ \eta(Y)\eta(\tilde{M}(X,Z)W) - g(X,Z)\eta(\tilde{M}(Y,\xi)W) + \eta(Z)\eta(\tilde{M}(Y,X)W) \\ &- g(X,W)\eta(\tilde{M}(Y,Z)\xi) + \eta(W)\eta(\tilde{M}(Y,Z)X) = 0. \quad (2.9.10) \end{aligned}$$

Then in view of (2.9.7) the above (2.9.9) becomes

$$[(\alpha^2-\rho)-\frac{(2\sigma+\alpha)}{2(n-1)}][g(Y,W)g(X,Z)-g(X,Y)g(Z,W)]+g(X,\tilde{M}(Y,Z)W)=0. \quad (2.9.11)$$

From (2.9.5) and (2.9.10) we get

$$[(\alpha^2-\rho)-\frac{(2\sigma+\alpha)}{2(n-1)}][g(Y,W)g(X,Z)-g(X,Y)g(Z,W)]+g(X,R(Y,Z)W)$$
$$-\frac{1}{2(n-1)}[S(Z,W)g(X,Y)-S(Y,W)g(X,Z)+g(Z,W)S(Y,X)-g(Y,W)S(Z,X)]=0. \quad (2.9.12)$$

Let us consider an orthonormal basis $\{e_i : 1 \le i \le n\}$ of the manifold (M, g). Then putting $X = Y = e_i$ in the equation (2.9.11) and summing over $1 \le i \le n$, we get

$$2nS(Z,W) = [2(n-1)^2(\alpha^2-\rho)-(n-1)(2\sigma+\alpha)-r]g(Z,W). \quad (2.9.13)$$

Again putting $Z = W = \xi$ in above and using equation (2.8.6) we get

$$2(n-1)^2(\alpha^2-\rho)-(5n-2)[\lambda-(\frac{p}{2}+\frac{1}{n})]+2n\alpha=0. \quad (2.9.14)$$

Now using (2.8.10) in the above equation (2.9.13) and after a simple calculation we arrive at

$$\lambda = (\frac{p}{2}+\frac{1}{n})-2\alpha. \quad (2.9.15)$$

Thus we have the following theorem

Theorem 2.9.2. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton and the manifold is ξ-$\tilde{M}$-projectively semi symmetric i.e; it satisfies the condition $R(\xi, X)\cdot\tilde{M} = 0$; ξ being the characteristic vector field, $\tilde{M}$ is the M-projective curvature tensor of the manifold. Then the soliton is shrinking, steady or expanding according as $p > (4\alpha - \frac{2}{n})$, $p = (4\alpha - \frac{2}{n})$ or $p < (4\alpha - \frac{2}{n})$.*

Next we prove an interesting result on $(LCS)_n$-manifold admitting a conformal Ricci soliton and satisfying the condition $R(\xi, X)\cdot\tilde{P} = 0$, where $\tilde{P}$ denotes the well-known Pseudo-projective curvature tensor. But before that let us recall some well-known results that will be used later in this section:

Theorem 2.9.3. *[69] If $S : g(x, y, z) = c$ is a surface in $\mathbb{R}^3$ then the gradient vector field ∇g (connected only at a point of S) is a non-vanishing normal vector field on the entire surface S.*

S.R. Ashoka et.al. in their paper [5] have given the higher dimensional version of the above theorem as follows:

Corollary 2.9.1. *[5] If $S : g(x, y, z) = c$ is a surface (abstract surface or manifold) in $\mathbb{R}^n$ then the gradient vector field ∇g (connected only at points of S) is a non-vanishing normal vector field on the entire surface (abstract surface or manifold) S.*

Then the above mentioned authors in [5] also gave the following remark from the above corollary as:

Remark 2.9.1. *[5] Taking a real valued scalar function α associated with an $(LCS)_n$-manifold with $M = \mathbb{R}^3$ and $g = \alpha$ in the above corollary we have, $\nabla\alpha$ as a non-vanishing normal vector field on $S \subset M$ and directional derivative of α with respect to ξ, $\xi\alpha = \xi$, $\nabla\alpha = |\xi||\nabla\alpha|cos(\hat{\xi, \nabla\alpha})$*

i) If ξ is tangent to S then $\xi\alpha = 0$.

ii) If ξ is tangent to M but not to S then $\xi\alpha \neq 0$.

iii) If the angle between ξ and $\nabla\alpha$ is acute then $0 < cos(\hat{\xi, \nabla\alpha}) < 1$, then $\xi\alpha = k|\nabla\alpha|$, $0 < k < 1$ and $\xi\alpha > 0$.

iv) If the angle between ξ and $\nabla\alpha$ is obtuse then $-1 < cos(\hat{\xi, \nabla\alpha}) < 0$, then $\xi\alpha = k|\nabla\alpha|$, $-1 < k < 0$ and $\xi\alpha < 0$.

Now we see the dependance of the conformal Ricci soliton on $\xi\alpha$ for $(LCS)_n$-manifolds satisfying $R(\xi, X) \cdot \tilde{P} = 0$. The Pseudo projective curvature tensor $\tilde{P}$ is defined by

$$\tilde{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}(\frac{a}{n-1} + b)[g(Y,Z)X - g(X,Z)Y], \quad (2.9.16)$$

where $a, b \neq 0$ are constants. Taking $Z = \xi$ in (2.9.15) we get

$$\tilde{P}(X,Y)\xi = aR(X,Y)\xi + b[S(Y,\xi)X - S(X,\xi)Y] - \frac{r}{n}(\frac{a}{n-1} + b)[\eta(Y)X - \eta(X)Y]. \quad (2.9.17)$$

Using (2.2.18) and (2.8.6) the above equation (2.9.16) yields

$$\tilde{P}(X,Y)\xi = [a(\alpha^2 - \rho) + b(\sigma + \alpha) - \frac{r}{n}(\frac{a}{n-1} + b)][\eta(Y)X - \eta(X)Y], \quad (2.9.18)$$

where σ is as described in the previous section. Again from (2.9.15) we can write

$$\eta(\tilde{P}(X,Y)Z) = a\eta(R(X,Y)Z) + b[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] \\ - \frac{r}{n}(\frac{a}{n-1} + b)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)].$$

Using (2.2.20) and (2.8.4) the above equation becomes

$$\eta(\tilde{P}(X,Y)Z) = [a(\alpha^2 - \rho) + b\sigma - \frac{r}{n}(\frac{a}{n-1} + b)][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \quad (2.9.19)$$

Now we assume the condition that $R(\xi, X)\cdot\tilde{P} = 0$. Then we have

$$R(\xi,X)\tilde{P}(U,V)W - \tilde{P}(R(\xi,X)U,V)W \\ - \tilde{P}(U,R(\xi,X)V)W - \tilde{P}(U,V)R(\xi,X)W = 0, \quad (2.9.20)$$

for any vector fields $X, U, V, W \in \chi(M)$. Using (2.2.19) in the above equation and then taking an inner product with respect to ξ we get

$$- g(X,\tilde{P}(U,V)W) - \eta(X)\eta(\tilde{P}(U,V)W) - g(X,U)\eta(\tilde{P}(\xi,V)W) \\ + \eta(U)\eta(\tilde{P}(X,V)W) - g(X,U)\eta(\tilde{P}(U,\xi)W) + \eta(V)\eta(\tilde{P}(U,X)W) \\ - g(X,W)\eta(\tilde{P}(U,V)\xi) + \eta(W)\eta(\tilde{P}(U,V)X) = 0.$$

Then using (2.9.17) and (2.9.18) the above equation becomes

$$[a(\alpha^2 - \rho) + b\sigma - \frac{r}{n}(\frac{a}{n-1} + b)][g(X,V)g(U,W) - g(X,U)g(V,W)] \\ + g(X,\tilde{P}(U,V)W) = 0. \quad (2.9.21)$$

Now in view of (2.9.15) and then using (2.8.4) in the equation (2.9.20) we get

$$ag(X,R(U,V)W) - b\alpha[\eta(V)\eta(W)g(X,U) - \eta(U)\eta(W)g(X,V)] \\ + a(\alpha^2 - \rho)[g(X,V)g(U,W) - g(X,U)g(V,W)] = 0. \quad (2.9.22)$$

Let us consider an orthonormal basis $\{e_i : 1 \le i \le n\}$ of the manifold (M, g). Then putting $X = U = e_i$ in the equation (2.9.21) and summing over $1 \le i \le n$, we get

$$aS(V,W) - b(n-1)\alpha\eta(V)\eta(W) - a(n-1)(\alpha^2 - \rho)g(V,W) = 0. \quad (2.9.23)$$

Again setting $V = W = \xi$ in (2.9.22) and after a few steps of simple calculations we get

$$\lambda = (n-1)[(\alpha^2 - \rho) - \frac{b}{a}\alpha] + (\frac{p}{2} + \frac{1}{n}). \quad (2.9.24)$$

Therefore in view of (2.9.24) and Remark 2.9.1 we can state the following:

Theorem 2.9.4. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$-manifold which admits a conformal Ricci soliton and the manifold is ξ-pseudo-projectively semi symmetric i.e; if it satisfies the condition $R(\xi, X) \cdot \tilde{P} = 0$; ξ being the characteristic vector field, $\tilde{P}$ is the pseudo-projective curvature tensor of the manifold and α is a positive function, then*

i) If ξ is orthogonal to $\nabla\alpha$; the soliton is expanding if $\alpha > \frac{b}{a}$, $p > -\frac{2}{n}$; steady if $\alpha = \frac{b}{a}$, $p = -\frac{2}{n}$ and shrinking if $\alpha < \frac{b}{a}$, $p < -\frac{2}{n}$.

ii) If the angle between ξ and $\nabla\alpha$ is acute; the soliton is expanding if $\alpha^2 + k|\nabla\alpha| > \frac{b}{a}\alpha$, $p > -\frac{2}{n}$; steady if $\alpha^2 + k|\nabla\alpha| = \frac{b}{a}\alpha$, $p = -\frac{2}{n}$ and shrinking if $\alpha^2 + k|\nabla\alpha| < \frac{b}{a}\alpha$, $p < -\frac{2}{n}$.

iii) If the angle between ξ and $\nabla\alpha$ is obtuse; the soliton is expanding if $\alpha^2 > k|\nabla\alpha| + \frac{b}{a}\alpha$, $p > -\frac{2}{n}$; steady if $\alpha^2 = k|\nabla\alpha| + \frac{b}{a}\alpha$, $p = -\frac{2}{n}$ and shrinking if $\alpha^2 < k|\nabla\alpha| + \frac{b}{a}\alpha$, $p < -\frac{2}{n}$.

2.10 Conformal Ricci soliton on warped product manifolds

This section deals with the study of conformal Ricci soliton on warped product manifolds. Basically here we want to see if a warped product manifold admits a conformal Ricci soliton then how its effect is on the base and the fiber i.e; we try to find out under which conditions they become conformal Ricci soliton.

So, let us assume that $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$ be an warped product of two Riemannian manifolds (B, g_B) and (F, g_F) with $\dim B = m$ and $\dim F = n$. Now let (M, g, μ, ξ) be a conformal Ricci soliton, where $\mu = [2\lambda - (p + \frac{2}{n})]$. Then from equation (1.2.3) we get

$$\mathcal{L}_\xi g + 2S = [2\lambda - (p + \frac{2}{n})]g = \mu g. \tag{2.10.1}$$

Again note that from the lemma 2.2.1 (for details see [8]) following two well-known formulas for warped product manifolds can easily be deduced

$$\mathcal{L}_\xi g = \mathcal{L}^B_{\xi_B} g_B + f^2 \mathcal{L}^F_{\xi_F} g_F + 2f\xi_B(f) g_F, \tag{2.10.2}$$

$$S = S^B - \frac{n}{f} H^f + S^F - \tilde{f} g_F, \tag{2.10.3}$$

where $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$. Now in equation (2.10.1) using the definition of the warped metric from equation (1.1.31) and then applying the values from the above two equations (2.10.2) and (2.10.3) we have

$$\begin{aligned}\mu(g_B + f^2 g_F) &= \mu g \\ &= \mathcal{L}_\xi g + 2S \\ &= \mathcal{L}^B_{\xi_B} g_B + f^2 \mathcal{L}^F_{\xi_F} g_F + 2f\xi_B(f) g_F + 2S^B \\ &\quad -2\frac{n}{f}H^f + 2S^F - 2\tilde{f} g_F, \end{aligned} \tag{2.10.4}$$

Again for all $U, V \in \chi(B)$, using the definition of Lie derivation we can write

$$(\mathcal{L}^B_{\xi_B} g_B)(U,V) = g_B(D^B_U \xi_B, V) + g_B(U, D^B_V \xi_B). \tag{2.10.5}$$

Now from the definition of Hessian and the above equation (2.10.5) we have

$$(\mathcal{L}^B_{\xi_B} g_B - 2\frac{n}{f}H^f)(U,V) = g_B(D^B_U \xi_B, V) + g_B(U, D^B_V \xi_B) - 2\frac{n}{f} g_B(D^B_U \nabla^B f, V).$$

The above equation can be rewritten as

$$\begin{aligned}(\mathcal{L}^B_{\xi_B} g_B - 2\frac{n}{f}H^f)(U,V) &= (g_B(D^B_U \xi_B, V) - \frac{n}{f} g_B(D^B_U \nabla^B f, V)) \\ &\quad +(g_B(U, D^B_V \xi_B) - \frac{n}{f} g_B(D^B_U \nabla^B f, V)) \\ &= g_B(D^B_U(\xi_B - n\nabla^B \ln f), V) \\ &\quad + g_B(U, D^B_V(\xi_B - n\nabla^B \ln f)). \end{aligned} \tag{2.10.6}$$

Using the definition of Lie derivative again equation (2.10.6) becomes

$$(\mathcal{L}^B_{\xi_B} g_B - 2\frac{n}{f}H^f)(U,V) = (\mathcal{L}^B_{\xi_B - n\nabla^B \ln f} g_B)(U,V), \forall U, V \in \chi(B).$$

Since the above equation is true for all $U, V \in \chi(B)$, in operator notation we can write

$$\mathcal{L}^B_{\xi_B} g_B - 2\frac{n}{f}H^f = \mathcal{L}^B_{\xi_B - n\nabla^B \ln f} g_B. \tag{2.10.7}$$

Now using the value from equation (2.10.7), the equation (2.10.4) finally yields

$$\begin{aligned}&(\mathcal{L}^B_{\xi_B - n\nabla^B \ln f} g_B + 2S^B) + (f^2 \mathcal{L}^F_{\xi_F} g_F + 2S^F) \\ &\qquad = \mu g_B + (\mu f^2 - 2f\xi_B(f) + 2\tilde{f}) g_F. \end{aligned} \tag{2.10.8}$$

Hence from the above discussion and equation (2.10.8) we have the following theorem

Theorem 2.10.1. *Let us consider that $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$ be an warped product of two Riemannian manifolds (B, g_B) and (F, g_F) with warping function f, $dimB = m$ and $dimF = n$. If (M, g, μ, ξ) be a conformal Ricci soliton, then the base $(B, g_B, \mu, \xi_B - n\nabla^B \ln f)$ and the fiber $(F, g_F, \mu f^2 - 2f\xi_B(f) + 2\tilde{f}, f^2\xi_F)$ are both conformal Ricci solitons; where $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$, $\mu = [2\lambda - (p + \frac{2}{n})]$, λ is the soliton constant and p is the conformal pressure.*

Now we study a special case when the soliton vector field ξ of the conformal Ricci soliton (M, g, μ, ξ) becomes gradient of some smooth function ϕ i.e; when $\xi = grad\phi = \nabla\phi$. In this case we call the soliton a conformal gradient Ricci soliton and the function ϕ is then called the potential function of the soliton. Also for notational purpose without any confusion we denote a conformal gradient Ricci soliton as (M, g, μ, ϕ), where the last term specifies the potential function of the soliton.

Let us assume that $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$ be an warped product of two Riemannian manifolds (B, g_B) and (F, g_F) with $\dim B = m$ and $\dim F = n$. Then if (M, g, μ, ϕ) be a conformal gradient Ricci soliton, for any vector fields $X, Y \in \chi(M)$, equation (1.2.4) implies

$$2H^\phi(X, Y) + 2S(X, Y) = [2\lambda - (p + \frac{2}{n})]g(X, Y) = \mu g(X, Y). \tag{2.10.9}$$

Now if we take $X = X_B$ and $Y = Y_B$, where X_B, Y_B are the lifts of the vector fields $X, Y \in \chi(B)$, then the equation (2.10.9) gives us

$$2H^\phi(X_B, Y_B) + 2S(X_B, Y_B) = \mu g(X_B, Y_B).$$

Using the value of the Ricci tensor for the base manifold from lemma 1.1, the above equation becomes

$$2H_B^{\phi_B}(X_B, Y_B) + 2S^B(X_B, Y_B) - 2\frac{n}{f}H_B^f(X_B, Y_B) = \mu g_B(X_B, Y_B),$$

where $\phi_B = \phi$ at a fixed point of the fiber F. Finally using the properties of Hessian in the above equation we get

$$2H_B^{\phi_B - n\ln f}(X_B, Y_B) + 2S^B(X_B, Y_B) = \mu g_B(X_B, Y_B). \tag{2.10.10}$$

This shows that $(B, g_B, \mu, \phi_B - n\ln f)$ is a conformal gradient Ricci soliton.

Again taking $X = X_F$ and $Y = Y_F$, where X_F, Y_F are the lifts of the vector fields $X, Y \in \chi(F)$, then the equation (2.10.9) gives us

$$2H^{\phi}(X_F, Y_F) + 2S(X_F, Y_F) = \mu g(X_F, Y_F).$$

Using equation (2.10.3) and lemma 1.1 the above equation becomes

$$2H_F^{\phi_F}(X_F, Y_F) + 2S^F(X_F, Y_F) - \tilde{f} g_F(X_F, Y_F) = \mu f^2 g_F(X_F, Y_F),$$

where $\phi_F = \phi$ at a fixed point of the base B and $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$. Thus finally we get from the above equation

$$2H_F^{\phi_F}(X_F, Y_F) + 2S^F(X_F, Y_F) = (\mu f^2 + \tilde{f}) g_F(X_F, Y_F).$$

Therefore if the warping function f is constant, the term $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$ vanishes from the right hand side of the above equation and we get the following

$$2H_F^{\phi_F}(X_F, Y_F) + 2S^F(X_F, Y_F) = \mu f^2 g_F(X_F, Y_F). \tag{2.10.11}$$

Thus $(F, g_F, \mu f^2, \phi_F)$ is a conformal gradient Ricci soliton. Hence from the above observations and equations (2.10.10) and (2.10.11) we can state the following

Theorem 2.10.2. *Let $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$ be an warped product of two Riemannian manifolds (B, g_B) and (F, g_F) with warping function f, $dimB = m$ and $dimF = n$. If (M, g, μ, ϕ) be a conformal gradient Ricci soliton, then*

i) the base $(B, g_B, \mu, \phi_B - n \ln f)$ is a conformal gradient Ricci soliton with $\phi_B = \phi$ at a fixed point of the fiber F.

ii) the fiber $(F, g_F, \mu f^2, \phi_F)$ is a conformal gradient Ricci soliton with $\phi_F = \phi$ at a fixed point of the base B, provided the warping function f is constant.

2.11 Effect of certain special types of vector fields on conformal Ricci soliton on warped product manifolds

The main purpose of this section is to study the effects of some special types of smooth vector fields on conformal Ricci solitons on warped product spaces. In particular, we focus on Killing vector fileds, conformal vector fields and concurrent vector fields.

Proposition 2.11.1. *Let $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$ be an warped product of two Riemannian manifolds (B, g_B) and (F, g_F) with warping function f, dim$B = m$ and dim$F = n$. If (M, g, μ, ξ) is a conformal Ricci soliton and any one of the following conditions holds*

i) $\xi = \xi_B$ and ξ_B is a Killing vector field on the base B.

ii) $\xi = \xi_F$ and ξ_F is a Killing vector field on the fiber F.

Then the manifold (M, g) becomes an Einstein manifold.

Proof. As per our assumption (M, g, μ, ξ) being a conformal Ricci soliton, it satisfies equation (1.2.3) and we get

$$\mathcal{L}_\xi g + 2S = \mu g, \tag{2.11.1}$$

Now let, $\xi = \xi_B$, and ξ_B is Killing on B, we get $\mathcal{L}^B_{\xi_B} g_B = 0$. Then using it in equation (2.10.2) we have $\mathcal{L}_\xi g = 0$. Therefore equation (2.11.1) gives us $S = \frac{\mu}{2} g$ and this implies (M, g) is Einstein manifold.

Again if $\xi = \xi_F$ and ξ_F is a Killing vector field on the fiber F, $\mathcal{L}^F_{\xi_F} g_F = 0$. Then using equations (2.10.2) and (2.11.1) and proceeding similarly as the first part of the proof, it can be easily shown that in this case also (M, g) is Einstein. □

Theorem 2.11.1. *Let $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$ be an warped product of two Riemannian manifolds (B, g_B) and (F, g_F) with warping function f, dim$B = m$ and dim$F = n$. If (M, g, μ, ξ) is a conformal Ricci soliton and ξ_B is Killing vector field on the base B; then the base $(B, g_B, \mu, -n \ln f)$ is a conformal gradient Ricci soliton; where ξ_B is the lift of the vector field ξ to $\chi(B)$.*

Proof. Since it is given that (M, g, μ, ξ) is a conformal Ricci soliton from theorem 2.1 it follows that the base $(B, g_B, \mu, \xi_B - n\nabla^B \ln f)$ is also a conformal Ricci soliton and hence it satisfies equation (1.2.3). Thus we can write

$$\mathcal{L}^B_{\xi_B - n\nabla^B \ln f} g_B + 2S^B = \mu g_B. \tag{2.11.2}$$

Again using equation (2.10.7) the above equation (2.11.2) becomes

$$\mathcal{L}^B_{\xi_B} g_B - 2\frac{n}{f} H^f + 2S^B = \mu g_B.$$

Now, as ξ_B is Killing vector field on the base B, we have $\mathcal{L}^B_{\xi_B} g_B = 0$. Thus with the help of this, the above equation gives us

$$-2\frac{n}{f}H^f + 2S^B = \mu g_B.$$

Thus using the properties of Hessian, the above equation finally yields

$$2H^{-n\ln f} + 2S^B = \mu g_B. \tag{2.11.3}$$

Hence comparing the above equation (2.11.3) with the conformal gradient Ricci soliton equation (2.10.9) completes the proof. □

We conclude this portion of study of Killing vector fields on conformal Ricci soliton warped product manifolds with the following result

Theorem 2.11.2. *Assume that $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$ be an warped product of two Riemannian manifolds (B, g_B) and (F, g_F) with warping function f, $dimB = m$ and $dimF = n$. Let (M, g, μ, ξ) be a conformal Ricci soliton and both the lifts ξ_B and ξ_F are Killing on the base B and the fiber F respectively. Then the manifold (M, g) is Einstein if $\xi_B(f) = 0$.*

Proof. Since it is given that both ξ_B and ξ_F are Killing, we have $\mathcal{L}^B_{\xi_B} g_B = 0$ and $\mathcal{L}^F_{\xi_F} g_F = 0$. Then using these values in equation (2.10.2) we get

$$\mathcal{L}_\xi g = 2f\xi_B(f)g_F. \tag{2.11.4}$$

Again as per our hypothesis (M, g, μ, ξ) being a conformal Ricci soliton, from equation (1.2.3) we get

$$\mathcal{L}_\xi g + 2S = \mu g.$$

Now using equation (2.11.4) in the above equation, gives us

$$2f\xi_B(f)g_F + 2S = \mu g. \tag{2.11.5}$$

Thus if $\xi_B(f) = 0$, the above equation (2.11.5) yields $S = \frac{\mu}{2}g$, which implies the manifold (M, g) is Einstein and this completes the proof. □

Now we shall focus on the effect of conformal vector fields on warped product manifolds admitting conformal Ricci solitons. In this direction a very immediate result is the following

Proposition 2.11.2. *Let $(M,g) = (B \times_f F, g_B \oplus f^2 g_F)$ be an warped product of two Riemannian manifolds (B, g_B) and (F, g_F) with warping function f, $dimB = m$ and $dimF = n$. Let (M, g, μ, ξ) is a conformal Ricci soliton. Then the manifold (M, g) becomes an Einstein manifold with factor $(\frac{\mu}{2} - \rho)$ if and only if the vector field ξ is conformal with factor 2ρ.*

Proof. (M, g, μ, ξ) being a conformal Ricci soliton, from (1.2.3) we can write

$$\mathcal{L}_\xi g + 2S = \mu g. \tag{2.11.6}$$

Since the vector field ξ is conformal with factor 2ρ, by definition we have $\mathcal{L}_\xi g = 2\rho g$, where ρ is a smooth function. Thus using this value in equation (2.11.6) finally we get

$$S = (\frac{\mu}{2} - \rho)g. \tag{2.11.7}$$

This implies (M, g) is an Einstein manifold. Similarly by reverse calculation process it can be shown that if (M, g) is an Einstein manifold with factor $(\frac{\mu}{2} - \rho)$ then ξ becomes conformal with factor 2ρ. This completes the proof. □

It is to be noted that in the above result we have discussed on conformal Ricci solitons with the vector field ξ is taken conformal. So it is natural to ask whether it is necessary to consider ξ conformal as a whole, or is there a weaker condition than this. The following theorem could put some light on it.

Theorem 2.11.3. *Assume that $(M,g) = (B \times_f F, g_B \oplus f^2 g_F)$ be an warped product of two Riemannian manifolds (B, g_B) and (F, g_F) with warping function f, $dimB = m$ and $dimF = n$. Let (M, g, μ, ξ) be a conformal Ricci soliton and both the lifts ξ_B and ξ_F are conformal on the base B and the fiber F with factors $2\rho_B$ and $2\rho_F$ respectively; where ρ_B and ρ_F are two smooth functions. Then the manifold (M, g) is Einstein provided $\rho_B = \rho_F + \xi_B(\ln f)$.*

Proof. Since ξ_B is conformal on the base B with factor $2\rho_B$, we have $\mathcal{L}^B_{\xi_B} g_B = 2\rho_B g_B$. Also ξ_F being conformal with factor $2\rho_F$, we get $\mathcal{L}^F_{\xi_F} g_F = 2\rho_F g_F$. Then using these two values in equation (2.10.2) we get

$$\mathcal{L}_\xi g = 2(\rho_B g_B + f^2 \rho_F g_F + f \xi_B(f) g_F). \tag{2.11.8}$$

Again (M, g, μ, ξ) being a conformal Ricci soliton, from equation (1.2.3) and the previous equation (2.11.8) we have

$$2(\rho_B g_B + f^2 \rho_F g_F + f\xi_B(f)g_F + S) = \mu g.$$

The above equation can be rewritten as

$$S = \frac{\mu}{2}g - \rho_B g_B - f^2(\rho_F + \xi_B(\ln f))g_F. \tag{2.11.9}$$

Hence if $\rho_B = \rho_F + \xi_B(\ln f)$, and using equation (1.1.31), the above equation (2.11.9) finally gives us $S = (\frac{\mu}{2} - \rho_B)g$. This implies (M, g) is Einstein and thus completes the proof. □

We end this section with our last theorem, which actually gives the converse part of the previous theorem. In the previous result we characterised the conformal Ricci soliton (M, g, μ, ξ) whereas our next result gives conditions under which a warped product manifold (M, g) admits a conformal Ricci soliton.

Theorem 2.11.4. *Let (B, g_B, μ, ξ_B) be a conformal Ricci soliton and (F, g_F) be an Einstein manifold with factor β, where $dimB = m$ and $dimF = n$. Let $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$ be an warped product of (B, g_B) and (F, g_F) with warping function f and ξ_F is conformal vector field with factor 2ρ. Then (M, g, μ, ξ) is a conformal Ricci soliton if $H^f = 0$ and the warping function f satisfies the quadratic equation*

$$(2\rho - \mu)f^2 + 2f\xi_B(f) + 2\beta + 2(1 - n)k^2 = 0,$$

where $k^2 = \|\nabla f\|_B^2 = g_B(\nabla f, \nabla f)$ for some real number k.

Proof. (B, g_B, μ, ξ_B) being a conformal Ricci soliton, from equation (1.2.3) we get

$$\mathcal{L}^B_{\xi_B} g_B + 2S^B = \mu g_B. \tag{2.11.10}$$

Again as (F, g_F) is an Einstein manifold with factor β, the Ricci tensor is given by $Ric^F = \beta g_F$. Using this value in equation (2.10.3) gives us

$$S = S^B - \frac{n}{f}H^f + \beta g_F - \tilde{f} g_F, \tag{2.11.11}$$

where $\tilde{f} = f\Delta f + (n - 1)\|\nabla f\|_B^2$. Now, using equation (2.11.10) in the equation (2.10.2) we get

$$\mathcal{L}_\xi g = \mu g_B - 2S^B + f^2 \mathcal{L}^F_{\xi_F} g_F + 2f\xi_B(f)g_F. \tag{2.11.12}$$

Multiplying both sides of the equation (2.11.11) by 2 and then adding it with equation (2.11.12) yields

$$\mathcal{L}_\xi g + 2S = \mu g_B + f^2\mathcal{L}^F_{\xi_F} g_F + 2f\xi_B(f)g_F + 2(-\frac{n}{f}H^f + \beta g_F - \tilde{f} g_F).$$

Now since the vector field ξ_F is conformal with factor 2ρ i.e; $\mathcal{L}^F_{\xi_F} g_F = 2\rho g_F$, the above equation becomes

$$\mathcal{L}_\xi g + 2S = \mu g_B + 2f^2\rho g_F + 2f\xi_B(f)g_F + 2(-\frac{n}{f}H^f + \beta g_F - \tilde{f} g_F). \tag{2.11.13}$$

As it is given that $H^f = 0$, then it implies that $\Delta f = 0$ and hence $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|^2_B$ becomes $\tilde{f} = (n-1)\|\nabla f\|^2_B = (n-1)k^2$, where $k^2 = \|\nabla f\|^2_B = g_B(\nabla f, \nabla f)$ for some real number k. Therefore using these results in the above equation (2.11.13) we get

$$\begin{aligned} \mathcal{L}_\xi g + 2S &= \mu g_B + 2f^2\rho g_F + 2f\xi_B(f)g_F + 2(\beta g_F - (n-1)k^2 g_F) \\ &= \mu(g_B + f^2 g_F) + (2f^2\rho - \mu f^+ 2f\xi_B(f) + 2(\beta - (n-1)k^2))g_F. \end{aligned}$$

Thus if $(2f^2\rho - \mu f^2 + 2f\xi_B(f) + 2(\beta - (n-1)k^2)) = 0$ i.e; if f satisfies the quadratic equation $(2\rho - \mu)f^2 + 2f\xi_B(f) + 2\beta + 2(1-n)k^2 = 0$; the above equation finally becomes

$$\mathcal{L}_\xi g + 2S = \mu(g_B + f^2 g_F) = \mu g. \tag{2.11.14}$$

Therefore from equation (2.11.14) we can conclude that (M, g, μ, ξ) is a conformal Ricci soliton and this completes the proof. □

2.12 Warped product manifolds admitting conformal Ricci soliton with concurrent vector field

In this section we study conformal Ricci solitons with the soliton vector field ξ being concircular (also, concurrent) vector field. Definitions of concircular and concurrent vector fields are discussed in chapter one. So in this direction our first result is as follows

Theorem 2.12.1. *Let (M, g, μ, ξ) be a conformal Ricci soliton on an n-dimensional Riemannian manifold (M, g) and the soliton vector field ξ is concircular with factor α, then*

i) the manifold (M, g) is an Einstein manifold with factor $(\mu - 2\alpha)$ and *ii) the soliton is expanding, steady or shrinking according as* $(p+2\alpha+\frac{1}{n})<0$, $(p+2\alpha+\frac{1}{n})=0$ *or* $(p+2\alpha+\frac{1}{n})>0$ *respectively.*

Proof. As per our assumption the soliton vector field ξ is concircular with factor α, then from equation (1.1.10) we get $\nabla_X\xi=\alpha X$. Then using it in the definition of Lie differentiation we get

$$\begin{aligned}(\mathcal{L}_\xi g)(X,Y) &= g(\nabla_X\xi,Y)+g(X,\nabla_Y\xi)\\ &= g(\alpha X,Y)+g(X,\alpha Y)\\ &= 2\alpha g(X,Y), \qquad (2.12.1)\end{aligned}$$

for all vector fields $X,Y\in\chi(M)$. Again, (M,g,μ,ξ) being a conformal Ricci soliton, using the value from (2.12.1) in the equation (1.2.3) we get

$$S(X,Y)=(\mu-2\alpha)g(X,Y), \qquad (2.12.2)$$

for all vector fields $X,Y\in\chi(M)$, and $\mu=[2\lambda-(p+\frac{2}{n})]$. Thus equation (2.12.2) proves that (M,g) Einstein with factor $(\mu-2\alpha)$ and this completes the first part of the theorem.

Again we know that for conformal Ricci flow, the scalar curvature $r(g)=-1$. So taking an orthonormal basis $\{e_i:1\leq i\leq n\}$ of the manifold M and summing over $1\leq i\leq n$ in both sides of the equation (2.12.2) gives us

$$-1=r(g)=n(\mu-2\alpha).$$

Finally using the value $\mu=[2\lambda-(p+\frac{2}{n})]$ in the above equation and after simplification we get

$$\lambda=\alpha+\frac{p}{2}+\frac{1}{2n}. \qquad (2.12.3)$$

We know that the soliton is expanding steady or shrinking if $\lambda<0$, $\lambda=0$ or $\lambda>0$, thus applying it in equation (2.12.3) completes the proof. □

Next, we have a result on concurrent vector field which immediately follows from the above theorem.

Corollary 2.12.1. (M,g,μ,ξ) *be a conformal Ricci soliton with the soliton vector field* ξ *is concurrent, then*

i) the manifold (M,g) *is an Einstein manifold with factor* $(\mu-2)$ *and* *ii) the soliton is expanding, steady or shrinking according as* $(p+2+\frac{1}{n}) < 0$, $(p+2+\frac{1}{n}) = 0$ *or* $(p+2+\frac{1}{n}) > 0$ *respectively.*

Proof. Proceeding similarly as theorem 4.1 and then putting $\alpha = 1$ in the equations (2.12.2) and (2.12.3) completes the proof. □

We conclude this section with the following theorem on concurrent vector field:

Theorem 2.12.2. *Assume that* $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$ *be an warped product of two Riemannian manifolds* (B, g_B) *and* (F, g_F) *with warping function* f*,* $dimB = m$ *and* $dimF = n$*. Let* (M, g, μ, ξ) *be a conformal Ricci soliton with concurrent vector field* ξ*. If* f *is constant and both the lifts* ξ_B *and* ξ_F *are concurrent on the base* B *and the fiber* F *then*

i) the soliton (M, g, μ, ξ) *is expanding, steady or shrinking according as* $(\frac{p}{2}+\frac{1}{n}+1) < 0$, $(\frac{p}{2}+\frac{1}{n}+1) = 0$ *or* $(\frac{p}{2}+\frac{1}{n}+1) > 0$ *respectively,*

ii) all the three manifolds M, B *and* F *are Ricci flat manifolds and*

iii) all the three manifolds M, B *and* F *admit conformal gradient Ricci solitons.*

Proof. Since (M, g, μ, ξ) is a conformal Ricci soliton on M with concurrent vector field ξ, from first part of the corollary 2.12.1 we can write

$$S(X, Y) = (\mu - 2)g(X, Y), \tag{2.12.4}$$

for all vector fields $X, Y \in \chi(M)$.
Now if we set $X = X_F$ and $Y = Y_F$, then from lemma 1.1 and equation (2.10.3) we get

$$S(X_F, Y_F) = S^F(X_F, Y_F) - \tilde{f} g_F(X_F, Y_F), \tag{2.12.5}$$

where $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$. Now using equation (2.12.4) and (1.1.31), in the above equation (2.12.5) yields

$$S^F(X_F, Y_F) = \tilde{f} g_F(X_F, Y_F) + (\mu - 2)f^2 g_F(X_F, Y_F),$$

where $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$. Since it is given that f is constant, say $f = c$ for some constant c, then it implies that $\tilde{f} = 0$ and thus the above equation becomes

$$S^F(X_F, Y_F) = c^2(\mu - 2)g_F(X_F, Y_F), \tag{2.12.6}$$

for all vector fields $X_F, Y_F \in \chi(F)$. Thus from the above equation (2.12.6) we can say that F is Einstein. Now as the equation (2.12.6) is true for any vector field in $\chi(F)$, by putting $X_F = Y_F = \xi_F$ in above we get

$$\begin{aligned} S^F(\xi_F, \xi_F) &= c^2(\mu - 2)g_F(\xi_F, \xi_F) \\ &= c^2(\mu - 2)\|\xi_F\|_F^2. \end{aligned} \tag{2.12.7}$$

Let $\{\xi_F, e_1, e_2, e_3, ..., e_{n-1}\}$ be an orthonormal basis of $\chi(F)$. Then the curvature tensor of the manifold F is given by

$$R^F(\xi_F, e_i, \xi_F, e_i) = g_F(R^F(\xi_F, e_i)\xi_F, e_i).$$

Using the well-known formula for curvature, the above equation can be rewritten as

$$R^F(\xi_F, e_i, \xi_F, e_i) = g_F(\nabla^F_{\xi_F}\nabla^F_{e_i}\xi_F - \nabla^F_{e_i}\nabla^F_{\xi_F}\xi_F - \nabla^F_{[\xi_F, e_i]}\xi_F, e_i). \tag{2.12.8}$$

Also since ξ_F is concurrent vector field, from equation (1.1.11) we have $\nabla_X\xi_F = X$, for all $X \in \chi(F)$ and using this in equation (2.12.8) we get

$$R^F(\xi_F, e_i, \xi_F, e_i) = g_F(\nabla^F_{\xi_F}e_i - \nabla^F_{e_i}\xi_F - [\xi_F, e_i], e_i) = 0.$$

This implies $Ric^F(\xi_F, \xi_F) = 0$ and then from equation (2.12.7) we get $\mu = 2$, i.e; $\mu = [2\lambda - (p + \frac{2}{n})] = 2$. After simplification this gives $\lambda = (\frac{p}{2} + \frac{1}{n} + 1)$ and the soliton is shrinking, steady or expanding according as $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$. This proves the first part of the theorem.

Now, using this value $\mu = 2$ in equations (2.12.6) and (2.12.4) we have $S = S^F = 0$. This proves that both the manifolds M and F are Ricci flat.
Again if we set $X = X_B$ and $Y = Y_B$, then from lemma 1.1 we can write

$$S(X_B, Y_B) = S^B(X_B, Y_B) - \frac{n}{f}H^f(X_B, Y_B),$$

for all $X_B, Y_B \in \chi(B)$. Now since we just proved $S = 0$, the above equation becomes

$$S^B(X_B, Y_B) = \frac{n}{f}H^f(X_B, Y_B). \tag{2.12.9}$$

Since we assumed that f is constant, it implies $H^f = 0$ and thus the above equation (2.12.9) finally gives us $S^B(X_B, Y_B) = 0$, for all $X_B, Y_B \in \chi(B)$. Therefore we get $S^B = 0$

and this proves that the manifold B is Ricci flat. This completes the proof of the second part of the theorem.

To prove the last part of the theorem, let us assume that $\phi = \frac{1}{2}g(\xi, \xi)$. Then

$$g(X, grad\phi) = X(\phi) = g(\nabla_X \xi, \xi), \tag{2.12.10}$$

for all $X \in \chi(M)$. Again ξ being concurrent, from equation (1.1.11) we have $\nabla_X \xi = X$ and using this value in equation (2.12.10) we get

$$g(X, grad\phi) = g(X, \xi),$$

for all $X \in \chi(M)$. Since the above equation is true for any vector field $X \in \chi(M)$, we can conclude that $\xi = grad\phi$. Hence (M, g) admits a conformal gradient Ricci soliton. Again taking $\phi_B = \frac{1}{2}g(\xi_B, \xi_B)$ and $\phi_F = \frac{1}{2}g(\xi_F, \xi_F)$ and proceeding similarly we can show that $\xi_B = grad\phi_B$ and $\xi_F = grad\phi_F$. Also from theorem 2.1 we know that since (M, g) is conformal Ricci soliton, B and F both are conformal Ricci soliton. Hence can conclude that both the manifolds B and F admit conformal gradient Ricci soliton. □

2.13 Application of conformal Ricci soliton on generalized Robertson-Walker spacetimes

This section deals with the study of conformal Ricci solitons on generalized Robertson-Walker spacetimes. The definition of a generalized Robertson-Walker spacetime is given in chapter one. Based on that definiton we consider a generalized Robertson-Walker spacetime and study the effect of conformal Ricci soliton on it. Our main result of this section is the following

Theorem 2.13.1. *Let $M = I \times_f F$ be a generalized Robertson-Walker spacetime endowed with the metric $g = -dt^2 \oplus f^2 g_F$ and let $\phi = \int_c^t f(z)dz$, for some constant $c \in I$. If (M, g, μ, ϕ) admits a conformal gradient Ricci soliton, then*

i) the generalized Robertson-Walker spacetime (M, g) becomes Ricci flat if the soliton constant λ satisfies the relation $\lambda = \dot{f} + \frac{p}{2} + \frac{1}{n}$ and

ii) the generalized Robertson-Walker spacetime (M, g) is an Einstein manifold if the warping function f is of the form $f(t) = at + b$, where a, b are constants.

Proof. As per our assumption, (M, g, μ, ϕ) being a conformal gradient Ricci soliton, setting $\xi = grad\phi$, from equation (1.2.3) we can write

$$(\mathcal{L}_\xi g)(X,Y) + 2S(X,Y) = \mu g(X,Y) = [2\lambda - (p + \frac{2}{n})]g(X,Y), \tag{2.13.1}$$

for all $X, Y \in \chi(M)$.
Again since $\phi = \int_c^t f(z)dz$, then $\xi = grad\phi$ implies that $\xi = f(t)\frac{\partial}{\partial t}$ and it can be seen that the vector field ξ is orthogonal to the manifold F.
Let us assume that $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\}$ be an orthonormal basis of $\chi(M)$. Then the Hessian of the function ϕ is given by

$$H^\phi(X,Y) = g(\nabla_X grad\phi, Y). \tag{2.13.2}$$

Now, we consider the following three cases.

Case 1: First let us consider $X = Y = \frac{\partial}{\partial t}$.
Then from equation (2.13.2) we get

$$\begin{aligned} H^\phi(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) &= g(\nabla_{\frac{\partial}{\partial t}} grad\phi, \frac{\partial}{\partial t}) \\ &= \dot{f} g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}). \end{aligned} \tag{2.13.3}$$

Case 2: Next we consider $X = \frac{\partial}{\partial t}$ and $Y = \frac{\partial}{\partial x_i}$ for $i = 1, 2, ..., n$.
Then in this case equation (2.13.2) implies

$$\begin{aligned} H^\phi(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_i}) &= g(\nabla_{\frac{\partial}{\partial t}} grad\phi, \frac{\partial}{\partial x_i}) \\ &= \dot{f} g(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_i}). \end{aligned} \tag{2.13.4}$$

Case 3: Finally we consider $X = \frac{\partial}{\partial x_i}$ and $Y = \frac{\partial}{\partial x_j}$ for $1 \leq i, j \leq n$.
Then from equation (2.13.2) we have

$$\begin{aligned} H^\phi(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) &= g(\nabla_{\frac{\partial}{\partial x_i}} grad\phi, \frac{\partial}{\partial x_j}) \\ &= f g(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial t}, \frac{\partial}{\partial x_j}) \\ &= f g(\frac{\dot{f}}{f} \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \\ &= \dot{f} g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}). \end{aligned} \tag{2.13.5}$$

Therefore combining equations (2.13.3), (2.13.4) and (2.13.5) and using it in (2.13.2) we get

$$H^{\phi}(X,Y) = \dot{f}g(X,Y). \tag{2.13.6}$$

Now, since $\xi = grad\phi$, using the definition of Lie differentiation we can write

$$\begin{aligned}(\mathcal{L}_\xi g)(X,Y) &= g(\nabla_X grad\phi, Y) + g(X, \nabla_Y grad\phi) \\ &= 2H^{\phi}(X,Y).\end{aligned}$$

Thus using equation (2.13.6), the above equation becomes

$$(\mathcal{L}_\xi g)(X,Y) = 2\dot{f}g(X,Y). \tag{2.13.7}$$

Using the value of equation (2.13.7) in the equation (2.13.1) and after simplification we get

$$S(X,Y) = [\lambda - \dot{f} - \frac{p}{2} - \frac{1}{n}]g(X,Y). \tag{2.13.8}$$

Thus if $\lambda = \dot{f} + \frac{p}{2} + \frac{1}{n}$, from equation (2.13.8), it implies that (M,g) is Ricci flat. This completes the first part of the theorem.

Again if $\dot{f}$ is a constant, say $\dot{f} = a$, i.e; if $df = adt$, i.e; if $f = at + b$ for some arbitrary constant b, then from equation (2.13.8) we can conclude that (M,g) is Einstein. This completes the proof. □

3

On conformal η-Ricci solitons

3.1 Introduction

In this chapter we study conformal η-Ricci solitons on ϵ-Kenmotsu manifold and Kählerian spacetime manifold. This chapter is divided into ten sections. In sections one and two we give introduction and preliminaries respectively.

In section three, we study ϵ-Kenmotsu manifold admitting conformal η-Ricci soliton and establish the relation between the soliton constants λ and μ. Section four deals with conformal η-Ricci soliton on ϵ-Kenmotsu manifold in terms of Codazzi type Ricci tensor, cyclic parallel Ricci tensor and cyclic η-recurrent Ricci tensor. Then Section five is devoted to the study of conformal η-Ricci soliton on ϵ-Kenmotsu manifold satisfying curvature conditions $R \cdot S = 0$, $C \cdot S = 0$, $Q \cdot C = 0$. In section six, we consider torse-forming vector field on ϵ-Kenmotsu manifold admitting conformal η-Ricci soliton. Section seven characterizes gradient conformal η-Ricci soliton on ϵ-Kenmotsu manifold. In section eight, we construct an example to illustrate the existence of conformal η-Ricci soliton on ϵ-Kenmotsu manifold.

In section nine, we characterize the nature of conformal η-Ricci soliton on projectively flat and conharmonically flat almost pseudo symmetric Kählerian spacetime manifold. Finally section ten is devoted to the study of gradient conformal η-Ricci soliton on Kählerian spacetime manifold.

3.2 Preliminaries

Here, we discuss some preliminaries of ϵ-Kenmotsu manifold and almost pseudo symmetric Kählerian spacetime manifold.

The defintion of ϵ-Kenmotsu manifold is given in chapter one. Furthermore, in an ϵ-Kenmotsu manifold (M, g) the following relations hold,

$$(\nabla_X \eta)(Y) = g(X,Y) - \epsilon\eta(X)\eta(Y), \tag{3.2.1}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{3.2.2}$$

$$R(\xi,X)Y = \eta(Y)X - \epsilon g(X,Y)\xi, \tag{3.2.3}$$

$$R(\xi,X)\xi = -R(X,\xi)\xi = X - \eta(X)\xi, \tag{3.2.4}$$

$$\eta(R(X,Y)Z) = \epsilon(g(X,Z)\eta(Y) - g(Y,Z)\eta(X)), \tag{3.2.5}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{3.2.6}$$

$$Q\xi = -\epsilon(n-1)\xi, \tag{3.2.7}$$

where R is the curvature tensor, S is the Ricci tensor and Q is the Ricci operator given by $g(QX,Y) = S(X,Y)$, for all $X, Y \in \chi(M)$.

Moreover, It is to be noted that for spacelike structure vector field ξ and $\epsilon = 1$, an ϵ-Kenmotsu manifold reduces to an usual Kenmotsu manifold.

The definition of Kählerian spacetime manifold is given in the introductory chapter one. So, combining equations (1.2.11) and (1.2.12) we can see that the Ricci tensor S becomes a functional combination of the metric tensor g and $\eta \otimes \eta$ satisfying

$$S(X,Y) = -\left(\omega - \frac{r}{2} - \kappa\rho\right) g(X,Y) + \kappa(\sigma+\rho)\eta(X)\eta(Y), \tag{3.2.8}$$

for all smooth vector fields $X, Y \in \chi(M)$. Recall that, manifold having such type of Ricci tensor is called quasi-Einstein manifold [23] and they arose during the study of exact solutions of Einstein field equations.

Furthermore, in a Kählerian spacetime manifold the Riemannian curvature tensor R and the Ricci tensor S satisfy the following

$$\tilde{R}(X,Y,U,W) = \tilde{R}(JX,JY,U,W), \tag{3.2.9}$$

$$S(X,Y) = S(JX,JY), \tag{3.2.10}$$

$$S(X,JY) + S(JX,Y) = 0, \tag{3.2.11}$$

for all $X, Y, U, W \in \chi(M)$ and $\tilde{R}(X, Y, U, W) = g(R(X, Y)U, W)$.

Let us consider an orthonormal frame field $\{E_i : 1 \leq i \leq 4\}$ such that $g(E_i, E_j) = \epsilon_{ij}\delta ij$ for all $1 \leq i, j \leq 4$; with $\epsilon_{ii} = -1$ and $\epsilon_{ij} = 0$ for $i \neq j$. Then assuming $\xi = \sum_{i=1}^{4} \xi^i E_i$ we can write

$$-1 = g(\xi, \xi) = \sum_{1 \leq i,j \leq 4} \xi^i \xi^j g(E_i, E_i) = \epsilon_{ii}(\xi^i)^2. \tag{3.2.12}$$

Also we can deduce the following

$$\eta(E_i) = g(E_i, \xi) = \sum_{j=1}^{4} \xi^j g(E_i, E_j) = \epsilon_{ii}\xi^j. \tag{3.2.13}$$

In Kählerian spacetime manifold, contracting the equation (3.2.8) over X and Y and using (3.2.12) we obtain

$$r = 4\omega + \kappa(\sigma - 3\rho). \tag{3.2.14}$$

Recalling equation (3.2.8) and making use of the above value in it yields

$$S(X, Y) = \left(\omega + \frac{\kappa(\sigma - \rho)}{2}\right) g(X, Y) + \kappa(\sigma + \rho)\eta(X)\eta(Y), \tag{3.2.15}$$

for all vector fields $X, Y \in \chi(M)$. Hence in view of the above, we can state the following

Proposition 3.2.1. *A Kählerian spacetime manifold is a quasi-Einstein manifold.*

Next, we discuss the notion of almost pseudo symmetric manifold which was introduced by U. C. De and A. K. Gazi [31] in 2008. Let (M, g) be a non-flat Riemannian manifold of dimension greater than three. Then it is said to be an almost pseudo symmetric manifold if its Riemannian curvature tensor satisfies

$$\begin{aligned}(\nabla_Z \tilde{R})(X, Y, U, W) &= [\mathcal{A}(Z) + \mathcal{B}(Z)]\tilde{R}(X, Y, U, W) + \mathcal{A}(X)\tilde{R}(Z, Y, U, W) \\ &\quad + \mathcal{A}(Y)\tilde{R}(X, Z, U, W) + \mathcal{A}(U)\tilde{R}(X, Y, Z, W) \\ &\quad + \mathcal{A}(W)\tilde{R}(X, Y, U, Z),\end{aligned} \tag{3.2.16}$$

where $\mathcal{A}$ and $\mathcal{B}$ are two non-zero 1-forms on M defined as

$$\mathcal{A}(Z) = g(Z, \mathcal{P}), \qquad \mathcal{B}(Z) = g(Z, \mathcal{N}). \tag{3.2.17}$$

After B. O'Neill [68] investigated the applications of semi-Riemannian geometry in general relativity, many authors have studied the curvature tensors and geometrical structures in spacetimes of general relativity [2]. Venkatesha and Kumara [94] studied Ricci solitons and geometrical structure in a perfect fluid spacetime with torse forming vector field. Blaga [12] investigated curvature properties and solitons in perfect fluid spacetime.

3.3 ϵ-Kenmotsu manifold admitting conformal η-Ricci soliton

Let us consider an ϵ-Kenmotsu manifold (M, g) admits a conformal η-Ricci soliton (g, ξ, λ, μ). Then from equation (1.2.5) we can write

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + [2\lambda - (p + \frac{2}{n})]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0, \quad (3.3.1)$$

for all $X, Y \in \chi(M)$.
Again from the well-known formula $(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)$ of Lie-derivative and using (1.1.38), we obtain for an ϵ-Kenmotsu manifold

$$(\mathcal{L}_\xi g)(X, Y) = 2\epsilon[g(X, Y) - \epsilon\eta(X)\eta(Y)]. \quad (3.3.2)$$

Now in view of the equations (3.3.1) and (3.3.2) we get

$$S(X, Y) = -[(\lambda + \epsilon) - (\frac{p}{2} + \frac{1}{n})]g(X, Y) - (\mu - 1)\eta(X)\eta(Y). \quad (3.3.3)$$

This shows that the manifold (M, g) is an η-Einstein manifold.
Also from equation (3.3.3) replacing $Y = \xi$ we find that

$$S(X, \xi) = [\epsilon(\frac{p}{2} + \frac{1}{n}) - (\epsilon\lambda + \mu)]\eta(X). \quad (3.3.4)$$

Comparing the above equation (3.3.4) with (3.2.6) yields

$$\epsilon\lambda + \mu = \epsilon(\frac{p}{2} + \frac{1}{n}) + (n - 1). \quad (3.3.5)$$

Thus the above discussion leads to the following

Theorem 3.3.1. *If an n-dimensional ϵ-Kenmotsu manifold (M, g) admits a conformal η-Ricci soliton (g, ξ, λ, μ), then (M, g) becomes an η-Einstein manifold and the scalars λ and μ are related by $\epsilon\lambda + \mu = \epsilon(\frac{p}{2} + \frac{1}{n}) + (n - 1)$.*

Furthermore if we consider $\mu = 0$ in particular, then from equations (3.3.3) and (3.3.5), we get

$$\begin{aligned} S(X, Y) &= -[(\lambda + \epsilon) - (\frac{p}{2} + \frac{1}{n})]g(X, Y) + \eta(X)\eta(Y), \\ \lambda &= (\frac{p}{2} + \frac{1}{n}) + \epsilon(n - 1). \end{aligned}$$

This leads us to write

Corollary 3.3.1. *If an n-dimensional ϵ-Kenmotsu manifold (M, g) admits a conformal Ricci soliton (g, ξ, λ), then (M, g) becomes an η-Einstein manifold and the scalars λ and μ are related by $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n-1)$. Moreover,*

1. *if ξ is spacelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) > (1-n)$, $(\frac{p}{2} + \frac{1}{n}) = (1-n)$ or $(\frac{p}{2} + \frac{1}{n}) < (1-n)$; and*
2. *if ξ is timelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) > (n-1)$, $(\frac{p}{2} + \frac{1}{n}) = (n-1)$ or $(\frac{p}{2} + \frac{1}{n}) < (n-1)$.*

Next we try to find a condition in terms of second order symmetric parallel tensor which will ensure when an ϵ-Kenmotsu manifold (M, g) admits a conformal η-Ricci soliton. So for this purpose let us consider the second order tensor T on the manifold (M, g) defined by

$$T := \mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta. \tag{3.3.6}$$

It is easy to see that the $(0, 2)$ tensor T is symmetric and also parallel with respect to the Levi-Civita connection.

Now in view of (3.3.2) and (3.3.3) the previous equation (3.3.6) we have

$$T(X, Y) = [(p + \frac{2}{n}) - 2\lambda]g(X, Y); \quad \forall X, Y \in TM. \tag{3.3.7}$$

Putting $X = Y = \xi$ in the above equation (3.3.7) we obtain

$$T(\xi, \xi) = \epsilon[(p + \frac{2}{n}) - 2\lambda]. \tag{3.3.8}$$

On the other hand, as T is a second order symmetric parallel tensor; i.e; $\nabla T = 0$, we can write

$$T(R(X, Y)Z, U) + T(Z, R(X, Y)U) = 0,$$

for all $X, Y, Z, U \in TM$. Then replacing $X = Z = U = \xi$ in above gives us

$$T(R(\xi, Y)\xi, \xi) + T(\xi, R(\xi, Y)\xi) = 0, \quad \forall Y \in TM. \tag{3.3.9}$$

Using (3.2.4) in the above equation (3.3.9) we get

$$T(Y, \xi) = T(\xi, \xi)\eta(Y). \tag{3.3.10}$$

Taking covariant differentiation of (3.3.10) in the direction of an arbitrary vector field X, and then in the resulting equation, again using the equation (3.3.10) we obtain

$$T(Y, \nabla_X \xi) = T(\xi, \xi)(\nabla_X \eta)Y + 2T(\nabla_X \xi, \xi)\eta(Y).$$

Then in view of (1.1.38) and (3.2.1), the above equation becomes

$$T(X, Y) = \epsilon T(\xi, \xi)g(X, Y), \quad \forall X, Y \in TM. \tag{3.3.11}$$

Now using (3.3.8) in the above equation (3.3.11) and in view of (3.3.6) finally we get

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + [2\lambda - (p + \frac{2}{n})]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0.$$

This leads us to the following:

Theorem 3.3.2. *Let (M, g) be an n-dimensional ϵ-Kenmotsu manifold. If the second order symmetric tensor $T := \mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta$ is parallel with respect to the Levi-Civita connection of the manifold, then the manifold (M, g) admits a conformal η-Ricci soliton (g, ξ, λ, μ).*

Now let us consider an ϵ-Kenmotsu manifold (M, g) and assume that it admits a conformal η-Ricci soliton (g, V, λ, μ) such that V is pointwise collinear with ξ, i.e; $V = \alpha\xi$, for some function α; then from the equation (1.2.5) it follows that

$$\begin{aligned} &\alpha g(\nabla_X \xi, Y) + (X\alpha)\eta(Y) + \alpha g(\nabla_Y \xi, X) + (Y\alpha)\eta(X) \\ &\qquad + 2S(X, Y) + [2\lambda - (p + \frac{2}{n})]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0. \end{aligned}$$

Then using the equation (1.1.38) in above we get

$$\begin{aligned} &2\epsilon\alpha g(X, Y) - 2\alpha\eta(X)\eta(Y) + (X\alpha)\eta(Y) + (Y\alpha)\eta(X) \\ &\quad + 2S(X, Y) + [2\lambda - (p + \frac{2}{n})]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0. \end{aligned} \tag{3.3.12}$$

Replacing $Y = \xi$ in the above equation yields

$$(X\alpha) + (\xi\alpha)\eta(X) + 2S(X, \xi) + \epsilon[2\lambda - (p + \frac{2}{n})]\eta(X) + 2\mu\eta(X) = 0. \tag{3.3.13}$$

By virtue of (3.2.6) the above equation (3.3.13) becomes

$$(X\alpha) + [(\xi\alpha) + \epsilon[2\lambda - (p + \frac{2}{n})] + 2\mu - 2(n-1)]\eta(X) = 0. \tag{3.3.14}$$

By taking $X = \xi$ in the above equation (3.3.14) gives us

$$(\xi\alpha) = (n-1) - \mu - \epsilon[\lambda - (\frac{p}{2} + \frac{1}{n})]. \tag{3.3.15}$$

Using this value from (3.3.15) in the equation (3.3.14) we can write

$$d\alpha = [(n-1) - \mu - \epsilon[\lambda - (\frac{p}{2} + \frac{1}{n})]]\eta. \tag{3.3.16}$$

Now taking exterior differentiation on both sides of (3.3.16) and using the famous Poincare's lemma i.e; $d^2 = 0$, finally we arrive at

$$[(n-1) - \mu - \epsilon[\lambda - (\frac{p}{2} + \frac{1}{n})]]d\eta = 0.$$

Since $d\eta \neq 0$ in ϵ-Kenmotsu manifold, the above equation implies

$$\mu + \epsilon[\lambda - (\frac{p}{2} + \frac{1}{n})] = (n-1). \tag{3.3.17}$$

In view of the above (3.3.17) the equation (3.3.16) gives us $d\alpha = 0$ i.e; the function α is constant. Then the equation (3.3.12) becomes

$$S(X,Y) = [(\frac{p}{2} + \frac{1}{n}) - \lambda - \epsilon\alpha]g(X,Y) + (\alpha - \mu)\eta(X)\eta(Y), \tag{3.3.18}$$

for all $X, Y \in \chi(M)$. This shows that the manifold is η-Einstein. Hence we have the following

Theorem 3.3.3. *If an n-dimensional ϵ-Kenmotsu manifold (M,g) admits a conformal η-Ricci soliton (g,V,λ,μ) such that V is pointwise collinear with ξ, then V is constant multiple of ξ and the manifold (M,g) is an η-Einstein manifold. Moreover the scalars λ and μ are related by $\mu + \epsilon[\lambda - (\frac{p}{2} + \frac{1}{n})] = (n-1)$.*

In particular if we put $\mu = 0$ in (3.3.17) and (3.3.18) we can conclude that

Corollary 3.3.2. *If an n-dimensional ϵ-Kenmotsu manifold (M,g) admits a conformal Ricci soliton (g,V,λ,μ) such that V is pointwise collinear with ξ, then V is constant multiple of ξ and the manifold (M,g) is an η-Einstein manifold, and the scalars λ and μ are related by $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n-1)$. Furthermore,*

i) if ξ is spacelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) + n > 1$, $(\frac{p}{2} + \frac{1}{n}) + n = 1$ or $(\frac{p}{2} + \frac{1}{n}) + n < 1$; and

ii) if ξ is timelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) + 1 > n$, $(\frac{p}{2} + \frac{1}{n}) + 1 = n$ or $(\frac{p}{2} + \frac{1}{n}) + 1 < n$.

3.4 Conformal η-Ricci soliton on ϵ-Kenmotsu manifold with certain special types of Ricci tensor

The purpose of this section is to study Conformal η-Ricci soliton on ϵ-Kenmotsu manifold admitting three special types of Ricci tensor namely codazzi type Ricci tensor, cyclic parallel Ricci tensor and cyclic η-recurrent Ricci tensor.

Let us consider that, an ϵ-Kenmotsu manifold having Codazzi type Ricci tensor admits a conformal η-Ricci soliton (g, ξ, λ, μ), then equations (3.3.3) and (1.1.7) hold. Now taking covariant differentiation of (3.3.3) and using equation (3.2.1) we obtain

$$(\nabla_X S)(Y, Z) = (1 - \mu)[g(X, Y)\eta(Z) + g(X, Z)\eta(Y) - 2\epsilon\eta(X)\eta(Y)\eta(Z)]. \tag{3.4.1}$$

Since the manifold has Codazzi type Ricci tensor, in view of (1.1.7) equation (3.4.1) yields

$$(1 - \mu)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)] = 0, \quad \forall X, Y, Z \in \chi(M).$$

The above equation implies that $\mu = 1$ and then from equation (3.3.5) it follows that $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n - 2)$. Therefore we can state the following

Theorem 3.4.1. *Let (M, g) be an n-dimensional ϵ-Kenmotsu manifold admitting a conformal η-Ricci soliton (g, ξ, λ, μ). If the Ricci tensor of the manifold is of Codazzi type then $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n - 2)$ and $\mu = 1$.*

Corollary 3.4.1. *Let an n-dimensional ϵ-Kenmotsu manifold admits a conformal η-Ricci soliton (g, ξ, λ, μ) and the manifold has Codazzi type Ricci tensor then*

i) if ξ is spacelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) + n > 2$, $(\frac{p}{2} + \frac{1}{n}) + n = 2$ or $(\frac{p}{2} + \frac{1}{n}) + n < 2$; and

ii) if ξ is timelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) + 2 > n$, $(\frac{p}{2} + \frac{1}{n}) + 2 = n$ or $(\frac{p}{2} + \frac{1}{n}) + 2 < n$.

Now, we consider an ϵ-Kenmotsu manifold, having cyclic parallel Ricci tensor, admits a conformal η-Ricci soliton (g, ξ, λ, μ), then equations (3.3.3) and (1.1.8) hold. Now taking covariant differentiation of (3.3.3) and using equation (3.2.1) we obtain

$$(\nabla_X S)(Y, Z) = (1 - \mu)[g(X, Y)\eta(Z) + g(X, Z)\eta(Y) - 2\epsilon\eta(X)\eta(Y)\eta(Z)]. \tag{3.4.2}$$

In a similar manner we can obtain the following relations

$$(\nabla_Y S)(Z,X) = (1-\mu)[g(X,Y)\eta(Z) + g(Y,Z)\eta(X) - 2\epsilon\eta(X)\eta(Y)\eta(Z)] \qquad (3.4.3)$$

and

$$(\nabla_Z S)(X,Y) = (1-\mu)[g(X,Z)\eta(Y) + g(Y,Z)\eta(X) - 2\epsilon\eta(X)\eta(Y)\eta(Z)]. \qquad (3.4.4)$$

Now using the values from (3.4.2), (3.4.3) and (3.4.4) in the equation (1.1.8) we get

$$2(1-\mu)[g(X,Y)\eta(Z) + g(Y,Z)\eta(X) + g(X,Z)\eta(Y) - 3\epsilon\eta(X)\eta(Y)\eta(Z)] = 0.$$

Replacing $Z = \xi$ in the above equation yields

$$2(1-\mu)[g(X,Y) - \epsilon\eta(X)\eta(Y)] = 0 \quad \forall X, Y \in TM.$$

The above equation implies that $\mu = 1$ and then from equation (3.3.5) it follows that $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n-2)$. Hence we have

Theorem 3.4.2. *Let (M, g) be an n-dimensional ϵ-Kenmotsu manifold admitting a conformal η-Ricci soliton (g, ξ, λ, μ). If the manifold has cyclic parallel Ricci tensor, then $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n-2)$ and $\mu = 1$.*

Next we focus on another important curvature tensor namely the cyclic-η-recurrent Ricci tensor. An ϵ-Kenmotsu manifold is said to have cyclic-η-recurrent Ricci tensor if its Ricci tensor S is non-zero and satisfies the following relation:

$$\begin{aligned} & (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\ = \; & \eta(X)S(Y,Z) + \eta(Y)S(Z,X) + \eta(Z)S(X,Y) \quad \forall X, Y, Z \in \chi(M). \qquad (3.4.5) \end{aligned}$$

Let us consider an ϵ-Kenmotsu manifold, having cyclic-η-recurrent Ricci tensor, that admits a conformal η-Ricci soliton (g, ξ, λ, μ), then equation (3.3.3) holds. Now taking covariant differentiation of (3.3.3) and using equation (3.2.1) and proceeding similarly as the previous theorem we arrive at equations (3.4.2), (3.4.3) and (3.4.4). Then putting these three values in (3.4.5) we get

$$\begin{aligned} & (2(1-\mu) - \beta)[g(X,Y)\eta(Z) + g(Y,Z)\eta(X) + g(X,Z)\eta(Y)] \\ & \qquad - (3 + 6\epsilon)(1-\mu)\eta(X)\eta(Y)\eta(Z) = 0, \qquad (3.4.6) \end{aligned}$$

where $\beta = (\frac{p}{2} + \frac{1}{n}) - (\lambda + \epsilon)$. Now putting $Y = Z = \xi$ in (3.4.6) we obtain

$$3(\epsilon\beta + (1 - \mu))\eta(X) = 0. \quad \forall X \in TM. \tag{3.4.7}$$

Since $\eta(X) \neq 0$ and replacing the value of β in (3.4.7), after simplification we get $\lambda = (\frac{p}{2} + \frac{1}{n}) - \epsilon\mu$. Therefore we can state

Theorem 3.4.3. *Let (M, g) be an n-dimensional ϵ-Kenmotsu manifold admitting a conformal η-Ricci soliton (g, ξ, λ, μ). If the manifold has cyclic-eta-parallel Ricci tensor, then $\lambda = (\frac{p}{2} + \frac{1}{n}) - \epsilon\mu$ and moreover*

i) if ξ is spacelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) > \mu$, $(\frac{p}{2} + \frac{1}{n}) = \mu$ or $(\frac{p}{2} + \frac{1}{n}) < \mu$; and

ii) if ξ is timelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) + \mu > 0$, $(\frac{p}{2} + \frac{1}{n}) + \mu = 0$ or $(\frac{p}{2} + \frac{1}{n}) + \mu < 0$.

Corollary 3.4.2. *Let (M, g) be an n-dimensional ϵ-Kenmotsu manifold admitting a conformal Ricci soliton (g, ξ, λ, μ). If the manifold has cyclic-eta-parallel Ricci tensor, then the soliton constant λ is given by $\lambda = (\frac{p}{2} + \frac{1}{n})$.*

3.5 Conformal η-Ricci soliton on ϵ-Kenmotsu manifold satisfying some curvature conditions

Let us consider an ϵ-Kenmotsu manifold which admits a conformal η-Ricci soliton (g, ξ, λ, μ) and also the manifold is Ricci semi symmetric i.e; the manifold satisfies the curvature condition $R(X, Y) \cdot S = 0$. Then for all $X, Y, Z, W \in \chi(M)$ we can write

$$S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = 0.$$

Taking $W = \xi$ in the above equation implies

$$\eta(R(X, Y)Z) + S(Z, R(X, Y)\xi) = 0. \tag{3.5.1}$$

Now using (3.2.2) and (3.2.5) in (3.5.1) we get

$$\eta(X)[S(Y, Z) - \epsilon g(Y, Z)] - \eta(Y)[S(X, Z) - \epsilon g(X, Z)] = 0.$$

In view of (3.3.3) the previous equation becomes

$$[(\frac{p}{2}+\frac{1}{n})-\lambda-2\epsilon][\eta(X)g(Y,Z)-\eta(Y)g(X,Z)]=0.$$

Putting $X=\xi$ in the above equation and then using (1.1.34) and (1.1.35) we finally obtain

$$[(\frac{p}{2}+\frac{1}{n})-\lambda-2\epsilon]g(\phi Y,\phi Z)=0. \tag{3.5.2}$$

Since $g(\phi Y,\phi Z)\neq 0$ always, we can conclude from the equation (3.5.2) that $[(\frac{p}{2}+\frac{1}{n})-\lambda-2\epsilon]=0$ i.e; $\lambda=(\frac{p}{2}+\frac{1}{n})-2\epsilon$. Then from the equation (3.3.5) we have $\mu=(n+1)$. Therefore we have the following

Theorem 3.5.1. *Let (M,g) be an n-dimensional ϵ-Kenmotsu manifold admitting a conformal η-Ricci soliton (g,ξ,λ,μ). If the manifold is Ricci semi symmetric i.e; if the manifold satisfies the curvature condition $R(X,Y)\cdot S=0$, then $\lambda=(\frac{p}{2}+\frac{1}{n})-2\epsilon$ and $\mu=(n+1)$. Moreover*

i) if ξ is spacelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2}+\frac{1}{n})>2$, $(\frac{p}{2}+\frac{1}{n})=2$ or $(\frac{p}{2}+\frac{1}{n})<2$; and

ii) if ξ is timelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2}+\frac{1}{n})+2>0$, $(\frac{p}{2}+\frac{1}{n})+2=0$ or $(\frac{p}{2}+\frac{1}{n})+2<0$.

Next we consider an n-dimensional ϵ-Kenmotsu manifold satisfying the curvature condition $C(\xi,X)\cdot S=0$ admitting a conformal η-Ricci soliton (g,ξ,λ,μ). Then we have

$$S(C(\xi,X)Y,Z)+S(Y,C(\xi,X)Z)=0 \quad \forall X,Y,Z\in\chi(M). \tag{3.5.3}$$

Now from equation (1.1.2) we can write

$$C(\xi,X)Y=R(\xi,X)Y-\frac{r}{n(n-1)}[g(X,Y)\xi-\epsilon\eta(Y)X].$$

Using (3.2.3) the above equation becomes

$$C(\xi,X)Y=[1+\frac{\epsilon r}{n(n-1)}][\eta(Y)X-\epsilon g(X,Y)\xi]. \tag{3.5.4}$$

In view of (3.5.4) the equation (3.5.3) yields

$$[1+\frac{\epsilon r}{n(n-1)}][S(X,Z)\eta(Y)-\epsilon g(X,Y)S(\xi,Z)$$
$$+S(Y,X)\eta(Z)-\epsilon g(X,Z)S(\xi,Y]=0.$$

By virtue of (3.2.6) the above equation eventually becomes

$$[1+\frac{\epsilon r}{n(n-1)}][S(X,Z)\eta(Y)+S(Y,X)\eta(Z)$$
$$+\epsilon(n-1)(g(X,Y)\eta(Z)+g(X,Z)\eta(Y))]=0. \tag{3.5.5}$$

Putting $Z=\xi$ in (3.5.5) and then using (1.1.34), (3.2.6) we arrive at

$$[1+\frac{\epsilon r}{n(n-1)}][S(X,Y)+\epsilon(n-1)g(X,Y)]=0.$$

Thus from the above we can conclude that either $r=-\epsilon n(n-1)$ or

$$S(X,Y)=-\epsilon(n-1)g(X,Y). \tag{3.5.6}$$

Combining (3.5.6) with (3.3.3) we get

$$[(\lambda+\epsilon)-(\frac{p}{2}+\frac{1}{n})-\epsilon(n-1)]g(X,Y)+(\mu-1)\eta(X)\eta(Y)=0.$$

Taking $Y=\xi$ in above gives us

$$[(n-\mu)+\epsilon(\frac{p}{2}+\frac{1}{n}-\lambda-\epsilon)]\eta(X)=0, \quad \forall X\in\chi(M).$$

Since $\eta(X)\neq 0$ always, from the above we have $\lambda=\epsilon(n-1)+(\frac{p}{2}+\frac{1}{n})-\epsilon\mu$. Therefore we can state

Theorem 3.5.2. *Let (M,g) be an n-dimensional ϵ-Kenmotsu manifold admitting a conformal η-Ricci soliton (g,ξ,λ,μ). If the manifold satisfies the curvature condition $C(\xi,X)\cdot S=0$, then either the scalar curvature of the manifold is constant or the manifold is an Einstein manifold of the form* (3.5.6) *and the scalars λ and μ are related by $\lambda=\epsilon(n-1)+(\frac{p}{2}+\frac{1}{n})-\epsilon\mu$.*

Next we prove two results on ξ-projectively flat and ξ-concircularly flat manifolds. For that let us first consider an ϵ-Kenmotsu manifold (M,g,ξ,ϕ,η) admitting a conformal η-Ricci soliton (g,ξ,λ,μ). We know from definition that the manifold is ξ-projectively flat if $P(X,Y)\xi=0, \quad \forall X,Y\in\chi(M)$. Then putting $Z=\xi$ in (1.1.1) we obtain

$$P(X,Y)\xi=R(X,Y)\xi-\frac{1}{n-1}[S(Y,\xi)X-S(X,\xi)Y]. \tag{3.5.7}$$

Now since it is given that (g,ξ,λ,μ) admits a conformal η-Ricci soliton, using (3.2.2) and (3.3.4) in the above (3.5.7), we obtain

$$P(X,Y)\xi=\left[1+\frac{\epsilon(\frac{p}{2}+\frac{1}{n})-\epsilon\lambda-\mu}{n-1}\right][\eta(X)Y-\eta(Y)X].$$

In view of (3.3.5) the above equation finally becomes $P(X, Y)\xi = 0$. Hence we have the following

Proposition 3.5.1. *An n-dimensional ϵ-Kenmotsu manifold (M, g, ξ, ϕ, η) admitting a conformal η-Ricci soliton (g, ξ, λ, μ) is ξ-projectively flat.*

Again consider an n-dimensional ϵ-Kenmotsu manifold (M, g, ξ, ϕ, η) admitting a conformal η-Ricci soliton (g, ξ, λ, μ). Then from definition we know that an ϵ-Kenmotsu manifold is ξ-concircularly flat if $C(X, Y)\xi = 0, \quad \forall X, Y \in \chi(M)$. So taking $Z = \xi$ in (1.1.2) we get

$$C(X, Y)\xi = R(X, Y)\xi - \frac{\epsilon r}{n(n-1)}[\eta(Y)X - \eta(X)Y]. \tag{3.5.8}$$

Using (3.2.2) in (3.5.8) we obtain

$$C(X, Y)\xi = [1 + \frac{\epsilon r}{n(n-1)}][\eta(X)Y - \eta(Y)X].$$

Thus from the above we can conclude that $C(X, Y)\xi = 0$ if and only if, $[1 + \frac{\epsilon r}{n(n-1)}] = 0$, i.e; if and only if, $r = -\epsilon n(n-1)$. Again since (g, ξ, λ, μ) is a conformal η-Ricci soliton, the equation (3.3.3) holds and thus contracting (3.3.3) we obtain $r = [(\frac{p}{2}+\frac{1}{n})-\lambda-\mu]n-(\mu-1)$. Thus combining both the values of r we have, $\lambda = (\frac{p}{2} + \frac{2}{n}) - \frac{\mu}{n} - 2\epsilon$. Therefore we can state

Proposition 3.5.2. *An n-dimensional ϵ-Kenmotsu manifold (M, g, ξ, ϕ, η) admitting a conformal η-Ricci soliton (g, ξ, λ, μ) is ξ-concircularly flat if and only if, $\lambda = (\frac{p}{2} + \frac{2}{n}) - \frac{\mu}{n} - 2\epsilon$.*

We now assume that an n-dimensional ϵ-Kenmotsu manifold (M, g, ξ, ϕ, η) admits a conformal η-Ricci soliton (g, ξ, λ, μ) which satisfies the curvature condition $Q \cdot C = 0$, where C denotes the concircular curvature tensor of the manifold. Then we can write

$$Q(C(X, Y)Z) - C(QX, Y)Z - C(X, QY)Z - C(X, Y)QZ = 0. \tag{3.5.9}$$

Using (1.1.2) in (3.5.9) yields

$$\begin{aligned} Q(R(X, Y)Z) - R(QX, Y)Z - R(X, QY)Z - R(X, Y)QZ \\ + \frac{2r}{n(n-1)}[S(Y, Z)X - S(X, Z)Y] = 0. \end{aligned} \tag{3.5.10}$$

Taking inner product of (3.5.10) with respect to the vector field ξ we get

$$\begin{aligned}&\eta(Q(R(X,Y)Z)) - \eta(R(QX,Y)Z) - \eta(R(X,QY)Z)\\ &\quad - \eta(R(X,Y)QZ) + \frac{2r}{n(n-1)}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] = 0.\end{aligned}$$

Putting $Z = \xi$ in above we obtain

$$\begin{aligned}&\eta(Q(R(X,\xi)Z)) - \eta(R(QX,\xi)Z) - \eta(R(X,Q\xi)Z)\\ &\quad - \eta(R(X,\xi)QZ) + \frac{2r}{n(n-1)}[S(\xi,Z)\eta(X) - S(X,Z)] = 0. \qquad (3.5.11)\end{aligned}$$

Again from (3.2.3) we can derive

$$\eta(Q(R(X,\xi)Z)) = \eta(R(X,Q\xi)Z) = (n-1)[\epsilon\eta(X)\eta(Z) - g(X,Z)], \qquad (3.5.12)$$

$$\eta(R(QX,\xi)Z) = \eta(R(X,\xi)QZ) = \epsilon[S(X,Z) + (n-1)\eta(X)\eta(Z)]. \qquad (3.5.13)$$

By virtue of (3.5.12) and (3.5.13), the equation (3.5.11) becomes

$$\epsilon[(n-1)\eta(X)\eta(Z) + S(X,Z)] - \frac{r}{n(n-1)}[S(\xi,Z)\eta(X) - S(X,Z)] = 0.$$

Using (3.2.6) in above we arrive at

$$[\epsilon + \frac{r}{n(n-1)}][(n-1)\eta(X)\eta(Z) + S(X,Z)] = 0.$$

Hence we can conclude that either $r = -\epsilon n(n-1)$ or,

$$S(X,Z) = -(n-1)\eta(X)\eta(Z). \qquad (3.5.14)$$

Now combining equations (3.5.14) and (3.3.3), we get

$$[(\lambda + \epsilon) - (\frac{p}{2} + \frac{1}{n})]g(X,Z) + (\mu - n)\eta(X)\eta(Z) = 0.$$

Taking $Z = \xi$ in above yields

$$[\epsilon(\lambda - (\frac{p}{2} + \frac{1}{n})) + (\mu + 1 - n)]\eta(X) = 0, \quad \forall X \in \chi(M).$$

Since $\eta(X) \neq 0$ always, from the above we can conclude that $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n - \mu - 1)$. Hence we can state the following

Theorem 3.5.3. *Let (M,g) be an n-dimensional ϵ-Kenmotsu manifold admitting a conformal η-Ricci soliton (g,ξ,λ,μ). If the manifold satisfies the curvature condition $Q\cdot C=0$, then either the scalar curvature of the manifold is constant or the manifold is a special type of η-Einstein manifold of the form (3.5.14) and the scalars λ and μ are related by $\lambda=(\frac{p}{2}+\frac{1}{n})+\epsilon(n-\mu-1)$.*

We conclude this section by this result on W_2-curvature tensor. For this let us consider an n-dimensional ϵ-Kenmotsu manifold admitting a conformal η-Ricci soliton (g,ξ,λ,μ) and assume that the manifold satisfies the curvature condition $W_2(\xi,Y)\cdot S=0$. Then we can write

$$S(W_2(\xi,Y)Z,U)+S(Z,W_2(\xi,Y)U)=0, \quad \forall Y,Z,U\in\chi(M).$$

Putting $U=\xi$ in above we get

$$S(W_2(\xi,Y)Z,\xi)+S(Z,W_2(\xi,Y)\xi)=0. \tag{3.5.15}$$

Now taking $X=\xi$ in (1.1.4) we obtain

$$W_2(\xi,Y)Z=R(\xi,Y)Z+\frac{\epsilon}{n-1}[\eta(Z)QY-\eta(Y)Q\xi].$$

Using (3.2.3) in above yields

$$W_2(\xi,Y)Z=\eta(Z)Y-\epsilon g(Y,Z)\xi+\frac{\epsilon}{n-1}[\eta(Z)QY-\eta(Y)Q\xi]. \tag{3.5.16}$$

putting $Z=\xi$ in (3.5.16) we arrive at

$$W_2(\xi,Y)\xi=Y-\eta(Y)\xi+\frac{\epsilon}{n-1}[QY-\eta(Y)Q\xi]. \tag{3.5.17}$$

Using (3.5.16) and (3.5.17) in the (3.5.15) we get

$$\begin{aligned}\eta(Z)S(Y,\xi)+\epsilon(n-1)g(Y,Z)+\frac{\epsilon}{n-1}[\eta(Z)S(QY,\xi)-\eta(Y)S(Q\xi,\xi)]\\ +S(Y,Z)-\eta(Y)S(Z,\xi)+\frac{\epsilon}{n-1}[S(Z,QY)-\eta(Y)S(Z,Q\xi)]=0.\end{aligned}$$

In view of (3.2.6) the above equation becomes

$$\begin{aligned}S(Y,Z)+\epsilon(n-1)g(Y,Z)+\epsilon[\eta(Y)\eta(Q\xi)-\eta(Z)\eta(QY)]\\ +\frac{\epsilon}{n-1}[S(Z,QY)-\eta(Y)S(Z,Q\xi)]=0.\end{aligned} \tag{3.5.18}$$

Taking $Y = \xi$ in (3.5.18) and then using (3.2.7) we have

$$S(Z,\xi) + (1+\epsilon)(n-1)\eta(Z) = (n-1). \tag{3.5.19}$$

Making use of (3.3.4) in (3.5.19) and then taking $Z = \xi$ we finally get

$$\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon[(n-1)(n-2) - \mu]. \tag{3.5.20}$$

Thus we arrive at the following

Theorem 3.5.4. *Let (M, g) be an n-dimensional ϵ-Kenmotsu manifold admitting a conformal η-Ricci soliton (g, ξ, λ, μ). If the manifold satisfies the curvature condition $W_2(\xi, Y)\cdot S = 0$, then the scalars λ and μ are related by $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon[(n-1)(n-2) - \mu]$.*

3.6 Conformal η-Ricci soliton on ϵ-Kenmotsu manifold with torse-forming vector field

This section is devoted to the study of conformal η-Ricci solitons on ϵ-Kenmotsu manifolds with torse-forming vector field.

Now let (M, g, ξ, ϕ, η) be an ϵ-Kenmotsu manifold admitting a conformal η-Ricci soliton (g, ξ, λ, μ) and assume that the potential vector field ξ of is a torse-forming vector field. Then ξ being a torse-forming vector field, by definiton from equation (1.1.12) we have

$$\nabla_X \xi = fX + \gamma(X)\xi, \tag{3.6.1}$$

$\forall X \in \chi(M)$, f being a smooth function and γ is a 1-form.
Recalling the equation (1.1.38) and taking inner product on both sides with ξ we can write

$$g(\nabla_X \xi, \xi) = \epsilon g(X, \xi) - \epsilon\eta(X)g(\xi, \xi),$$

which, in view of (1.1.34), reduces to

$$g(\nabla_X \xi, \xi) = 0. \tag{3.6.2}$$

Again from the equation (3.6.1), applying inner product with ξ we obtain

$$g(\nabla_X \xi, \xi) = \epsilon f\eta(X) + \epsilon\gamma(X). \tag{3.6.3}$$

Combining (3.6.2) and (3.6.3) we get, $\gamma = -f\eta$. Thus for torse-forming vector field ξ in ϵ-Kenmotsu manifolds, we have

$$\nabla_X \xi = f(X - \eta(X)\xi). \tag{3.6.4}$$

Since (g, ξ, λ, μ) is a conformal η-Ricci soliton, from (1.2.5) we can write

$$g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) + 2S(X, Y) + [2\lambda - (p + \frac{2}{n})]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0.$$

In view of (3.6.4) the above becomes

$$S(X, Y) = [(\frac{p}{2} + \frac{1}{n}) - (\lambda + f)]g(X, Y) + (\epsilon f - \mu)\eta(X)\eta(Y). \tag{3.6.5}$$

This implies that the manifold is an η-Einstein manifold. Therefore we have the following

Theorem 3.6.1. *Let (g, ξ, λ, μ) be a conformal η-Ricci soliton on an n-dimensional ϵ-Kenmotsu manifold (M, g), with torse-forming vector field ξ, then the manifold becomes an η-Einstein manifold of the form* (3.6.5)*.*

In particular if ξ is spacelike, i.e; $\epsilon = 1$, then for $\mu = f$, the equation (3.6.5) reduces to

$$S(X, Y) = [(\frac{p}{2} + \frac{1}{n}) - (\lambda + f)]g(X, Y), \tag{3.6.6}$$

which implies that the manifold is an Einstein manifold. Similarly for ξ timelike and for $\mu = -f$, from (3.6.5) we can say that the manifold becomes an Einstein manifold. Therefore we can state

Corollary 3.6.1. *Let (g, ξ, λ, μ) be a conformal η-Ricci soliton on an n-dimensional ϵ-Kenmotsu manifold (M, g), with torse-forming vector field ξ, then the manifold becomes an Einstein manifold according as ξ is spacelike and $\mu = f$, or; ξ is timelike and $\mu = -f$.*

3.7 Gradient conformal η-Ricci soliton on ϵ-Kenmotsu manifold

This section is devoted to the study of ϵ-Kenmotsu manifolds admitting gradient conformal η-Ricci solitons and we try to characterize the potential vector field of the soliton. First, we prove the following lemma which will be used later in this section.

Lemma 3.7.1. *On an n-dimensional ϵ-Kenmotsu manifold (M, g, ϕ, ξ, η), the following relations hold*

$$g((\nabla_Z Q)X, Y) = g((\nabla_Z Q)Y, X), \qquad (3.7.1)$$
$$(\nabla_Z Q)\xi = -\epsilon QZ - (n-1)Z, \qquad (3.7.2)$$

for all smooth vector fields $X, Y, Z \in \chi(M)$.

Proof. Since we know that the Ricci tensor is symmetric, we have $g(QX, Y) = g(X, QY)$. Covariantly differentiating this relation along Z and using $g(QX, Y) = S(X, Y)$ we can easily obtain (3.7.1).
To prove the second part, let us recall equation (3.2.7) and taking its covariant derivative in the direction of an arbitrary smooth vector field Z we get

$$(\nabla_Z Q)\xi + Q(\nabla_Z \xi) + \epsilon(n-1)\nabla_Z \xi = 0. \qquad (3.7.3)$$

In view of (1.1.38) and (3.2.7), the previous equation gives the desired result (3.7.2). This completes the proof. □

Now, we consider ϵ-Kenmotsu manifolds admitting gradient conformal η-Ricci solitons i.e.; when the vector field V is gradient of some smooth function f on M. Thus if $V = Df$, where $Df = gradf$, then the conformal η-Ricci soliton equation becomes

$$Hessf + S + [\lambda - (\frac{p}{2} + \frac{1}{n})]g + \mu\eta \otimes \eta = 0, \qquad (3.7.4)$$

where $Hessf$ denotes the Hessian of the smooth function f. In this case the vector field V is called the potential vector field and the smooth function f is called the potential function.

Lemma 3.7.2. *If (g, V, λ, μ) is a gradient conformal η-Ricci soliton on an n-dimensional ϵ-Kenmotsu manifold (M, g, ϕ, ξ, η), then the Riemannian curvature tensor R satisfies*

$$R(X, Y)Df = [(\nabla_Y Q)X - (\nabla_X Q)Y] + \epsilon\mu[\eta(X)Y - \eta(Y)X]. \qquad (3.7.5)$$

Proof. Since the data (g, V, λ, μ) is a gradient conformal η-Ricci soliton, equation (3.7.4) holds and it can be rewritten as

$$\nabla_X Df = -QX - [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]X - \mu\eta(X)\xi, \qquad (3.7.6)$$

for all smooth vector field X on M and for some smooth function f such that $V = Df = gradf$. Covariantly diffrentiating the previous equation along an arbitrary vector field Y and using (1.1.38) we obtain

$$\begin{aligned}\nabla_Y\nabla_X Df &= -\nabla_Y(QX) - [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]\nabla_Y X \\ &\quad -\mu[(\nabla_Y\eta(X))\xi + \epsilon(Y - \eta(Y)\xi)\eta(X)]. \qquad (3.7.7)\end{aligned}$$

Interchanging X and Y in (3.7.7) gives

$$\begin{aligned}\nabla_X\nabla_Y Df &= -\nabla_X(QY) - [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]\nabla_X Y \\ &\quad -\mu[(\nabla_X\eta(Y))\xi + \epsilon(X - \eta(X)\xi)\eta(Y)]. \qquad (3.7.8)\end{aligned}$$

Again in view of (3.7.6) we can write

$$\begin{aligned}\nabla_{[X,Y]}Df &= -Q(\nabla_X Y - \nabla_Y X) - \mu\eta(\nabla_X Y - \nabla_Y X)\xi \\ &\quad -[\lambda - (\frac{p}{2} + \frac{1}{2n+1})](\nabla_X Y - \nabla_Y X). \qquad (3.7.9)\end{aligned}$$

Therefore substituting the values from (3.7.7), (3.7.8) (3.7.9) in (1.1.5) we obtain our desired expression (3.7.5). This completes the proof. □

Now we proceed to prove our main result of this section.

Theorem 3.7.1. *Let (M, g, ϕ, ξ, η) be an n-dimensional ϵ-Kenmotsu manifold admitting a gradient conformal η-Ricci soliton (g, V, λ, μ), then the potential vector field V is pointwise collinear with the characteristic vector field ξ.*

Proof. Recalling the equation (3.2.2) and taking its inner product with Df yields

$$g(R(X,Y)\xi, Df) = (Yf)\eta(X) - (Xf)\eta(Y).$$

Again we know that $g(R(X,Y)\xi, Df) = -g(R(X,Y)Df, \xi)$ and in view of this the previous equation becomes

$$g(R(X,Y)Df, \xi) = (Xf)\eta(Y) - (Yf)\eta(X). \qquad (3.7.10)$$

Now taking inner product of (3.7.5) with ξ and using (3.7.2) we obtain

$$g(R(X,Y)Df, \xi) = 0. \qquad (3.7.11)$$

Thus combining (3.7.10) and (3.7.11) we arrive at

$$(Xf)\eta(Y) = (Yf)\eta(X).$$

Taking $Y = \xi$ in the foregoing equation gives us $(Xf) = (\xi f)\eta(X)$, which essentially implies $g(X, Df) = g(X, \epsilon(\xi f)\xi)$. Since this equation is true for all X, we can conclude that

$$V = Df = \epsilon(\xi f)\xi. \tag{3.7.12}$$

Hence, V is pointwise collinear with ξ and this completes the proof. □

Corollary 3.7.1. *If (g, V, λ, μ) is a gradient conformal η-Ricci soliton on an n-dimensional ϵ-Kenmotsu manifold (M, g, ϕ, ξ, η), then the direction of the potential vector field V is same or opposite to the direction of the characteristic vector field ξ, according as ξ is spacelike or timelike vector field.*

Again covariantly differentiating (3.7.12) and then combining it with (3.7.6), and after that taking $X = \xi$ in the derived expression we obtain

$$\nabla^2{}_\xi f = \lambda + \mu - (\frac{p}{2} + \frac{1}{n}) - \epsilon(n-1).$$

Hence we can conclude the following

Corollary 3.7.2. *If $(g, V = Df, \lambda, \mu)$ is a gradient conformal η-Ricci soliton on an n-dimensional ϵ-Kenmotsu manifold (M, g, ϕ, ξ, η), then at the particular point ξ, the potential function f satisfies the Laplace's equation $\nabla^2 f = 0$, if and only if,*

$$\lambda + \mu = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n-1).$$

3.8 Example of a 5-dimensional ϵ-Kenmotsu manifold admitting conformal η-Ricci soliton

In this section an example of a conformal η-Ricci soliton on a 5-dimensional ϵ-Kenmotsu manifold is constructed.

Example 3.8.1. *Let us consider the 5-dimensional manifold* $M = \{(u_1, u_2, v_1, v_2, w) \in \mathbb{R}^5 : w \neq 0\}$. *Define a set of vector fields* $\{e_i : 1 \leq i \leq 5\}$ *on the manifold* M *given by*

$$e_1 = \epsilon w\frac{\partial}{\partial u_1}, \quad e_2 = \epsilon w\frac{\partial}{\partial u_2}, \quad e_3 = \epsilon w\frac{\partial}{\partial v_1}, \quad e_4 = \epsilon w\frac{\partial}{\partial v_2}, \quad e_5 = -\epsilon w\frac{\partial}{\partial w}.$$

Let us define the indefinite metric g *on* M *by*

$$g(e_i, e_j) = \begin{cases} \epsilon, & for \quad i = j \\ 0, & for \quad i \neq j \end{cases}$$

for all $i, j = 1, 2, 3, 4, 5$. *Now considering* $e_5 = \xi$, *let us take the* 1*-form* η, *on the manifold* M, *defined by*

$$\eta(U) = \epsilon g(U, e_5) = \epsilon g(U, \xi), \quad \forall U \in \chi(M).$$

Then it can be observed that $\eta(e_5) = 1$. *Let us define the* $(1, 1)$ *tensor field* ϕ *on* M *as*

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = e_4, \quad \phi(e_4) = -e_3, \quad \phi(e_5) = 0.$$

Then using the linearity of g *and* ϕ *it can be easily checked that*

$$\phi^2(U) = -U + \eta(U)\xi, \quad g(\phi U, \phi V) = g(U, V) - \epsilon\eta(U)\eta(V), \quad \forall U, V \in \chi(M).$$

Hence the structure $(\phi, \xi, \eta, g, \epsilon)$ *defines an indefinite almost contact structure on the manifold* M.

Now, using the definitions of Lie bracket, direct computations give us
$[e_i, e_5] = \epsilon e_i; \quad \forall i = 1, 2, 3, 4, 5$ *and all other* $[e_i, e_j]$ *vanishes. Again the Riemannian connection* ∇ *of the metric* g *is defined by the well-known Koszul's formula which is given by*

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Y g(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]). \end{aligned}$$

Using the above formula one can easily calculate that
$\nabla_{e_i} e_i = -\epsilon e_5$, $\nabla_{e_i} e_5 = -\epsilon e_i$; *for i=1,2,3,4 and all other* $\nabla_{e_i} e_j$ *vanishes. Thus it follows that* $\nabla_X \xi = \epsilon(X - \eta(X)\xi), \quad \forall X \in \chi(M)$. *Therefore the manifold* (M, g) *is a* 5*-dimensional* ϵ*-Kenmotsu manifold.*

Then, the non-vanishing components of the Riemannian curvature tensor R are

$$R(e_1, e_2)e_2 = R(e_1, e_3)e_3 = R(e_1, e_4)e_4 = R(e_1, e_5)e_5 = -e_1,$$
$$R(e_1, e_2)e_1 = e_2, \quad R(e_1, e_3)e_1 = R(e_1, e_3)e_2 = R(e_1, e_3)e_5 = e_3,$$
$$R(e_1, e_2)e_3 = R(e_1, e_2)e_4 = R(e_1, e_2)e_5 = -e_2, \quad R(e_1, e_2)e_4 = -e_3,$$
$$R(e_1, e_2)e_2 = R(e_1, e_2)e_1 = R(e_1, e_2)e_4 = R(e_1, e_2)e_3 = e_5,$$
$$R(e_1, e_2)e_1 = R(e_1, e_2)e_2 = R(e_1, e_2)e_3 = R(e_1, e_2)e_5 = e_4.$$

From the above values of the curvature tensor, we obtain the components of the Ricci tensor as follows

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = S(e_4, e_4) = S(e_5, e_5) = -4. \tag{3.8.1}$$

Therefore using (3.8.1) *in the equation* (3.3.3) *we can calculate* $\lambda = 3\epsilon + (\frac{p}{2} + \frac{1}{5})$ *and* $\mu = 1$. *Hence we can say that for* $\lambda = 3\epsilon + (\frac{p}{2} + \frac{1}{5})$ *and* $\mu = 1$, *the data* (g, ξ, λ, μ) *defines a* 5*-dimensional conformal* η*-Ricci soliton on the manifold* (M, g, ϕ, ξ, η).

3.9 Conformal η-Ricci soliton on almost pseudo symmetric Kählerian spacetime manifold

This section is devoted to the study of a four dimensional almost pseudo symmetric Kählerian spacetime manifold admitting conformal η-Ricci solitons and we try to characterize the nature of the soliton. For this, we will make use of two types of curvature conditions namely the projective curvature tensor and the conharmonic curvature tensor.

First, we consider the projective curvature tensor in a Kählerian spacetime manifold (M, g) of dimension 4 and hence taking $n = 4$ in equation (1.1.1) we have

$$\tilde{P}(X, Y, U, W) = \tilde{R}(X, Y, U, W) + \frac{1}{3}[g(X, U)S(Y, W) - g(Y, U)S(X, W)], \tag{3.9.1}$$

for all vector fields $X, Y, U, W \in \chi(M)$ and $\tilde{P}(X, Y, U, W) = g(P(X, Y)U, W)$, where P is the projective curvature tensor.

Now, let us consider a projectively flat almost pseudo symmetric Kählerian spacetime manifold of dimension 4 and study the nature of the conformal η-Ricci soliton within this framework. In this regard our first result is

Theorem 3.9.1. *A conformal η-Ricci soliton (g, ξ, λ, μ), on a projectively flat almost pseudo symmetric Kählerian spacetime manifold, is shrinking if $p < 8\omega + 2\kappa(\sigma - 3\rho) - \frac{1}{2}$, steady if $p = 8\omega + 2\kappa(\sigma - 3\rho) - \frac{1}{2}$ and expanding if $p > 8\omega + 2\kappa(\sigma - 3\rho) - \frac{1}{2}$; provided ξ is solenoidal.*

Proof. Since by hypothesis the manifold is projectively flat, i.e; $P(X,Y)U = 0$, from (3.9.1) we can write

$$\tilde{R}(X,Y,U,W) = -\frac{1}{3}[g(X,U)S(Y,W) - g(Y,U)S(X,W)], \tag{3.9.2}$$

for all $X, Y, U, W \in \chi(M)$.

Now, covariantly differentiating the relation (3.2.9) along the arbitrary vector field Z and using (1.1.28), (1.1.29) we obtain

$$(\nabla_Z \tilde{R})(JX, JY, U, W) = (\nabla_Z \tilde{R})(X, Y, U, W), \tag{3.9.3}$$

for all $X, Y, U, W \in \chi(M)$.

Making use of equation (3.2.16) in above (3.9.1) and recalling (3.9.2) yields

$$\begin{aligned} & \mathcal{A}(X)\tilde{R}(Z,Y,U,W) + \mathcal{A}(Y)\tilde{R}(X,Z,U,W) \\ = \; & \mathcal{A}(JX)\tilde{R}(Z,JY,U,W) + \mathcal{A}(JY)\tilde{R}(JX,Z,U,W). \end{aligned} \tag{3.9.4}$$

By virtue of (3.9.2) the foregoing equation becomes

$$\begin{aligned} & \mathcal{A}(X)[g(U,Z)S(Y,W) - g(Y,U)S(Z,W)] \\ & + \mathcal{A}(Y)[g(U,X)S(Z,W) - g(Z,U)S(X,W)] \\ = \; & \mathcal{A}(JX)[g(U,Z)S(JY,W) - g(JY,U)S(Z,W)] \\ & + \mathcal{A}(JY)[g(U,JX)S(Y,W) - g(Z,U)S(JX,W)]. \end{aligned} \tag{3.9.5}$$

Taking an orthonormal basis $\{E_i : 1 \leq i \leq 4\}$ and putting $X = \mathcal{P} = E_i$ in (3.9.5), then after contraction we get

$$g(Z,U)S(Y,W) - g(Y,U)S(Z,W) = 0. \tag{3.9.6}$$

On contracting the above equation over $Z = W = E_i$ we arrive at

$$S(Y,U) = rg(Y,U) \quad \forall\, U, Y \in \chi(M). \tag{3.9.7}$$

Since, (g, ξ, λ, μ) is a conformal η-Ricci soliton, from (1.2.5) we can write

$$\begin{aligned} g(\nabla_X \xi) + g(X, \nabla_Y \xi) + 2S(X,Y) + \left[2\lambda - \left(p + \frac{1}{2}\right)\right] g(X,Y) & \\ +2\mu\eta(X)\eta(Y) &= 0. \end{aligned} \tag{3.9.8}$$

Plugging in the value from (3.9.7) in the above equation (3.9.8) we get

$$g(\nabla_X \xi) + g(X, \nabla_Y \xi) + \left[2r + 2\lambda - \left(p + \frac{1}{2}\right)\right] g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{3.9.9}$$

Multiplying both sides of (3.9.9) by ϵ_{ii}, then taking summation over $1 \leq i \leq 4$ for $X = Y = E_i$ and making use of equations (3.2.12), (3.2.13) yields

$$div\xi + 2\left[2r + 2\lambda - \left(p + \frac{1}{2}\right)\right] - \mu = 0. \tag{3.9.10}$$

Again setting $X = Y = \xi$ in (3.9.9) and recalling (3.2.12), (3.2.13) we get

$$\left[2r + 2\lambda - \left(p + \frac{1}{2}\right)\right] - 2\mu = 0. \tag{3.9.11}$$

From equations (3.9.10) and (3.9.11) after solving we finally obtain

$$\lambda = \left(\frac{p}{2} + \frac{1}{4}\right) - r - \frac{div\xi}{3}, \tag{3.9.12}$$

$$\mu = -\frac{div\xi}{3}. \tag{3.9.13}$$

Therefore using the condition ξ is solenoidal, i.e; $div\xi = 0$ in (3.9.12) and then by virtue of (3.2.14) we get

$$\lambda = \left(\frac{p}{2} + \frac{1}{4}\right) - 4\omega - \kappa(\sigma - 3\rho). \tag{3.9.14}$$

Hence from the above the soliton is shrinking, steady and expanding according to $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ respectively. This completes the proof. □

Now, note that from (3.9.13) it is evident that if $div\xi = 0$ then $\mu = 0$ and vice versa. Therefore we have the following

Corollary 3.9.1. *A conformal η-Ricci soliton (g, ξ, λ, μ), on a projectively flat almost pseudo symmetric Kählerian spacetime manifold, reduces to a conformal Ricci soliton (g, ξ, λ) if and only if, the potential vector field ξ is solenoidal.*

Remark 3.9.1. *We know that, for radiation fluid* $\sigma = 3\rho$*, then from* (3.9.14) *it follows that,* $\lambda = \left(\frac{p}{2} + \frac{1}{4}\right) - 4\omega$*. Hence in this case the soliton is shrinking, steady or expanding according to* $p < 8\omega - \frac{1}{2}$*,* $p = 8\omega - \frac{1}{2}$ *or* $p > 8\omega - \frac{1}{2}$ *respectively.*

Next, we consider another important curvature tensor, namely the conharmonic curvature tensor [99] in a Kählerian spacetime manifold (M, g) of dimension 4 and hence taking $n = 4$ in equation (1.1.3) we have

$$\begin{aligned} \tilde{\mathcal{H}}(X,Y,U,W) &= \tilde{R}(X,Y,U,W) + \frac{1}{2}[g(X,U)S(Y,W) - g(Y,U)S(X,W) \\ &\quad + S(X,U)g(Y,W) - S(Y,U)g(X,W)], \end{aligned} \tag{3.9.15}$$

for all vector fields $X, Y, U, W \in \chi(M)$ and $\tilde{H}(X,Y,U,W) = g(H(X,Y)U,W)$, where H denotes the conharmonic curvature tensor.

Now, we focus our study on the nature of the conformal η-Ricci soliton on a conharmonicly flat almost pseudo symmetric Kählerian spacetime manifold. We precisely prove the following

Theorem 3.9.2. *A conformal* η*-Ricci soliton* (g, ξ, λ, μ)*, on a conharmonically flat almost pseudo symmetric Kählerian spacetime manifold, is shrinking if* $p < -4\omega - \kappa(\sigma - 3\rho) - \frac{1}{2}$*, steady if* $p = -4\omega - \kappa(\sigma - 3\rho) - \frac{1}{2}$ *and expanding if* $p > -4\omega - \kappa(\sigma - 3\rho) - \frac{1}{2}$*; provided* ξ *is solenoidal.*

Proof. Since by hypothesis the manifold is conharmonically flat, i.e; $H(X,Y)U = 0$, from (3.9.15) we can write

$$\begin{aligned} \tilde{R}(X,Y,U,W) &= -\frac{1}{2}[g(X,U)S(Y,W) - g(Y,U)S(X,W) \\ &\quad + S(X,U)g(Y,W) - S(Y,U)g(X,W)], \end{aligned} \tag{3.9.16}$$

for all $X, Y, U, W \in \chi(M)$.

Then, making use of (3.9.16) in (3.9.4) we obtain

$$\begin{aligned} & \tilde{A}(X)[g(U,Z)S(Y,W)-g(Y,U)S(Z,W) \\ & +S(U,Z)g(Y,W)-S(Y,U)g(Z,W)] \\ & +\tilde{A}(Y)[g(U,X)S(Z,W)-g(Z,U)S(X,W) \\ & +S(U,X)g(Z,W)-S(Z,U)g(X,W)] \\ = & \tilde{A}(JX)[g(U,Z)S(JY,W)-g(JY,U)S(Z,W) \\ & +S(U,Z)g(JY,W)-S(JY,U)g(Z,W)] \\ & +\tilde{A}(JY)[g(U,JX)S(Z,W)-g(Z,U)S(JX,W) \\ & +S(U,JX)g(Z,W)-S(Z,U)g(JX,W)]. \end{aligned} \tag{3.9.17}$$

On contracting the above equation over $X = \mathcal{P} = E_i$ we get

$$g(Z,U)S(Y,W)-g(Y,U)S(Z,W)+S(Z,U)g(Y,W)-S(Y,U)g(Z,W)=0. \tag{3.9.18}$$

Setting $Z = W = E_i$ in (3.9.18) we arrive at

$$S(Y,U) = -\frac{r}{2}g(Y,U) \quad \forall\, U,Y \in \chi(M). \tag{3.9.19}$$

As (g, ξ, λ, μ) is a conformal η-Ricci soliton, from (1.2.5) we can write

$$\begin{aligned} g(\nabla_X\xi)+g(X,\nabla_Y\xi)+2S(X,Y)+\left[2\lambda-\left(p+\frac{1}{2}\right)\right]g(X,Y) & \\ +2\mu\eta(X)\eta(Y) &= 0. \end{aligned} \tag{3.9.20}$$

Using (3.9.19) in the previous equation (3.9.20) we obtain

$$g(\nabla_X\xi)+g(X,\nabla_Y\xi)+\left[2\lambda-r-\left(p+\frac{1}{2}\right)\right]g(X,Y)+2\mu\eta(X)\eta(Y)=0. \tag{3.9.21}$$

Multiplying both sides of (3.9.21) by ϵ_{ii}, then taking summation over $1 \leq i \leq 4$ for $X = Y = E_i$ and making use of equations (3.2.12), (3.2.13) yields

$$div\xi + 2\left[2\lambda-r-\left(p+\frac{1}{2}\right)\right]-\mu=0. \tag{3.9.22}$$

Again setting $X = Y = \xi$ in (3.9.21) and recalling (3.2.12), (3.2.13) we get

$$\left[2\lambda-r-\left(p+\frac{1}{2}\right)\right]-2\mu=0. \tag{3.9.23}$$

Solving equations (3.9.22) and (3.9.23) we finally obtain

$$\lambda = \frac{r}{2} + \left(\frac{p}{2} + \frac{1}{4}\right) - \frac{div\xi}{3}, \tag{3.9.24}$$

$$\mu = -\frac{div\xi}{3}. \tag{3.9.25}$$

Therefore using the condition ξ is solenoidal, i.e; $div\xi = 0$ in (3.9.24) and then by virtue of (3.2.14) we get

$$\lambda = 2\omega + \frac{\kappa}{2}(\sigma - 3\rho) + \left(\frac{p}{2} + \frac{1}{4}\right). \tag{3.9.26}$$

Hence completes the proof. □

Now, note that from (3.9.25) it follows that $\mu = 0$ if and only if $div\xi = 0$. Therefore we can state the following:

Corollary 3.9.2. *A conformal η-Ricci soliton (g, ξ, λ, μ), on a conharmonically flat almost pseudo symmetric Kählerian spacetime manifold, reduces to a conformal Ricci soliton (g, ξ, λ) if and only if, the potential vector field ξ is solenoidal.*

Remark 3.9.2. *We know that, for radiation fluid $\sigma = 3\rho$, then from* (3.9.26) *it follows that, $\lambda = 2\omega + \left(\frac{p}{2} + \frac{1}{4}\right)$. Hence in this case the soliton is shrinking, steady or expanding according to $p < -4\omega - \frac{1}{2}$, $p = -4\omega - \frac{1}{2}$ or $p > -4\omega - \frac{1}{2}$ respectively.*

3.10 Gradient conformal η-Ricci soliton on almost pseudo symmetric Kählerian spacetime manifold

This section is devoted to the study of almost pseudo symmetric Kählerian spacetime manifolds admitting gradient conformal η-Ricci soliton.

On a Kählerian spacetime manifold (M, g) a conformal η-Ricci soliton (g, V, λ, μ) is said to be a gradient conformal η-Ricci soliton if the potential vector field V is the gradient of some smooth function f on the manifold M. Hence in this case $V = Df$, where Df denotes the gradient of f and then equation (1.2.5) becomes

$$Hessf + S + \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] g + \mu\eta \otimes \eta = 0, \tag{3.10.1}$$

where $Hessf$ denotes the Hessian operator of the function f. Moreover, the gradient conformal η-Ricci soliton is called proper if $\mu \neq 0$.

Now we prove our main theorem of this section on gradient conformal η-Ricci soliton on almost pseudo symmetric Kählerian spacetime manifolds.

Theorem 3.10.1. *Let (g, Df, λ, μ) be a proper gradient conformal η-Ricci soliton on a projectively flat almost pseudo symmetric Kählerian spacetime manifold of non-zero scalar curvature. Then the integral curves generated by the velocity vector field ξ are geodesics if and only if the potential function f is constant.*

Proof. Given that (g, Df, λ, μ) is a proper gradient conformal η-Ricci soliton, the equation (3.10.1) can be rewritten as

$$\nabla_X Df + QX + \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] X + \mu\eta(X)\xi = 0, \tag{3.10.2}$$

for all $X \in \chi(M)$ and Q is the Ricci operator given by $g(QX, Y) = S(X, Y)$.
Taking covariant differentiation of (3.10.2) along Y we get

$$\begin{aligned} \nabla_Y \nabla_X Df &= -(\nabla_Y Q)(X) - Q(\nabla_Y X) - \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] \nabla_Y X \\ &\quad -\mu[(\nabla_Y \eta)(X)\xi + \eta(\nabla_Y X)\xi + \eta(X)\nabla_Y \xi]. \end{aligned} \tag{3.10.3}$$

Interchanging X and Y in (3.10.3) we obtain

$$\begin{aligned} \nabla_X \nabla_Y Df &= -(\nabla_X Q)(Y) - Q(\nabla_X Y) - \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] \nabla_X Y \\ &\quad -\mu[(\nabla_X \eta)(Y)\xi + \eta(\nabla_X Y)\xi + \eta(Y)\nabla_X \xi]. \end{aligned} \tag{3.10.4}$$

Again from (3.10.2) we can write

$$\begin{aligned} \nabla_{[X,Y]} Df &= -Q(\nabla_X Y) + Q(\nabla_Y X) - \left[\lambda - \left(\frac{p}{2} + \frac{1}{4}\right)\right] (\nabla_X Y - \nabla_Y X) \\ &\quad -\mu[\eta(\nabla_X Y)\xi - \eta(\nabla_Y X)\xi]. \end{aligned} \tag{3.10.5}$$

Now using (3.10.3), (3.10.4), and (3.10.5) in (1.1.5) we arrive at

$$\begin{aligned} R(X, Y)Df &= (\nabla_Y Q)(X) - (\nabla_X Q)(Y) - \mu[(\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi] \\ &\quad +\mu[(\nabla_Y \eta)(X)\xi + \eta(X)\nabla_Y \xi]. \end{aligned} \tag{3.10.6}$$

Contracting the foregoing equation (3.10.6) along X and in view of (3.2.12), (3.2.13) we obtain

$$S(Y, Df) = -\mu[(\nabla_\xi \eta)(Y) + \eta(Y) div\xi]. \tag{3.10.7}$$

Since by our hypothesis, the manifold is projectively flat, from (3.9.7) we can write

$$S(Y, Df) = rg(Y, Df). \tag{3.10.8}$$

Combining the equations (3.10.7) and (3.10.8) we get

$$-\mu[(\nabla_\xi \eta)(Y) + \eta(Y) div\xi] = rg(Y, Df). \tag{3.10.9}$$

Finally putting $Y = \xi$ in (3.10.9) and using (3.2.12) we arrive at

$$\nabla_\xi \xi = \frac{r}{\mu} Df. \tag{3.10.10}$$

Hence $\nabla_\xi \xi = 0$ if and only if $Df = 0$ i.e; if and only if f is constant. This completes the proof. □

4

On some Ricci solitons and Yamabe solitons

4.1 Introduction

In this chapter we consider Ricci soliton and some types of Yamabe soliton on some differentiable manifolds. This chapter is divided into six sections. In sections one and two we give introduction and preliminaries respectively.

In section three, we study some curvature properties of 3-dimensional quasi-Sasakian manifolds with respect to Zamkovoy connection. Then section four characterizes the Ricci soliton on 3-dimensional quasi-Sasakian manifolds with respect to Zamkovoy connection.

Section five deals with the study η-Ricci-Yamabe soliton on almost pseudo symmetric Kählerian spacetime manifolds. Finally in section six, we characterize quasi-Yamabe soliton within the framework of generalized Sasakian spaceform.

4.2 Preliminaries

In this section we discuss some basic results From now on we assume that the manifold M is a 3-dimensional quasi-Sasakian manifold. According to Z. Olszak [66], an almost contact metric manifold M of dimension 3 is quasi-Sasakian if and only if

$$\nabla_X \xi = -\beta\phi(X), \tag{4.2.1}$$

for all $X \in \chi(M)$ and for some smooth function β on the manifold M, ∇ being the Levi-Civita connection of M. This function β is called the structure function of M [67].

For a 3-dimensional quasi-Sasakian manifold it is known that the structure function β satisfies the relation $\xi\beta = 0$. In particular, a quasi-Sasakian structure with $\beta = 0$ is a cosymplectic manifold of rank 1.

Olszak [67] established the necessary and sufficient condition for a three-dimensional quasi-Sasakian manifold to be conformally flat with the help of its structure function β. In [36], the authors studied 3-dimensional quasi-Sasakian manifolds with semi-symmetric non-metric connection.

In a 3-dimensional quasi-Sasakian manifold the following relations hold [67]

$$(\nabla_X\phi)(Y) = \beta(g(X,Y)\xi - \eta(Y)X), \tag{4.2.2}$$

$$(\nabla_X\eta)(Y) = -\beta g(\phi(X),Y), \tag{4.2.3}$$

$$S(X,Y) = (\frac{r}{2} - \beta^2)g(X,Y) + (3\beta^2 - \frac{r}{2})\eta(X)\eta(Y) - \eta(X)d\beta(\phi Y) - \eta(Y)d\beta(\phi X), \tag{4.2.4}$$

$$R(X,Y)\xi = \beta^2(\eta(Y)X - \eta(X)Y) - (X\beta)\phi Y + (Y\beta)\phi X, \tag{4.2.5}$$

$$S(X,\xi) = 2\beta^2\eta(X) - d\beta(\phi X), \tag{4.2.6}$$

for all $X, Y \in \chi(M)$, S denotes the Ricci tensor, R is the curvature tensor of M and $df(X) = g(gradf, X)$ relates the gradient of a function f to the exterior derivative df.

4.3 Geometry of quasi-Sasakian 3-manifold with respect to the Zamkovoy connection

This section is devoted to the study of some geometrical properties of a 3-dimensional quasi-Sasakian manifold M admitting Zamkovoy connection. First we try to establish the relations between the Riemannian curvature tensors and Ricci tensors of M with respect to the Levi-Civita connection and the Zamkovoy connection.

Using equations (4.2.1) and (4.2.3) in (1.1.20), we obtain the expression for the Zamkovoy connection in a 3-dimensional Quasi-Sasakian manifold M as follows

$$\nabla^*_X Y = \nabla_X Y - \beta(g(\phi(X),Y)\xi - \eta(Y)\phi X) + \eta(X)\phi Y, \tag{4.3.1}$$

for all $X, Y \in \chi(M)$, where ∇ is the Levi-Civita connection on M.

Furthermore, it can be easily seen that in this case the torsion tensor with respect to the Zamkovoy connection is

$$T^*(X,Y) = (1-\beta)(\eta(X)\phi Y - \eta(Y)\phi X) + 2\beta g(X,\phi Y), \tag{4.3.2}$$

for all $X, Y \in \chi(M)$.

Again, in a 3-dimensional quasi-Sasakian manifold M we can easily calculate that, with respect to the Zamkovoy connection, $\nabla^* g = 0$, i.e., the Zamkovoy connection is a metric compatible connection on the manifold M.

Now, in view of (4.3.1), it can be easily obtained that,

$$\begin{aligned}
\nabla^*_X \nabla^*_Y Z &= \nabla_X \nabla_Y Z + \eta(\nabla_X Y)\phi Z + \eta(Y)\phi(\nabla_X Z) - g(\phi Y, Z)(X\beta)\xi \\
&\quad + \eta(Z)(X\beta)\phi Y + \eta(X)\phi(\nabla_Y Z) - \eta(X)\eta(Y)(Z - \eta(Z)\xi) \\
&\quad + \beta[\eta(\nabla_X Z)\phi Y + \eta(Z)\phi(\nabla_X Y) + \eta(\nabla_Y Z)\phi X - \eta(Z)\eta(X)Y \\
&\quad + g(\nabla_X Y, \phi Z)\xi - \eta(Z)\eta(Y)X - g(\phi X, \nabla_Y Z)\xi \\
&\quad + 2\eta(X)\eta(Y)\eta(Z)\xi - g(\phi X, Y)\phi Z] \\
&\quad + \beta^2[g(X,Z)\eta(Y)\xi - g(\phi X, Z)\phi Y - \eta(Y)\eta(Z)X \\
&\quad + \eta(X)\eta(Y)\eta(Z)\xi - g(X,Y)\eta(Z)\xi],
\end{aligned} \tag{4.3.3}$$

for all $X, Y \in \chi(M)$, where ∇ is the Levi-Civita connection on M. Again, interchanging X and Y in (4.3.3) we get the value of $\nabla^*_Y \nabla^*_X Z$.

Recalling the following curvature formula

$$R^*(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z$$

and then using the values of (4.3.3) and $\nabla^*_Y \nabla^*_X Z$, after simplification we obtain

$$\begin{aligned}
R^*(X,Y)Z &= R(X,Y)Z + [(X\beta)\phi Y - (Y\beta)\phi X]\eta(Z) + 2\beta g(X,\phi Y)\phi Z \\
&\quad + [g(\phi X, Z)(Y\beta) - g(\phi Y, Z)(X\beta)]\xi + \beta^2[g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y] \\
&\quad + \beta^2[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi + \beta^2[\eta(X)Y - \eta(Y)X]\eta(Z),
\end{aligned} \tag{4.3.4}$$

for all $X, Y \in \chi(M)$, where $R(X,Y)Z$ is the Riemannian curvature tensor with respect to the Levi-Civita connection on M.

Contracting (4.3.4) yields the Ricci tensor S^* of a quasi-Sasakian 3-manifold with respect to Zamkovoy connection

$$S^*(X,Y) = S(X,Y) + \eta(Y)d\beta(\phi X) + 2\beta g(X,Y) - 2\beta(1+\beta)\eta(X)\eta(Y), \tag{4.3.5}$$

for all $X, Y \in \chi(M)$.

Again contraction of (4.3.5) gives the scalar curvature r^* of a quasi-Sasakian 3-manifold with respect to Zamkovoy connection

$$r^* = r + 2\beta(2-\beta). \tag{4.3.6}$$

Now, taking inner product of (4.3.4) with an arbitrary vector field W, we obtain the following two relations

$$g(R^*(X,Y)Z,W) + g(R^*(Y,X)Z,W) = 0 \tag{4.3.7}$$

and

$$g(R^*(X,Y)Z,W) + g(R^*(Y,X)W,Z) = 0, \tag{4.3.8}$$

for all $X, Y, Z, W \in \chi(M)$.

Therefore, summarizing all the discussions above, we have the following

Theorem 4.3.1. *Let (M, g) be a quasi-Sasakian 3-manifold with respect to the Zamkovoy connection, then*

i) The Riemannian curvature tensor R^*, the Ricci tensor S^* and the scalar curvature r^* are given by (4.3.4), (4.3.5) and (4.3.6) respectively,

ii) The Ricci tensor S^* is not symmetric,

iii) The Riemannian curvature tensor R^* satisfies

(a) $g(R^*(X,Y)Z,W) = -g(R^*(Y,X)Z,W)$ and

(b) $g(R^*(X,Y)Z,W) = -g(R^*(X,Y)W,Z)$,

for all $X, Y, Z, W \in \chi(M)$.

Now, we focus on two important curvature tensors in a quasi-Sasakian 3-manifold with respect to the Zamkovoy connection.

First, let us consider a quasi-Sasakian 3-manifold with concircular curvature tensor. Then equation (1.1.2) holds and taking $n = 3$ in it, we can write the expression for the concircular curvature tensor C^* in a quasi-Sasakian 3-manifold with respect to Zamkovoy connection as follows:

$$C^*(X,Y)Z = R^*(X,Y)Z - \frac{r^*}{6}(g(Y,Z)X - g(X,Z)Y), \tag{4.3.9}$$

for all $X, Y, Z \in \chi(M)$.

The manifold is called ξ-concircularly flat if $C^*(X,Y)\xi = 0$ for all $X, Y \in \chi(M)$.

Using (4.3.4) and (4.3.6) in (4.3.9) we get

$$\begin{aligned} C^*(X,Y)Z \;&=\; R(X,Y)Z + [(X\beta)\phi Y - (Y\beta)\phi X]\eta(Z) + 2\beta g(X,\phi Y)\phi Z \\ &\quad + [g(\phi X,Z)(Y\beta) - g(\phi Y,Z)(X\beta)]\xi + \beta^2[g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y] \\ &\quad + \beta^2[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi + \beta^2[\eta(X)Y - \eta(Y)X)]\eta(Z), \\ &\quad - \frac{r + 2\beta(2-\beta)}{6}[g(Y,Z)X - g(X,Z)Y], \end{aligned}$$

for all $X, Y, Z \in \chi(M)$.

Replacing $Z = \xi$ in the previous equation yields

$$\begin{aligned} C^*(X,Y)\xi \;&=\; R(X,Y)\xi + [(X\beta)\phi Y - (Y\beta)\phi X] + \beta^2[\eta(X)Y - \eta(Y)X)]\eta(Z) \\ &\quad - \frac{r + 2\beta(2-\beta)}{6}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{4.3.10}$$

In view of (4.2.5), the above (4.3.10) becomes

$$C^*(X,Y)\xi = \frac{r + 2\beta(2-\beta)}{6}[\eta(X)Y) - \eta(Y)X], \tag{4.3.11}$$

for all $X, Y \in \chi(M)$. Hence, $C^*(X,Y)\xi = 0$ if and only if $r = 2\beta(2-\beta)$. Therefore, we can conclude the following

Theorem 4.3.2. *A quasi-Sasakian 3-manifold with respect to the Zamkovoy connection is ξ-concircularly flat, if and only if, the scalar curvature with respect to the Levi-Civita connection is given by $r = 2\beta(2-\beta)$.*

Corollary 4.3.1. *Let (M, g) be a ξ-concircularly flat quasi-Sasakian 3-manifold with respect to the Zamkovoy connection. Then the manifold M is Ricci flat with respect to the Levi-Civita connection if and only if, either it is a cosymplectic manifold or the structure function $\beta = 2$.*

Next, we study another curvature tensor, namely the conharmonic curvature tensor in a quasi-Sasakian 3-manifold. Then equation (1.1.3) holds and using it we can write the expression for the conharmonic curvature tensor H^* in a quasi-Sasakian 3-manifold w.r.to Zamkovoy connection as follows:

$$\begin{aligned} H^*(X,Y)Z &= R^*(X,Y)Z - (g(Y,Z)Q^*X - g(X,Z)Q^*Y \\ &\quad + S^*(Y,Z)X - S^*(X,Z)Y), \end{aligned} \tag{4.3.12}$$

for all $X, Y, Z \in \chi(M)$. The manifold (M, g) is called ξ-conharmonically flat if $H^*(X,Y)\xi = 0$ for all $X, Y \in \chi(M)$.

Now, from (4.3.4), taking $Z = \xi$ we obtain

$$R^*(X,Y)\xi = R(X,Y)\xi + [(X\beta)\phi Y - (Y\beta)\phi X] + \beta^2[\eta(X)Y - \eta(Y)X]. \tag{4.3.13}$$

Again (4.3.5) can be rewritten as

$$Q^*X = QX + d\beta(\phi X)\xi + 2\beta X - 2\beta(1+\beta)\eta(X)\xi. \tag{4.3.14}$$

So, if we consider a ξ-conharmonically flat quasi-Sasakian 3-manifold with respect to the Zamkovoy connection, we have $H^*(X,Y)\xi = 0$ and then in view of (4.3.12), (4.3.13) and (4.3.14) we have

$$\begin{aligned} R(X,Y)\xi &= [(Y\beta)\phi X - (X\beta)\phi Y] + \beta^2[\eta(Y)X - \eta(X)Y] \\ &\quad + \eta(Y)[QX + d\beta(\phi X)\xi + 2\beta X - 2\beta(1+\beta)\eta(X)\xi] \\ &\quad - \eta(X)[QY + d\beta(\phi Y)\xi + 2\beta Y - 2\beta(1+\beta)\eta(Y)\xi] \\ &\quad + S(Y,\xi)X - 2\beta^2\eta(Y)X - S(X,\xi)Y + 2\beta^2\eta(X)Y. \end{aligned}$$

Recalling, (4.2.5) and (4.2.6), then using them in the foregoing equation gives

$$\begin{aligned} &[\eta(Y)QX - \eta(X)QY] + [Y + \eta(Y)\xi]d\beta(\phi X) \\ &= [X + \eta(X)\xi]d\beta(\phi Y) - 2\beta[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{4.3.15}$$

Taking inner product of (4.3.15) with an arbitrary vector field Z we obtain

$$\begin{aligned} & [\eta(Y)S(X,Z)-\eta(X)S(Y,Z)]+[g(Y,Z)+\eta(Y)\eta(Z)]d\beta(\phi X) \\ = \ & [g(X,Z)+\eta(X)\eta(Z)]d\beta(\phi Y)-2\beta[\eta(Y)g(X,Z)-\eta(X)g(Y,Z)]. \end{aligned}$$

Putting $Y = Z = \xi$ in the previous equation yields

$$4\beta^2\eta(X) = 0,$$

for all $X \in \chi(M)$. Since, $\eta(X) \neq 0$ always, we can conclude that $\beta = 0$. Hence we can state the following:

Theorem 4.3.3. *Let (M, g) be a ξ-conharmonically flat quasi-Sasakian 3-manifold with respect to the Zamkovoy connection. Then the manifold M becomes a cosymplectic manifold.*

4.4 Ricci soliton on quasi-Sasakian 3-manifold with respect to the Zamkovoy connection

In this section we focus on characterizing a Ricci soliton (g, V, λ) on a quasi-Sasakian 3-manifold with respect to the Zamkovoy connection. We say that the manifold (M, g) admits a Ricci soliton (g, V, λ) with respect to Zamkovoy connection if

$$\mathcal{L}^*{}_V g + 2S^* = 2\lambda g, \tag{4.4.1}$$

where $\mathcal{L}^*{}_V$ denotes the Lie derivative with respect to Zamkovoy connection along the direction of the vector field V and is given by $(\mathcal{L}^*{}_V g)(X,Y) = g(\nabla^*_X V, Y) + g(X, \nabla^*_Y V)$, for all $X, Y \in \chi(M)$.

Now, let us consider a quasi-Sasakian 3-manifold (M, g) that admits a Ricci soliton (g, V, λ) with respect to the Zamkovoy connection. Then from (4.4.1) we can infer that

$$g(\nabla^*_X V, Y) + g(X, \nabla^*_Y V) + 2S^*(X,Y) = 2\lambda g(X,Y), \tag{4.4.2}$$

for all $X, Y \in \chi(M)$.

Recalling (4.3.1) and (4.3.5) and using them in the previous equation gives

$$\begin{aligned} 2S(X,Y)+g(\nabla_X V,Y)+g(X,\nabla_Y V) &= 2(\lambda-2\beta)g(X,Y)-2d\beta(\phi X)\eta(Y) \\ &\quad +\beta(g(\phi X,V)\eta(Y)+g(\phi Y,V)\eta(X)) \\ &\quad -(g(\phi V,X)\eta(Y)+g(\phi V,Y)\eta(X)) \\ &\quad +4\beta(1+\beta)\eta(X)\eta(Y). \end{aligned} \tag{4.4.3}$$

Moreover, If (g,V,λ) is also a Ricci soliton with respect to the Levi-Civita connection, then (1.2.1) holds, and in view of that equation (4.4.3) reduces to

$$\begin{aligned} \beta(g(\phi X,V)\eta(Y)+g(\phi Y,V)\eta(X)) &= 4\beta(g(X,Y)-(1+\beta)\eta(X)\eta(Y))+2d\beta(\phi X)\eta(Y) \\ &\quad +(g(\phi V,X)\eta(Y)+g(\phi V,Y)\eta(X)). \end{aligned} \tag{4.4.4}$$

Taking $X=Y=\xi$ in (4.4.4), we obtain $\beta=0$ and this implies that the manifold (M,g) is a cosymplectic manifold. Hence we can state the following

Theorem 4.4.1. *Let (M,g) be a quasi-Sasakian 3-manifold which admits a Ricci soliton (g,V,λ) with respect to the Zamkovoy connection. If (g,V,λ) is also a Ricci soliton with respect to the Levi-Civita connection, then the manifold M becomes a cosymplectic manifold.*

Again, if (g,V,λ) is a Ricci soliton with respect to the Zamkovoy connection on a quasi-Sasakian 3-manifold (M,g), then from (4.4.3) we can write

$$\begin{aligned} 2S(X,Y)+g(\nabla_X V,Y)+g(X,\nabla_Y V) &= 2(\lambda-2\beta)g(X,Y)-2d\beta(\phi X)\eta(Y) \\ &\quad +\beta(g(\phi X,V)\eta(Y)+g(\phi Y,V)\eta(X)) \\ &\quad -(g(\phi V,X)\eta(Y)+g(\phi V,Y)\eta(X)) \\ &\quad +4\beta(1+\beta)\eta(X)\eta(Y). \end{aligned}$$

Considering an orthonormal basis $\{e_1,e_2,e_3\}$ of (M,g) and replacing $X=Y=e_i$ in the foregoing equation, then summing over $i=1,2,3$ we get

$$r = 3\lambda+2\beta^2-4\beta-divV, \tag{4.4.5}$$

where r is the scalar curvature of M and $divV$ denotes the divergence of the vector field V. Therefore, we can write

Theorem 4.4.2. *Let (M,g) be a quasi-Sasakian 3-manifold admitting a Ricci soliton (g, V, λ) with respect to the Zamkovoy connection, then the scalar curvature is given by $r = 3\lambda + 2\beta^2 - 4\beta - divV$.*

Corollary 4.4.1. *Let (M, g) be a quasi-Sasakian 3-manifold admitting a Ricci soliton (g, V, λ) with respect to the Zamkovoy connection. If V is a solenoidal vector field, then the soliton is*

i) shrinking if $r > 2\beta(\beta - 2)$,

ii) steady if $r = 2\beta(\beta - 2)$ and

iii) expanding if $r < 2\beta(\beta - 2)$.

Next, we consider $V = gradf$, for some $f \in C^\infty(M)$ and we focus our study on the Laplacian equation satisfied by the function f. Laplace equation is a second order partial differential equation which is frequently used in physics. Solution of Laplace equation is widely known as harmonic functions and they appear in various physical problems in magnetic and gravitational potentials of steady state temperature. For example, real and imaginary parts u and v of a complex analytic function $f = u+iv$ both satisfy the Laplace equation.

Let us consider a a Ricci soliton (g, V, λ) with respect to the Zamkovoy connection on a quasi-Sasakian 3-manifold (M, g) and assume that the potential vector field V is gradient of a smooth function f on M. Then from (4.4.1) we can write

$$g(\nabla^*_X Df, Y) + g(X, \nabla^*_Y Df) + 2S^*(X, Y) = 2\lambda g(X, Y), \tag{4.4.6}$$

for all $X, Y \in \chi(M)$ and $Df = gradf = V$. Then using (4.3.1) and (4.3.5) in (4.4.6) yields

$$\begin{aligned} 2S(X, Y) + g(\nabla_X Df, Y) + g(X, \nabla_Y Df) &= 2(\lambda - 2\beta)g(X, Y) - 2d\beta(\phi X)\eta(Y) \\ &\quad +\beta(g(\phi X, Df)\eta(Y) + g(\phi Y, Df)\eta(X)) \\ &\quad -(g(\phi(Df), X)\eta(Y) + g(\phi(V), Y)\eta(X)) \\ &\quad +4\beta(1 + \beta)\eta(X)\eta(Y). \end{aligned}$$

Putting $X = Y = e_i$ in the foregoing equation, where $\{e_1, e_2, e_3\}$ constitute a local orthonormal basis of (M, g) and then summing over $i = 1, 2, 3$ we get

$$divV + 4\beta + r = 3\lambda + 2\beta^2, \tag{4.4.7}$$

where r is the scalar curvature of M and $divV$ denotes the divergence of the vector field V. Since, our potential vector field V is the gradient of a smooth function f on M, the above equation (4.4.7) becomes

$$\Delta(f) = 3\lambda - r - 4\beta + 2\beta^2,$$

where $\Delta(f) = div(gradf)$ denotes the Laplacian operator of f. Hence we have the following

Theorem 4.4.3. *Let (M, g) be a quasi-Sasakian 3-manifold admitting a Ricci soliton (g, V, λ) with respect to the Zamkovoy connection. If the potential vector field V is gradient of some smooth function f on M, then the Laplacian equation satisfied by f becomes*

$$\Delta(f) = 3\lambda - r - 4\beta + 2\beta^2. \tag{4.4.8}$$

Remark 4.4.1. *Laplace equation has applications in the theory of gravity also. If the gravitational acceleration field is represented as the gradient of a scalar potential function h, then the Poisson's equation for gravitational field is given by*

$$\nabla^2 h = -4\pi G\rho,$$

where G denotes the universal gravitational constant and ρ denotes the mass density. This physical significance is equivalent to the above Theorem 4.4.3 and the equation (4.4.8) of this section, which is a Laplace equation with potential vector field of gradient type.

Finally, we conclude this section with our last result on Ricci soliton and for this we consider the case when the potential vector field is same as the characteristic vector field, i.e., $V = \xi$.

So, let us assume that (g, ξ, λ), be a Ricci soliton on a quasi-Sasakian 3-manifold (M, g) with respect to the Zamkovoy connection. Then by (4.4.1) we have

$$g(\nabla^*_X \xi, Y) + g(X, \nabla^*_Y \xi) + 2S^*(X, Y) = 2\lambda g(X, Y), \tag{4.4.9}$$

for all $X, Y \in \chi(M)$. Using (4.3.1) and (4.3.5) in (4.4.9) and then recalling (4.2.1) we arrive at

$$S(X,Y) = (\lambda - 2\beta)g(X,Y) + 2\beta\eta(X)\eta(Y) - d\beta(\phi X)\eta(Y). \tag{4.4.10}$$

Taking $X = Y = \xi$ in the previous equation (4.4.10) gives

$$S(\xi,\xi) = \lambda + 2\beta^2. \tag{4.4.11}$$

Again, replacing $X = \xi$ in equation (4.2.6) we get

$$S(\xi,\xi) = 2\beta^2. \tag{4.4.12}$$

Thus, combining (4.4.11) and (4.4.12), we can conclude that $\lambda = 0$ and this implies that the Ricci soliton is steady. Hence, we can state

Theorem 4.4.4. *Let (M, g) be a quasi-Sasakian 3-manifold admitting a Ricci soliton (g, ξ, λ) with respect to the Zamkovoy connection, then the soliton is a steady Ricci soliton.*

4.5 Almost pseudo symmetric Kählerian spacetime manifold admitting η-Ricci-Yamabe soliton

In this section we investigate the nature of η-Ricci-Yamabe soliton on four dimensional almost pseudo symmetric Kählerian spacetime manifolds which are projectively flat and conharmonicly flat respectively. First we prove the following

Theorem 4.5.1. *If $(g, \xi, \lambda, \mu, \alpha, \beta)$ is an η-Ricci-Yamabe soliton on a projectively flat almost pseudo symmetric Kählerian spacetime manifold, then the scalars λ and μ are related by $\lambda = 2\mu + \left(\frac{\beta}{2} - \alpha\right)[4\omega + \kappa(\sigma - 3\rho)]$. Moreover, the η-Ricci-Yamabe soliton reduces to the Ricci-Yamabe soliton $(g, \xi, \lambda, \alpha, \beta)$ if and only if the potential vector vector field ξ is solenoidal.*

Proof. Since the data (g, ξ, λ, μ) is an η-Ricci-Yamabe soliton, from (1.2.12) we can write

$$g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) + 2\alpha S(X,Y) + (2\lambda - \beta r)g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{4.5.1}$$

By hypothesis, as the manifold is projectively flat, we proceed similarly as Theorem-3.9.1 and then using the equation (3.9.7) in the above equation (4.5.1) we get

$$g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) + (2r\alpha + 2\lambda - \beta r)g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{4.5.2}$$

Multiplying both sides of (4.5.2) by ϵ_{ii}, then considering an orthonormal basis $\{E_i : 1 \leq i \leq 4\}$ and taking summation over $1 \leq i \leq 4$ for $X = Y = E_i$, and making use of equations (3.2.12), (3.2.13) yields

$$div\xi + (4r\alpha + 4\lambda - 2\beta r) - \mu = 0. \tag{4.5.3}$$

Again setting $X = Y = \xi$ in (4.5.2) and recalling (3.2.12), (3.2.13) we get

$$(2r\alpha + 2\lambda - \beta r) - 2\mu = 0. \tag{4.5.4}$$

From equations (4.5.3) and (4.5.4) after solving we finally obtain

$$\lambda = \left(\frac{\beta}{2} - \alpha\right) r - \frac{2div\xi}{3}, \tag{4.5.5}$$

$$\mu = -\frac{div\xi}{3}. \tag{4.5.6}$$

Therefore using (3.2.14) and (4.5.6) in the equation (4.5.5) we get

$$\lambda = 2\mu + \left(\frac{\beta}{2} - \alpha\right)[4\omega + \kappa(\sigma - 3\rho)]. \tag{4.5.7}$$

This completes the proof of the first part.

Again from (4.5.6) we can see that $\mu = 0$ if and only if $div\xi = 0$ i.e; if and only if ξ is solenoidal. This completes the proof. □

Now taking $\mu = 0$ and $(\alpha, \beta) = (1, 0)$ in (4.5.7) we obtain $\lambda = -4\omega - \kappa(\sigma - 3\rho)$. Hence we can state the following

Corollary 4.5.1. *A Ricci soliton (g, ξ, λ) on a projectively flat almost pseudo symmetric Kählerian spacetime manifold is shrinking if $\omega > -\frac{\kappa}{4}(\sigma - 3\rho)$, steady if $\omega = -\frac{\kappa}{4}(\sigma - 3\rho)$ and expanding if $\omega < -\frac{\kappa}{4}(\sigma - 3\rho)$.*

Again for $\mu = 0$ and $(\alpha, \beta) = (0, 1)$ in (4.5.7) we arrive at $\lambda = 2\omega - \frac{\kappa}{2}(\sigma - 3\rho)$. Therefore we have

Corollary 4.5.2. *A Yamabe soliton (g, ξ, λ) on a projectively flat almost pseudo symmetric Kählerian spacetime manifold is shrinking if $\omega < -\frac{\kappa}{4}(\sigma - 3\rho)$, steady if $\omega = -\frac{\kappa}{4}(\sigma - 3\rho)$ and expanding if $\omega > -\frac{\kappa}{4}(\sigma - 3\rho)$.*

Also setting $\mu = 0$ and $(\alpha, \beta) = (1, -1)$ in (4.5.7) we get $\lambda = -6\omega - \frac{3\kappa}{2}(\sigma - 3\rho)$. Hence we can state the following

Corollary 4.5.3. *An Einstein soliton (g, ξ, λ) on a projectively flat almost pseudo symmetric Kählerian spacetime manifold is shrinking if $\omega > -\frac{\kappa}{4}(\sigma - 3\rho)$, steady if $\omega = -\frac{\kappa}{4}(\sigma - 3\rho)$ and expanding if $\omega < -\frac{\kappa}{4}(\sigma - 3\rho)$.*

Next we focus on conharmonically flat almost pseudo symmetric Kählerian spacetime manifold and we study the nature of the η-Ricci-Yamabe soliton within this framework. We precisely prove the following

Theorem 4.5.2. *If $(g, \xi, \lambda, \mu, \alpha, \beta)$ is an η-Ricci-Yamabe soliton on a conharmonically flat almost pseudo symmetric Kählerian spacetime manifold, then the scalars λ and μ are related by $\lambda = \mu + \left(\frac{\alpha+\beta}{2}\right)[4\omega + \kappa(\sigma - 3\rho)]$. Moreover, the η-Ricci-Yamabe soliton reduces to the Ricci-Yamabe soliton $(g, \xi, \lambda, \alpha, \beta)$ if and only if, the potential vector vector field ξ is solenoidal.*

Proof. Since the data (g, ξ, λ, μ) is an η-Ricci-Yamabe soliton, from (1.2.12) we can write

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2\alpha S(X, Y) + (2\lambda - \beta r)g(X, Y) + 2\mu\eta(X)\eta(Y) = 0. \quad (4.5.8)$$

By hypothesis, as the manifold is conharmonically flat, we proceed similarly as Theorem-3.9.2 and then using the equation (3.9.19) in the previous equation (4.5.8) we get

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + (2\lambda - \alpha r - \beta r)g(X, Y) + 2\mu\eta(X)\eta(Y) = 0. \quad (4.5.9)$$

Multiplying both sides of (4.5.9) by ϵ_{ii}, then taking summation over $1 \leq i \leq 4$ for $X = Y = E_i$ and making use of equations (3.2.12), (3.2.13) yields

$$div\xi + 2(2\lambda - \alpha r - \beta r) - \mu = 0. \quad (4.5.10)$$

Again setting $X = Y = \xi$ in (4.5.9) and recalling (3.2.12), (3.2.13) we get

$$(2\lambda - \alpha r - \beta r) - 2\mu = 0. \quad (4.5.11)$$

Solving equations (4.5.10) and (4.5.11) we obtain

$$\lambda = \left(\frac{\alpha + \beta}{2}\right) r - \frac{div\xi}{3}, \quad (4.5.12)$$

$$\mu = -\frac{div\xi}{3}. \quad (4.5.13)$$

Therefore using (3.2.14) and (4.5.13) in the equation (4.5.12) we get

$$\lambda = \mu + \left(\frac{\alpha + \beta}{2}\right)[4\omega + \kappa(\sigma - 3\rho)]. \tag{4.5.14}$$

This completes the proof of the first part.

Again from (4.5.13) we can see that $\mu = 0$ if and only if $div\xi = 0$ i.e; if and only if ξ is solenoidal. This completes the proof. □

Now taking $\mu = 0$ and for different values of $(\alpha, \beta) = (1, 0)$ in (4.5.14) we obtain $\lambda = 2\omega - \frac{\kappa}{2}(\sigma - 3\rho)$. Hence we can state the following

Corollary 4.5.4. *A Ricci soliton (g, ξ, λ) on a conharmonicly flat almost pseudo symmetric Kählerian spacetime manifold is shrinking if $\omega < -\frac{\kappa}{4}(\sigma - 3\rho)$, steady if $\omega = -\frac{\kappa}{4}(\sigma - 3\rho)$ and expanding if $\omega > -\frac{\kappa}{4}(\sigma - 3\rho)$.*

Again for $\mu = 0$ and $(\alpha, \beta) = (0, 1)$ in (4.5.14) we arrive at $\lambda = 2\omega - \frac{\kappa}{2}(\sigma - 3\rho)$. Therefore we have

Corollary 4.5.5. *A Yamabe soliton (g, ξ, λ) on a conharmonicly flat almost pseudo symmetric Kählerian spacetime manifold is shrinking if $\omega < -\frac{\kappa}{4}(\sigma - 3\rho)$, steady if $\omega = -\frac{\kappa}{4}(\sigma - 3\rho)$ and expanding if $\omega > -\frac{\kappa}{4}(\sigma - 3\rho)$.*

Also setting $\mu = 0$ and $(\alpha, \beta) = (1, -1)$ in (4.5.14) we get $\lambda = 0$. Hence we can state the following

Corollary 4.5.6. *An Einstein soliton (g, ξ, λ) on a conharmonicly flat almost pseudo symmetric Kählerian spacetime manifold is always steady.*

4.6 Quasi-Yamabe soliton on generalized Sasakian space form

This section is devoted to the study of generalized Sasakian space form admitting quasi-Yamabe soliton whose the potential vector field is pointwise collinear with the Reeb vector field. In this regard, our main result of this section is as follows:

Theorem 4.6.1. *If a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a quasi-Yamabe soliton (g, V, σ, μ) with the potential vector field V pointwise collinear with the Reeb vector field ξ, then* *i) M becomes a manifold of constant scalar curvature,*

ii) the soliton reduces to the Yamabe soliton (g, V, σ),

iii) V becomes a constant multiple of ξ and

iv) V is a strict infinitesimal contact transformation.

Proof. Let us assume that (g, V, σ, μ) is a quasi-Yamabe soliton on the generalized Sasakian space form $M(f_1, f_2, f_3)$ such that the potential vector field V is pointwise collinear with ξ, the there exists a non-zero smmooth function b on M such that $V = b\xi$. Then from the equation (1.2.8) we can write

$$\frac{1}{2}(\mathcal{L}_{b\xi}g)(X,Y) = (r-\sigma)g(X,Y) + \mu b^2\eta(X)\eta(Y), \tag{4.6.1}$$

for all vector fields $X, Y \in \chi(M)$.
Again, from the definition of Lie derivative we have

$$\begin{aligned}(\mathcal{L}_{b\xi}g)(X,Y) &= g(\nabla_X b\xi, Y) + g(X, \nabla_Y b\xi) \\ &= bg(\nabla_X\xi, Y) + X(b)\eta(Y) + bg(X, \nabla_Y\xi) + Y(b)\eta(X),\end{aligned}$$

which in view of (2.2.1) and (1.1.16) reduces to

$$(\mathcal{L}_{b\xi}g)(X,Y) = X(b)\eta(Y) + Y(b)\eta(X). \tag{4.6.2}$$

Substituting (4.6.2) in (4.6.1) infers that

$$X(b)\eta(Y) + Y(b)\eta(X) = 2(r-\sigma)g(X,Y) + 2\mu b^2\eta(X)\eta(Y). \tag{4.6.3}$$

Now taking $Y = \xi$ in (4.6.3) we obtain

$$X(b) = [2(r-\sigma+\mu b^2) - \xi(b)]\eta(X). \tag{4.6.4}$$

Again replacing X by ξ in the foregoing equation yields

$$\xi(b) = (r-\sigma) + \mu b^2. \tag{4.6.5}$$

Now consider an orthonormal basis $\{e_i : 1 \le i \le (2n+1)\}$ of the tangent space at each point of the manifold. Then setting $X = Y = e_i$ in (4.6.3) and summing over $1 \le i \le (2n+1)$ we get

$$\xi(b) = (r-\sigma)(2n+1) + \mu b^2. \tag{4.6.6}$$

Equating (4.6.5) with (4.6.6) we arrive at

$$r = \sigma = constant, \tag{4.6.7}$$

which implies that M is a manifold of constant scalar curvature and this proves part $i)$ of the theorem.

Again, in view of (4.6.5), the equation (4.6.4) gives us

$$db = \mu b^2 \eta. \tag{4.6.8}$$

Taking exterior differentiation on the above equation and using Poincare lemma $d^2 = 0$, we obtain $\mu b^2 = 0$. This eventually implies that $\mu = 0$ and hence the quasi-Yamabe soliton reduces to the Yamabe soliton (g, V, σ). This proves part $ii)$ of the theorem.

Now using $\mu = 0$ in the equation (4.6.8) we get $db = 0$, which implies b is constant. Therefore V is a constant multiple of ξ and this proves part $iii)$ of the theorem.

Again setting $Y = \xi$ in (4.6.1) and using (1.1.14) we obtain

$$(\mathcal{L}_V g)(X, \xi) = 2(r - \sigma + \mu b^2)\eta(X). \tag{4.6.9}$$

Using (4.6.7) and the fact that $\mu = 0$ in (4.6.9) yields $(\mathcal{L}_V g)(X, \xi) = 0$, which implies

$$(\mathcal{L}_V \eta)(X) = g(X, \mathcal{L}_V \xi). \tag{4.6.10}$$

Recalling that $V = b\xi$ and b is a constant it can be easily deduced that $\mathcal{L}_V \xi = 0$. Therefore from (4.6.10) finally we obtain $(\mathcal{L}_V \eta)(X) = 0$ for any vector field X on M. Hence the potential vector field V is a strict infinitesimal contact transformation. This proves part $iv)$ of the theorem and hence completes the proof. □

According to Corollary 1.1 of [74], in a generalized Sasakian space form $M(f_1, f_2, f_3)$ with the Yamabe soliton metric, the scalar curvature is harmonic. Thus in view of this and from our previous theorem we can conclude the following

Corollary 4.6.1. *If a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a quasi-Yamabe soliton (g, V, λ), whose potential vector field V is pointwise collinear with the Reeb vector field ξ, then the scalar curvature is harmonic.*

5

On η-Einstein solitons

5.1 Introduction

This chapter deals with 3-dimensional trans-Sasakian manifolds admitting η-Einstein solitons and the chapter is divided into ten sections. Chapter one and two are introduction and preliminaries respectively.

In section three, we characterize the nature of η-Einstein soliton on a 3-dimensional trans-Sasakian manifold and find the conditions when the soliton is shrinking, steady and expanding. Then in section four, we construct an example of a trans-Sasakian 3-manifold admitting an η-Einstein soliton and we verify some of our results. Next section five is devoted to the study of η-Einstein solitons on 3-dimensional trans-Sasakian manifolds with Codazzi type and cyclic parallel Ricci tensor and the nature of the manifold is characterized. In Sections six to nine, we study some curvature conditions $R \cdot S = 0$, $W_2 \cdot S = 0$, $R \cdot E = 0$, $B \cdot S = 0$, $S \cdot R = 0$ admitting η-Einstein solitons on 3-dimensional trans-Sasakian manifold. Finally in section ten, we consider 3-dimensional trans-Sasakian manifolds admitting η-Einstein solitons with torse forming vector field.

5.2 Preliminaries

The definition of trans-Sasakian manifold has been given in the introductory Chapter one. In what follows, by a trans-Sasakian 3-manifold, we mean a 3-dimensional trans-Sasakian manifold (M, g, ϕ, ξ, η) of type (α, β). Now recalling the definition of trans-

Sasakian manifold, from the expression (1.1.19), it can be derived that

$$\nabla_X\xi = -\alpha\phi(X) + \beta(X - \eta(X)\xi), \tag{5.2.1}$$

$$(\nabla_X\eta)(Y) = -\alpha g(\phi(X), Y) + \beta g(\phi(X), \phi(Y)), \tag{5.2.2}$$

for all vector fields $X, Y \in \chi(M)$. Again from equation (20) of corollary 4.2. in the paper [35], the Riemannian curvature tensor in a trans-Sasakian 3-manifold (M, g) is given by

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2))[g(Y,Z)X - g(X,Z)Y] \\ &\quad -g(Y,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\xi \\ &\quad -\eta(X)(\phi(grad\alpha) - grad\beta) + (X\beta + (\phi X)\alpha)\xi] \\ &\quad +g(X,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\xi \\ &\quad -\eta(Y)(\phi(grad\alpha) - grad\beta) + (Y\beta + (\phi Y)\alpha)\xi] \\ &\quad -[(Z\beta + (\phi Z)\alpha)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(Z) \\ &\quad +(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\eta(Z)]X \\ &\quad +[(Z\beta + (\phi Z)\alpha)\eta(X) + (X\beta + (\phi X)\alpha)\eta(Z) \\ &\quad +(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\eta(Z)]Y. \end{aligned} \tag{5.2.3}$$

Furthermore, if the functions α, β are constants then, in a trans-Sasakian 3-manifold (M, g) the following relations hold,

$$R(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y], \tag{5.2.4}$$

$$R(\xi,X)Y = (\alpha^2 - \beta^2)[g(X,Y)\xi - \eta(Y)X], \tag{5.2.5}$$

$$R(\xi,X)\xi = (\alpha^2 - \beta^2)[\eta(X)\xi - X], \tag{5.2.6}$$

$$S(X,Y) = [\frac{r}{2} - (\alpha^2 - \beta^2)]g(X,Y) - [\frac{r}{2} - 3(\alpha^2 - \beta^2)]\eta(X)\eta(Y), \tag{5.2.7}$$

$$S(X,\xi) = 2(\alpha^2 - \beta^2)\eta(X), \tag{5.2.8}$$

for all smooth vector fields $X, Y \in \chi(M)$, where R is the curvature tensor and S is the Ricci tensor of the manifold M.

5.3 η-Einstein soliton on trans-Sasakian 3-manifold

Let us consider a trans-Sasakian 3-manifold (M, g) admitting an η-Einstein soliton (g, ξ, λ, μ). Then from equation (1.2.10) we can write

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + [2\lambda - r]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0, \qquad (5.3.1)$$

for all $X, Y \in \chi(M)$.
Again from the well-known formula $(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)$ of Lie-derivative and using (5.2.1), we obtain for a trans-Sasakian 3-manifold

$$(\mathcal{L}_\xi g)(X, Y) = 2\beta[g(X, Y) - 2\beta\eta(X)\eta(Y)]. \qquad (5.3.2)$$

Now in view of the equations (5.3.1) and (5.3.2) we get

$$S(X, Y) = (\frac{r}{2} - \lambda - \beta)g(X, Y) + (\beta - \mu)\eta(X)\eta(Y). \qquad (5.3.3)$$

This shows that the manifold (M, g) is an η-Einstein manifold.
Also from equation (5.3.3) replacing $Y = \xi$ we find that

$$S(X, \xi) = (\frac{r}{2} - \lambda - \mu)\eta(X). \qquad (5.3.4)$$

Comparing the above equation (5.3.4) with (5.2.8) yields

$$r = 4(\alpha^2 - \beta^2) + 2\lambda + 2\mu. \qquad (5.3.5)$$

Again, considering an orthonormal basis $\{e_1, e_2, e_3\}$ of (M, g) and then setting $X = Y = e_i$ in equation (5.3.3) and summing over $i = 1, 2, 3$ we get

$$r = 6\lambda + 4\beta + 2\mu. \qquad (5.3.6)$$

Finally combining equations (5.3.5) and (5.3.6) we arrive at

$$\lambda = (\alpha^2 - \beta^2) - \beta. \qquad (5.3.7)$$

Thus the above discussion leads to the following

Theorem 5.3.1. *If a trans-Sasakian 3-manifold (M, g) admits an η-Einstein soliton (g, ξ, λ, μ), then the manifold (M, g) becomes an η-Einstein manifold of constant scalar curvature $r = 6\lambda + 4\beta + 2\mu$. Furthermore, the soliton is shrinking, steady or expanding according as; $\alpha^2 < \beta(\beta + 1)$, $\alpha^2 = \beta(\beta + 1)$, $\alpha^2 > \beta(\beta + 1)$ respectively.*

Next we consider a trans-Sasakian 3-manifold (M, g) and assume that it admits an η-Einstein soliton (g, V, λ, μ) such that V is pointwise collinear with ξ, i.e; $V = b\xi$, for some function b; then from the equation (1.2.10) it follows that

$$bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) \\ + 2S(X,Y) + (2\lambda - r)g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Then using the equation (5.2.1) in above we arrive at

$$(2b\beta + 2\lambda - r)g(X,Y) + (Xb)\eta(Y) + (Yb)\eta(X) \\ + 2S(X,Y) + 2(b\beta + \mu)\eta(X)\eta(Y) = 0. \tag{5.3.8}$$

Replacing $Y = \xi$ in the above equation yields

$$(Xb) + (\xi b)\eta(X) + 2S(X,\xi) + (2\lambda + 2\mu - r)\eta(X) = 0. \tag{5.3.9}$$

Again taking $X = \xi$ in (5.3.9) and by virtue of (5.2.8) we arrive at

$$2(\xi b) = (r - 2\lambda - 2\mu) - 4(\alpha^2 - \beta^2). \tag{5.3.10}$$

Using this value from (5.3.10) in the equation (5.3.9) and recalling (5.2.8) we can write

$$db = [\frac{r}{2} - \lambda - \mu - 2(\alpha^2 - \beta^2)]\eta. \tag{5.3.11}$$

Now taking exterior differentiation on both sides of (5.3.11) and using the famous Poincare's lemma i.e; $d^2 = 0$, finally we arrive at

$$r = 2\lambda + 2\mu + 4(\alpha^2 - \beta^2). \tag{5.3.12}$$

In view of the above (5.3.12) the equation (5.3.11) gives us $db = 0$ i.e; the function b is constant. Then the equation (5.3.8) reduces to

$$S(X,Y) = (\frac{r}{2} - \lambda - b\beta)g(X,Y) + (b\beta - \mu)\eta(X)\eta(Y), \tag{5.3.13}$$

for all $X, Y \in TM$. Hence we can state the following

Theorem 5.3.2. *If a trans-Sasakian 3-manifold (M, g) admits an η-Einstein soliton (g, V, λ, μ) such that V is pointwise collinear with ξ, then V is constant multiple of ξ and the manifold (M, g) becomes an η-Einstein manifold of constant scalar curvature $r = 2\lambda + 2\mu + 4(\alpha^2 - \beta^2)$.*

5.4 Example of an η-Einstein soliton on a trans-Sasakian 3-manifold

In this section we construct an example of a trans-Sasakian 3-manifold admitting an η-Einstein soliton and we verify some of our results.

Example 5.4.1. *Let us consider the 3-dimensional manifold* $M = \{(u, v, w) \in \mathbb{R}^3 : w \neq 0\}$*. Define a linearly independent set of vector fields* $\{e_i : 1 \leq i \leq 3\}$ *on the manifold* M *given by*

$$e_1 = e^{2w}\frac{\partial}{\partial u}, \qquad e_2 = e^{2w}\frac{\partial}{\partial v}, \qquad e_3 = \frac{\partial}{\partial w}.$$

Let us define the Riemannian metric g *on* M *by*

$$g(e_i, e_j) = \begin{cases} 1, & for \quad i = j \\ 0, & for \quad i \neq j \end{cases}$$

for all $i, j = 1, 2, 3$*. Now considering* $e_3 = \xi$*, let us take the* 1*-form* η*, on the manifold* M*, defined by*

$$\eta(U) = g(U, e_3), \quad \forall U \in \chi(M).$$

Then it can be observed that $\eta(\xi) = 1$*. Let us define the* $(1, 1)$ *tensor field* ϕ *on* M *as*

$$\phi(e_1) = e_2, \qquad \phi(e_2) = -e_1, \qquad \phi(e_3) = 0.$$

Using the linearity of g *and* ϕ *it can be easily checked that*

$$\phi^2(U) = -U + \eta(U)\xi, \quad g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \quad \forall U, V \in \chi(M).$$

Hence the structure (g, ϕ, ξ, η) *defines an almost contact metric structure on the manifold* M*. Now, using the definitions of Lie bracket, after some direct computations we get*

$$[e_1, e_2] = 0, \qquad [e_1, e_3] = -2e_1, \qquad [e_2, e_3] = -2e_2.$$

Again the Riemannian connection ∇ *of the metric* g *is defined by the well-known Koszul's formula which is given by*

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]). \end{aligned}$$

Using the above formula one can easily calculate that

$$\nabla_{e_1} e_1 = 2e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -2e_1,$$
$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = 2e_3, \quad \nabla_{e_2} e_3 = -2e_2,$$
$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

Thus from the above relations it follows that the manifold (M, g) *is a trans-Sasakian* 3-*manifold. Now using* (1.1.5) *the non-vanishing components of the Riemannian curvature tensor* R *can be easily obtained as*

$$R(e_1, e_2)e_2 = R(e_1, e_3)e_3 = -4e_1,$$
$$R(e_2, e_3)e_3 = R(e_3, e_1)e_1 = -4e_2,$$
$$R(e_3, e_2)e_2 = 4e_2, \quad R(e_2, e_1)e_1 = 4e_3.$$

Hence we can calculate the components of the Ricci tensor as follows

$$S(e_1, e_1) = 0, \quad S(e_2, e_2) = 0, \quad S(e_3, e_3) = -8.$$

Therefore in view of the above values of the Ricci tensor, from the equation (1.2.10) *we can calculate* $\lambda = -2$ *and* $\mu = 6$*. Hence we can say that the data* $(g, \xi, -2, 6)$ *defines an* η*-Einstein soliton on the trans-Sasakian* 3*-manifold* (M, g)*. Also we can see that the manifold* (M, g) *is a manifold of constant scalar curvature* $r = -8$ *and hence the Theorem* 5.3.1 *is verified.*

5.5 η-Einstein soliton on trans-Sasakian 3-manifold with Codazzi type and cyclic parallel Ricci tensor

The purpose of this section is to study η-Einstein solitons in trans-Sasakian 3-manifolds having certain special types of Ricci tensor namely codazzi type Ricci tensor and cyclic parallel Ricci tensor.

Let us consider a trans-Sasakian 3-manifold having Codazzi type Ricci tensor and admits an η-Einstein soliton (g, ξ, λ, μ), then equation (5.3.3) holds. Now covariantly differentiating the equation (5.3.3) with respect to an arbitrary vector field X and then using (5.2.2) we get

$$(\nabla_X S)(Y,Z) = 2(\beta-\mu)[\eta(Y)(-\alpha g(\phi X,Z)+\beta g(\phi X,\phi Z)) +\eta(Z)(-\alpha g(\phi X,Y)+\beta g(\phi X,\phi Y))]. \tag{5.5.1}$$

Similarly we can compute

$$(\nabla_Y S)(X,Z) = 2(\beta-\mu)[\eta(X)(-\alpha g(\phi Y,Z)+\beta g(\phi Y,\phi Z)) +\eta(Z)(-\alpha g(\phi Y,X)+\beta g(\phi Y,\phi X))]. \tag{5.5.2}$$

Since the manifold has Codazzi type Ricci tensor, using (5.5.1) and (5.5.2) in the equation (1.1.7) and then recalling (1.1.15) we arrive at

$$2(\beta-\mu)[\eta(Y)(-\alpha g(\phi X,Z)+\beta g(X,Z))-\eta(X)(-\alpha g(\phi Y,Z) +\beta g(Y,Z))-2\alpha\eta(Z)g(\phi X,Y)] = 0. \tag{5.5.3}$$

Putting $Z=\xi$ in above and in view of (1.1.14) we finally obtain

$$4\alpha(\mu-\beta)g(\phi X,Y)=0, \tag{5.5.4}$$

forall $X,Y\in\chi(M)$. Therefore from (5.5.3) we can conclude that either $\alpha=0$ or $\mu=\beta$. Hence we have the following

Theorem 5.5.1. *Let (M,g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g,ξ,λ,μ). If the Ricci tensor of the manifold is of Codazzi type then the manifold becomes a β-Kenmotsu manifold provided $\mu\neq\beta$.*

Now using $\alpha=0$ in equation (5.3.7) we get $\lambda=-\beta(\beta+1)$. Thus we can state the following

Corollary 5.5.1. *Let (M,g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g,ξ,λ,μ) with $\mu\neq\beta$. If the Ricci tensor of the manifold is of Codazzi type then the soliton is shrinking if $\beta<-1$ or, $\beta>0$; steady if $\beta=-1$ or $\beta=0$; and expanding if $-1<\beta<0$ respectively.*

Again from the equation (5.5.3) we can write that $\mu=\beta$ if $\alpha\neq 0$. Then from equation (5.3.3) we obtain

$$S(X,Y)=(\frac{r}{2}-\lambda-\beta)g(X,Y), \tag{5.5.5}$$

for all $X,Y\in\chi(M)$. Then contracting the equation (5.5.4) we get $r=6\lambda+6\beta$. Hence in view of this and equation (5.5.4) we have the following

Theorem 5.5.2. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ). If the Ricci tensor of the manifold is of Codazzi type then the manifold becomes an Einstein manifold of constant scalar curvature $r = 6\lambda + 6\beta$ provided $\alpha \neq 0$.*

Let us now consider a trans-Sasakian 3-manifold, having cyclic parallel Ricci tensor, admits an η-Einstein soliton (g, ξ, λ, μ), then equation (5.3.3) holds. Now taking covariant differentiation of (5.3.3) and using equation (5.2.2) we obtain relations (5.5.1) and (5.5.2). In a similar manner we get the following

$$\begin{aligned}(\nabla_Z S)(X, Y) &= 2(\beta - \mu)[\eta(X)(-\alpha g(\phi Z, Y) + \beta g(\phi Z, \phi Y)) \\ &\quad + \eta(Y)(-\alpha g(\phi Z, X) + \beta g(\phi Z, \phi X))]. \end{aligned} \tag{5.5.6}$$

Now since the manifold has cyclic parallel Ricci tensor, using the values from (5.5.1), (5.5.2) and (5.5.6) in the equation (1.1.8) and then making use of (1.1.15) we arrive at

$$4\beta(\beta - \mu)[\eta(X)g(\phi Y, \phi Z) + \eta(Y)g(\phi Z, \phi X) + \eta(Z)g(\phi X, \phi Y)] = 0. \tag{5.5.7}$$

Replacing $Z = \xi$ in the above equation (5.5.7) yields

$$4\beta(\beta - \mu)g(\phi X, \phi Y) = 0, \tag{5.5.8}$$

for all $X, Y \in \chi(M)$. Since $g(\phi X, \phi Y) \neq 0$ always, the above equation (5.5.8) implies that either $\beta = 0$ or, $\mu = \beta$. Thus we can state the following

Theorem 5.5.3. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ). If the manifold has cyclic parallel Ricci tensor, then the manifold becomes an α-Sasakian manifold provided $\mu \neq \beta$.*

Now using $\beta = 0$ in equation (5.3.7) we get $\lambda = \alpha^2 > 0$. Therefore we have

Corollary 5.5.2. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ) with $\mu \neq \beta$. If the manifold has cyclic parallel Ricci tensor then the soliton is expanding.*

Again if $\beta \neq 0$ then from (5.5.8) it follows that $\mu = \beta$. Therefore after a similar calculation like equation (5.5.5) we can state

Theorem 5.5.4. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ). If the manifold has cyclic parallel Ricci tensor, then the manifold becomes an Einstein manifold of constant scalar curvature $r = 6\lambda + 6\beta$ provided $\beta \neq 0$.*

5.6 η-Einstein soliton on trans-Sasakian 3-manifold satisfying $R(\xi, X) \cdot S = 0$ and $W_2(\xi, X) \cdot S = 0$

Let us first consider a trans-Sasakian 3-manifold which admits an η-Einstein soliton (g, ξ, λ, μ) and the manifold satisfies the curvature condition $R(\xi, X) \cdot S = 0$. Then $\forall X, Y, Z \in \chi(M)$ we can write

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0. \tag{5.6.1}$$

Now using (5.3.3) in (5.6.1) we get

$$\begin{aligned}&(\frac{r}{2} - \lambda - \beta)g(R(\xi, X)Y, Z) + (\beta - \mu)\eta(R(\xi, X)Y)\eta(Z)\\&+ (\frac{r}{2} - \lambda - \beta)g(R(\xi, X)Z, Y) + (\beta - \mu)\eta(R(\xi, X)Z)\eta(Y) = 0.\end{aligned} \tag{5.6.2}$$

In view of (5.2.5) the previous equation becomes

$$(\alpha^2 - \beta^2)(\beta - \mu)[g(X, Y)\eta(Z) + g(X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0. \tag{5.6.3}$$

Putting $Z = \xi$ in the above equation (5.6.3) and recalling (1.1.15) obtain

$$(\alpha^2 - \beta^2)(\beta - \mu)g(\phi X, \phi Y) = 0, \tag{5.6.4}$$

for all $X, Y \in \chi(M)$. Since $g(\phi X, \phi X) \neq 0$ always and for non-trivial case $\alpha^2 \neq \beta^2$, we can conclude from the equation (5.6.4) that $\mu = \beta$. Then from equation (5.3.3) we obtain

$$S(X, Y) = (\frac{r}{2} - \lambda - \beta)g(X, Y), \tag{5.6.5}$$

for all $X, Y \in \chi(M)$. Then contracting the equation (5.6.5) we get $r = 6\lambda + 6\beta$. Hence in view of this and equation (5.6.5) we have the following

Theorem 5.6.1. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ). If the manifold satisfies the curvature condition $R(\xi, X) \cdot S = 0$, then the manifold becomes an Einstein manifold of constant scalar curvature $r = 6\lambda + 6\beta$.*

Our next result is on W_2-curvature tensor, which is defined in the introductory chapter one. So, the equation (1.1.4) holds and taking $n = 3$ in it, we can write the expression for the W_2-curvature tensor in a trans-Sasakian 3-manifold as follows:

$$W_2(X, Y)Z = R(X, Y)Z + \frac{1}{2}[g(X, Z)QY - g(Y, Z)QX]. \tag{5.6.6}$$

Now assume that (M, g) is a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ) and also the manifold satisfies the curvature condition $W_2(\xi, X) \cdot S = 0$. Then we can write

$$S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z) = 0, \quad \forall X, Y, Z \in TM. \tag{5.6.7}$$

In view of (5.3.3) the above equation (5.6.7) becomes

$$\begin{aligned}&(\frac{r}{2} - \lambda - \beta)[g(W_2(\xi, X)Y, Z) + g(W_2(\xi, X)Z, Y)] \\ &+ (\beta - \mu)[\eta(W_2(\xi, X)Y)\eta(Z) + \eta(W_2(\xi, X)Z)\eta(Y)] = 0.\end{aligned} \tag{5.6.8}$$

Again from (5.3.3) it follows that

$$QX = (\frac{r}{2} - \lambda - \beta)X + (\beta - \mu)\eta(X)\xi, \tag{5.6.9}$$

which implies

$$Q\xi = (\frac{r}{2} - \lambda - \mu)\xi. \tag{5.6.10}$$

Replacing $X = \xi$ in (5.6.6) and then using equations (5.2.5), (5.6.9) and (5.6.10) we obtain

$$W_2(\xi, Y)Z = Bg(Y, Z)\xi - A\eta(Z)Y + (A - B)\eta(Y)\eta(Z), \tag{5.6.11}$$

where $A = (\alpha^2 - \beta^2) - \frac{1}{2}(\frac{r}{2} - \lambda - \beta)$ and $B = (\alpha^2 - \beta^2) - \frac{1}{2}(\frac{r}{2} - \lambda - \mu)$. Taking inner product of (5.11) with respect to the vector field ξ yields

$$\eta(W_2(\xi, Y)Z) = B[g(Y, Z) - \eta(Y)\eta(Z)]. \tag{5.6.12}$$

Using (5.6.11) and (5.6.12) in the equation (5.6.8) and then taking $Z = \xi$ we arrive at

$$(A - B)[2B - (\frac{r}{2} - \lambda - \beta)][g(X, Y) - \eta(X)\eta(Y)] = 0,$$

which in view of (1.1.15) implies

$$(A - B)[2B - (\frac{r}{2} - \lambda - \beta)]g(\phi X, \phi Y) = 0, \tag{5.6.13}$$

for all $X, Y \in \chi(M)$. Since $g(\phi X, \phi X) \neq 0$ always, we can conclude from the equation (5.6.13) that either $A = B$ or, $2B = \frac{r}{2} - \lambda - \beta$. Thus recalling the values of A and B it implies that either $\mu = \beta$ or,

$$2(\alpha^2 - \beta^2) = r - 2\lambda - \mu - \beta. \tag{5.6.14}$$

Now for the case $\mu = \beta$, proceeding similarly as the equation (5.6.5) we can say that the manifold becomes an Einstein manifold. Again combining (5.6.14) with (5.3.5) we get

$$r = 2\lambda + 2\beta. \tag{5.6.15}$$

Therefore we can state the following

Theorem 5.6.2. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ). If the manifold satisfies the curvature condition $W_2(\xi, X) \cdot S = 0$, then either the manifold becomes an Einstein manifold or it is a manifold of constant scalar curvature $r = 2\lambda + 2\beta$.*

Again in view of (5.3.6), the equation (5.6.15) implies $\lambda = -\frac{1}{2}(\mu + \beta)$. Hence we have

Corollary 5.6.1. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ) with $\mu \neq \beta$. If the manifold satisfies the curvature condition $W_2(\xi, X) \cdot S = 0$, then the soliton is expanding, steady or shrinking according as $\mu < -\beta$, $\mu = -\beta$ or, $\mu > -\beta$ respectively.*

5.7 Einstein semi-symmetric trans-Sasakian 3-manifold admitting η-Einstein soliton

This section is devoted to the study of η-Einstein solitons on Einstein semi-symmetric trans-Sasakian 3-manifolds.

A trans-Sasakian 3-manifold (M, g) is called Einstein semi-symmetric [89] if $R.E = 0$, where E is the Einstein tensor given by

$$E(X, Y) = S(X, Y) - \frac{r}{3} g(X, Y), \tag{5.7.1}$$

for all vector fields $X, Y \in \chi(M)$ and r is the scalar curvature of the manifold.

Now consider a trans-Sasakian 3-manifold is Einstein semi-symmetric i.e; the manifold satisfies the curvature condition $R.E = 0$. Then for all vector fields $X, Y, Z, W \in \chi(M)$ we can write

$$E(R(X, Y)Z, W) + E(Z, R(X, Y)W) = 0. \tag{5.7.2}$$

In view of (5.7.1) the equation (5.7.2) becomes

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = \frac{r}{3}[g(R(X,Y)Z,W) + g(Z,R(X,Y)W)]. \quad (5.7.3)$$

Replacing $X = Z = \xi$ in the above equation (5.7.3) and then using (5.2.5), (5.2.6) we arrive at

$$(\alpha^2 - \beta^2)S(Y,W) = (\alpha^2 - \beta^2)[\eta(Y)S(\xi,W) + \eta(W)S(\xi,Y) - g(Y,W)S(\xi,\xi)]. \quad (5.7.4)$$

So, now in view of (5.2.8) the above equation (5.7.4) finally yields

$$S(Y,W) = -2(\alpha^2 - \beta^2)g(Y,W) + 4(\alpha^2 - \beta^2)\eta(Y)\eta(W), \quad (5.7.5)$$

for all $Y, W \in \chi(M)$. This implies that the manifold is an η-Einstein manifold. Hence we have the following

Lemma 5.7.1. *An Einstein semi-symmetric trans-Sasakian* 3*-manifold is an* η*-Einstein manifold.*

Now let us assume that the Einstein semi-symmetric trans-Sasakian 3-manifold (M,g) admits an η-Einstein soliton (g,ξ,λ,μ). Then equation (5.3.3) holds and combining (5.3.3) with the above equation (5.7.5) we get

$$r = 2\lambda + \mu + \beta. \quad (5.7.6)$$

Again recalling the equation (5.3.6) in the above (5.7.6) we have

$$\lambda = -\frac{1}{4}(\mu + 3\beta). \quad (5.7.7)$$

Therefore we can state the following

Theorem 5.7.1. *Let* (M,g) *be a trans-Sasakian* 3*-manifold admitting an* η*-Einstein soliton* (g,ξ,λ,μ)*. If the manifold is Einstein semi-symmetric, then the manifold becomes an* η*-Einstein manifold of constant scalar curvature* $r = 2\lambda + \mu + \beta$ *and the soliton is expanding, steady or shrinking according as* $\mu < 3\beta$*,* $\mu = 3\beta$ *or,* $\mu > 3\beta$ *respectively.*

5.8 η-Einstein soliton on trans-Sasakian 3-manifold satisfying $B(\xi, X) \cdot S = 0$

In 1949, S. Bochner [16] introduced the concept of the well-known Bochner curvature tensor merely as a Kähler analogue of the Weyl conformal curvature tensor but the geometric significance of it in the light of Boothby-Wangs fibration was presented later by D. E. Blair [13]. The notion of C-Bochner curvature tensor in a Sasakian manifold was introduced by M. Matsumoto, G. Chūman [62] in 1969. The C-Bochner curvature tensor in trans-Sasakian 3-manifold (M, g) is given by

$$\begin{aligned}
B(X,Y)Z \;=\;& R(X,Y)Z + \frac{1}{6}[g(X,Z)QY - S(Y,Z) - g(Y,Z)QX \\
&+S(X,Z)Y + g(\phi X,Z)Q\phi Y - S(\phi Y,Z)\phi X - g(\phi Y,Z)Q\phi X \\
&+S(\phi X,Z)\phi Y + 2S(\phi X,Y)\phi Z + 2g(\phi X,Y))Q\phi Z \\
&+\eta(Y)\eta(Z)QX - \eta(Y)S(X,Z)\xi + \eta(X)S(Y,Z)\xi - \eta(X)\eta(Z)QY] \\
&-\frac{D+2}{6}[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z] \\
&+\frac{D}{6}[\eta(Y)g(X,Z)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y - \eta(X)g(Y,Z)\xi] \\
&-\frac{D-4}{6}[g(X,Z)Y - g(Y,Z)X], \qquad (5.8.1)
\end{aligned}$$

where $D = \frac{r+2}{4}$.

Let us consider a trans-Sasakian 3-manifold (M, g) which admits an η-Einstein soliton (g, ξ, λ, μ) and also the manifold satisfies the curvature condition $B(\xi, X) \cdot S = 0$. Then $\forall X, Y, Z \in \chi(M)$ we can write

$$S(B(\xi, X)Y, Z) + S(Y, B(\xi, X)Z) = 0. \qquad (5.8.2)$$

Now using (5.3.3) in (5.8.2) we get

$$\begin{aligned}
&(\frac{r}{2} - \lambda - \beta)[g(B(\xi, X)Y, Z) + g(B(\xi, X)Z, Y)] \\
&+ (\beta - \mu)[\eta(B(\xi, X)Y)\eta(Z) + \eta(B(\xi, X)Z)\eta(Y)] = 0. \qquad (5.8.3)
\end{aligned}$$

Again from (5.3.3) it follows that

$$QX = (\frac{r}{2} - \lambda - \beta)X + (\beta - \mu)\eta(X)\xi, \qquad (5.8.4)$$

which implies

$$Q\xi = (\frac{r}{2} - \lambda - \mu)\xi. \tag{5.8.5}$$

Also taking $X = \xi$ in (5.8.1) we obtain

$$\begin{aligned} B(\xi, Y)Z &= R(\xi, Y)Z\frac{1}{6}[S(\xi, Z)Y - g(Y, Z)Q\xi + \eta(Y)\eta(Z)Q\xi \\ &\quad -\eta(Y)S(\xi, Z)\xi] + \frac{4}{6}[\eta(Z)Y - g(Y, Z)\xi]. \end{aligned} \tag{5.8.6}$$

Using equations (5.2.5), (5.3.4) and (5.8.5) in (5.8.6) yields

$$B(\xi, Y)Z = [(\alpha^2 - \beta^2) - \frac{1}{6}(\frac{r}{2} - \lambda - \mu) - \frac{4}{6}][g(Y, Z)\xi - \eta(Z)Y]. \tag{5.8.7}$$

In view of (5.8.7) the equation (5.8.3) becomes

$$\begin{aligned} &[(\alpha^2 - \beta^2) - \frac{1}{6}(\frac{r}{2} - \lambda - \mu) - \frac{4}{6}](\beta - \mu)[g(X, Y)\eta(Z) \\ &\quad +g(X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0. \end{aligned}$$

Replacing $Z = \xi$ in the above equation and recalling (1.1.15), finally we arrive at

$$[(\alpha^2 - \beta^2) - \frac{1}{6}(\frac{r}{2} - \lambda - \mu) - \frac{4}{6}](\beta - \mu)g(\phi X, \phi Y) = 0, \tag{5.8.8}$$

for all vector fields $X, Y \in \chi(M)$. Hence from (5.8.8) we can conclude that either

$$[(\alpha^2 - \beta^2) - \frac{1}{6}(\frac{r}{2} - \lambda - \mu) - \frac{4}{6}] = 0, \tag{5.8.9}$$

or, $\mu = \beta$. Also for $\mu = \beta$ proceeding similarly as equation (5.5.5) it can be easily shown that the manifold becomes an Einstein manifold. Again if $\mu \neq \beta$ using (5.3.7) in the equation (5.8.9) we have

$$r = 10\lambda + 2\mu + 12\beta - 8, \tag{5.8.10}$$

which implies that the manifold becomes a manifold of constant scalar curvature. Therefore we can state the following

Theorem 5.8.1. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ). If the manifold satisfies the curvature condition $B(\xi, X) \cdot S = 0$, then either the manifold is an Einstein manifold or it is a manifold of constant scalar curvature $r = 10\lambda + 2\mu + 12\beta - 8$.*

Now for the case $\mu \neq \beta$, using the equation (5.3.6) in (5.8.10) we obtain $\lambda = 2(1-\beta)$. Hence we have

Corollary 5.8.1. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ) with $\mu \neq \beta$. If the manifold satisfies the curvature condition $B(\xi, X) \cdot S = 0$, then the soliton is expanding, steady or shrinking according as $\beta < 1$, $\beta = 1$ or, $\beta > 1$ respectively.*

5.9 η-Einstein soliton on trans-Sasakian 3-manifold satisfying $S(\xi, X) \cdot R = 0$

In this section we study the curvature condition $S(\xi, X) \cdot R = 0$, where by $\cdot$ we denote the derivation of the tensor algebra at each point of the tangent space as follows:

$$\begin{aligned} S((\xi, X) \cdot R)(Y, Z)W &:= ((\xi \wedge_S X) \cdot R)(Y, Z)W \\ &:= (\xi \wedge_S X)R(Y, Z)W + R((\xi \wedge_S X)Y, Z)W \\ &\quad +R(Y, (\xi \wedge_S X)Z)W + R(Y, Z)(\xi \wedge_S X)W, \end{aligned} \tag{5.9.1}$$

where the endomorphism $X \wedge_S Y$ is defined by

$$(X \wedge_S Y)Z := S(Y, Z)X - S(X, Z)Y.$$

Now let us consider a trans-Sasakian 3-manifold (M, g) which admits an η-Einstein soliton (g, ξ, λ, μ) and also the manifold satisfies the curvature condition $S(\xi, X) \cdot R = 0$. Then using this condition and the equation (5.9.1) we can write

$$\begin{aligned} &S(X, R(Y, Z)W)\xi - S(\xi, R(Y, Z)W)X + S(X, Y)R(\xi, Z)W \\ &-S(\xi, Y)R(X, Z)W + S(X, Z)R(Y, \xi)W - S(\xi, Z)R(Y, X)W \\ &+S(X, W)R(Y, Z)\xi - S(\xi, W)R(Y, Z)X = 0, \end{aligned} \tag{5.9.2}$$

for all vector fields $X, Y, Z, W \in \chi(M)$. Taking inner product of the above (5.9.2) with the vector field ξ and then replacing $W = \xi$ we obtain

$$\begin{aligned} &S(X, R(Y, Z)\xi) - S(\xi, R(Y, Z)\xi)\eta(X) + S(X, Y)\eta(R(\xi, Z)\xi) \\ &-S(\xi, Y)\eta(R(X, Z)\xi) + S(X, Z)\eta(R(Y, \xi)\xi) - S(\xi, Z)\eta(R(Y, X)\xi) \\ &+S(X, \xi)\eta(R(Y, Z)\xi) - S(\xi, \xi)\eta(R(Y, Z)X) = 0, \end{aligned}$$

In view of (5.2.4) and (5.2.6) the above equation becomes

$$
\begin{aligned}
(\alpha^2 - \beta^2)[S(X,Y)\eta(Z) - S(X,Z)\eta(Y) - S(\xi,Y)\eta(X)\eta(Z) \\
+S(\xi,Z)\eta(X)\eta(Y)] - S(\xi,\xi)\eta(R(Y,Z)X) &= 0. \qquad (5.9.3)
\end{aligned}
$$

Putting $Y = \xi$ in (5.9.3) and then recalling (5.2.5) we get

$$
\begin{aligned}
(\alpha^2 - \beta^2)[S(X,\xi)\eta(Z) - S(X,Z) - S(\xi,\xi)\eta(X)\eta(Z) \\
+S(\xi,Z)\eta(X) - S(\xi,\xi)[g(X,Z) - \eta(X)\eta(Z)]] &= 0. \qquad (5.9.4)
\end{aligned}
$$

Using equations (5.3.3) and (5.3.4) in the above (5.9.4) yields

$$
(\alpha^2 - \beta^2)[(r - 2\lambda - 2\mu + \beta)\eta(X)\eta(Z) - (r - 2\lambda - \mu - \beta)g(X,Z)] = 0.
$$

Replacing $X = \xi$ in above we arrive at

$$
(\alpha^2 - \beta^2)(2\beta - \mu)\eta(X) = 0, \qquad \forall X \in TM. \qquad (5.9.5)
$$

Since for non-trivial case $\alpha^2 \neq \beta^2$, from the above equation (5.9.5) it follows that $\mu = 2\beta$. Therefore in view of this and recalling (5.3.6) we finally obtain $r = 6\lambda + 8\beta$. Therefore we can state the following

Theorem 5.9.1. *Let (M,g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g,ξ,λ,μ). If the manifold satisfies the curvature condition $S(\xi,X)\cdot R = 0$, then it becomes a manifold of constant scalar curvature $r = 6\lambda + 8\beta$.*

5.10 η-Einstein soliton on trans-Sasakian 3-manifold with torse-forming vector field

This section is devoted to study the nature of η-Einstein solitons on trans-Sasakian 3-manifolds with torse-forming vector field. The definition of a torse-forming vector field is given in the introductory chapter one and hence (1.1.12) holds.

Now let us consider that (g,ξ,λ,μ) be an η-Einstein soliton on a trans-Sasakian 3-manifold (M,g) and assume that the Reeb vector field ξ of the manifold is a torse-forming vector field. Then ξ being a torse-forming vector field, from equation (1.1.12) we have

$$
\nabla_X \xi = fX + \gamma(X)\xi, \qquad (5.10.1)
$$

$\forall X \in \chi(M)$, f being a smooth function and γ is a 1-form.
Recalling the equation (5.2.1) and taking inner product on both sides with ξ we can write

$$g(\nabla_X \xi, \xi) = (\beta - 1)\eta(X). \tag{5.10.2}$$

Again from the equation (5.10.1), applying inner product with ξ we obtain

$$g(\nabla_X \xi, \xi) = f\eta(X) + \gamma(X). \tag{5.10.3}$$

Combining (5.10.2) and (5.10.3) we get, $\gamma = (\beta - 1 - f)\eta$. Thus from (5.10.1) it implies that, for torse-forming vector field ξ in a trans-Sasakian 3-manifold, we have

$$\nabla_X \xi = f(X - \eta(X)\xi) + (\beta - 1)\eta(X)\xi. \tag{5.10.4}$$

Now from the formula of Lie differentiation and using (5.10.4) yields

$$\begin{aligned} (\mathcal{L}_\xi g)(X, Y) &= g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) \\ &= 2f[g(X, Y) - \eta(X)\eta(Y)] + 2(\beta - 1)\eta(X)\eta(Y). \end{aligned} \tag{5.10.5}$$

Since (g, ξ, λ, μ) is an η-Einstein soliton, the equation (1.2.10) holds. So in view of (5.10.5), the equation (1.2.10) reduces to

$$S(X, Y) = (\frac{r}{2} - \lambda + f)g(X, Y) + (f - \mu - \beta + 1)\eta(X)\eta(Y). \tag{5.10.6}$$

This implies that the manifold is an η-Einstein manifold. Again putting $Y = \xi$ in (5.10.6) we get

$$S(X, \xi) = (\frac{r}{2} - \lambda - \mu - \beta + 1)\eta(X). \tag{5.10.7}$$

Combining (5.10.7) with the equation (5.2.8) implies

$$(\frac{r}{2} - \lambda - \mu - \beta + 1) = 2(\alpha^2 - \beta^2). \tag{5.10.8}$$

Again tracing out the equation (5.10.6) we obtain

$$r = 6\lambda + 2\mu + 4f + 2\beta - 2. \tag{5.10.9}$$

Using the above equation (5.10.9) in (5.10.8), finally we get $\lambda = f - (\alpha^2 - \beta^2)$. Therefore we have the following

Theorem 5.10.1. *Let (g, ξ, λ, μ) be an η-Einstein soliton on a trans-Sasakian 3-manifold (M, g), with torse-forming vector field ξ, then the manifold becomes an η-Einstein manifold and the soliton is expanding, steady or shrinking according as $f > (\alpha^2 - \beta^2)$, $f = (\alpha^2 - \beta^2)$ or, $f < (\alpha^2 - \beta^2)$ respectively.*

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Conformal Ricci soliton and quasi-Yamabe soliton on generalized Sasakian space form

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ABSTRACT

The present paper is devoted to generalized Sasakian space forms admitting conformal Ricci soliton and Quasi-Yamabe soliton. Nature of the conformal Ricci soliton is characterized on generalized Sasakian space form with various types of the potential vector field, and conditions for the conformal Ricci soliton to be shrinking, steady, or expanding are also given. Then it is shown that, depending on the nature of the structure functions of a generalized Sasakian space form, the potential function of a conformal gradient Ricci soliton is constant. Next, it is proved that under certain conditions, a quasi-Yamabe soliton reduces to a Yamabe soliton on generalized Sasakian space forms. Finally, an illustrative example of a generalized Sasakian space form is discussed to verify our results.

1. Introduction

In 1982, R. S. Hamilton [15] introduced the Ricci soliton as a self similar solution to the Ricci flow equation given by $\frac{\partial}{\partial t}(g(t)) = -2Ric(g(t))$, where $g(t)$ is an one parameter family of metrics on a certain manifold.

A Riemannian metric g defined on a smooth manifold M of dimension n is said to be a Ricci soliton if, for some constant λ, there exists a smooth vector field V on M satisfying the equation

$$Ric + \frac{1}{2}\mathcal{L}_V g = \lambda g, \tag{1.1}$$

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where $\mathcal{L}_V$ denotes the Lie derivative in the direction of V and Ric is the Ricci tensor. The Ricci soliton is called shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$. Ricci solitons can also be viewed as natural generalizations of Einstein metrics which moves only by a one-parameter group of diffeomorphisms and scaling [16].

In [12] A. E. Fischer introduced conformal Ricci flow equations a modified version of Hamilton's Ricci flow equation that modifies the volume constraint such that equation scalar curvature constraint. The conformal Ricci flow equations on a smooth closed connected oriented manifold M of dimension n is given by

$$\begin{aligned} &\frac{\partial g}{\partial t} + 2(Ric + \frac{g}{n}) = -pg, \\ &r(g) = -1, \end{aligned} \tag{1.2}$$

where p is a non-dynamical (time-dependent) scalar field and $r(g)$ is the scalar curvature of the manifold. The term $-pg$ acts as the constraint force to maintain the scalar curvature constraint. Thus these evolution equations are analogous to famous Navier-Stokes equations in fluid mechanics where the constraint is divergence-free. That is why sometimes p is also called the conformal pressure.

N. Basu et al. [3] introduced the concept of conformal Ricci soliton as a generalization of the classical Ricci soliton.

A Riemannian metric g on a smooth manifold M of dimension $(2n+1)$ is called a conformal Ricci soliton if there exists a constant λ and a vector field V such that

$$\mathcal{L}_V g + 2Ric = [2\lambda - (p + \frac{2}{2n+1})]g, \tag{1.3}$$

where $S = Ric$ is the Ricci tensor, λ is a constant, and p is the conformal pressure. It can be easily checked that the above soliton equation satisfies the conformal Ricci flow equation (1.2). Further, Ganguly and A. Bhattacharyya studied the conformal Ricci soliton within the framework of almost co-Kähler manifolds [13] and $(LCS)_n$-manifolds [14] and authors generalized conformal Ricci soliton and obtained some interesting results in [11]. M. D. Siddiqui [20] proved that if a compact Lagrangian submanifold in a complex space form under certain conditions admits a conformal Ricci soliton, then it is either totally geodesic or flat with parallel mean curvature vector field.

Furthermore, if the soliton vector field is a gradient of some smooth function on the manifold, i.e., $V = grad\ f = \nabla f$, for some smooth function f then the soliton is called conformal gradient Ricci soliton [13]. In this case, the soliton equation (1.2) becomes

$$S + \nabla\nabla f = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]g, \tag{1.4}$$

where ∇ is the Riemannian connection on the manifold M^n.

The notion of Yamabe flow was introduced by R. S. Hamilton [16] as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian metrics on a Riemannian manifold of dimension greater than or equal to three. The Yamabe flow on a smooth Riemannian manifold (M, g) is defined as the evolution equation of the Riemannian metric $g = g(t)$ as follows

$$\frac{\partial}{\partial t}(g(t)) = -r(g(t)), \tag{1.5}$$

where r denotes the scalar curvature of the manifold. It should be noted that two dimensional the Yamabe flow is equivalent to the Ricci flow. Still for dimensions greater than two, the Yamabe flow and the Ricci flow do not agree, as the Yamabe flow preserves the metric's conformal class whereas the Ricci flow does not. The Yamabe flow corresponds to the fast diffusion case of the plasma equation inmathematical physics.

Let (M, g) be an n-dimensional complete Riemannian manifold. If the Riemannian metric g satisfies

$$\frac{1}{2}\mathcal{L}_V g = (r - \sigma)g, \tag{1.6}$$

for some smooth vector field V and some $\sigma \in \mathbb{R}$, then it is known as a Yamabe soliton [16]. The Yamabe soliton is said to be shrinking, steady, or expanding according to $\sigma < 0$, $\sigma = 0$ or $\sigma > 0$ respectively. Like Ricci solitons are self-similar solutions of the Ricci flow, Yamabe solitons are also self-similar solutions to the Yamabe flow, which moves by a one-parameter family diffeomorphism and scaling. Over the years, many authors have studied Yamabe solitons [4,6,7,17,10,18,21].

As a generalization of Yamabe soliton, recently, B. Y. Chen and S. Deshmukh [7] introduced the notion of quasi-Yamabe soliton. A Riemannian metric (M, g) is said to be a quasi-yamabe soliton if

$$\frac{1}{2}\mathcal{L}_V g = (r - \sigma)g + \mu V^* \otimes V^*, \tag{1.7}$$

for some smooth function μ, real constant σ and V^* is the dual 1-form of V. If $\mu = 0$ then the quasi-Yamabe soliton (g, V, σ, μ) reduces to the Yamabe soliton (g, V, σ). Chen-Deshmukh [7] proved that a Euclidean hypersurface is totally umbilical if and only if it admits a Yamabe soliton with the tangential component of the position vector field as the soliton vector field. In [9], authors showed that if a contact metric manifold admits a quasi-Yamabe soliton whose soliton vector field is a V-Ric vector field, then the Ricci operator Q and ϕ commutes with each other. A. M. Blaga [5] investigated almost quasi-yamabe solitons on warped product manifolds and derived a Bochner-type formula for a gradient almost quasi-Yamabe soliton.

Motivated by the above studies, we study conformal Ricci soliton and quasi-Yamabe soliton in generalized Sasakian space forms. The paper is organized as follows: After the introduction, Section-2, we discuss some basic notions and curvature formulas of generalized Sasakian space forms. Section-3, we study generalized Sasakian space forms admitting conformal Ricci soliton and conformal gradient Ricci soliton. Finally, Section-4, we investigate quasi-Yamabe solitons on generalized Sasakian space forms.

2. Generalized Sasakian space form

A $2n+1$-dimensional smooth Riemannian manifold (M, g) is said to be an almost contact metric manifold [5] if it admits a $(1,1)$ tensor field ϕ, a characteristic vector field ξ, a global 1-form η and a Riemannian metric g on M satisfying the following relations

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$\eta(X) = g(X, \xi), \quad \phi(\xi) = 0, \quad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \phi Y) + g(Y, \phi X) = 0, \tag{2.4}$$

for all vector fields $X, Y \in TM$, where TM is the tangent bundle of the manifold M.

A Sasakian manifold with constant ϕ-sectional curvature c is called a Sasakian space form. Similarly, a Kenmotsu space form is a Kenmotsu manifold with constant ϕ-sectional curvature c. As a natural generalization of these spaces, P. Alegre, D. E. Blair and A. Carriazo [1] introduced the concept of generalized Sasakian space form. An almost contact metric manifold (M, g, ϕ, ξ, η) is called a generalized Sasakian space form if there exist three smooth functions $f_1,\ f_2,\ f_3$ on M such that the curvature tensor R satisfies

$$\begin{aligned} R(X, Y)Z = {} & f_1[g(Y, Z)X - g(X, Z)Y] \\ & + f_2[g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z] \\ & + f_3[g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\ & + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X], \end{aligned} \tag{2.5}$$

for all vector fields $X, Y, Z \in \Gamma(TM)$. In particular, for $f_1 = \frac{c+3}{4},\ f_2 = f_3 = \frac{c-1}{4}$ M becomes a Sasakian space form. Again, if $f_1 = \frac{c-3}{4},\ f_2 = f_3 = \frac{c+1}{4}$ then M is a Kenmotsu space form. M is a cosymplectic space form if $f_1 = f_2 = f_3 = \frac{c}{4}$. In [2] the authors constructed various examples of generalized Sasakian space forms and showed that any three dimensional trans-Sasakian manifold under certain conditions is a generalized Sasakian space form. U. C. De and A. Sarkar [8] proved that a conformally flat generalized Sasakian space form is locally ϕ-symmetric if and only if f_1 is constant. From hereon, throughout this article, the notation $M(f_1, f_2, f_3)$ will be used to denote a $(2n+1)$-dimensional generalized Sasakian space form with $f_1 \neq f_3$ in general.

Now a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$, we have the following relations from [1];

$$\nabla_X \xi = (f_3 - f_1)\phi(X), \tag{2.6}$$

$$(\nabla_X \eta)(Y) = (f_3 - f_1)g(\phi(X), Y), \tag{2.7}$$

$$(\nabla_X \phi)(Y) = (f_3 - f_1)[\eta(Y)X - g(X, Y)\xi], \tag{2.8}$$

$$R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.9}$$

$$R(\xi, X)Y = (f_3 - f_1)[\eta(Y)X - g(X, Y)\xi], \tag{2.10}$$

$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.11}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.12}$$

$$S(X, \xi) = 2n(f_1 - f_3)\eta(X), \tag{2.13}$$

$$Q\xi = 2n(f_1 - f_3)\xi, \tag{2.14}$$

for all vector fields X, Y in TM and where R is the curvature tensor, S is the Ricci tensor and Q is the Ricci operator respectively.

Definition 2.1. A smooth vector field V on a $(2n+1)$-dimensional Riemannian manifold (M, g) is said to be a conformal vector field [23,24] if

$$\mathcal{L}_V g = 2\rho g, \tag{2.15}$$

for some smooth function ρ on M.

Definition 2.2. A smooth vector field V on a contact metric manifold is said to be an infinitesimal contact transformation [22] if $\mathcal{L}_V \eta = h\eta$ for some smooth function h and $\mathcal{L}_V \eta$ denotes the Lie derivative of η by V. In particular, if $h = 0$, then V is said to be a strict infinitesimal contact transformation.

3. Conformal Ricci soliton on a generalized Sasakian space form

In this section we characterize a generalized Sasakian space form admitting a conformal Ricci soliton with various conditions on the potential vector field. Then we study conformal gradient Ricci soliton on a generalized Sasakian space form. So let us first state our first result of this section:

Theorem 3.1. *If a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a conformal Ricci soliton (g, V, λ) then the soliton is*

1. *shrinking if $p < [4n(f_3 - f_1) - \frac{2}{2n+1}]$,*
2. *steady if $p = [4n(f_3 - f_1) - \frac{2}{2n+1}]$ and*
3. *expanding if $p > [4n(f_3 - f_1) - \frac{2}{2n+1}]$.*

Proof. Then for all vector fields X, Y in TM, from (1.3) we have

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) = [2\lambda - (p + \frac{2}{2n+1})]g(X, Y). \tag{3.1}$$

Now consider a $(0, 2)$ tensor field defined by

$$\mathfrak{T} = \mathcal{L}_V g + 2S. \tag{3.2}$$

It can be easily seen that the tensor field $\mathfrak{T}$ is a symmetric tensor field. Again since g is a metric connection, we have $\nabla g = 0$ and hence from (3.1) note that $\mathcal{L}_V g + 2S$ is parallel with the Levi-Civita connection. Therefore (3.2) implies that $\mathfrak{T}$ is a parallel, symmetric $(0, 2)$ tensor field. Thus we have $\nabla\mathfrak{T} = 0$, which can be written as

$$\mathfrak{T}(R(X, Y)Z, W) + \mathfrak{T}(Z, R(X, Y)W) = 0$$

Putting $X = W = Z = \xi$ in the above equation and using (2.10) we get

$$\mathfrak{T}(Y, \xi) = \mathfrak{T}(\xi, \xi)\eta(Y) \tag{3.3}$$

Taking covariant differentiation of (3.3) along arbitrary vector field X, then recalling (2.6) and (2.7) we obtain

$$\mathfrak{T}(\nabla_X Y, \xi) + (f_3 - f_1)\mathfrak{T}(Y, \phi X) = \mathfrak{T}(\xi, \xi)[(f_3 - f_1)g(\phi X, Y) + \eta(\nabla_X Y)].$$

In view of (3.3) the above equation reduces to

$$\mathfrak{T}(Y, \phi X) = \mathfrak{T}(\xi, \xi)g(\phi X, Y).$$

Replacing X by ϕX in the foregoing equation and then using (3.3) we arrive at

$$\mathfrak{T}(X, Y) = \mathfrak{T}(\xi, \xi)g(X, Y). \tag{3.4}$$

Again from (3.2) we can write

$$\mathfrak{T}(X, Y) = (\mathcal{L}_V g)(X, Y) + 2S(X, Y).$$

Taking $X = Y = \xi$ in above, the using (2.7) and (2.13) yields

$$\mathfrak{T}(\xi, \xi) = 4n(f_1 - f_3).$$

Using the above value in (3.4) and then recalling (3.2) we get

$$(\mathcal{L}_V g)(X,Y)+2S(X,Y)=4n(f_1-f_3)g(X,Y). \tag{3.5}$$

Finally equating (3.1) and (3.5) we obtain

$$\lambda=2n(f_1-f_3)+(\frac{p}{2}+\frac{1}{2n+1}). \tag{3.6}$$

Hence the soliton is shrinking if $\lambda>0$, steady if $\lambda=0$ and expanding if $\lambda<0$. This completes the proof. □

Again for a Sasakian space form $f_1=\frac{c+3}{4}$ and $f_3=\frac{c-1}{4}$, then from (3.6) we get $\lambda=2n+(\frac{p}{2}+\frac{1}{2n+1})$. Thus we have the following

Corollary 3.2. *A conformal Ricci soliton in a* $(2n+1)$*-dimensional Sasakian space form is shrinking if* $(p+4n+\frac{2}{2n+1})<0$, *steady if* $(p+4n+\frac{2}{2n+1})=0$ *and expanding if* $(p+4n+\frac{2}{2n+1})>0$.

Similarly in a Kenmotsu space form $f_1=\frac{c-3}{4}$ and $f_3=\frac{c+1}{4}$, then from (3.6) we deduce $\lambda=-2n+(\frac{p}{2}+\frac{1}{2n+1})$. Thus we have the following

Corollary 3.3. *A conformal Ricci soliton in a* $(2n+1)$*-dimensional Kenmotsu space form is shrinking if* $(p-4n+\frac{2}{2n+1})<0$, *steady if* $(p-4n+\frac{2}{2n+1})=0$ *and expanding if* $(p-4n+\frac{2}{2n+1})>0$.

Now we consider a conformal Ricci soliton (g,V,λ) with V pointwise collinear with the Reeb vector field ξ. In this regard our next theorem is

Theorem 3.4. *Let* $M(f_1,f_2,f_3)$ *be a* $(2n+1)$*-dimensional generalized Sasakian space form admitting a conformal Ricci soliton* (g,V,λ), *whose potential vector field* V *is pointwise collinear with the Reeb vector field* ξ. *Then* V *is a constant multiple of* ξ *and* M *is an Einstein manifold of scalar curvature* $r=2n(2n+1)(f_1-f_3)$.

Proof. Let us assume that $V=b\xi$ for some smooth function b, then from (1.3) we can write

$$bg(\nabla_X\xi,Y)+X(b)\eta(Y)+bg(X,\nabla_Y\xi)+Y(b)\eta(X)=[2\lambda-(p+\frac{2}{2n+1})]g(X,Y)-2S(X,Y).$$

Using (2.6) the foregoing equation reduces to

$$X(b)\eta(Y)+Y(b)\eta(X)+2S(X,Y)=[2\lambda-(p+\frac{2}{2n+1})]g(X,Y). \tag{3.7}$$

Replacing Y by ξ in (3.7) and recalling (2.13) we have

$$X(b)=[2\lambda-(p+\frac{2}{2n+1})-4n(f_1-f_3)-\xi(b)]\eta(X). \tag{3.8}$$

Taking $X=\xi$ in the previous equation yields

$$\xi(b)=[\lambda-(\frac{p}{2}+\frac{1}{2n+1})-2n(f_1-f_3)]. \tag{3.9}$$

In view of (3.9), the equation (3.8) becomes

$$db=[\lambda-(\frac{p}{2}+\frac{1}{2n+1})-2n(f_1-f_3)]\eta. \tag{3.10}$$

Operating d on both sides of (3.10) and using Poincare lemma $d^2=0$ we get

$$[\lambda-(\frac{p}{2}+\frac{1}{2n+1})-2n(f_1-f_3)]d\eta=0.$$

But as $d\eta\neq 0$, the foregoing equation gives us

$$\lambda=2n(f_1-f_3)+(\frac{p}{2}+\frac{1}{2n+1}). \tag{3.11}$$

Now using (3.11) in (3.11) we obtain $db=0$, which eventually implies that

$$b=constant. \tag{3.12}$$

Therefore V is a constant multiple of ξ. This proves first part of the theorem.

Again considering an orthonormal basis $\{e_i : 1 \leq i \leq (2n+1)\}$ of the tangent space at each point of the manifold and then putting $X = Y = e_i$ in (3.7) and summing over $1 \leq i \leq (2n+1)$ we get

$$\xi(b) + r = (2n+1)[\lambda - (\frac{p}{2} + \frac{1}{2n+1})].$$

Using (3.12) in the previous equation infers that

$$r = (2n+1)[\lambda - (\frac{p}{2} + \frac{1}{2n+1})]. \tag{3.13}$$

Combining (3.11) and (3.13) we obtain

$$r = 2n(2n+1)(f_1 - f_3). \tag{3.14}$$

Also recalling (3.7) and then using (3.12) we get

$$S(X, Y) = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]g(X, Y). \tag{3.15}$$

Thus in view of (3.14) and (3.15) we can conclude that the manifold M is an Einstein manifold of scalar curvature $r = 2n(2n+1)(f_1 - f_3)$, which proves the second part of the theorem. Hence completes the proof. □

Corollary 3.5. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional Sasakian space form admitting a conformal Ricci soliton (g, V, λ), whose potential vector field V is pointwise collinear with the Reeb vector field ξ. Then V is a constant multiple of ξ and the manifold M is an Einstein manifold of constant scalar curvature $r = 2n(2n+1)$.*

Corollary 3.6. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional Kenmotsu space form admitting a conformal Ricci soliton (g, V, λ), whose potential vector field V is pointwise collinear with the Reeb vector field ξ. Then V is a constant multiple of ξ and the manifold M is an Einstein manifold of constant scalar curvature $r = -2n(2n+1)$.*

Next we characterize the potential vector field V of a conformal Ricci soliton (g, V, λ) on a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ that satisfies the Ricci semi-symmetric curvature condition. Regarding this we prove the following:

Theorem 3.7. *If a $(2n+1)$-dimensional Ricci semi-symmetric generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a conformal Ricci soliton (g, V, λ), then M is an Einstein manifold and the potential vector field V is a conformal vector field.*

Proof. Let us assume that $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian space form admitting a conformal Ricci soliton (g, V, λ), and the manifold is Ricci semi-symmetric. Then we have $R(X, Y) \cdot S = 0$, which can be written as

$$S(R(X, Y)Z, U) + S(Z, R(X, Y)U) = 0.$$

Replacing U by ξ in above yields

$$S(R(X, Y)Z, \xi) + S(Z, R(X, Y)\xi) = 0.$$

Using (2.9) and (2.14) in the previous equation we obtain

$$2n(f_1 - f_3)\eta(R(X, Y)Z) + (f_1 - f_3)\eta(Y)S(X, Z) - (f_1 - f_3)\eta(X)S(Y, Z) = 0.$$

Taking $X = \xi$ in the foregoing equation, then recalling (2.10) and (2.13) infers that

$$S(Y, Z) = 2n(f_1 - f_3)g(Y, Z), \tag{3.16}$$

which implies that the manifold is an Einstein manifold and this proves the first part of the theorem.

Again, as (g, V, λ) is a conformal Ricci soliton on the $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$, recalling the soliton equation (1.3) we have

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) = [2\lambda - (p + \frac{2}{2n+1})]g(X, Y).$$

Now in view of (3.16) the previous equation reduces to

$$(\mathcal{L}_V g)(X, Y) = [2\lambda - 4n(f_1 - f_3) - (p + \frac{2}{2n+1})]g(X, Y), \tag{3.17}$$

for all vector fields X, Y in TM. Thus from (3.17) it can be written that

$$\mathcal{L}_V g = 2\rho g,$$

where $\rho = [\lambda - 2n(f_1 - f_3) - (\frac{p}{2} + \frac{1}{2n+1})]$. Thus in view of the equation (2.15), we can conclude that V is a conformal vector field. This completes the proof. □

In the last part of this section, we now concentrate on the generalized Sasakian space form $M(f_1, f_2, f_3)$ admitting conformal Ricci soliton whose potential vector field V is gradient of some smooth function f. But before proving our main theorem in this direction, let us first prove the following

Lemma 3.8. *If a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a conformal gradient Ricci soliton $(g, \nabla f, \lambda)$, then the curvature tensor R satisfies*

$$\begin{aligned} R(X,Y)\nabla f &= (2ndf_1 + 3df_2 - df_3)(Y)X - (3df_2 + (2n-1)df_3)(Y)\eta(X)\xi \\ &+ (3df_2 + (2n-1)df_3)(X)\eta(Y)\xi - (2ndf_1 + 3df_2 - df_3)(X)Y \\ &+ (f_1 - f_3)(3f_2 + (2n-1)f_3)[g(X,\phi Y)\xi - g(\phi X, Y)\xi \\ &+ \eta(X)\phi Y - \eta(Y)\phi X], \end{aligned} \tag{3.18}$$

for all vector fields X, Y on the manifold.

Proof. Let us assume that $(g, \nabla f, \lambda)$ be a conformal gradient Ricci soliton on $M(f_1, f_2, f_3)$. Then from the conformal gradient Ricci soliton equation (1.4) we can write

$$\nabla_X \nabla f = [\lambda - 2n(f_1 - f_3) - (\frac{p}{2} + \frac{1}{2n+1})]X - QX, \tag{3.19}$$

for any vector field X on M and Q is the Ricci operator.
Taking covariant differentiation of (3.19) along an arbitrary vector field Y we obtain

$$\nabla_Y \nabla_X \nabla f = [\lambda - 2n(f_1 - f_3) - (\frac{p}{2} + \frac{1}{2n+1})]\nabla_Y X - \nabla_Y QX \tag{3.20}$$

Interchanging X and Y in the foregoing equation infers that

$$\nabla_X \nabla_Y \nabla f = [\lambda - 2n(f_1 - f_3) - (\frac{p}{2} + \frac{1}{2n+1})]\nabla_X Y - \nabla_X QY. \tag{3.21}$$

Also from (3.19) it can be written that

$$\nabla_{[X,Y]} \nabla f = [\lambda - 2n(f_1 - f_3) - (\frac{p}{2} + \frac{1}{2n+1})](\nabla_X Y - \nabla_Y X) - Q(\nabla_X Y - \nabla_Y X). \tag{3.22}$$

Now using (3.20)–(3.22) in $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ we obtain

$$R(X,Y)\nabla f = (\nabla_Y Q)X - (\nabla_X Q)Y. \tag{3.23}$$

Again recalling (2.12) and covariantly differentiating it along Y yields

$$\begin{aligned} \nabla_Y QX &= (2nf_1 + 3f_2 - f_3)\nabla_Y X + (2ndf_1 + 3df_2 - df_3)(Y)X \\ &- (3f_2 + (2n-1)f_3)[\nabla_Y \eta(X)\xi + \eta(X)\nabla_Y \xi] \\ &- (3df_2 + (2n-1)df_3)(Y)\eta(X)\xi. \end{aligned} \tag{3.24}$$

Also from (2.12) we can write

$$Q(\nabla_Y X) = (2nf_1 + 3f_2 - f_3)\nabla_Y X - (3f_2 + (2n-1)f_3)\eta(\nabla_Y X)\xi. \tag{3.25}$$

Using (3.24) and (3.25) in $(\nabla_Y Q)X = \nabla_Y QX - Q(\nabla_Y X)$, then recalling (2.6) and (2.7) we obtain

$$\begin{aligned}(\nabla_Y Q)X = &(2ndf_1 + 3df_2 - df_3)(Y)X\\ &-(3df_2 + (2n-1)df_3)(Y)\eta(X)\xi\\ &+(f_1 - f_3)(3f_2 + (2n-1)f_3)[g(X, \phi Y)\xi + \eta(X)\phi Y].\end{aligned} \tag{3.26}$$

Interchanging X and Y in (3.26) yields

$$\begin{aligned}(\nabla_X Q)Y = &(2ndf_1 + 3df_2 - df_3)(X)Y\\ &-(3df_2 + (2n-1)df_3)(X)\eta(Y)\xi\\ &+(f_1 - f_3)(3f_2 + (2n-1)f_3)[g(\phi X, Y)\xi + \eta(Y)\phi X].\end{aligned} \tag{3.27}$$

Finally making use of (3.26) and (3.27) in the equation (3.23) completes the proof. □

Now we conclude this section with our main result on conformal gradient Ricci soliton which is the following:

Theorem 3.9. *If a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a conformal gradient Ricci soliton $(g, \nabla f, \lambda)$ then the potential function f is constant, provided f_1 and f_3 are both constants. Furthermore, the soliton is shrinking if $p < [2f_3 - 6f_2 - 2nf_1 - \frac{2}{2n+1}]$, steady if $p = [2f_3 - 6f_2 - 2nf_1 - \frac{2}{2n+1}]$ or, expanding if $p > [2f_3 - 6f_2 - 2nf_1 - \frac{2}{2n+1}]$.*

Proof. Let us assume that $(g, \nabla f, \lambda)$ is a conformal gradient Ricci soliton on the generalized Sasakian space form $M(f_1, f_2, f_3)$. Now putting $X = \xi$ in (3.18) of Lemma (3.8) and then taking inner product with arbitrary vector field Z we obtain

$$\begin{aligned}g(R(\xi, Y)\nabla f, Z) = &2n(df_1 - df_3)(Y)\eta(Z)\\ &-2n(df_1 - df_3)(\xi)[g(Y, Z) - \eta(Y)\eta(Z)]\\ &+(f_1 - f_3)(3f_2 + (2n-1)f_3)g(\phi Y, Z),\end{aligned} \tag{3.28}$$

for all vector fields Y, Z on the manifold.
Again in view of (2.10) and making use of the curvature property $g(R(\xi, Y)\nabla f, Z) = -g(R(\xi, Y)Z, Df)$, we can write

$$g(R(\xi, Y)Df, Z) = (f_3 - f_1)[g(Y, Z)(\xi f) - \eta(Z)(Yf)]. \tag{3.29}$$

Equating (3.28) and (3.29) we deduce

$$\begin{aligned}(f_3 - f_1)[g(Y, Z)(\xi f) - \eta(Z)(Yf)] = &2n(df_1 - df_3)(Y)\eta(Z)\\ &-2n(df_1 - df_3)(\xi)[g(Y, Z) - \eta(Y)\eta(Z)]\\ &+(f_1 - f_3)(3f_2 + (2n-1)f_3)g(\phi Y, Z).\end{aligned}$$

Now replacing Z by ξ, the foregoing equation infers that

$$2n(df_1 - df_3)(Y) = (f_3 - f_1)[\eta(Y)(\xi f) - (Yf)],$$

which reduces to

$$[\eta(Y)(\xi f) - (Yf)] = 0,$$

provided f_1 and f_3 are both constants. Also this can be rewritten as

$$g(Y, (\xi f)\xi) = g(Y, \nabla f).$$

Since the above holds for all vector field Y on the manifold, we obtain

$$\nabla f = (\xi f)\xi. \tag{3.30}$$

Differentiating (3.30) covariantly along arbitrary vector field X and then using 2.6 yields

$$\nabla_X \nabla f = (X(\xi f))\xi + (f_3 - f_1)(\xi f)\phi X. \tag{3.31}$$

Equating (3.31) with (3.19) we deduce

$$QX = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]X - (X(\xi f))\xi - (f_3 - f_1)(\xi f)\phi X. \tag{3.32}$$

Comparing the coefficients of ϕX from (3.32) and (2.12) we get $(\xi f) = 0$. Using this in (3.30) infers that $Df = grad\ f = 0$, which eventually implies that f is constant. This proves first part of the theorem.

Again comparing the coefficients of X from (3.32) and (2.12) we obtain

$$\lambda = [(\frac{p}{2} + \frac{1}{2n+1}) + (2nf_1 - 3f_2 - f_3)]. \tag{3.33}$$

Hence the soliton is shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$. This completes the proof. □

4. Quasi-Yamabe soliton on generalized Sasakian space form

This section is devoted to the study of generalized Sasakian space form admitting quasi-Yamabe soliton whose the potential vector field is pointwise collinear with the Reeb vector field. In this regard, our main result of this section is as follows:

Theorem 4.1. *If a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a quasi-Yamabe soliton (g, V, σ, μ) with the potential vector field V pointwise collinear with the Reeb vector field ξ, then*

1. *M becomes a manifold of constant scalar curvature,*
2. *the soliton reduces to the Yamabe soliton (g, V, σ),*
3. *V becomes a constant multiple of ξ and*
4. *V is a strict infinitesimal contact transformation.*

Proof. Let us assume that (g, V, σ, μ) is a quasi-Yamabe soliton on the generalized Sasakian space form $M(f_1, f_2, f_3)$ such that the potential vector field V is pointwise collinear with ξ, the there exists a non-zero smooth function b on M such that $V = b\xi$. Then from the equation (1.7) we can write

$$\frac{1}{2}(\mathcal{L}_{b\xi} g)(X, Y) = (r - \sigma)g(X, Y) + \mu b^2\eta(X)\eta(Y), \tag{4.1}$$

for all vector fields X, Y in TM.
Again from the definition of Lie derivative we have

$$\begin{aligned}(\mathcal{L}_{b\xi} g)(X, Y) &= g(\nabla_X b\xi, Y) + g(X, \nabla_Y b\xi)\\ &= bg(\nabla_X \xi, Y) + X(b)\eta(Y) + bg(X, \nabla_Y \xi) + Y(b)\eta(X),\end{aligned}$$

which in view of (2.6) and (2.4) reduces to

$$(\mathcal{L}_{b\xi} g)(X, Y) = X(b)\eta(Y) + Y(b)\eta(X). \tag{4.2}$$

Substituting (4.2) in (4.1) infers that

$$X(b)\eta(Y) + Y(b)\eta(X) = 2(r - \sigma)g(X, Y) + 2\mu b^2\eta(X)\eta(Y). \tag{4.3}$$

Now taking $Y = \xi$ in (4.3) we obtain

$$X(b) = [2(r - \sigma + \mu b^2) - \xi(b)]\eta(X). \tag{4.4}$$

Again replacing X by ξ in the foregoing equation yields

$$\xi(b) = (r - \sigma) + \mu b^2. \tag{4.5}$$

Now consider an orthonormal basis $\{e_i : 1 \le i \le (2n+1)\}$ of the tangent space at each point of the manifold. Then setting $X = Y = e_i$ in (4.3) and summing over $1 \le i \le (2n+1)$ we get

$$\xi(b) = (r - \sigma)(2n+1) + \mu b^2. \tag{4.6}$$

Equating (4.5) with (4.6) we arrive at

$$r = \sigma = constant, \tag{4.7}$$

which implies that M is a manifold of constant scalar curvature and this proves (1).

Again, in view of (4.5), the equation (4.4) gives us

$$db = \mu b^2 \eta. \tag{4.8}$$

Taking exterior differentiation on the above equation and using Poincare lemma $d^2 = 0$, we obtain $\mu b^2 = 0$. This eventually implies that $\mu = 0$ and hence the quasi-Yamabe soliton reduces to the Yamabe soliton (g, V, σ). This proves (2) of the theorem.

Now using $\mu = 0$ in the equation (4.8) we get $db = 0$, which implies b is constant. Therefore V is a constant multiple of ξ and this proves (3).

Again setting $Y = \xi$ in (4.1) and using (2.2) we obtain

$$(\mathcal{L}_V g)(X, \xi) = 2(r - \sigma + \mu b^2)\eta(X). \tag{4.9}$$

Using (4.7) and the fact that $\mu = 0$ in (4.9) yields $(\mathcal{L}_V g)(X, \xi) = 0$, which implies

$$(\mathcal{L}_V \eta)(X) = g(X, \mathcal{L}_V \xi). \tag{4.10}$$

Recalling that $V = b\xi$ and b is a constant it can be easily deduced that $\mathcal{L}_V \xi = 0$. Therefore from (4.10) finally we obtain $(\mathcal{L}_V \eta)(X) = 0$ for any vector field X on M. Hence in the sense of definition (2.2), the potential vector field V is a strict infinitesimal contact transformation. This proves (4) and hence completes the proof. □

According to Corollary 1.1 of [17], in a generalized Sasakian space form $M(f_1, f_2, f_3)$ with the Yamabe soliton metric, the scalar curvature is harmonic. Thus in view of this and from our previous theorem we can conclude the following

Corollary 4.2. *If a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a quasi-Yamabe soliton (g, V, λ), whose potential vector field V is pointwise collinear with the Reeb vector field ξ, then the scalar curvature is harmonic.*

5. Examples of generalized Sasakian space form

In this section we discuss examples of generalized Sasakian space form admitting conformal Ricci soliton and Yamabe soliton.

Example 5.1. P. Alegre, D. E. Blair and A. Carriazo in their seminal work [1] constructed an example of generalized Sasakian space form on the manifold $\mathbb{R} \times \mathbb{C}^m$ endowed with three smooth functions given as follows:

$$f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f'}, \tag{5.1}$$

for some smooth real valued function $f = f(t)$ and f' denotes the derivative of f with respect to t.

Now if we consider $f(t) = e^{\alpha t}$, for some real number α, then from equation (3.6) we can compute $\lambda = -m\alpha + \frac{p}{2} + \frac{1}{2m+1}$. Therefore we can comment that the generalized Sasakian space form $(\mathbb{R} \times \mathbb{C}^m, f_1, f_2, f_3)$ admits a conformal Ricci soliton with the soliton constant λ as computed above. Furthermore the conformal Ricci soliton is shrinking if $p < [2m\alpha - \frac{2}{m+1}]$, steady if $p = [2m\alpha - \frac{2}{m+1}]$ and expanding if $p > [2m\alpha - \frac{2}{m+1}]$.

Example 5.2. Here we give a non-trivial example of a conformal Ricci soliton on a three dimensional generalized Sasakian space form as constructed in [19]. Let us consider the 3-dimensional manifold $M = \{(u, v, w) \in \mathbb{R}^3 \setminus \{(0,0,0)\}\}$. Define a linearly independent set of vector fields $\{E_i : 1 \le i \le 3\}$ on the manifold M given by

$$E_1 = \frac{\partial}{\partial u} - v\frac{\partial}{\partial w}, \quad E_2 = \frac{\partial}{\partial v}, \quad e_3 = \frac{\partial}{\partial w}.$$

Let us define the Riemannian metric g on M by

$$g(E_i, E_j) = \begin{cases} 1, & \text{for} \quad i = j \\ 0, & \text{for} \quad i \neq j \end{cases}$$

for all $i, j = 1, 2, 3$. Now considering $E_3 = \xi$, let us take the 1-form η, on the manifold M, defined by

$$\eta(U) = g(U, E_3), \quad \forall U \in TM.$$

Then it can be observed that $\eta(\xi) = 1$. Let us define the $(1,1)$ tensor field ϕ on M as

$$\phi(E_1) = -E_2, \quad \phi(E_2) = E_1, \quad \phi(E_3) = 0.$$

Using the linearity of g and ϕ it can be easily checked that

$$\phi^2(U) = -U + \eta(U)\xi, \quad g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V), \quad \forall U, V \in TM.$$

Hence the structure (g, ϕ, ξ, η) defines an almost contact metric structure on the manifold M. Now, using the definitions of Lie bracket, after some direct computations we get $[E_1, E_2] = E_3$ and $[e_1, e_3] = [e_2, e_3] = 0$. Again the Riemannian connection ∇ of the metric g is defined by the well-known Koszul's formula which is given by

$$\begin{aligned}2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y)\\ &\quad - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]).\end{aligned}$$

Using the above formula one can easily calculate that

$$\begin{aligned}&\nabla_{E_1}E_1 = 0, \quad \nabla_{E_1}E_2 = \frac{1}{2}E_3, \quad \nabla_{E_1}E_3 = -\frac{1}{2}E_2,\\ &\nabla_{E_2}E_1 = -\frac{1}{2}E_3, \quad \nabla_{E_2}E_2 = 0, \quad \nabla_{E_2}E_3 = \frac{1}{2}E_1,\\ &\nabla_{E_3}E_1 = -\frac{1}{2}E_2, \quad \nabla_{E_3}E_2 = \frac{1}{2}E_1, \quad \nabla_{E_3}E_3 = 0.\end{aligned}$$

Thus from the above relations and using the well-known formula $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ the non-vanishing components of the Riemannian curvature tensor R can easily be computed as

$$\begin{aligned}&R(E_1,E_2)E_1 = \frac{3}{4}E_2, \quad R(E_1,E_3)E_1 = -\frac{1}{4}E_3, \quad R(E_2,E_2)E_3 = \frac{1}{4}E_2,\\ &R(E_1,E_2)E_2 = -\frac{3}{4}E_1, \quad R(E_2,E_3)E_2 = -\frac{1}{4}E_3, \quad R(E_1,E_3)E_3 = \frac{1}{4}E_1.\end{aligned}$$

Hence we can calculate the non-vanishing components of the Ricci tensor as follows

$$S(E_1,E_1) = -\frac{1}{2}, \quad S(E_2,E_2) = -\frac{1}{2}, \quad S(E_3,E_3) = \frac{1}{2}.$$

Therefore in view of the above values of the Ricci tensor, we can say that the manifold M is a generalized Sasakian space form with the functions $f_1 = -\frac{1}{4}$, $f_2 = 0$ and $f_3 = -\frac{1}{3}$.

Now if we take the soliton vector field $V = \xi = E_3$, then from the equation (1.3) we obtain $\lambda = (\frac{p}{2} - \frac{1}{6})$. Hence for this value of λ the data (g, ξ, λ) defines a conformal Ricci soliton on the generalized Sasakian space form $M(f_1, f_2, f_3)$. Moreover we can see that (M, g) is a manifold of constant scalar curvature $r = -\frac{1}{2} = 2 \times 3 \times (f_1 - f_3)$ and hence the theorem (3.4) is verified.

Again on this generalized Sasakian space form $M(f_1, f_2, f_3)$, considering $V = \xi$ in the equation (1.7), we compute that $\sigma = -\frac{1}{2}$ and $\mu = 0$. Therefore for this values of σ and μ the data (g, ξ, σ, μ) defines a quasi-Yamabe soliton, which eventually reduces to a Yamabe soliton as $\mu = 0$ and hence the theorem (4.1) is verified.

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CONFORMAL RICCI SOLITON ON ALMOST CO-KÄHLER MANIFOLD

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ABSTRACT. In this paper, we study almost coKähler manifolds admitting the conformal Ricci soliton and determine the value of the soliton constant λ and hence the condition for the soliton to be shrinking, steady or expanding. Then we find the condition on the conformal pressure p under which, a conformal Ricci soliton on a (k,μ)-almost coKähler manifold becomes expanding. Finally we show that a (k,μ)-almost coKähler manifold, with the potential vector field V pointwise collinear with the Reeb vector field ξ, does not admit conformal gradient Ricci soliton.

1. INTRODUCTION

A Riemannian metric g defined on a smooth manifold M^n,of dimension n, is said to be a Ricci soliton if for some constant λ, there exists a smooth vector field X on M satisfying the equation

$$Ric + \frac{1}{2}\mathcal{L}_V g = \lambda g, \tag{1.1}$$

where $\mathcal{L}_V$ denotes the Lie derivative in the direction of V and Ric is the Ricci tensor. The Ricci soliton is called shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$. In 1982, R.S. Hamilton [11] first studied the Ricci soliton as a self similar solution to the Ricci flow equation given by: $\frac{\partial}{\partial t}(g(t)) = -2Ric(g(t))$, where $g(t)$ is a one parameter family of metrics on M^{2n+1}.

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Ricci solitons can also be viewed as natural generalizations of Einstein metrics which moves only by a one-parameter group of diffeomorphisms and scaling [12]. Again a Ricci soliton is called a gradient Ricci soliton [3] if the concerned vector field V in the equation (1.1), is the gradient of some smooth function f, i.e; if $V = Df$, where D is the gradient operator of g. This function f is called the potential function of the Ricci soliton.

A.E. Fisher, in 2005, introduced [9] conformal Ricci flow equation which is a modified version of the Hamilton's Ricci flow equation that modifies the volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow equations on a smooth closed connected oriented manifold M^n, of dimension n, are given by

$$\frac{\partial g}{\partial t} + 2\left(Ric + \frac{g}{n}\right) = -pg, \tag{1.2}$$

$$r(g) = -1,$$

where p is a non-dynamical(time dependent) scalar field and $r(g)$ is the scalar curvature of the manifold. The term $-pg$ acts as the constraint force to maintain the scalar curvature constraint. Thus these evolution equations are analogous to famous Navier-Stokes equations in fluid mechanics where the constraint is divergence free. That is why sometimes p is also called the conformal pressure.

Recently, in 2015, N. Basu and A. Bhattacharyya [2] introduced the concept of conformal Ricci soliton as a generalization of the classical Ricci soliton.

Definition 1.1. *A Riemannian metric g on a smooth manifold M^n,of dimension n, is called a conformal Ricci soliton if there exists a constant λ and a vector field V such that*

$$\mathcal{L}_V g + 2S = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g, \tag{1.3}$$

where $S = Ric$ is the Ricci tensor, λ is a constant and p is the conformal pressure.

It can be easily checked that the above soliton equation satisfies the conformal Ricci flow equation (1.2). Later, T. Dutta. et.al. [7] studied this conformal Ricci soliton in the framework of Lorentzian α-Sasakian manifolds. Moreover, if the vector field V is the gradient of some smooth function f on M^n, we call the soliton a conformal gradient Ricci soliton and then the soliton equation (1.2) becomes

$$S + \nabla\nabla f = \left[\lambda - \left(\frac{p}{2} + \frac{1}{n}\right)\right] g, \tag{1.4}$$

where ∇ is the Riemannian connection on the manifold M^n.

Motivated by the above studies, here we study conformal Ricci soliton in the framework of almost coKähler manifold and on its various versions. We find conditions to determine the nature of the soliton for different cases. The paper is organised as follows: in section-2, we discuss some preliminary concepts of almost coKähler manifolds. Then in section-3, we study almost coKähler manifolds admitting the conformal Ricci soliton and we calculate the value of the soliton constant λ and hence we find the condition for the soliton to be shrinking, steady or expanding. After that in section-4, we find the condition on the conformal pressure p under which, a conformal Ricci soliton on a (k,μ)-almost coKähler manifold becomes expanding. Finally in section-5, we show that a (k,μ)-almost coKähler manifold, with the potential vector field V pointwise collinear with the Reeb vector field ξ, does not admit conformal gradient Ricci soliton.

2. Preliminaries on almost coKähler manifolds

The geometry of coKähler manifolds as a special case of almost contact manifolds was studied primarily as an odd-dimensional analogy of the Kähler manifolds in complex geometry. So, let us first recall some preliminaries on almost coKähler manifolds. A smooth $(2n+1)$ dimensional manifold M^{2n+1} is said to admit an almost contact structure (ϕ, ξ, η) if there exist a $(1,1)$ tensor field ϕ, a vector field ξ and a global 1-form η on M^{2n+1} such that

$$\phi^2 = -I + \eta \otimes \xi \; and \; \eta(\xi) = 1, \tag{2.1}$$

where I is the identity endomorphism on M. Then the manifold M equipped with this almost contact structure (ϕ, ξ, η) is called an almost contact manifold (see [1]) and is denoted as $(M^{2n+1}\phi, \xi, \eta)$. The vector field ξ is called the characteristic vector field or Reeb vector field.

From (2.1) it can easily be seen that, for an almost contact structure the following relations hold; $\phi(\xi) = 0$ and $\eta \circ \phi = 0$.

Furthermore, on an almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ if there exists a Riemannian metric g satisfying;

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y in TM, where TM is tangent bundle of M, then the metric g is called compatible with the almost contact structure. The manifold M^{2n+1} together with the almost contact metric structure (ϕ, ξ, η, g) is called an almost contact metric manifold and we denote it as $(M^{2n+1}, g, \phi, \xi, \eta)$.

We define the fundamental 2-form Φ on an almost contact metric manifold as

$$\Phi(X, Y) = g(X, \phi Y) = d\eta(X, Y), \tag{2.2}$$

for all vector fields X, Y in TM. Now, it is known that on the product manifold $M^{2n+1} \times \mathbb{R}$, if we define a structure J as;

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

for all X in TM, where t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^{2n+1} \times \mathbb{R}$: then J becomes an almost complex structure and if this almost complex structure J is integrable we say that the almost contact structure $(M^{2n+1}, \phi, \xi, \eta)$ is normal. Again, D.E. Blair [1] expressed the condition for normality of an almost contact structure as: $[\phi, \phi] = -2d\eta \otimes \xi$; where the Nijenhuis tensor $[\phi, \phi]$ is defined as

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y],$$

for all X, Y in TM and [X,Y] is the Lie bracket operation. Now we are in a position to define the concept of coKähler manifold [see [1], [4]] and almost coKähler manifold.

Definition 2.1. *An almost contact metric manifold is called an almost coKähler manifold if both the* 1*-form* η *and the fundamental* 2*-form* Φ *(as defined by equation (2.2)) are closed.*

In particular, if the associated almost contact structure is normal or equivalently $\nabla\phi = 0$ or $\nabla\Phi = 0$: then the almost coKähler manifold is called a coKähler manifold. Also, it is to be noted that, examples (see [5], [13]) of almost coKähler manifolds exist, which are not globally the product of a almost Kähler manifold and the real line.

Next, we set two symmetric operators h and h' given by, $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' = h \circ \phi$ on the almost coKähler manifold $(M^{2n+1}, g, \phi, \xi, \eta)$. Then the following relations can be obtained (see [13], [6])

$$h\xi = 0,\ h\phi + \phi h = 0,\ tr(h) = tr(h') = 0, \tag{2.3}$$

$$\nabla_\xi \phi = 0,\ \nabla \xi = h',\ div\xi = 0, \tag{2.4}$$

$$S(\xi, \xi) + \|h\|^2 = 0, \tag{2.5}$$

$$\phi l \phi - l = 2h^2, \tag{2.6}$$

$$\nabla_\xi h = -h^2\phi - \phi l,$$

where we set $l := R(., \xi)\xi$ and R is the Riemannian curvature tensor defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.7}$$

for all vector fields $X, Y, Z \in TM$.

3. Conformal Ricci Soliton on Almost CoKähler Manifold

Let us consider $(M^{2n+1}, g, \phi, \xi, \eta)$ be an almost coKähler manifold which satisfies the conformal Ricci soliton equation given in equation (1.3); then for all vector fields X, Y in TM i.e; we have

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) = \left[2\lambda - (p + \frac{2}{2n+1})\right] g(X, Y). \tag{3.1}$$

Now, let the vector field V be pointwise collinear with the Reeb vector field ξ, i.e; $V = \beta\xi$, where $\beta$is a non-zero smooth function on the corresponding manifold. Then taking covariant differentiation of both sides of $V = \beta\xi$, along the direction of X we get

$$\nabla_X V = X(\beta)\xi + \beta \nabla_X \xi,$$

and using $\nabla\xi = h'$ from equation (2.4) the above equation eventually becomes

$$\nabla_X V = X(\beta)\xi + \beta h' X. \tag{3.2}$$

On the other hand, from the definiton of Lie derivative it follows from equation (3.1) that

$$g(\nabla_Y \beta\xi, Z) + g(Y, \nabla_Z \beta\xi) + 2S(Y, Z) = \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(Y, Z), \tag{3.3}$$

for all Y, Z in TM. Then using equation (3.2) in the above equation (3.3) we get

$$g(Y\beta\xi + \beta h'Y, Z) + g(Y, Z\beta\xi + \beta h'Z) + 2S(Y, Z) = [2\lambda - (p + \frac{2}{2n+1})]g(Y, Z).$$

Again using from the fact that h' is symmetric and after simplification the above equation finally becomes

$$Y(\beta)\eta(Z) + Z(\beta)\eta(Y) + 2\beta g(h'Y, Z) + 2S(Y, Z) = [2\lambda - (p + \frac{2}{2n+1})]g(Y, Z). \tag{3.4}$$

Next, we consider a local ϕ-basis $\{e_j : 1 \leq j \leq 2n+1\}$ on the tangent space T_pM for each point $p \in M^{2n+1}$. Then putting $Y = Z = e_j$ in the equation (3.4) and taking summation over $1 \leq j \leq 2n+1$ and also using $tr(h') = 0$ from equation (2.3) we get

$$\xi(\beta) + r = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})](2n+1). \tag{3.5}$$

Again putting $Z = \xi$ in the equation (3.4) and using symmetry of h' we have

$$Y(\beta) + \xi(\beta)\eta(Y) + 2S(Y, \xi) = [2\lambda - (p + \frac{2}{2n+1})]\eta(Y). \tag{3.6}$$

Now, combining equations (3.5) and (3.6) and after some calculations we get

$$Y(\beta) + 2S(Y, \xi) = [[\lambda - (\frac{p}{2} + \frac{1}{2n+1})](1-2n) + r]\eta(Y).$$

Thus, from the above it is easily seen that

$$\xi(\beta) + 2S(\xi, \xi) = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})](1-2n) + r. \tag{3.7}$$

Eliminating $\xi(\beta)$ from equations (3.5) and (3.7) and after simplification we arrive at

$$2n[\lambda - (\frac{p}{2} + \frac{1}{2n+1})] - r + S(\xi, \xi) = 0.$$

Using equation (2.5) in the above equation and using the fact that for conformal Ricci soliton the scalar curvature $r = -1$ (see equation(1.2)) and then simplifying we get the value of the soliton constant as

$$\lambda = \frac{\|h\|^2 - 1}{2n} + (\frac{p}{2} + \frac{1}{2n+1}). \tag{3.8}$$

Therefore in view of the fact that the soliton is shrinking, steady or expanding according as $\lambda > 0$, $\lambda = 0$ or, $\lambda < 0$; from the above equation (3.8) we can state the following theorem

Theorem 3.1. *Let* $(M^{2n+1}, g, \phi, \xi, \eta)$ *be an almost coKähler manifold such that the metric* g *is a conformal Ricci soliton. If the potential vector field* V *be non-zero pointwise collinear with the Reeb vector field* ξ, *then*

(i) *the soliton is shrinking if,* $p > \frac{1-(2n+1)\|h\|^2}{(2n^2+n)}$,
(ii) *the soliton is steady if,* $p = \frac{1-(2n+1)\|h\|^2}{(2n^2+n)}$,
(iii) *the soliton is expanding if,* $p < \frac{1-(2n+1)\|h\|^2}{(2n^2+n)}$.

Again if we have $S = [\frac{\|h\|^2-1}{2n}]g$, then from equation (3.1) and using value of the soliton constant λ from (3.8) we have $\mathcal{L}_V g = 0$. Therefore we can see that $V = \beta\xi$ is a Killing vector field and hence the soliton becomes trivial. Hence we can state the following corollary.

Corollary 3.1. *Let* $(M^{2n+1}, g, \phi, \xi, \eta)$ *be an almost coKähler manifold such that the metric* g *is a conformal Ricci soliton. If the potential vector field* V *be non-zero pointwise collinear with the Reeb vector field* ξ *and the Ricci tensor* S *be a constant multiple of the metric* g, *with the constant* $\frac{\|h\|^2-1}{2n}$, *(i.e; if* $S = [\frac{\|h\|^2-1}{2n}]g$*), then the soliton is trivial.*

4. Conformal Ricci Soliton on (k, μ)-Almost CoKähler Manifold

In recent years, many authors studied (k, μ)-contact metric manifolds as a generalization of Sasakian and K-contact metric manifolds. Also R. Sharma [15], and later A. Ghosh [10] proved some interesting results in the field of Ricci solitons on (k, μ)-contact metric manifolds. Let us now give the definition (k, μ)-almost coKähler manifold.

Definition 4.1. *An almost coKähler manifold is said to be a* (k, μ)*-almost coKähler manifold if the characteristic vector field* ξ *belongs to the generalised* (k, μ)*-nullity distribution i.e; if the Riemannian curvature tensor* R *satisfies*

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{4.1}$$

for all X, Y *in* TM *and for some smooth functions* (k, μ).

Remark 4.1. *Here, in this paper, we call a* (k, μ)*-almost coKähler manifold with* $k < 0$, *a proper* (k, μ)*-almost coKähler manifold . Proper almost coKähler manifolds with* k *and* μ *being constants were introduced by H. Endo [8] and later Dacko and Olszak [6] further studied it in generalised cases.*

Now, putting $Y=\xi$ in (4.1) we get

$$R(X,\xi)\xi = k[X-\eta(X)\xi]+\mu[hX-\eta(X)h\xi].$$

Then using the definition of $l := R(.,\xi)\xi$ and from equation (2.3) using the fact that $h\xi = 0$, we can write

$$l = -k\phi^2 + \mu h.$$

Combining the equation (2.6) and the above equation and after brief calculations we get $h^2 = k\phi^2$. Thus, it is clear that the manifold M^{2n+1} is K-almost coKähler if and ony if, $k=0$. According to Dacko and Olszak [6] a (k,μ,ν)-almost coKähler manifold with $k<0$ becomes a $(-1,\frac{\mu}{\sqrt{-k}})$-almost coKähler manifold, under some D-homothetic deformation.

Now, we state a lemma [for proof see Lemma 4.1 of [16]] which will be used in the later theorems.

Lemma 4.1. *Let $(M^{2n+1},g,\phi,\xi,\eta)$ be a (k,μ)-almost coKähler manifold of dimension greater than 3 with $k<0$. Then the Ricci operator is given by*

$$Q = \mu h + 2nk\eta\otimes\xi, \tag{4.2}$$

where k is a non-zero constant and μ is a smooth function satisfying $d\mu\wedge\eta = 0$.

Now let us consider the metric g of the (k,μ)-almost coKähler manifold admits a conformal Ricci soliton. Then from the soliton equation (1.3) and using the definition of the Lie derivative we can write

$$g(\nabla_X V,Y)+g(X,\nabla_Y V)+2S(X,Y) = [2\lambda-(p+\frac{2}{2n+1})]g(X,Y). \tag{4.3}$$

Then, substituting $V=\xi$ in the above equation (4.3) and using the result $\nabla\xi = h'$ from (2.4) we get

$$g(h'X,Y)+g(X,h'Y)+2S(X,Y) = [2\lambda-(p+\frac{2}{2n+1})]g(X,Y).$$

Again as h' is symmetric the above equation implies

$$g(h'X,Y)+g(QX,Y) = [\lambda-(\frac{p}{2}+\frac{1}{2n+1})]g(X,Y). \tag{4.4}$$

Now, in view of the Lemma 4.1 putting value of the Ricci operator Q, from equation (4.2), in the above equation (4.4) we get

$$g(h'X,Y)+g(\mu hX,Y)+2nk\eta(X)\eta(Y) = [\lambda-(\frac{p}{2}+\frac{1}{2n+1})]g(X,Y). \tag{4.5}$$

Thus putting $Y = \xi$ in the above (4.5) and using $h\phi + \phi h = 0$ from equation (2.3) we finally get

$$2nk = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]. \tag{4.6}$$

Now, as it is mentioned in the Lemma 4.1 that $k < 0$, so from the above relation (4.6) we can conclude that $[\lambda - (\frac{p}{2} + \frac{1}{2n+1})] < 0$ that is; $\lambda < (\frac{p}{2} + \frac{1}{2n+1})$. Thus if $(\frac{p}{2} + \frac{1}{2n+1}) \leq 0$, i.e; if, $p \leq \frac{-2}{2n+1}$ then $\lambda < 0$ and therefore the soliton is expanding. So, in view of the above we have the following theorem.

Theorem 4.1. *Let $(M^{2n+1}, g, \phi, \xi, \eta)$ be a (k, μ)-almost coKähler manifold of dimension greater than 3 with $k < 0$ and the metric g admits a conformal Ricci soliton. Then the soliton is expanding if the conformal pressure p satisfy the inequality $p \leq \frac{-2}{2n+1}$.*

5. CONFORMAL GRADIENT RICCI SOLITON ON (k, μ)-ALMOST COKÄHLER MANIFOLD

This section is devoted to the study of conformal gradient Ricci soliton on (k, μ)-almost coKähler manifold. So, let us first give the statement of our main theorem of this section.

Theorem 5.1. *Let $(M^{2n+1}, g, \phi, \xi, \eta)$ be a (k, μ)-almost coKähler manifold of dimension greater than 3 with $k < 0$. Then there exist no conformal gradient Ricci soliton on the manifold, with the potential vector field V pointwise collinear with the Reeb vector field ξ.*

Proof. We prove this theorem by the method of contradiction. So, let us assume that the manifold admits a conformal gradient Ricci soliton. Then from equation (1.4) we have

$$S + \nabla\nabla f = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]g.$$

Now as the soliton is gradient, i.e; $V = Df$ for some smooth function f and here D is the gradient operator. Thus for any vector field $X \in TM$, the above equation is equivalent to

$$\nabla_X Df + QX = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]X. \tag{5.1}$$

Replacing X by Y in the above (5.1) we get

$$\nabla_Y Df + QY = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]Y. \tag{5.2}$$

Similarly replacing X by $[X,Y]$ in the equation (5.1) we get

$$\nabla_{[X,Y]} Df + Q[X,Y] = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})][X,Y]. \tag{5.3}$$

Now from the well-known formula for Riemannian curvature, using (2.7) we can write

$$R(X,Y)Df = \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df. \tag{5.4}$$

Using equations (5.1), (5.2) and (5.3) in the equation (5.4) and after some simple calculations we get

$$R(X,Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y. \tag{5.5}$$

Again for any vector fields X, Y in TM, using equation (4.2) of Lemma 4.1 we obtain

$$\begin{aligned}(\nabla_Y Q)X - (\nabla_X Q)Y &= \mu((\nabla_Y h)X - (\nabla_X h)Y) \\ &+ 2nk(\eta(X)h'Y - \eta(Y)h'X) + Y(\mu)hX - X(\mu)hY.\end{aligned} \tag{5.6}$$

Now we shall use an equation from Proposition-9 of the paper [14]. The result is, for any vector fields X, Y in TM,

$$\begin{aligned}(\nabla_X h)Y - (\nabla_Y h)X &= k(\eta(Y)\phi X - \eta(X)\phi Y \\ &+ 2g(\phi X, Y)\xi) + \mu(\eta(X)h'Y - \eta(Y)h'X).\end{aligned} \tag{5.7}$$

Then using (5.6) in (5.5) and then using (5.7), a simple computation gives that

$$\begin{aligned}R(X,Y)Df &= k\mu(\eta(X)\phi Y - \eta(Y)\phi X + 2g(X,\phi Y)\xi) + Y(\mu)hX \\ &- X(\mu)hY - \mu^2(\eta(X)h'Y - \eta(Y)h'X) + 2nk(\eta(X)h'Y - \eta(Y)h'X),\end{aligned} \tag{5.8}$$

for any vector fields X, Y in TM. Putting $X = \xi$ in the above equation (5.8) we get

$$R(\xi, Y)Df = k\mu(\phi Y) - \xi(\mu)hY - \mu^2(h'Y) + 2nk(h'Y).$$

Replacing Y by X in the above equation and then taking inner product with respect to arbitrary vector Y gives us

$$\begin{aligned}g(R(\xi,X)Df, Y) =& k\mu g(\phi X, Y) - \xi(\mu)g(hX,Y) - \mu^2 g(h'X,Y) \\ &+2nkg(h'X,Y).\end{aligned} \tag{5.9}$$

Again for a (k,μ)-almost coKähler manifold, using equation (4.1) we can write

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX].$$

Taking inner-product of the equation with respect to the vector field Df and using the fact that $g(X,Df) = (Xf)$ we get

$$g(R(\xi,X)Y,Df) = k[g(X,Y)(\xi f) - \eta(Y)(Xf)] + \mu[g(hX,Y)(\xi f) - \eta(Y)((hX)f)]. \tag{5.10}$$

Now combining (5.9) and (5.10) and applying the property $g(R(X,Y)Z,U) = -g(R(X,Y)U,Z)$, for any vector fields X,Y,Z,U in TM, yields

$$k\mu g(\phi X,Y) - \xi(\mu)g(hX,Y) - \mu^2 g(h'X,Y) + 2nkg(h'X,Y) = - kg(X,Y)(\xi f) + k\eta(Y)(Xf) - \mu g(hX,Y)(\xi f) + \mu\eta(Y)((hX)f). \tag{5.11}$$

Antisymmetrizing the above equation we get

$$k\mu[g(\phi X,Y) - g(X,\phi Y)] = k[\eta(Y)(Xf) - \eta(X)(Yf)] + \mu[\eta(Y)((hX)f)] - \eta(X)((hY)f). \tag{5.12}$$

Now as per our assumption $V = b\xi$, it is easy to see that $h'(Df) = 0$. This again implies, $(h'X)f = g(h'X,Df) = g(X,h'(Df)) = 0$. Similarly $(h'Y)f = 0$. Thus

$$(h(\phi X))f = 0,\ (h(\phi Y))f = 0. \tag{5.13}$$

Using antisymmetry of ϕ and then putting $X = \phi X$ in equation (5.11) and using (5.12) we get

$$-2\mu g(X,Y) + \mu\eta(X)\eta(Y) = \eta(Y)((\phi X)f). \tag{5.14}$$

Putting $Y = \xi$ in the above (5.13) yields

$$-\mu g(X,\xi) = g(\phi X,Df). \tag{5.15}$$

Then again using $X = \phi X$ in the above equation (5.14) we get $g(X,Df) = g(X,\xi(\xi f))$. This gives us

$$Df = (\xi f)\xi. \tag{5.16}$$

Covariant differentiation of the equation (5.15) along the direction of X we get

$$\nabla_X Df = (X(\xi f))\xi + (\xi f)(h'X). \tag{5.17}$$

Again from the equation (5.1) we have

$$\nabla_X Df = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]X - QX. \tag{5.18}$$

Thus combining equations (5.16) and (5.17) we get

$$QX = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]X - (X(\xi f))\xi - (\xi f)(h'X). \tag{5.19}$$

Again, the value of Q from Lemma 4.1 gives us

$$QX = \mu hX + 2nk\eta(X)\xi.$$

Now, compairing right hand sides of (5.18) and (5.19) we get $(X(\xi f)) = -2nk\eta(X)$ i.e; $D(\xi f) = -2nk\xi$ or equivalently, $d^2 f = -2nk$, where d is the exterior derivative of f. Again from the well-known Poincare lemma of exterior differentiation we know that, $d^2 = 0$ and this implies, $-2nk = 0$, which is a contradiction to our assumtion that $k < 0$. This completes the proof. □

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A study on conformal Ricci solitons in the framework of $(LCS)_n$-manifolds

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Abstract

The main aim of this paper is to study Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) admitting the conformal Ricci soliton and to characterize when the soliton is shrinking, steady or expanding. Next we establish some results on the $(LCS)_n$-manifold whose metric is a conformal Ricci soliton. Finally some interesting results have been obtained by applying certain curvature conditions on $(LCS)_n$-manifolds admitting conformal Ricci solitons.

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Keywords: Ricci soliton, Conformal Ricci soliton, $(LCS)_n$-manifold, pseudo-projective curvature tensor, concircular curvature tensor.

1 Introduction

In 1982, R.S. Hamilton [7] introduced the Ricci soliton as a self similar solution to the Ricci flow equation given by: $\frac{\partial}{\partial t}(g(t)) = -2Ric(g(t))$, where $g(t)$ is an one parameter family of metrics on the manifold.

A Riemannian metric g defined on a smooth manifold M,of dimension n, is said to be a Ricci soliton if for some constant λ, there exists a smooth vector field V on M satisfying the equation

$$Ric + \frac{1}{2}\mathcal{L}_V g = \lambda g, \tag{1.1}$$

where $\mathcal{L}_V$ denotes the Lie derivative in the direction of V and Ric is the Ricci tensor. The Ricci soliton is called shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$. Ricci solitons can also be viewed as natural generalizations of Einstein metrics which moves only by an one-parameter group of diffeomorphisms and scaling [8]. After Hamilton's work many authors have studied Ricci flow and a rigorous literature on this topic can be found in [4, 17].

A.E. Fischer [6] in 2005, introduced conformal Ricci flow equation which is a modified version of the Hamilton's Ricci flow equation that modifies the volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow equations on a smooth closed connected oriented manifold M, of dimension n, are given by

$$\frac{\partial g}{\partial t} + 2(Ric + \frac{g}{n}) = -pg, \tag{1.2}$$

$$r(g) = -1,$$

where p is a non-dynamical (time dependent) scalar field and $r(g)$ is the scalar curvature of the manifold. The term $-pg$ acts as the constraint force to maintain the scalar curvature constraint. Thus these evolution equations are analogous to famous Navier-Stokes equations in fluid mechanics where the constraint is divergence free. That is why sometimes p is also called the conformal pressure.

Recently, in 2015, N.Basu et.al. [3] introduced the concept of conformal Ricci soliton as a generalization of the classical Ricci soliton.

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Definition 1. *A Riemannian metric g on a smooth manifold M, of dimension n, is called a conformal Ricci soliton if there exists a constant λ and a vector field V such that*

$$\mathcal{L}_V g + 2S = [2\lambda - (p + \frac{2}{n})]g, \tag{1.3}$$

where $S = Ric$ is the Ricci tensor, λ is a constant and p is the conformal pressure.

It can be easily checked that the above soliton equation satisfies the conformal Ricci flow equation(1.2). Later, T. Dutta et.al. [5] studied the conformal Ricci soliton in the framework of Lorentzian α-Sasakian manifolds.

A.A. Shaikh [14] in 2003, introduced the study of Lorentzian concircular structure manifolds (or, briefly, $(LCS)_n$-manifolds) which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [10]. After that, a lot of study has been carried out on $(LCS)_n$-manifolds and on locally ϕ-symmetric $(LCS)_n$-manifolds [16]. Moreover, in 2005, A.A. Shaikh et.al. [15] have shown the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology.

Motivated by the above studies, here we study conformal Ricci soliton in the framework of $(LCS)_n$-manifold. We find conditions to determine the nature of the soliton for different cases. The paper is organised as follows: After introduction, we discuss some preliminary concepts of $(LCS)_n$-manifolds, in section-2. Then in section-3, we study $(LCS)_n$-manifolds admitting the conformal Ricci soliton and we calculate the value of the soliton constant λ and hence we find the condition for the soliton to be shrinking, steady or expanding. After that, we prove that, if a $(LCS)_n$-manifold admits conformal Ricci soliton then it is ξ-projectively flat. In this section we also find conditions for a $(LCS)_n$-manifold admitting conformal Ricci soliton to be ξ-conharmonically flat and ξ-concircularly flat. Finally, in section-4 and section-5 we obtain some interesting results on conformal Ricci soliton on $(LCS)_n$-manifolds satisfying curvature conditions $R(\xi, X) \cdot \tilde{P} = 0$ and $R(\xi, X) \cdot \tilde{M} = 0$; where R is the Riemann curvature tensor, $\tilde{P}$ is the pseudo-projective curvature tensor and $\tilde{M}$ is the M-projective curvature tensor.

2 Brief overview of $(LCS)_n$-manifolds

A smooth connected paracompact Hausdorff n dimensional manifold (M, g) is said to be a Lorentzian manifold if the metric g is Lorentzian metric, i.e; M admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, +, ..., +)$, where T_pM denotes the tangent space of the manifold M at point p and $\mathbb{R}$ is the real line. A non-zero vector $v \in T_pM$ is said to be timelike(respectively; non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (respectively;$\leq 0, = 0, > 0$) [11].

Next, we give the definition of a concircular vector field in a Lorentzian manifold, which is essential for the study of $(LCS)_n$-manifolds.

Definition 2. *Let (M, g) be a Lorentzian manifold and P is a vector field in M defined by $g(U, P) = B(U)$, for any vector field U in M. Then the vector field P is said to be a concircular vector field if*

$$(\nabla_U B)(Y) = \alpha[g(U, Y) + \omega(U)B(Y)],$$

where α is a non-zero scalar and ω is closed 1-form and ∇ denotes the covariant differentiation operator of the manifold M with respect to the Lorentzian metric g.

Let (M, g) be a Lorentzian manifold of dimension n and let M admits a unit timelike concircular vector field ξ satisfying $g(\xi, \xi) = -1$. The vector field ξ is called the characteristic vector field of the manifold (M, g). Then ξ being unit concircular vector field, there exists a non-zero 1-form η such that

$$g(X, \xi) = \eta(X) \quad \textit{and} \quad (\nabla_X \eta)(Y) = \alpha[g(X, Y) + \eta(X)\eta(Y)], \quad \alpha \neq 0. \tag{2.1}$$

Also the non-zero scalar α satisfies the equation

$$(\nabla_X \alpha) = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.2}$$

where ρ is a scalar function given by $\rho = -(\xi\alpha)$ and ∇ denotes the covariant differentiation operator of the manifold M with respect to the Lorentzian metric g. Now we consider a $(1, 1)$ tensor field ϕ given by, $\phi X = \frac{1}{\alpha}\nabla_X \xi$. Therefore it is to be noted that the tensor field ϕ also satisfies $\phi X = X + \eta(X)\xi$ and this implies that ϕ is a symmetric $(1, 1)$ tensor field, called the structure tensor of the manifold.

So now, we are in a position to define $(LCS)_n$-manifolds, introduced by A.A. Shaikh [14] to generalize the notion of LP-Sasakian manifolds of Matsumoto [10].

Definition 3. *Let (M, g) be an n-dimensional Lorentzian manifold. Then the manifold (M, g) together with the unit timelike concircular vector field ξ, associated 1-form η an $(1, 1)$ tensor field ϕ and the non-zero scalar function α is said to be a Lorentzian concircular structure manifold $(M, g, \xi, \eta, \phi, \alpha)$(briefly, $(LCS)_n$-manifold) [14].*

It is to be noted that, if we consider the scalar function $\alpha = 1$, then we can obtain the LP-Sasakian structure introduced by Matsumoto [10]. So, in that sense $(LCS)_n$-manifolds are a generalization of LP-Sasakian manifolds. Furthermore, in a $(LCS)_n$-manifold $(n > 2)$, the following relations hold [14, 15, 16]:

$$\phi^2 X = X + \eta(X)\xi \quad , \quad \eta(\xi) = -1, \tag{2.3}$$
$$\phi(\xi) = 0 \quad , \quad \eta \circ \phi = 0, \tag{2.4}$$
$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.5}$$
$$R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.6}$$
$$R(\xi, X)Y = (\alpha^2 - \rho)[g(X, Y)\xi - \eta(Y)X], \tag{2.7}$$
$$\eta(R(X, Y)Z) = (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.8}$$
$$S(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{2.9}$$

for all vector fields X, Y, Z in TM, where TM is tangent bundle of M. Here R is the Riemannian curvature tensor of the manifold M defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and S is the Ricci tensor defined by $S(X, Y) = g(QX, Y)$, where Q is the Ricci operator.

Next, we discuss an illustrative example of an $(LCS)_n$-manifold of dimension $n = 3$ as follows:

Example: Let us consider the manifold $M = \{(u, v, w) \in \mathbb{R}^3 : u \neq 0\}$, where $\{u, v, w\}$ are usual Euclidean coordinates in $\mathbb{R}^3$. Now we choose a set $\{E_i : 1 \leq i \leq 3\}$ of linearly independent vector fields on the manifold M as follows,

$$E_1 = u\frac{\partial}{\partial u}, \quad E_2 = u\frac{\partial}{\partial v}, \quad E_3 = u\frac{\partial}{\partial w}.$$

Define the Lorentzian metric g on M as,

$$g(E_1, E_1) = -1, \quad g(E_2, E_2) = g(E_3, E_3) = 1; \qquad g(E_i, E_j) = 0, \forall i \neq j.$$

Now if we choose $\xi = E_1$ and define a 1-form η on M by, $\eta(X) = g(X, E_1)$, $\forall X \in TM$, where TM is the tangent bundle of M, then it is easy to see that $\eta(\xi) = -1$.
Next let us define a $(1, 1)$ tensor field ϕ on M as,

$$\phi(E_1) = 0, \quad \phi(E_2) = E_3, \quad \phi(E_3) = E_2.$$

Again as g and ϕ are both linear maps, for all $X, Y \in TM$, from the above one can easily check that,

$$\begin{aligned} \phi^2(X) &= X + \eta(X)\xi, \\ g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y). \end{aligned}$$

Now, it is well known that the connection ∇ of the metric g is given by the Koszul's formula,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= \nabla_X g(Y,Z) + \nabla_Y g(X,Z) - \nabla_Z g(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \end{aligned}$$

for all $X, Y, Z \in TM$ and the Lie bracket operation $[X,Y]$ is given by $[X,Y] = \nabla_X Y - \nabla_Y X$. Then one can easily calculate $[E_1, E_2] = E_2$, $[E_2, E_3] = 0$, $[E_1, E_3] = E_3$. Again using the above Koszul's formula and after a straightforward calculation we get,

$$\begin{aligned} \nabla_{E_1} E_1 = 0, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = 0, \\ \nabla_{E_2} E_1 = -E_2, \quad \nabla_{E_2} E_2 = -E_1, \quad \nabla_{E_2} E_3 = 0, \\ \nabla_{E_3} E_1 = -E_3, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = -E_1. \end{aligned}$$

Thus from the above we can easily verify that for $\alpha = -1$, the relation $\phi X = \frac{1}{\alpha} \nabla_X \xi$ holds for all $X \in TM$. Hence we can conclude that $(M, g, \xi, \eta, \phi, \alpha)$ is an $(LCS)_n$-manifold of dimension $n = 3$.

3 Conformal Ricci soliton on $(LCS)_n$-manifolds

Let us consider $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$-manifold. Again we know that, for all vector fields X, Y in TM, the 1-form η satisfies the equation

$$(\nabla_X \eta)(Y) = \nabla_X \eta(Y) - \eta(\nabla_X Y). \tag{3.1}$$

Using the equation (2.1) in the above equation (3.1), after a simple calculation, we get

$$(\mathcal{L}_\xi g)(X,Y) = 2\alpha[g(X,Y) + \eta(X)\eta(Y)]. \tag{3.2}$$

Now applying the conformal Ricci soliton equation (1.3) in the above equation (3.2) we have

$$S(X,Y) = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]g(X,Y) - \alpha\eta(X)\eta(Y). \tag{3.3}$$

Let us take $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$. Then we can rewrite the above equation (3.3) as

$$S(X,Y) = \sigma g(X,Y) - \alpha\eta(X)\eta(Y). \tag{3.4}$$

which shows that the manifold is an η-Einstein manifold.

Now since the above is true for all vector fields X and Y, using the relation $S(X,Y) = g(QX,Y)$ in the above equation (3.4) we have

$$QX = \sigma X - \alpha\eta(X)\xi. \tag{3.5}$$

Again taking $Y = \xi$ in the equation (3.4) we get

$$S(X, \xi) = (\sigma + \alpha)\eta(X). \tag{3.6}$$

Let us consider an orthonormal basis $\{e_i : 1 \leq i \leq n\}$ of the manifold (M, g). Then putting $X = Y = e_i$ in the equation (3.4) and summing over $1 \leq i \leq n$, we have $r(g) = n\sigma + \alpha$. But we know that for conformal Ricci flow, $r(g) = -1$, which leads us to get $\sigma = -(\frac{\alpha+1}{n})$. Again we have $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$, using this in the previous result we get

$$\lambda = \frac{p}{2} + (1 - \frac{1}{n})\alpha. \tag{3.7}$$

So, from the above discussions, using equations (3.4) and (3.7), we can state the following theorem

Theorem 1. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton. Then*

a) The manifold becomes an η-Einstein manifold.

b) The value of the soliton scalar λ is equal to $\lambda = \frac{p}{2} + (1 - \frac{1}{n})\alpha$.

c) The soliton is shrinking, steady or expanding according as the conformal pressure $p < 2(\frac{1-n}{n})\alpha$, $p = 2(\frac{1-n}{n})\alpha$ or $p > 2(\frac{1-n}{n})\alpha$.

Next, we discuss about the projective curvature tensor which plays an important role in the study of differential geometry. The projective curvature has an one-to-one correspondence between each coordinate neighbourhood of an n-dimensional Riemannian manifold and a domain of Euclidean space such that there is an one-to-one correspondence between geodesics of the Riemannian manifold with the straight lines in the Euclidean space. The projective curvature tensor in an n-dimensional Riemannian manifold (M, g) is defined by [19]

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)}[g(QY, Z)X - g(QX, Z)Y], \tag{3.8}$$

for any vector fields $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of the manifold M, R is the Riemannian curvature tensor of M and Q is the Ricci operator.
The manifold (M, g) is called ξ-projectively flat if $P(X, Y)\xi = 0$ for any vector fields $X, Y \in \chi(M)$ and ξ is the characteristic vector field of the manifold. Now for an $(LCS)_n$-manifold of dimension n, putting $Z = \xi$ in (3.8) we get

$$P(X, Y)\xi = R(X, Y)\xi - \frac{1}{(n-1)}[S(Y, \xi)X - S(X, \xi)Y].$$

Using (2.6) and (3.6) the above equation becomes

$$P(X, Y)\xi = [(\alpha^2 - \rho) - \frac{\sigma + \alpha}{(n-1)}][\eta(Y)X - \eta(X)Y]. \tag{3.9}$$

Again combining equations (2.9) and (3.6) we have

$$[(\alpha^2-\rho)(n-1)-\sigma-\alpha]\eta(X)=0, \tag{3.10}$$

which essentially gives us

$$[(\alpha^2-\rho)(n-1)]=(\sigma+\alpha). \tag{3.11}$$

Now in view of (3.11), the equation (3.9) yields us $P(X,Y)\xi=0$ for any vector fields $X,Y\in\chi(M)$. Thus we have the following

Theorem 2. *If $(M,g,\xi,\eta,\phi,\alpha)$ is an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton, then the manifold becomes ξ-projectively flat, ξ being the characteristic vector field of the manifold.*

A transformation of a Riemannian manifold of dimension n, which transforms every geodesic circle of the manifold M into a geodesic circle, is called a concircular transformation [18]. Here a geodesic circle is a curve in M whose first curvature is constant and second curvature (that is, torsion) is identically equal to zero. The concircular curvature tensor in a Riemannian manifold (M,g) of dimension n is defined by [13, 18]

$$C(X,Y)Z=R(X,Y)Z-\frac{r}{n(n-1)}[g(Y,Z)X-g(X,Z)Y], \tag{3.12}$$

for any vector fields $X,Y,Z\in\chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of the manifold M and r is the scalar curvature of M.
The manifold (M,g) is called ξ-concircularly flat if $C(X,Y)\xi=0$ for any vector fields $X,Y\in\chi(M)$ and ξ is the characteristic vector field of the manifold. Now for an $(LCS)_n$-manifold of dimension n, putting $Z=\xi$ in (3.12) we get

$$C(X,Y)\xi=R(X,Y)\xi-\frac{r}{n(n-1)}[\eta(Y)X-\eta(X)Y].$$

Using (2.6) the above equation becomes

$$C(X,Y)\xi=[(\alpha^2-\rho)-\frac{r}{n(n-1)}][\eta(Y)X-\eta(X)Y]. \tag{3.13}$$

Again in view of equation (3.11), the above equation (3.13) becomes

$$C(X,Y)\xi=[\frac{(\sigma+\alpha)}{(n-1)}-\frac{r}{n(n-1)}][\eta(Y)X-\eta(X)Y]. \tag{3.14}$$

Now in view of equation (3.14), we can say that $C(X,Y)\xi=0$ iff $r=n(\sigma+\alpha)$. Again using the fact that for conformal Ricci flow $r=-1$ and using $\sigma=[(\lambda-\alpha)-(\frac{p}{2}+\frac{1}{n})]$ we eventually get $C(X,Y)\xi=0$ iff $\lambda=\frac{p}{2}$. This leads to the following theorem

Theorem 3. *If $(M, g, \xi, \eta, \phi, \alpha)$ is an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton, then the manifold becomes ξ-concircularly flat iff $\lambda = \frac{p}{2}$, ξ being the characteristic vector field of the manifold and p is the conformal pressure.*

The conharmonic curvature tensor plays an important role in the study of manifolds. The conharmonic curvature tensor of an n-dimensional Riemannian manifold (M, g) is defined as [9]
(3.15)

$$H(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y]$$

for any vector fields $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of the manifold M, R is the Riemannian curvature tensor of M, S is the Ricci tensor and Q is the Ricci operator.
The manifold (M, g) is called ξ-conharmonically flat if $H(X,Y)\xi = 0$ for any vector fields $X, Y \in \chi(M)$ and ξ is the characteristic vector field of the manifold. Now for an $(LCS)_n$-manifold of dimension n, putting $Z = \xi$ in (3.15) we have

$$H(X,Y)\xi = R(X,Y)\xi - \frac{1}{(n-2)}[\eta(Y)QX - \eta(X)QY + S(Y,\xi)X - S(X,\xi)Y].$$

Using (2.6), (3.5) and (3.6) the above equation yields

$$H(X,Y)\xi = [(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{(n-2)}][\eta(Y)X - \eta(X)Y]. \tag{3.16}$$

Again in view of equation (3.11), the above equation (3.16) becomes

$$H(X,Y)\xi = [\frac{(-n\sigma - \alpha)}{(n-1)(n-2)}][\eta(Y)X - \eta(X)Y]. \tag{3.17}$$

Thus from the above (3.17) we can conclude that $H(X,Y)\xi = 0$ iff $n\sigma = -\alpha$. Moreover, using the value $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$ and after few steps of calculations we have $H(X,Y)\xi = 0$ iff $\lambda = \frac{p}{2} + \frac{1}{n} + (1 - \frac{1}{n})\alpha$. Thus we can state the following:

Theorem 4. *If $(M, g, \xi, \eta, \phi, \alpha)$ is an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton, then the manifold becomes ξ-conharmonically flat iff $\lambda = \frac{p}{2} + \frac{1}{n} + (1 - \frac{1}{n})\alpha$, ξ being the characteristic vector field of the manifold and p is the conformal pressure.*

Next, let us consider a conformal Ricci soliton (g, V, λ) on an n-dimensional $(LCS)_n$-manifold M as

$$\mathcal{L}_V g(X,Y) + 2S(X,Y) = [2\lambda - (p + \frac{2}{n})]g(X,Y), \tag{3.18}$$

where $\mathcal{L}_V g$ denotes the Lie derivative of the Lorentzian metric g in the direction of the vector field V. This vector field V is also called the potential vector field.
Now assume that the vector field V be pointwise collinear with the characteristic

vector field ξ, that is, $V = b\xi$, where b is a smooth function on the manifold M. Then for any vector fields $X, Y \in \chi(M$, the equation (3.18) implies

$$\mathcal{L}_{b\xi}g(X,Y) + 2S(X,Y) = [2\lambda - (p + \frac{2}{n})]g(X,Y). \tag{3.19}$$

Again from the property of the Lie derivative of the Levi-Civita connection we know that $\mathcal{L}_Z g(X,Y) = g(\nabla_X Z, Y) + g(\nabla_Y Z, X)$. Applying this formula in the above equation (3.19) and then using $\phi X = \frac{1}{\alpha}\nabla_X \xi$ we get

(3.20)

$$b\alpha g(\phi X,Y)+(Xb)\eta(Y)+b\alpha g(\phi Y,X)+(Yb)\eta(X)+2S(X,Y) = [2\lambda-(p+\frac{2}{n})]g(X,Y).$$

Putting $Y = \xi$ in (3.20) and using the equations (2.4) we obtain

$$2S(X,\xi) - (Xb) + (\xi b)\eta(X) = [2\lambda - (p + \frac{2}{n})]\eta(X). \tag{3.21}$$

Using equation (3.6) in the above(3.21) and then putting the value $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$ gives us

$$(Xb) = (\xi b)\eta X. \tag{3.22}$$

Again putting $X = \xi$ in the equation (3.21) we have

$$S(\xi,\xi) - (\xi b) + [\lambda - (\frac{p}{2} + \frac{1}{n})] = 0. \tag{3.23}$$

Now, in view of equation (3.6) and $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$, the above equation (3.23) yields $(\xi b) = 0$. Furthermore, using $(\xi b) = 0$ in equation (3.22) we can conclude that (Xb)=0, for any vector field $X \in \chi(M)$. And this implies that the function b is constant and hence V is a constant multiple of ξ. Therefore we have the following theorem

Theorem 5. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$-manifold which admits a conformal Ricci soliton (g, V, λ), V being the potential vector field of the manifold. If the potential vector field V is pointwise collinear with the characteristic vector field ξ, i.e; if $V = b\xi$, then b is constant, i.e; V becomes constant multiple of ξ.*

Next, we study an important curvature property called ξ-Ricci semi symmetry.

Let $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$-manifold. Then we say that the manifold M is ξ-Ricci semi symmetric if $R(\xi, X) \cdot S = 0$ in M, where ξ is the characteristic vector field, R is the Riemannian curvature tensor, S is the Ricci tensor.

Let us start with the known formula that for any vector fields X, Y, Z on M,

$$R(\xi,X) \cdot S = S(R(\xi,X)Y,Z) + S(Y,R(\xi,X)Z). \tag{3.24}$$

Now, using (2.7) the above equation (3.24) yields

$$R(\xi,X)\cdot S=(\alpha^2-\rho)[g(X,Y)S(\xi,Z)-\eta(Y)S(X,Z)+S(Y,\xi)g(X,Z)-\eta(Z)S(Y,X)].$$

Using (2.9) in the above equation and after few steps we get

$$R(\xi,X)\cdot S=\alpha(\alpha^2-\rho)[g(X,Y)\eta(Z)+g(X,Z)\eta(Y)+2\eta(X)\eta(Y)\eta(Z)]. \tag{3.25}$$

Now note that $(\alpha^2-\rho)=0$ implies $\lambda=\frac{p}{2}+\frac{1}{n}$, which is the trivial case. Thus for non-triviality we assume $(\alpha^2-\rho)\neq 0$. Again as α is a non-zero scalar, from (3.25) we can state the following:

Theorem 6. *If $(M,g,\xi,\eta,\phi,\alpha)$ is an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton, then the manifold becomes ξ-Ricci semi symmetric, i.e; $R(\xi,X)\cdot S=0$ iff the Lorentzian metric g satisfies the relation*

$$g(X,Y)\eta(Z)+g(X,Z)\eta(Y)+2\eta(X)\eta(Y)\eta(Z)=0$$

for any vector fields X,Y,Z on M, ξ being the characteristic vector field, R is the Riemannian curvature tensor and S is the Ricci tensor.

4 Conformal Ricci soliton on $(LCS)_n$-manifolds satisfying certain curvature conditions

First let (M,g) be an n-dimensional $(LCS)_n$-manifold. Then from equation (3.15) the conharmonic curvature tensor on M is given by
(4.1)

$$H(X,Y)Z=R(X,Y)Z-\frac{1}{n-2}[g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X-S(X,Z)Y].$$

Interchanging Z and X and the putting $Z=\xi$, we can rewrite the above equation (4.1) as

$$H(\xi,X)Y=R(\xi,X)Y-\frac{1}{n-2}[S(X,Y)\xi-S(\xi,Y)X+g(X,Y)Q\xi-g(\xi,Y)QX].$$

Using (2.7), (3.4), (3.5) and (3.6) in the above we get

$$H(\xi,X)Y=[(\alpha^2-\rho)-\frac{(2\sigma+\alpha)}{(n-2)}][g(X,Y)\xi-\eta(Y)X]. \tag{4.2}$$

Also from (4.2) we can write

$$\eta(H(\xi,X)Y)=-[(\alpha^2-\rho)-\frac{(2\sigma+\alpha)}{(n-2)}][g(X,Y)+\eta(X)\eta(Y)]. \tag{4.3}$$

Now we assume that $H(\xi, X) \cdot S = 0$ holds. Then we have

$$S(H(\xi, X)Y, Z) + S(Y, H(\xi, X)Z) = 0. \tag{4.4}$$

In view of (3.4) the above (4.4) yields

$$\sigma[g(H(\xi, X)Y, Z)+g(Y, H(\xi, X)Z)]-\alpha[\eta(H(\xi, X)Z)\eta(Y)+\eta(H(\xi, X)Y)\eta(Z)] = 0.$$

Using (4.2) and (4.3) in the above equation we get

$$\alpha[(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{(n-2)}][g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)] = 0. \tag{4.5}$$

Finally taking $Z = \xi$ in equation (4.5) and then using (2.5) we arrive at

$$\alpha[(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{(n-2)}]g(\phi X, \phi Y) = 0. \tag{4.6}$$

Since α is non-zero and $g(\phi X, \phi Y) \neq 0$ always; then $[(\alpha^2 - \rho) - \frac{(2\sigma+\alpha)}{(n-2)}] = 0$ i.e; $\lambda = \frac{p}{2} + \frac{1}{n} + (1 - \frac{1}{n})\alpha$. Therefore we can state the following theorem:

Theorem 7. *If $(M, g, \xi, \eta, \phi, \alpha)$ is an n-dimensional $(LCS)_n$-manifold which admits a conformal Ricci soliton, and satisfies the condition $H(\xi, X) \cdot S = 0$ i.e; the manifold is ξ-Ricci conharomnically symmetric. Then the soliton constant is given by $\lambda = \frac{p}{2} + \frac{1}{n} + (1 - \frac{1}{n})\alpha$; where H is the conharmonic curvature tensor and S is the Ricci tensor of the manifold and ξ is the characteristic vector field.*

Next we study another important curvature tensor called $\tilde{M}$-projective curvature tensor. The $\tilde{M}$-projective curvature tensor on an $(LCS)_n$-manifold is defined by [1]

(4.7)

$$\tilde{M}(X, Y)Z = R(X, Y)Z - \frac{1}{2(n-1)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY].$$

Taking inner product with respect to the vector field ξ, the above (4.6) yields

$$\eta(\tilde{M}(X, Y)Z) = \eta(R(X, Y)Z) - \frac{1}{2(n-1)}[S(Y, Z)\eta(X) - S(X, Z)\eta(Y) + g(Y, Z)\eta(QX) - g(X, Z)\eta(QY)]. \tag{4.8}$$

Using (2.8), (3.4) and (3.5) in the above equation we get

$$\eta(\tilde{M}(X, Y)Z) = [(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{2(n-1)}][g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \tag{4.9}$$

Now we assume the condition that $R(\xi, X) \cdot \tilde{M} = 0$. Then we have

$$R(\xi, X)\tilde{M}(Y, Z)W - \tilde{M}(R(\xi, X)Y, Z)W - \tilde{M}(Y, R(\xi, X)Z)W - \tilde{M}(Y, Z)R(\xi, X)W = 0. \tag{4.10}$$

Using (2.7) in (4.9) and then taking an inner product with respect to ξ we get

$$(4.11)\quad -g(X,\tilde{M}(Y,Z)W)-\eta(X)\eta(\tilde{M}(Y,Z)W)-g(X,Y)\eta(\tilde{M}(\xi,Z)W) +\eta(Y)\eta(\tilde{M}(X,Z)W)-g(X,Z)\eta(\tilde{M}(Y,\xi)W)+\eta(Z)\eta(\tilde{M}(Y,X)W) -g(X,W)\eta(\tilde{M}(Y,Z)\xi)+\eta(W)\eta(\tilde{M}(Y,Z)X)=0.$$

Then in view of (4.8) the above (4.10) becomes

$$(4.12)\quad [(\alpha^2-\rho)-\frac{(2\sigma+\alpha)}{2(n-1)}][g(Y,W)g(X,Z)-g(X,Y)g(Z,W)]+g(X,\tilde{M}(Y,Z)W)=0.$$

From (4.6) and (4.11) we get

$$(4.13)\quad [(\alpha^2-\rho)-\frac{(2\sigma+\alpha)}{2(n-1)}][g(Y,W)g(X,Z)-g(X,Y)g(Z,W)]+g(X,R(Y,Z)W) -\frac{1}{2(n-1)}[S(Z,W)g(X,Y)-S(Y,W)g(X,Z)+g(Z,W)S(Y,X)-g(Y,W)S(Z,X)]=0.$$

Let us consider an orthonormal basis $\{e_i : 1 \leq i \leq n\}$ of the manifold (M,g). Then putting $X=Y=e_i$ in the equation (4.12) and summing over $1 \leq i \leq n$, we get

$$(4.14)\quad 2nS(Z,W)=[2(n-1)^2(\alpha^2-\rho)-(n-1)(2\sigma+\alpha)-r]g(Z,W).$$

Again putting $Z=W=\xi$ in above and using equation (3.6) we get

$$(4.15)\quad 2(n-1)^2(\alpha^2-\rho)-(5n-2)[\lambda-(\frac{p}{2}+\frac{1}{n})]+2n\alpha=0.$$

Now using (3.11) in the above equation (4.14) and after a simple calculation we arrive at

$$(4.16)\quad \lambda=(\frac{p}{2}+\frac{1}{n})-2\alpha.$$

Thus we have the following theorem

Theorem 8. *Let $(M,g,\xi,\eta,\phi,\alpha)$ be an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton and the manifold is ξ-$\tilde{M}$-projectively semi symmetric i.e; it satisfies the condition $R(\xi,X)\cdot\tilde{M}=0$; ξ being the characteristic vector field, $\tilde{M}$ is the M-projective curvature tensor of the manifold. Then the soliton is shrinking, steady or expanding according as $p>(4\alpha-\frac{2}{n})$, $p=(4\alpha-\frac{2}{n})$ or $p<(4\alpha-\frac{2}{n})$*

Next we prove an interesting result on $(LCS)_n$-manifold admitting a conformal Ricci soliton and satisfying the condition $R(\xi,X)\cdot\tilde{P}=0$, where $\tilde{P}$ denotes the well-known Pseudo-projective curvature tensor. But before that let us recall some well-known results that will be used later in this section:

Theorem 9. *[11] If $S : g(x, y, z) = c$ is a surface in $\mathbb{R}^3$ then the gradient vector field ∇g (connected only at a point of S) is a non-vanishing normal vector field on the entire surface S.*

S.R. Ashoka et.al. in their paper [1] have given the higher dimensional version of the above theorem as follows:

Corollary 1. *[1] If $S : g(x, y, z) = c$ is a surface (abstract surface or manifold) in $\mathbb{R}^n$ then the gradient vector field ∇g (connected only at points of S) is a non-vanishing normal vector field on the entire surface (abstract surface or manifold) S.*

Then the above mentioned authors in [1] also gave the following remark from the above corollary as:

Remark 1. *[1] Taking a real valued scalar function α associated with an $(LCS)_n$-manifold with $M = \mathbb{R}^3$ and $g = \alpha$ in the above corollary we have, $\nabla\alpha$ as a non-vanishing normal vector field on $S \subset M$ and directional derivative of α with respect to ξ, $\xi\alpha = \xi,\ \nabla\alpha = |\xi||\nabla\alpha|cos(\hat{\xi}, \nabla\alpha)$*
1) If ξ is tangent to S then $\xi\alpha = 0$.
2) If ξ is tangent to M but not to S then $\xi\alpha \neq 0$.
3) If the angle between ξ and $\nabla\alpha$ is acute then $0 < cos(\hat{\xi}, \nabla\alpha) < 1$, then $\xi\alpha = k|\nabla\alpha|$, $0 < k < 1$ and $\xi\alpha > 0$.
4) If the angle between ξ and $\nabla\alpha$ is obtuse then $-1 < cos(\hat{\xi}, \nabla\alpha) < 0$, then $\xi\alpha = k|\nabla\alpha|$, $-1 < k < 0$ and $\xi\alpha < 0$.

Now we see the dependance of the conformal Ricci soliton on $\xi\alpha$ for $(LCS)_n$-manifolds satisfying $R(\xi, X) \cdot \tilde{P} = 0$. The Pseudo projective curvature tensor $\tilde{P}$ is defined by

$$\begin{aligned} \tilde{P}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] \\ - \frac{r}{n}(\frac{a}{n-1} + b)[g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{4.17}$$

where $a, b \neq 0$ are constants. Taking $Z = \xi$ in (4.16) we get

$$\begin{aligned} \tilde{P}(X, Y)\xi = aR(X, Y)\xi + b[S(Y, \xi)X - S(X, \xi)Y] \\ - \frac{r}{n}(\frac{a}{n-1} + b)[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{4.18}$$

Using (2.6) and (3.6) the above equation (4.17) yields

$$\tilde{P}(X, Y)\xi = [a(\alpha^2 - \rho) + b(\sigma + \alpha) - \frac{r}{n}(\frac{a}{n-1} + b)][\eta(Y)X - \eta(X)Y], \tag{4.19}$$

where σ is as described in the previous section. Again from (4.16) we can write

$$\begin{aligned} \eta(\tilde{P}(X, Y)Z) = a\eta(R(X, Y)Z) + b[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)] \\ - \frac{r}{n}(\frac{a}{n-1} + b)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \end{aligned}$$

Using (2.8) and (3.4) the above equation becomes

$$\eta(\tilde{P}(X,Y)Z) = [a(\alpha^2-\rho)+b\sigma-\frac{r}{n}(\frac{a}{n-1}+b)][g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]. \tag{4.20}$$

Now we assume the condition that $R(\xi, X)\cdot\tilde{P} = 0$. Then we have

$$\begin{aligned} R(\xi,X)\tilde{P}(U,V)W - \tilde{P}(R(\xi,X)U,V)W \\ - \tilde{P}(U,R(\xi,X)V)W - \tilde{P}(U,V)R(\xi,X)W = 0, \end{aligned} \tag{4.21}$$

for any vector fields X, U, V, W on M. Using (2.7) in the above equation and then taking an inner product with respect to ξ we get

$$\begin{aligned} - g(X,\tilde{P}(U,V)W) - \eta(X)\eta(\tilde{P}(U,V)W) - g(X,U)\eta(\tilde{P}(\xi,V)W) \\ + \eta(U)\eta(\tilde{P}(X,V)W) - g(X,U)\eta(\tilde{P}(U,\xi)W) + \eta(V)\eta(\tilde{P}(U,X)W) \\ - g(X,W)\eta(\tilde{P}(U,V)\xi) + \eta(W)\eta(\tilde{P}(U,V)X) = 0. \end{aligned}$$

Then using (4.18) and (4.19) the above equation becomes

$$\begin{aligned} [a(\alpha^2-\rho)+b\sigma-\frac{r}{n}(\frac{a}{n-1}+b)][g(X,V)g(U,W) - g(X,U)g(V,W)] \\ + g(X,\tilde{P}(U,V)W) = 0. \end{aligned} \tag{4.22}$$

Now in view of (4.16) and then using (3.4) in the equation (4.21) we get

$$\begin{aligned} ag(X,R(U,V)W) - b\alpha[\eta(V)\eta(W)g(X,U) - \eta(U)\eta(W)g(X,V)] \\ + a(\alpha^2-\rho)[g(X,V)g(U,W) - g(X,U)g(V,W)] = 0. \end{aligned} \tag{4.23}$$

Let us consider an orthonormal basis $\{e_i : 1 \leq i \leq n\}$ of the manifold (M, g). Then putting $X = U = e_i$ in the equation (4.22) and summing over $1 \leq i \leq n$, we get

$$aS(V,W) - b(n-1)\alpha\eta(V)\eta(W) - a(n-1)(\alpha^2-\rho)g(V,W) = 0. \tag{4.24}$$

Again setting $V = W = \xi$ in (4.23) and after a few steps of simple calculations we get

$$\lambda = (n-1)[(\alpha^2-\rho) - \frac{b}{a}\alpha] + (\frac{p}{2}+\frac{1}{n}). \tag{4.25}$$

Therefore in view of the above equation (4.24) and Remark-4.1 we can state the following :

Theorem 10. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$-manifold which admits a conformal Ricci soliton and the manifold is ξ-pseudo-projectively semi symmetric i.e; if it satisfies the condition $R(\xi, X)\cdot\tilde{P} = 0$; ξ being the characteristic vector field, $\tilde{P}$ is the pseudo-projective curvature tensor of the manifold and α is a positive function; then*

1) If ξ is orthogonal to $\nabla\alpha$; the soliton is expanding if $\alpha > \frac{b}{a}$, $p > -\frac{2}{n}$; steady if $\alpha = \frac{b}{a}$, $p = -\frac{2}{n}$ and shrinking if $\alpha < \frac{b}{a}$, $p < -\frac{2}{n}$.

2)If the angle between ξ and $\nabla\alpha$ is acute; the soliton is expanding if $\alpha^2 + k|\nabla\alpha| > \frac{b}{a}\alpha$, $p > -\frac{2}{n}$; steady if $\alpha^2 + k|\nabla\alpha| = \frac{b}{a}\alpha$, $p = -\frac{2}{n}$ and shrinking if $\alpha^2 + k|\nabla\alpha| < \frac{b}{a}\alpha$, $p < -\frac{2}{n}$.

3) If the angle between ξ and $\nabla\alpha$ is obtuse; the soliton is expanding if $\alpha^2 > k|\nabla\alpha| + \frac{b}{a}\alpha$, $p > -\frac{2}{n}$; steady if $\alpha^2 = k|\nabla\alpha| + \frac{b}{a}\alpha$, $p = -\frac{2}{n}$ and shrinking if $\alpha^2 < k|\nabla\alpha| + \frac{b}{a}\alpha$, $p < -\frac{2}{n}$.

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Research article

Conformal η-Ricci solitons within the framework of indefinite Kenmotsu manifolds

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Abstract: The present paper is to deliberate the class of ϵ-Kenmotsu manifolds which admits conformal η-Ricci soliton. Here, we study some special types of Ricci tensor in connection with the conformal η-Ricci soliton of ϵ-Kenmotsu manifolds. Moving further, we investigate some curvature conditions admitting conformal η-Ricci solitons on ϵ-Kenmotsu manifolds. Next, we consider gradient conformal η-Ricci solitons and we present a characterization of the potential function. Finally, we develop an illustrative example for the existence of conformal η-Ricci soliton on ϵ-Kenmotsu manifold.

Keywords: Ricci soliton; conformal Ricci soliton; conformal η-Ricci soliton; ϵ-Kenmotsu manifold; concircular curvature tensor; codazzi type Ricci tensor
Mathematics Subject Classification: 53C15, 53C25, 53D10

1. Introduction

The scientists and mathematicians across many disciplines have always been fascinated to study indefinite structures on manifolds. When a manifold is endowed with a geometric structure, we have more opportunities to explore its geometric properties. There are different classes of submanifolds such as warped product submanifolds, biharmonic submanifolds and singular submanifolds, etc., which motivates further exploration and attracts many researchers from different research areas [26–37, 40–50]. After A. Bejancu et al. [7] in 1993, introduced the concept of an idefinite manifold namely ϵ-Sasakian manifold, it gained attention of various researchers and it was established by X. Xufeng et al. [53] that the class of ϵ-Sasakian manifolds are real hypersurfaces of indefinite Kaehlerian manifolds. On the other hand K. Kenmotsu [25] introduced a special class of contact Riemannian manifolds, satisfying certain conditions, which was later named as Kenmotsu manifold. Later on U. C. De et al. [14] introduced the concept of ϵ-Kenmotsu manifolds and further proved that the existence of the new indefinite structure on the manifold influences the curvatures of the manifold. After that several authors [20, 21, 52] studied ϵ-Kenmotsu manifolds and many interesting results have been obtained on this indefinite structure.

A smooth manifold M equipped with a Riemannian metric g is said to be a Ricci soliton, if for some constant λ, there exist a smooth vector field V on M satisfying the equation

$$S + \frac{1}{2}\mathcal{L}_V g = \lambda g,$$

where $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V and S is the Ricci tensor. The Ricci soliton is called shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$. In 1982, R. S. Hamilton [22] initiated the study of Ricci flow as a self similar solution to the Ricci flow equation given by

$$\frac{\partial g}{\partial t} = -2S.$$

Ricci soliton also can be viewed as natural generalization of Einstein metric which moves only by an one-parameter group of diffeomorphisms and scaling [11, 23]. After Hamilton, the significant work on Ricci flow has been done by G. Perelman [38] to prove the well known Thurston's geometrization conjecture.

A. E. Fischer [16] in 2005, introduced conformal Ricci flow equation which is a modified version of the Hamilton's Ricci flow equation that modifies the volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow equations on a smooth closed connected oriented n-manifold, $n \geq 3$, are given by

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg, \quad r(g) = -1,$$

where p is a non-dynamical (time dependent) scalar field and $r(g)$ is the scalar curvature of the manifold. The term-pg acts as the constraint force to maintain the scalar curvature constraint in the above equation. Note that these evolution equations are analogous to famous Navier-Stokes equations where the constraint is divergence free. The non-dynamical scalar p is also called the conformal pressure. At the equilibrium points of the conformal Ricci flow equations (i.e., Einstein metrices with Einstein constant $-\frac{1}{n}$) the conformal pressure p is equal to zero and strictly positive otherwise.

Later in 2015, N. Basu and A. Bhattacharyya [6] introduced the concept of conformal Ricci soliton as a generalization of the classical Ricci soliton and is given by the equation

$$\mathcal{L}_V g + 2S = [2\lambda - (p + \frac{2}{n})]g, \tag{1.1}$$

where λ is a constant and p is the conformal pressure. It is to be noted that the conformal Ricci soliton is a self-similar solution of the Fisher's conformal Ricci flow equation. After that several authors have studied conformal Ricci solitons on various geometric structures like Lorentzian α-Sasakian Manifolds [15] and f-Kenmotsu manifods [24]. Since the introduction of these geometric flows, the respective solitons and their generalizations etc. have been a great centre of attention of many geometers viz. [1–5, 8, 9, 13, 17, 40–47] who have provided new approaches to understand the geometry of different kinds of Riemannian manifold. Recently Sarkar et al. [48–50] studied $*$-conformal η-Ricci soliton and $*$-conformal Ricci soliton within the frame work of contact geometry and obtaind some beautiful results.

Again a Ricci soliton is called a gradient Ricci soliton [11] if the concerned vector field X in the Eq (1.1) is the gradient of some smooth function f. This function f is called the potential function of the Ricci soliton. J. T. Cho and M. Kimura [12] introduced the concept of η-Ricci soliton and later C. Calin and M. Crasmareanu [10] studied it on Hopf hypersufaces in complex space forms. A Riemannian manifold (M, g) is said to admit an η-Ricci soliton if for a smooth vector field V, the metric g satisfies the following equation

$$\mathcal{L}_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0,$$

where $\mathcal{L}_V$ is the Lie derivative along the direction of V, S is the Ricci tensor and λ, μ are real constants. It is to be noted that for $\mu = 0$ the η-Ricci soliton becomes a Ricci soliton.

Very recently M. D. Siddiqi [51] introduced the notion of conformal η-Ricci soliton given by the following equation

$$\mathcal{L}_V g + 2S + [2\lambda - (p + \frac{2}{n})]g + 2\mu\eta \otimes \eta = 0, \tag{1.2}$$

where $\mathcal{L}_V$ is the Lie derivative along the direction of V, S is the Ricci tensor, n is the dimension of the manifold, p is the non-dynamical scalar field (conformal pressure) and λ, μ are real constants. In particular if $\mu = 0$ the conformal η-Ricci soliton reduces to the conformal Ricci soliton.

The outline of the article goes as follows: In Section 2, after a brief introduction, we give some notes on ϵ-Kenmotsu manifolds. Section 3 deals with ϵ-Kenmotsu manifolds admitting conformal η-Ricci solitons and establish the relation between λ and μ. In Section 4, we have contrived conformal η-Ricci solitons in ϵ-Kenmotsu manifolds in terms of Codazzi type Ricci tensor, cyclic parallel Ricci tensor and cyclic η-recurrent Ricci tensor. Section 5 is devoted to the study of conformal η-Ricci solitons on ϵ-Kenmotsu manifolds satisfying curvature conditions $R \cdot S = 0$, $C \cdot S = 0$, $Q \cdot C = 0$. In Section 6, we have studied torse-forming vector field on ϵ-Kenmotsu manifolds admitting conformal η-Ricci solitons. Section 7 is devoted to the study of gradient conformal η-Ricci soliton on ϵ-Kenmotsu manifold. Lastly, we have constructed an example to illustrate the existence of conformal η-Ricci soliton on ϵ-Kenmotsu manifold.

2. Preliminaries

An n-dimensional smooth manifold (M, g) is said to be an ϵ-almost contact metric manifold [7] if it admits a $(1, 1)$ tensor field ϕ, a characteristic vector field ξ, a global 1-form η and an indefinite metric g on M satisfying the following relations

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$\eta(X) = \epsilon g(X, \xi), \quad g(\xi, \xi) = \epsilon, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.3}$$

for all vector fields $X, Y \in TM$, where TM is the tangent bundle of the manifold M. Here the value of the quantity ϵ is either $+1$ or -1 according as the characteristic vector field ξ is spacelike or timelike vector field. Also it can be easily seen that rank of ϕ is $(n-1)$ and $\phi(\xi) = 0$, $\eta \circ \phi = 0$. Now if we define

$$d\eta(X, Y) = g(X, \phi Y), \tag{2.4}$$

for all $X, Y \in TM$, then the manifold (M, g) is called an ϵ-contact metric manifold.

If the Levi-Civita connection ∇ of an ϵ-contact metric manifold satisfies

$$(\nabla_X \phi)(Y) = -g(X, \phi Y) - \epsilon\eta(Y)\phi X, \tag{2.5}$$

for all $X, Y \in TM$, then the manifold (M, g) is called an ϵ-Kenmotsu manifold [14].

Again an ϵ-almost contact metric manifold is an ϵ-Kenmotsu manifold if and only if it satisfies

$$\nabla_X \xi = \epsilon(X - \eta(X)\xi), \quad \forall X \in TM. \tag{2.6}$$

Furthermore in an ϵ-Kenmotsu manifold (M, g) the following relations hold,

$$(\nabla_X \eta)(Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.7}$$
$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$
$$R(\xi, X)Y = \eta(Y)X - \epsilon g(X, Y)\xi, \tag{2.9}$$
$$R(\xi, X)\xi = -R(X, \xi)\xi = X - \eta(X)\xi, \tag{2.10}$$
$$\eta(R(X, Y)Z) = \epsilon(g(X, Z)\eta(Y) - g(Y, Z)\eta(X)), \tag{2.11}$$
$$S(X, \xi) = -(n-1)\eta(X), \tag{2.12}$$
$$Q\xi = -\epsilon(n-1)\xi, \tag{2.13}$$

where R is the curvature tensor, S is the Ricci tensor and Q is the Ricci operator given by $g(QX, Y) = S(X, Y)$, for all $X, Y \in TM$.

Moreover, it is to be noted that for spacelike structure vector field ξ and $\epsilon = 1$, an ϵ-Kenmotsu manifold reduces to an usual Kenmotsu manifold.

Next, we discuss about the projective curvature tensor which plays an important role in the study of differential geometry. The projective curvature has an one-to-one correspondence between each coordinate neighbourhood of an n-dimensional Riemannian manifold and a domain of Euclidean space such that there is a one-to-one correspondence between geodesics of the Riemannian manifold with the straight lines in the Euclidean space.

Definition 2.1. The projective curvature tensor in an n-dimensional ϵ-Kenmotsu manifold (M, g) is defined by [55]

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)}[g(QY, Z)X - g(QX, Z)Y], \tag{2.14}$$

for any vector fields $X, Y, Z \in TM$ and Q is the Ricci operator.
The manifold (M, g) is called ξ-projectively flat if $P(X, Y)\xi = 0$, for all $X, Y \in TM$.

A transformation of a Riemannian manifold of dimension n, which transforms every geodesic circle of the manifold M into a geodesic circle, is called a concircular transformation [54]. Here a geodesic circle is a curve in M whose first curvature is constant and second curvature (that is, torsion) is identically equal to zero.

Definition 2.2. The concircular curvature tensor in an ϵ-Kenmotsu manifold (M, g) of dimension n is defined by [54]

$$C(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y], \tag{2.15}$$

for any vector fields $X, Y, Z \in TM$, and r is the scalar curvature of M.
The manifold (M, g) is called ξ-concircularly flat if $C(X, Y)\xi = 0$ for any vector fields $X, Y \in TM$.

Another important curvature tensor is W_2-curvature tensor which was introduced in 1970 by Pokhariyal and Mishra [39].

Definition 2.3. The W_2-curvature tensor in an n-dimensional ϵ-Kenmotsu manifold (M, g) is defined as

$$W_2(X, Y)Z = R(X, Y)Z + \frac{1}{n-1}[g(X, Z)QY - g(Y, Z)QX]. \tag{2.16}$$

Definition 2.4. An ϵ-Kenmotsu manifold (M, g) is said to be an η-Einstein manifold if its Ricci tensor S satisfies

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.17}$$

for all $X, Y \in TM$ and smooth functions a, b on the manifold (M, g).

3. ϵ-Kenmotsu manifolds admitting conformal η-Ricci solitons

Let us consider an ϵ-Kenmotsu manifold (M, g) admits a conformal η-Ricci soliton (g, ξ, λ, μ). Then from Eq (1.2) we can write

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + [2\lambda - (p + \frac{2}{n})]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{3.1}$$

for all $X, Y \in TM$.
Again from the well-known formula $(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)$ of Lie-derivative and using (2.6), we obtain for an ϵ-Kenmotsu manifold

$$(\mathcal{L}_\xi g)(X, Y) = 2\epsilon[g(X, Y) - \epsilon\eta(X)\eta(Y)]. \tag{3.2}$$

Now in view of the Eqs (3.1) and (3.2) we get

$$S(X, Y) = -[(\lambda + \epsilon) - (\frac{p}{2} + \frac{1}{n})]g(X, Y) - (\mu - 1)\eta(X)\eta(Y). \tag{3.3}$$

This shows that the manifold (M, g) is an η-Einstein manifold.
Also from Eq (3.3) replacing $Y = \xi$ we find that

$$S(X, \xi) = [\epsilon(\frac{p}{2} + \frac{1}{n}) - (\epsilon\lambda + \mu)]\eta(X). \tag{3.4}$$

Compairing the above Eq (3.4) with (2.12) yields

$$\epsilon\lambda + \mu = \epsilon(\frac{p}{2} + \frac{1}{n}) + (n-1). \quad (3.5)$$

Thus the above discussion leads to the following

Theorem 3.1. *If an n-dimensional ϵ-Kenmotsu manifold (M, g) admits a conformal η-Ricci soliton (g, ξ, λ, μ), then (M, g) becomes an η-Einstein manifold and the scalars λ and μ are related by $\epsilon\lambda + \mu = \epsilon(\frac{p}{2} + \frac{1}{n}) + (n-1)$.*

Furthermore if we consider $\mu = 0$ in particular, then from Eqs (3.3) and (3.5), we get

$$\begin{aligned} S(X, Y) &= -[(\lambda + \epsilon) - (\frac{p}{2} + \frac{1}{n})]g(X, Y) + \eta(X)\eta(Y), \\ \lambda &= (\frac{p}{2} + \frac{1}{n}) + \epsilon(n-1). \end{aligned}$$

This leads us to write

Corollary 3.2. *If an n-dimensional ϵ-Kenmotsu manifold (M, g) admits a conformal Ricci soliton (g, ξ, λ), then (M, g) becomes an η-Einstein manifold and the scalar λ satisfies $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n-1)$. Moreover,*

1. *if ξ is spacelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) > (1-n)$, $(\frac{p}{2} + \frac{1}{n}) = (1-n)$ or $(\frac{p}{2} + \frac{1}{n}) < (1-n)$; and*
2. *if ξ is timelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) > (n-1)$, $(\frac{p}{2} + \frac{1}{n}) = (n-1)$ or $(\frac{p}{2} + \frac{1}{n}) < (n-1)$.*

Next we try to find a condition in terms of second order symmetric parallel tensor which will ensure when an ϵ-Kenmotsu manifold (M, g) admits a conformal η-Ricci soliton. So for this purpose let us consider the second order tensor T on the manifold (M, g) defined by

$$T := \mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta. \quad (3.6)$$

It is easy to see that the $(0, 2)$ tensor T is symmetric and also parallel with respect to the Levi-Civita connection.
Now in view of (3.2) and (3.3) the above Eq (3.6) we have

$$T(X, Y) = [(p + \frac{2}{n}) - 2\lambda]g(X, Y); \quad \forall X, Y \in TM. \quad (3.7)$$

Putting $X = Y = \xi$ in the above Eq (3.7) we obtain

$$T(\xi, \xi) = \epsilon[(p + \frac{2}{n}) - 2\lambda]. \quad (3.8)$$

On the other hand, as T is a second order symmetric parallel tensor; i.e., $\nabla T = 0$, we can write

$$T(R(X, Y)Z, U) + T(Z, R(X, Y)U) = 0,$$

for all $X, Y, Z, U \in TM$. Then replacing $X = Z = U = \xi$ in above gives us

$$T(R(\xi, Y)\xi, \xi) + T(\xi, R(\xi, Y)\xi) = 0, \quad \forall Y \in TM. \tag{3.9}$$

Using (2.10) in the above Eq (3.9) we get

$$T(Y, \xi) = T(\xi, \xi)\eta(Y). \tag{3.10}$$

Taking covariant differentiation of (3.10) in the direction of an arbitrary vector field X, and then in the resulting equation, again using the Eq (3.10) we obtain

$$T(Y, \nabla_X\xi) = T(\xi, \xi)(\nabla_X\eta)Y + 2T(\nabla_X\xi, \xi)\eta(Y).$$

Then in view of (2.6) and (2.7), the above equation becomes

$$T(X, Y) = \epsilon T(\xi, \xi)g(X, Y), \quad \forall X, Y \in TM. \tag{3.11}$$

Now using (3.8) in the above Eq (3.11) and in view of (3.6) finally we get

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + [2\lambda - (p + \frac{2}{n})]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0.$$

This leads us to the following

Theorem 3.3. *Let (M, g) be an n-dimensional ϵ-Kenmotsu manifold. If the second order symmetric tensor $T := \mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta$ is parallel with respect to the Levi-Civita connection of the manifold, then the manifold (M, g) admits a conformal η-Ricci soliton (g, ξ, λ, μ).*

Now let us consider an ϵ-Kenmotsu manifold (M, g) and assume that it admits a conformal η-Ricci soliton (g, V, λ, μ) such that V is pointwise collinear with ξ, i.e., $V = \alpha\xi$, for some function α; then from the Eq (1.2) it follows that

$$\alpha g(\nabla_X\xi, Y) + \epsilon(X\alpha)\eta(Y) + \alpha g(\nabla_Y\xi, X) + \epsilon(Y\alpha)\eta(X) + 2S(X, Y) + [2\lambda - (p + \frac{2}{n})]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Then using the Eq (2.6) in above we get

$$2\epsilon\alpha g(X, Y) - 2\epsilon\alpha\eta(X)\eta(Y) + \epsilon(X\alpha)\eta(Y) + \epsilon(Y\alpha)\eta(X) + 2S(X, Y) + [2\lambda - (p + \frac{2}{n})]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{3.12}$$

Replacing $Y = \xi$ in the above equation yields

$$\epsilon(X\alpha) + \epsilon(\xi\alpha)\eta(X) + 2S(X, \xi) + \epsilon[2\lambda - (p + \frac{2}{n})]\eta(X) + 2\mu\eta(X) = 0. \tag{3.13}$$

By virtue of (2.12) the above Eq (3.13) becomes

$$\epsilon(X\alpha) + \epsilon[(\xi\alpha) + 2\lambda - (p + \frac{2}{n})]\eta(X) + 2[\mu - (n - 1)]\eta(X) = 0. \tag{3.14}$$

By taking $X = \xi$ in the above Eq (3.14) gives us

$$\epsilon(\xi\alpha) = (n-1) - \mu - \epsilon[\lambda - (\frac{p}{2} + \frac{1}{n})]. \tag{3.15}$$

Using this value from (3.15) in the Eq (3.14) we can write

$$\epsilon d\alpha = [(n-1) - \mu - \epsilon[\lambda - (\frac{p}{2} + \frac{1}{n})]]\eta. \tag{3.16}$$

Now taking exterior differentiation on both sides of (3.16) and using the famous Poincare's lemma, i.e., $d^2 = 0$, finally we arrive at

$$[(n-1) - \mu - \epsilon[\lambda - (\frac{p}{2} + \frac{1}{n})]]d\eta = 0.$$

Since $d\eta \neq 0$ in ϵ-Kenmotsu manifold, the above equation implies

$$\mu + \epsilon[\lambda - (\frac{p}{2} + \frac{1}{n})] = (n-1). \tag{3.17}$$

In view of the above (3.17) the Eq (3.16) gives us $d\alpha = 0$ i.e., the function α is constant. Then the Eq (3.12) becomes

$$S(X, Y) = [(\frac{p}{2} + \frac{1}{n}) - \lambda - \epsilon\alpha]g(X, Y) + (\alpha - \mu)\eta(X)\eta(Y), \tag{3.18}$$

for all $X, Y \in TM$. This shows that the manifold is η-Einstein. Hence we have the following

Theorem 3.4. *If an n-dimensional ϵ-Kenmotsu manifold (M, g) admits a conformal η-Ricci soliton (g, V, λ, μ) such that V is pointwise collinear with ξ, then V is constant multiple of ξ and the manifold (M, g) is an η-Einstein manifold. Moreover the scalars λ and μ are related by $\mu + \epsilon[\lambda - (\frac{p}{2} + \frac{1}{n})] = (n-1)$.*

In particular if we put $\mu = 0$ in (3.17) and (3.18) we can conclude that

Corollary 3.5. *If an n-dimensional ϵ-Kenmotsu manifold (M, g) admits a conformal Ricci soliton (g, V, λ, μ) such that V is pointwise collinear with ξ, then V is constant multiple of ξ and the manifold (M, g) is an η-Einstein manifold, and the scalars λ and μ are related by $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n-1)$. Furthermore,*

1. *if ξ is spacelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) + n > 1$, $(\frac{p}{2} + \frac{1}{n}) + n = 1$ or $(\frac{p}{2} + \frac{1}{n}) + n < 1$; and*
2. *if ξ is timelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) + 1 > n$, $(\frac{p}{2} + \frac{1}{n}) + 1 = n$ or $(\frac{p}{2} + \frac{1}{n}) + 1 < n$.*

4. Conformal η-Ricci solitons on ϵ-Kenmotsu manifolds with certain special types of Ricci tensor

The purpose of this section is to study conformal η-Ricci solitons in ϵ-Kenmotsu manifolds admitting three special types of Ricci tensor namely Codazzi type Ricci tensor, cyclic parallel Ricci tensor and cyclic η-recurrent Ricci tensor.

Definition 4.1. [19] An ϵ-Kenmotsu manifold is said to have Codazzi type Ricci tensor if its Ricci tensor S is non-zero and satisfies the following relation

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z), \quad \forall X,Y,Z \in TM. \tag{4.1}$$

Let us consider an ϵ-Kenmotsu manifold having Codazzi type Ricci tensor admits a conformal η-Ricci soliton (g,ξ,λ,μ), then Eq (3.3) holds. Now taking covariant differentiation of (3.3) and using Eq (2.7) we obtain

$$(\nabla_X S)(Y,Z) = (1-\mu)[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\epsilon\eta(X)\eta(Y)\eta(Z)]. \tag{4.2}$$

Since the manifold has Codazzi type Ricci tensor, in view of (4.1) Eq (4.2) yields

$$(1-\mu)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] = 0, \quad \forall X,Y,Z \in TM.$$

The above equation implies that $\mu = 1$ and then from Eq (3.5) it follows that $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n-2)$. Therefore we can state the following

Theorem 4.2. *Let (M,g) be an n-dimensional ϵ-Kenmotsu manifold admitting a conformal η-Ricci soliton (g,ξ,λ,μ). If the Ricci tensor of the manifold is of Codazzi type then $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n-2)$ and $\mu = 1$.*

Corollary 4.3. *Let an n-dimensional ϵ-Kenmotsu manifold admits a conformal η-Ricci soliton (g,ξ,λ,μ) and the manifold has Codazzi type Ricci tensor then*

1. *if ξ is spacelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) + n > 2$, $(\frac{p}{2} + \frac{1}{n}) + n = 2$ or $(\frac{p}{2} + \frac{1}{n}) + n < 2$; and*
2. *if ξ is timelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) + 2 > n$, $(\frac{p}{2} + \frac{1}{n}) + 2 = n$ or $(\frac{p}{2} + \frac{1}{n}) + 2 < n$.*

Definition 4.4. [19] An ϵ-Kenmotsu manifold is said to have cyclic parallel Ricci tensor if its Ricci tensor S is non-zero and satisfies the following relation

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0 \quad \forall X,Y,Z \in TM. \tag{4.3}$$

Let us consider an ϵ-Kenmotsu manifold, having cyclic parallel Ricci tensor, admits a conformal η-Ricci soliton (g,ξ,λ,μ), then Eq (3.3) holds. Now taking covariant differentiation of (3.3) and using Eq (2.7) we obtain

$$(\nabla_X S)(Y,Z) = (1-\mu)[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\epsilon\eta(X)\eta(Y)\eta(Z)]. \tag{4.4}$$

In a similar manner we can obtain the following relations

$$(\nabla_Y S)(Z,X) = (1-\mu)[g(X,Y)\eta(Z) + g(Y,Z)\eta(X) - 2\epsilon\eta(X)\eta(Y)\eta(Z)]. \tag{4.5}$$

and

$$(\nabla_Z S)(X,Y) = (1-\mu)[g(X,Z)\eta(Y) + g(Y,Z)\eta(X) - 2\epsilon\eta(X)\eta(Y)\eta(Z)]. \tag{4.6}$$

Now using the values from (4.4), (4.5) and (4.6) in the Eq (4.3) we get

$$2(1-\mu)[g(X,Y)\eta(Z) + g(Y,Z)\eta(X) + g(X,Z)\eta(Y) - 3\epsilon\eta(X)\eta(Y)\eta(Z)] = 0.$$

Replacing $Z = \xi$ in the above equation yields

$$2(1-\mu)[g(X,Y) - \epsilon\eta(X)\eta(Y)] = 0 \quad \forall X, Y \in TM.$$

The above equation implies that $\mu = 1$ and then from Eq (3.5) it follows that $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n-2)$. Hence we have

Theorem 4.5. *Let (M, g) be an n-dimensional ϵ-Kenmotsu manifold admitting a conformal η-Ricci soliton (g, ξ, λ, μ). If the manifold has cyclic parallel Ricci tensor, then $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n-2)$ and $\mu = 1$.*

Definition 4.6. An ϵ-Kenmotsu manifold is said to have cyclic-η-recurrent Ricci tensor if its Ricci tensor S is non-zero and satisfies the following relation

$$\begin{aligned} &(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\ = \; &\eta(X)S(Y,Z) + \eta(Y)S(Z,X) + \eta(Z)S(X,Y) \quad \forall X, Y, Z \in TM. \end{aligned} \tag{4.7}$$

Let us consider an ϵ-Kenmotsu manifold, having cyclic-η-recurrent Ricci tensor, admits a conformal η-Ricci soliton (g, ξ, λ, μ), then Eq (3.3) holds. Now taking covariant differentiation of (3.3) and using Eq (2.7) and proceeding similarly as the previous theorem we arrive at Eqs (4.4)–(4.6). Then putting these three values in (4.7) we get

$$\begin{aligned} (2(1-\mu) - \beta)[g(X,Y)\eta(Z) + g(Y,Z)\eta(X) + g(X,Z)\eta(Y)] \\ -(3+6\epsilon)(1-\mu)\eta(X)\eta(Y)\eta(Z) \; = \; 0, \end{aligned} \tag{4.8}$$

where $\beta = (\frac{p}{2} + \frac{1}{n}) - (\lambda + \epsilon)$. Now putting $Y = Z = \xi$ in (4.8) we obtain

$$3(\epsilon\beta + (1-\mu))\eta(X) = 0. \quad \forall X \in TM. \tag{4.9}$$

Since $\eta(X) \neq 0$ and replacing the value of β in (4.9), after simplification we get $\lambda = (\frac{p}{2} + \frac{1}{n}) - \epsilon\mu$. Therefore we can state

Theorem 4.7. *Let (M, g) be an n-dimensional ϵ-Kenmotsu manifold admitting a conformal η-Ricci soliton (g, ξ, λ, μ). If the manifold has cyclic-eta-parallel Ricci tensor, then $\lambda = (\frac{p}{2} + \frac{1}{n}) - \epsilon\mu$ and moreover*

1. *if ξ is spacelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) > \mu$, $(\frac{p}{2} + \frac{1}{n}) = \mu$ or $(\frac{p}{2} + \frac{1}{n}) < \mu$; and*
2. *if ξ is timelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) + \mu > 0$, $(\frac{p}{2} + \frac{1}{n}) + \mu = 0$ or $(\frac{p}{2} + \frac{1}{n}) + \mu < 0$.*

Corollary 4.8. *Let (M, g) be an n-dimensional ϵ-Kenmotsu manifold admitting a conformal Ricci soliton (g, ξ, λ, μ). If the manifold has cyclic-eta-parallel Ricci tensor, then the soliton constant λ is given by $\lambda = (\frac{p}{2} + \frac{1}{n})$.*

5. Conformal η-Ricci solitons on ϵ-Kenmotsu manifolds satisfying some curvature conditions

Let us consider an ϵ-Kenmotsu manifold which admits a conformal η-Ricci soliton (g, ξ, λ, μ) and also the manifold is Ricci semi symmetric i.e., the manifold satisfies the curvature condition $R(X, Y) \cdot S = 0$. Then $\forall X, Y, Z, W \in TM$ we can write

$$S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = 0.$$

Putting $W = \xi$ in above and taking (2.12) into account, we have

$$-(n-1)\eta(R(X, Y)Z) + S(Z, R(X, Y)\xi) = 0. \tag{5.1}$$

Now using (2.8) and (2.11) in (5.1) we get

$$\eta(X)[S(Y, Z) - \epsilon(n-1)g(Y, Z)] - \eta(Y)[S(X, Z) - \epsilon(n-1)g(X, Z)] = 0.$$

In view of (3.3) the previous equation becomes

$$[(\frac{p}{2} + \frac{1}{n}) - \lambda + \epsilon(n-2)][\eta(X)g(Y, Z) - \eta(Y)g(X, Z)] = 0.$$

Putting $X = \xi$ in the above equation and then using (2.2) and (2.3) we finally obtain

$$[(\frac{p}{2} + \frac{1}{n}) - \lambda + \epsilon(n-2)]g(\phi Y, \phi Z) = 0. \tag{5.2}$$

Since $g(\phi Y, \phi Z) \neq 0$ always, we can conclude from the Eq (5.2) that $[(\frac{p}{2} + \frac{1}{n}) - \lambda + \epsilon(n-2)] = 0$ i.e., $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n-2)$. Then from the Eq (3.5) we have $\mu = 1$. Therefore we have the following

Theorem 5.1. *Let (M, g) be an n-dimensional ϵ-Kenmotsu manifold admitting a conformal η-Ricci soliton (g, ξ, λ, μ). If the manifold is Ricci semi symmetric i.e., if the manifold satisfies the curvature condition $R(X, Y) \cdot S = 0$, then $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n-2)$ and $\mu = 1$. Moreover*

1. *if ξ is spacelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) > (2-n)$, $(\frac{p}{2} + \frac{1}{n}) = (2-n)$ or $(\frac{p}{2} + \frac{1}{n}) < (2-n)$; and*
2. *if ξ is timelike then the soliton is expanding, steady or shrinking according as, $(\frac{p}{2} + \frac{1}{n}) + (2-n) > 0$, $(\frac{p}{2} + \frac{1}{n}) + (2-n) = 0$ or $(\frac{p}{2} + \frac{1}{n}) + (2-n) < 0$.*

Next we consider an n-dimensional ϵ-Kenmotsu manifold satisfying the curvature condition $C(\xi, X) \cdot S = 0$ admitting a conformal η-Ricci soliton (g, ξ, λ, μ). Then we have

$$S(C(\xi, X)Y, Z) + S(Y, C(\xi, X)Z) = 0 \quad \forall X, Y, Z \in TM. \tag{5.3}$$

Now from Eq (2.15) we can write

$$C(\xi, X)Y = R(\xi, X)Y - \frac{r}{n(n-1)}[g(X, Y)\xi - \epsilon\eta(Y)X].$$

Using (2.9) the above equation becomes

$$C(\xi, X)Y = [1 + \frac{\epsilon r}{n(n-1)}][\eta(Y)X - \epsilon g(X, Y)\xi]. \tag{5.4}$$

In view of (5.4) the Eq (5.3) yields

$$[1 + \tfrac{\epsilon r}{n(n-1)}][S(X, Z)\eta(Y) - \epsilon g(X, Y)S(\xi, Z) + S(Y, X)\eta(Z) - \epsilon g(X, Z)S(\xi, Y] = 0.$$

By virtue of (2.12) the above equation eventually becomes

$$[1 + \tfrac{\epsilon r}{n(n-1)}][S(X, Z)\eta(Y) + S(Y, X)\eta(Z) + \epsilon(n-1)(g(X, Y)\eta(Z) + g(X, Z)\eta(Y))] = 0. \tag{5.5}$$

Putting $Z = \xi$ in (5.5) and then using (2.2), (2.12) we arrive at

$$[1 + \frac{\epsilon r}{n(n-1)}][S(X, Y) + \epsilon(n-1)g(X, Y)] = 0.$$

Thus from the above we can conclude that either $r = -\epsilon n(n-1)$ or

$$S(X, Y) = -\epsilon(n-1)g(X, Y). \tag{5.6}$$

Combining (5.6) with (3.3) we get

$$[(\lambda + \epsilon) - (\frac{p}{2} + \frac{1}{n}) - \epsilon(n-1)]g(X, Y) + (\mu - 1)\eta(X)\eta(Y) = 0.$$

Taking $Y = \xi$ in above gives us

$$[(n - \mu) + \epsilon(\frac{p}{2} + \frac{1}{n} - \lambda - \epsilon)]\eta(X) = 0, \quad \forall X \in TM.$$

Since $\eta(X) \neq 0$ always, from the above we have $\lambda = \epsilon(n-1) + (\frac{p}{2} + \frac{1}{n}) - \epsilon\mu$. Therefore we can state

Theorem 5.2. *Let (M, g) be an n-dimensional ϵ-Kenmotsu manifold admitting a conformal η-Ricci soliton (g, ξ, λ, μ). If the manifold satisfies the curvature condition $C(\xi, X) \cdot S = 0$, then either the scalar curvature of the manifold is constant or the manifold is an Einstein manifold of the form* (5.6) *and the scalars λ and μ are related by $\lambda = \epsilon(n-1) + (\frac{p}{2} + \frac{1}{n}) - \epsilon\mu$.*

Next we prove two results on ξ-projectively flat and ξ-concircularly flat manifolds. For that let us first consider an ϵ-Kenmotsu manifold (M, g, ξ, ϕ, η) admitting a conformal η-Ricci soliton (g, ξ, λ, μ). We know from definition 2.1 that the manifold is ξ-projectively flat if $P(X, Y)\xi = 0, \quad \forall X, Y \in TM$. Then putting $Z = \xi$ in (2.14) we obtain

$$P(X, Y)\xi = R(X, Y)\xi - \frac{1}{n-1}[S(Y, \xi)X - S(X, \xi)Y]. \tag{5.7}$$

Now since it is given that (g, ξ, λ, μ) admits a conformal η-Ricci soliton, using (2.8) and (3.4) in the above (5.7), we obtain

$$P(X, Y)\xi = \left[1 + \frac{\epsilon(\frac{p}{2} + \frac{1}{n}) - \epsilon\lambda - \mu}{n-1}\right][\eta(X)Y - \eta(Y)X].$$

In view of (3.5) the above equation finally becomes $P(X, Y)\xi = 0$. Hence we have the following

Proposition 5.3. *An n-dimensional ϵ-Kenmotsu manifold (M, g, ξ, ϕ, η) admitting a conformal η-Ricci soliton (g, ξ, λ, μ) is ξ-projectively flat.*

Again consider an n-dimensional ϵ-Kenmotsu manifold (M, g, ξ, ϕ, η) admitting a conformal η-Ricci soliton (g, ξ, λ, μ). Then from definition 2.2 we know that an ϵ-Kenmotsu manifold is ξ-concircularly flat if $C(X, Y)\xi = 0, \quad \forall X, Y \in TM$. So taking $Z = \xi$ in (2.15) we get

$$C(X, Y)\xi = R(X, Y)\xi - \frac{\epsilon r}{n(n-1)}[\eta(Y)X - \eta(X)Y]. \tag{5.8}$$

Using (2.8) in (5.8) we obtain

$$C(X, Y)\xi = [1 + \frac{\epsilon r}{n(n-1)}][\eta(X)Y - \eta(Y)X].$$

Thus from the above we can conclude that $C(X, Y)\xi = 0$ if and only if, $[1 + \frac{\epsilon r}{n(n-1)}] = 0$, i.e., if and only if, $r = -\epsilon n(n-1)$. Again since (g, ξ, λ, μ) is a conformal η-Ricci soliton, the Eq (3.3) holds and thus contracting (3.3) we obtain $r = [(\frac{p}{2} + \frac{1}{n}) - \lambda - \mu]n - (\mu - 1)$. Thus combining both the values of r we have, $\lambda = (\frac{p}{2} + \frac{2}{n}) - \frac{\mu}{n} - 2\epsilon$. Therefore we can state

Proposition 5.4. *An n-dimensional ϵ-Kenmotsu manifold (M, g, ξ, ϕ, η) admitting a conformal η-Ricci soliton (g, ξ, λ, μ) is ξ-concircularly flat if and only if, $\lambda = (\frac{p}{2} + \frac{2}{n}) - \frac{\mu}{n} - 2\epsilon$.*

We now assume that an n-dimensional ϵ-Kenmotsu manifold (M, g, ξ, ϕ, η) admits a conformal η-Ricci soliton (g, ξ, λ, μ) which satisfies the curvature condition $Q \cdot C = 0$, where C denotes the concircular curvature tensor of the manifold. Then we can write

$$Q(C(X, Y)Z) - C(QX, Y)Z - C(X, QY)Z - C(X, Y)QZ = 0. \tag{5.9}$$

Using (2.15) in (5.9) yields

$$Q(R(X, Y)Z) - R(QX, Y)Z - R(X, QY)Z - R(X, Y)QZ + \frac{2r}{n(n-1)}[S(Y, Z)X - S(X, Z)Y] = 0. \tag{5.10}$$

Taking inner product of (5.10) with respect to the vector field ξ we get

$$\eta(Q(R(X, Y)Z)) - \eta(R(QX, Y)Z) - \eta(R(X, QY)Z) - \eta(R(X, Y)QZ) + \frac{2r}{n(n-1)}[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)] = 0.$$

Putting $Z = \xi$ in above we obtain

$$\eta(Q(R(X, \xi)Z)) - \eta(R(QX, \xi)Z) - \eta(R(X, Q\xi)Z) - \eta(R(X, \xi)QZ) + \frac{2r}{n(n-1)}[S(\xi, Z)\eta(X) - S(X, Z)] = 0. \tag{5.11}$$

Again from (2.9) we can derive

$$\eta(Q(R(X,\xi)Z)) = \eta(R(X,Q\xi)Z) = (n-1)[\epsilon\eta(X)\eta(Z) - g(X,Z)], \tag{5.12}$$

$$\eta(R(QX,\xi)Z) = \eta(R(X,\xi)QZ) = \epsilon[S(X,Z) + (n-1)\eta(X)\eta(Z)]. \tag{5.13}$$

By virtue of (5.12) and (5.13), the Eq (5.11) becomes

$$\epsilon[(n-1)\eta(X)\eta(Z) + S(X,Z)] - \frac{r}{n(n-1)}[S(\xi,Z)\eta(X) - S(X,Z)] = 0.$$

Using (2.12) in above we arrive at

$$[\epsilon + \frac{r}{n(n-1)}][(n-1)\eta(X)\eta(Z) + S(X,Z)] = 0.$$

Hence we can conclude that either $r = -\epsilon n(n-1)$ or,

$$S(X,Z) = -(n-1)\eta(X)\eta(Z). \tag{5.14}$$

Now combining Eqs (5.14) and (3.3), we get

$$[(\lambda + \epsilon) - (\frac{p}{2} + \frac{1}{n})]g(X,Z) + (\mu - n)\eta(X)\eta(Z) = 0.$$

Taking $Z = \xi$ in above yields

$$[\epsilon(\lambda - (\frac{p}{2} + \frac{1}{n})) + (\mu + 1 - n)]\eta(X) = 0, \quad \forall X \in TM.$$

Since $\eta(X) \neq 0$ always, from the above we can conclude that $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n - \mu - 1)$. Hence we can state the following

Theorem 5.5. *Let (M,g) be an n-dimensional ϵ-Kenmotsu manifold admitting a conformal η-Ricci soliton (g,ξ,λ,μ). If the manifold satisfies the curvature condition $Q \cdot C = 0$, then either the scalar curvature of the manifold is constant or the manifold is a special type of η-Einstein manifold of the form* (5.14) *and the scalars λ and μ are related by $\lambda = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n - \mu - 1)$.*

We conclude this section by this result on W_2-curvature tensor. For this let us consider an n-dimensional ϵ-Kenmotsu manifold admitting a conformal η-Ricci soliton (g,ξ,λ,μ) and assume that the manifold satisfies the curvature condition $W_2(\xi, Y) \cdot S = 0$. Then we can write

$$S(W_2(\xi,Y)Z, U) + S(Z, W_2(\xi,Y)U) = 0, \quad \forall Y, Z, U \in TM.$$

Putting $U = \xi$ in above we get

$$S(W_2(\xi,Y)Z,\xi) + S(Z, W_2(\xi,Y)\xi) = 0. \tag{5.15}$$

Now taking $X = \xi$ in (2.16) we obtain

$$W_2(\xi,Y)Z = R(\xi,Y)Z + \frac{1}{n-1}[\epsilon\eta(Z)QY - g(Y,Z)Q\xi].$$

Using (2.9) in above yields

$$W_2(\xi, Y)Z = \eta(Z)Y - \epsilon g(Y, Z)\xi + \frac{1}{n-1}[\epsilon\eta(Z)QY - g(Y, Z)Q\xi]. \tag{5.16}$$

putting $Z = \xi$ in (5.16) we arrive at

$$W_2(\xi, Y)\xi = Y - \eta(Y)\xi + \frac{\epsilon}{n-1}[QY - \eta(Y)Q\xi]. \tag{5.17}$$

Using (5.16) and (5.17) in the Eq (5.15) and taking (2.1), (2.12) into account, we get

$$\begin{aligned} S(Y, Z) + \tfrac{\epsilon}{n-1}[S(Z, QY) - \eta(Y)S(Z, Q\xi)] \\ + \epsilon(n-1)g(Y, Z) - \epsilon\eta(Z)\eta(QY) + g(Y, Z)\eta(Q\xi) = 0. \end{aligned} \tag{5.18}$$

Taking $Y = \xi$ and taking (2.12) and (2.13) into account, the previous equation identically satisfies:

$$\epsilon(n-1)g(\xi, Z) + (n-1)\eta(Z) - \epsilon(n-1)g(\xi, Z) + S(Z, \xi) - S(Z, \xi)(n-1)eta(Z) = 0. \tag{5.19}$$

Thus we arrive at the following

Theorem 5.6. *Every n-dimensional ϵ-Kenmotsu manifold (M, g) admitting a conformal η-Ricci soliton (g, ξ, λ, μ) satisfies the curvature condition $W_2(\xi, Y) \cdot S = 0$.*

6. Conformal η-Ricci solitons on ϵ-Kenmotsu manifolds with torse-forming vector field

A vector field V on an n-dimensional ϵ-Kenmotsu manifold is said to be torse-forming vector field [56] if

$$\nabla_X V = fX + \gamma(X)V, \tag{6.1}$$

where f is a smooth function and γ is a 1-form.

Now let (g, ξ, λ, μ) be a conformal η-Ricci soliton on an ϵ-Kenmotsu manifold (M, g, ξ, ϕ, η) and assume that the Reeb vector field ξ of the manifold is a torse-forming vector field. Then ξ being a torse-forming vector field, by definiton from Eq (6.1) we have

$$\nabla_X \xi = fX + \gamma(X)\xi, \tag{6.2}$$

$\forall X \in TM$, f being a smooth function and γ is a 1-form.
Recalling the Eq (2.6) and taking inner product on both sides with ξ we can write

$$g(\nabla_X \xi, \xi) = \epsilon g(X, \xi) - \epsilon\eta(X)g(\xi, \xi),$$

which, in view of (2.2), reduces to

$$g(\nabla_X \xi, \xi) = 0. \tag{6.3}$$

Again from the Eq (6.2), applying inner product with ξ we obtain

$$g(\nabla_X \xi, \xi) = \epsilon f\eta(X) + \epsilon\gamma(X). \tag{6.4}$$

Combining (6.3) and (6.4) we get, $\gamma = -f\eta$. Thus for torse-forming vector field ξ in ϵ-Kenmotsu manifolds, we have

$$\nabla_X \xi = f(X - \eta(X)\xi). \tag{6.5}$$

Since (g, ξ, λ, μ) is a conformal η-Ricci soliton, from (1.2) we can write

$$g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) + 2S(X, Y) + [2\lambda - (p + \frac{2}{n})]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0.$$

In view of (6.5) the above becomes

$$S(X, Y) = [(\frac{p}{2} + \frac{1}{n}) - (\lambda + f)]g(X, Y) + (\epsilon f - \mu)\eta(X)\eta(Y). \tag{6.6}$$

This implies that the manifold is an η-Einstein manifold. Therefore we have the following

Theorem 6.1. *Let (g, ξ, λ, μ) be a conformal η-Ricci soliton on an n-dimensional ϵ-Kenmotsu manifold (M, g), with torse-forming vector field ξ, then the manifold becomes an η-Einstein manifold of the form* (6.6).

In particular if ξ is spacelike, i.e., $\epsilon = 1$, then for $\mu = f$, the Eq (6.6) reduces to

$$S(X, Y) = [(\frac{p}{2} + \frac{1}{n}) - (\lambda + f)]g(X, Y), \tag{6.7}$$

which implies that the manifold is an Einstein manifold. Similarly for ξ timelike and for $\mu = -f$, from (6.6) we can say that the manifold becomes an Einstein manifold. Therefore we can state

Corollary 6.2. *Let (g, ξ, λ, μ) be a conformal η-Ricci soliton on an n-dimensional ϵ-Kenmotsu manifold (M, g), with torse-forming vector field ξ, then the manifold becomes an Einstein manifold according as ξ is spacelike and $\mu = f$, or ξ is timelike and $\mu = -f$.*

7. Gradient conformal η-Ricci soliton on ϵ-Kenmotsu manifold

This section is devoted to the study of ϵ-Kenmotsu manifolds admitting gradient conformal η-Ricci solitons and we try to characterize the potential vector field of the soliton. First, we prove the following lemma which will be used later in this section.

Lemma 7.1. *On an n-dimensional ϵ-Kenmotsu manifold (M, g, ϕ, ξ, η), the following relations hold*

$$g((\nabla_Z Q)X, Y) = g((\nabla_Z Q)Y, X), \tag{7.1}$$

$$(\nabla_Z Q)\xi = -\epsilon QZ - (n-1)Z, \tag{7.2}$$

for all smooth vector fields X, Y, Z on M.

Proof. Since we know that the Ricci tensor is symmetric, we have $g(QX, Y) = g(X, QY)$. Covariantly differentiating this relation along Z and using $g(QX, Y) = S(X, Y)$ we can easily obtain (7.1).
To prove the second part, let us recall Eq (2.13) and taking its covariant derivative in the direction of an arbitrary smooth vector field Z we get

$$(\nabla_Z Q)\xi + Q(\nabla_Z \xi) + \epsilon(n-1)\nabla_Z \xi = 0. \tag{7.3}$$

In view of (2.6) and (2.13), the previous equation gives the desired result (7.2). This completes the proof. □

Now, we consider ϵ-Kenmotsu manifolds admitting gradient conformal η-Ricci solitons i.e., when the vector field V is gradient of some smooth function f on M. Thus if $V = Df$, where $Df = gradf$, then the conformal η-Ricci soliton equation becomes

$$Hessf + S + [\lambda - (\frac{p}{2} + \frac{1}{n})]g + \mu\eta \otimes \eta = 0, \tag{7.4}$$

where $Hessf$ denotes the Hessian of the smooth function f. In this case the vector field V is called the potential vector field and the smooth function f is called the potential function.

Lemma 7.2. *If (g, V, λ, μ) is a gradient conformal η-Ricci soliton on an n-dimensional ϵ-Kenmotsu manifold (M, g, ϕ, ξ, η), then the Riemannian curvature tensor R satisfies*

$$R(X, Y)Df = [(\nabla_Y Q)X - (\nabla_X Q)Y] + \epsilon\mu[\eta(X)Y - \eta(Y)X]. \tag{7.5}$$

Proof. Since the data (g, V, λ, μ) is a gradient conformal η-Ricci soliton, Eq (7.4) holds and it can be rewritten as

$$\nabla_X Df = -QX - [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]X - \mu\eta(X)\xi, \tag{7.6}$$

for all smooth vector field X on M and for some smooth function f such that $V = Df = gradf$. Covariantly diffrentiating the previous equation along an arbitrary vector field Y and using (2.6) we obtain

$$\begin{aligned} \nabla_Y\nabla_X Df &= -\nabla_Y(QX) - [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]\nabla_Y X \\ &\quad -\mu[(\nabla_Y\eta(X))\xi + \epsilon(Y - \eta(Y)\xi)\eta(X)]. \end{aligned} \tag{7.7}$$

Interchanging X and Y in (7.7) gives

$$\begin{aligned} \nabla_X\nabla_Y Df &= -\nabla_X(QY) - [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]\nabla_X Y \\ &\quad -\mu[(\nabla_X\eta(Y))\xi + \epsilon(X - \eta(X)\xi)\eta(Y)]. \end{aligned} \tag{7.8}$$

Again in view of (7.6) we can write

$$\begin{aligned} \nabla_{[X,Y]} Df &= -Q(\nabla_X Y - \nabla_Y X) - \mu\eta(\nabla_X Y - \nabla_Y X)\xi \\ &\quad -[\lambda - (\frac{p}{2} + \frac{1}{2n+1})](\nabla_X Y - \nabla_Y X). \end{aligned} \tag{7.9}$$

Therefore substituting the values from (7.7), (7.8) (7.9) in the following well-known Riemannian curvature formula

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

we obtain our desired expression (7.5). This completes the proof. □

Remark 7.3. A particular case of the above result for the case $\epsilon = 1$ is proved in Lemma 4.1 in the paper [18].

Now we proceed to prove our main result of this section.

Theorem 7.4. *Let (M, g, ϕ, ξ, η) be an n-dimensional ϵ-Kenmotsu manifold admitting a gradient conformal η-Ricci soliton (g, V, λ, μ), then the potential vector field V is pointwise collinear with the characteristic vector field ξ.*

Proof. Recalling the Eq (2.8) and taking its inner product with Df yields

$$g(R(X, Y)\xi, Df) = (Yf)\eta(X) - (Xf)\eta(Y).$$

Again we know that $g(R(X, Y)\xi, Df) = -g(R(X, Y)Df, \xi)$ and in view of this the previous equation becomes

$$g(R(X, Y)Df, \xi) = (Xf)\eta(Y) - (Yf)\eta(X). \tag{7.10}$$

Now taking inner product of (7.5) with ξ and using (7.2) we obtain

$$g(R(X, Y)Df, \xi) = 0. \tag{7.11}$$

Thus combining (7.10) and (7.11) we arrive at

$$(Xf)\eta(Y) = (Yf)\eta(X).$$

Taking $Y = \xi$ in the foregoing equation gives us $(Xf) = (\xi f)\eta(X)$, which essentially implies $g(X, Df) = g(X, \epsilon(\xi f)\xi)$. Since this equation is true for all X, we can conclude that

$$V = Df = \epsilon(\xi f)\xi. \tag{7.12}$$

Hence, V is pointwise collinear with ξ and this completes the proof. □

Remark 7.5. Since, the above result is independent of ϵ, it is also true for $\epsilon = 1$, i.e., for the case of Kenmotsu manifold (for details see [18]).

Corollary 7.6. *If (g, V, λ, μ) is a gradient conformal η-Ricci soliton on an n-dimensional ϵ-Kenmotsu manifold (M, g, ϕ, ξ, η), then the direction of the potential vector field V is same or opposite to the direction of the characteristic vector field ξ, according as ξ is spacelike or timelike vector field.*

Again covariantly differentiating (7.12) and then combining it with (7.6), and after that taking $X = \xi$ in the derived expression we obtain

$$\nabla^2{}_{\xi} f = \lambda + \mu - (\frac{p}{2} + \frac{1}{n}) - \epsilon(n - 1).$$

Hence we can conclude the following

Corollary 7.7. *If $(g, V = Df, \lambda, \mu)$ is a gradient conformal η-Ricci soliton on an n-dimensional ϵ-Kenmotsu manifold (M, g, ϕ, ξ, η), then at the particular point ξ, the potential function f satisfies the Laplace's equation $\nabla^2 f = 0$, if and only if,*

$$\lambda + \mu = (\frac{p}{2} + \frac{1}{n}) + \epsilon(n - 1).$$

8. Example of a 5-dimensional ϵ-Kenmotsu manifold admitting conformal η-Ricci soliton

Let us consider the 5-dimensional manifold $M = \{(u_1, u_2, v_1, v_2, w) \in \mathbb{R}^5 : w \neq 0\}$. Define a set of vector fields $\{e_i : 1 \leq i \leq 5\}$ on the manifold M given by

$$e_1 = \epsilon w\frac{\partial}{\partial u_1},\ \ e_2 = \epsilon w\frac{\partial}{\partial u_2},\ \ e_3 = \epsilon w\frac{\partial}{\partial v_1},\ \ e_4 = \epsilon w\frac{\partial}{\partial v_2},\ \ e_5 = -\epsilon w\frac{\partial}{\partial w}.$$

Let us define the indefinite metric g on M by

$$g(e_i, e_j) = \begin{cases} \epsilon, & \text{for} \quad i = j \\ 0, & \text{for} \quad i \neq j \end{cases}$$

for all $i, j = 1, 2, 3, 4, 5$. Now considering $e_5 = \xi$, let us take the 1-form η, on the manifold M, defined by

$$\eta(U) = \epsilon g(U, e_5) = \epsilon g(U, \xi), \quad \forall U \in TM.$$

Then it can be observed that $\eta(e_5 = 1)$. Let us define the $(1, 1)$ tensor field ϕ on M as

$$\phi(e_1) = e_2,\ \ \phi(e_2) = -e_1,\ \ \phi(e_3) = e_4,\ \ \phi(e_4) = -e_3,\ \ \phi(e_5) = 0.$$

Then using the linearity of g and ϕ it can be easily checked that

$$\phi^2(U) = -U + \eta(U)\xi,\ \ g(\phi U, \phi V) = g(U, V) - \epsilon\eta(U)\eta(V), \quad \forall U, V \in TM.$$

Hence the structure $(\phi, \xi, \eta, g, \epsilon)$ defines an indefinite almost contact structure on the manifold M.
Now, using the definitions of Lie bracket, direct computations give us
$[e_i, e_5] = \epsilon e_i;\ \ \forall i = 1, 2, 3, 4, 5$ and all other $[e_i, e_j]$ vanishes. Again the Riemannian connection ∇ of the metric g is defined by the well-known Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) \;\; &= \;\; Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]). \end{aligned}$$

Using the above formula one can easily calculate that
$\nabla_{e_i} e_i = -\epsilon e_5$, $\nabla_{e_i} e_5 = -\epsilon e_i$; for i=1,2,3,4 and all other $\nabla_{e_i} e_j$ vanishes. Thus it follows that $\nabla_X \xi = \epsilon(X - \eta(X)\xi), \quad \forall X \in TM$. Therefore the manifold (M, g) is a 5-dimensional ϵ-Kenmotsu manifold.
Now using the well-known formula $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ the non-vanishing components of the Riemannian curvature tensor R can be easily obtained as

$$\begin{aligned} &R(e_1, e_2)e_2 = R(e_1, e_3)e_3 = R(e_1, e_4)e_4 = R(e_1, e_5)e_5 = -e_1, \\ &R(e_1, e_2)e_1 = e_2,\ R(e_1, e_3)e_1 = R(e_1, e_3)e_2 = R(e_1, e_3)e_5 = e_3, \\ &R(e_1, e_2)e_3 = R(e_1, e_2)e_4 = R(e_1, e_2)e_5 = -e_2,\ R(e_1, e_2)e_4 = -e_3, \\ &R(e_1, e_2)e_2 = R(e_1, e_2)e_1 = R(e_1, e_2)e_4 = R(e_1, e_2)e_3 = e_5, \\ &R(e_1, e_2)e_1 = R(e_1, e_2)e_2 = R(e_1, e_2)e_3 = R(e_1, e_2)e_5 = e_4. \end{aligned}$$

From the above values of the curvature tensor, we obtain the components of the Ricci tensor as follows

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = S(e_4, e_4) = S(e_5, e_5) = -4. \tag{8.1}$$

Therefore using (8.1) in the Eq (3.3) we can calculate $\lambda = 3\epsilon + (\frac{p}{2} + \frac{1}{5})$ and $\mu = 1$. Hence we can say that for $\lambda = 3\epsilon + (\frac{p}{2} + \frac{1}{5})$ and $\mu = 1$, the data (g, ξ, λ, μ) defines a 5-dimensional conformal η-Ricci soliton on the manifold (M, g, ϕ, ξ, η).

9. Conclusions

The effect of conformal η-Ricci solitons have been studied within the framework of ϵ-Kenmotsu manifolds. Here we have characterized ϵ-Kenmotsu manifolds, which admit conformal η-Ricci soliton, in terms of Einstein and η-Einstein manifolds. It is well-known that for $\epsilon = 1$ and spacelike Reeb vector field ξ, the ϵ-Kenmotsu manifold becomes a Kenmotsu manifold. Also we know that Einstein manifolds, Kenmotsu manifolds are very important classes of manifolds having extensive use in mathematical physics and general relativity. Hence it is interesting to investigate conformal η-Ricci solitons on Sasakian manifolds as well as in other contact metric manifolds. Also there is further scope of research in this direction within the framework of various complex manifolds like Kaehler manifolds, Hopf manifolds etc.

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Conflict of interest

The authors declare no conflict of interest.

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On trans-Sasakian 3-manifolds as η-Einstein solitons

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The present paper is to deliberate the class of 3-dimensional trans-Sasakian manifolds which admits η-Einstein solitons. We have studied η-Einstein solitons on 3-dimensional trans-Sasakian manifolds where the Ricci tensors are Codazzi type and cyclic parallel. We have also discussed some curvature conditions admitting η-Einstein solitons on 3-dimensional trans-Sasakian manifolds and the vector field is torse-forming. We have also shown an example of 3-dimensional trans-Sasakian manifold with respect to η-Einstein soliton to verify our results.

Key words and phrases: Einstein soliton, η-Einstein soliton, trans-Sasakian manifold, Codazzi type Ricci tensor, C-Bochner curvature tensor.

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Introduction

In 2016, G. Catino and L. Mazzieri [7] introduced the notion of Einstein soliton, which can be viewed as a self-similar solution to the Einstein flow

$$\frac{\partial g}{\partial t} = -2\Big(S - \frac{r}{2}g\Big),$$

where g is the Riemannian metric, S is the Ricci tensor and r is the scalar curvature.

It can be easily seen that the Einstein soliton is analogous to the Ricci soliton, which is also generated by a self-similar solution to the very famous geometric revolution equation Ricci flow. The term "Ricci soliton" [11] arose as a need for a more general self-similar solution, to the Ricci flow equation, than the uniformly shrinking or expanding solutions in case of Einstein manifolds. It is a well-known fact now that the study of Ricci soliton has tremendous contribution in solving the longstanding Thurston's geometric conjecture. Similarly it is also interesting to study the Einstein soliton from various directions to solve many physical and geometrical problems. In [7], the authors characterized the nature of complete three-dimensional, positively curved, Riemannian manifold satisfying gradient Einstein soliton equation. Motivated from their work, in this paper we consider a slight perturbation of the Einstein soliton by $\eta \otimes \eta$, called the η-Einstein soliton. The mathematical expression for the η-Einstein soliton [1] is given by the following equation

$$\mathcal{L}_\xi g + 2S + (2\lambda - r)g + 2\mu\eta \otimes \eta = 0, \tag{1}$$

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where $\mathcal{L}_\xi$ denotes the Lie derivative along the direction of the vector field ξ, S is the Ricci tensor, r is the scalar curvature and λ, μ are real constants. The η-Einstein soliton is called shrinking if $\lambda < 0$, steady if $\lambda = 0$ and expanding if $\lambda > 0$. In particular, if $\mu = 0$, the η-Einstein soliton reduces to the Einstein soliton (g, ξ, λ).

J. T. Cho and M. Kimura [6] introduced the concept of η-Ricci soliton and later C. Calin and M. Crasmareanu [5] studied it on Hopf hypersufaces in complex space forms. A Riemannian manifold (M, g) is said to admit an η-Ricci soliton if for a smooth vector field V, the metric g satisfies the following equation

$$\mathcal{L}_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{2}$$

where $\mathcal{L}_\xi$ is the Lie derivative along the direction of ξ, S is the Ricci tensor and λ, μ are real constants. It is to be noted that if the manifold has constant scalar curvature, then the data $(g, \xi, \lambda - \frac{r}{2}, \mu)$ of the equation (1) satisfies the equation (2), i.e. the η-Einstein soliton reduces to an η-Ricci soliton. Hence we can remark that the two notions are different for the manifolds of non-constant scalar curvature and if the scalar curvature of the manifold is constant then the concepts of η-Ricci soliton and η-Einstein soliton coincide.

In [18], the authors studied Ricci solitons within the framework of three-dimensional trans-Sasakian manifolds. They proved that if a compact three-dimensional trans-Sasakian manifold with constant scalar curvature admits Ricci soliton, then the manifold is either α-Sasakian or β-Kenmotsu. Later T. Dutta et al. [8] investigated three-dimensional trans-Sasakian manifolds, which admits conformal Ricci soliton. Furthermore they showed that on a three-dimensional trans-Sasakian manifold, under some condition on the potential vector field, almost conformal Ricci soliton reduces to conformal Ricci soliton. Very recently, in [15] the author studied η-Ricci soliton on three-dimensional trans-Sasakian manifolds satisfying various tensorial conditions $S \cdot R = 0$, $R \cdot S = 0$, $W_2 \cdot S = 0$ and $S \cdot W_2 = 0$.

Motivated by the above papers, here we propose to study various geometric aspects of three-dimensional trans-Sasakian manifolds admitting η-Einstein solitons.

The paper is organised as follows. After a brief introduction, in Section 2, we recall some basic knowledge on trans-Sasakian manifolds. Section 3 deals with 3-dimensional trans-Sasakian manifolds admitting η-Einstein solitons and also the nature of the soliton is dicussed. In this section, we have constructed an example of a 3-dimensional trans-Sasakian manifold satisfying η-Einstein soliton. In Section 4, we have contrived η-Einstein solitons in 3-dimensional trans-Sasakian manifolds in terms of Codazzi type and cyclic parallel Ricci tensor and characterized the nature of the manifold. Sections $5, 6, 7, 8$ are devoted to the study of some curvature conditions $R \cdot S = 0$, $W_2 \cdot S = 0$, $R \cdot E = 0$, $B \cdot S = 0$, $S \cdot R = 0$ admitting η-Einstein solitons in 3-dimensional trans-Sasakian manifold. In last section we have studied torse forming vector field when 3-dimensional trans-Sasakian manifolds admitting η-Einstein solitons.

1 Preliminaries

An n-dimensional smooth Riemannian manifold (M, g) is said to be an almost contact metric manifold [3] if it admits a $(1, 1)$ tensor field ϕ, a characteristic vector field ξ, a global 1-form η and an indefinite metric g on M satisfying the following relations

$$\begin{aligned} \phi^2 &= -I + \eta \otimes \xi, \\ \eta(\xi) &= 1, \qquad (3) \\ \eta(X) &= g(X, \xi), \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \qquad (4) \\ g(X, \phi Y) + g(Y, \phi X) &= 0, \end{aligned}$$

for all vector fields $X, Y \in TM$, where TM is the tangent bundle of the manifold M. Also it can be easily seen that $\phi(\xi) = 0$, $\eta(\phi X) = 0$ and rank of ϕ is $(n-1)$.

The geometry of the almost Hermitian manifold $(M \times \mathbb{R}, G, J)$ gives rise to the geometry of the almost contact metric manifold (M, g, ϕ, ξ, η), where G is product metric of the product manifold $M \times \mathbb{R}$ with the complex structure J defined by

$$J(X, f\frac{d}{dt}) = \Big(\phi X - f\xi, \eta(X)\frac{d}{dt}\Big),$$

for all vector fields X on the manifold M and smooth function f on the product manifold $M \times \mathbb{R}$. An almost contact metric manifold (M, g, ϕ, ξ, η) is called a trans-Sasakian manifold if the product manifold $(M \times \mathbb{R}, G, J)$ belongs to the class W_4 [10]. The notion of trans-Sasakian manifolds was introduced by J.A. Oubina [14] and later J.C. Marrero [12] completely characterized the local structures of trans-Sasakian manifolds of dimension $n \geq 5$. The expression for which an almost contact metric manifold (M, g, ϕ, ξ, η) becomes a trans-Sasakian manifold is given by

$$(\nabla_X \phi)(Y) = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X], \qquad (5)$$

for all $X, Y \in TM$ and for some smooth functions α, β on the manifold M. Then such kind of manifold is called a trans-Sasakian manifold of type (α, β). In particular trans-Sasakian manifolds of type $(0,0)$, $(\alpha, 0)$ and $(0, \beta)$ are called cosymplectic, α-Sasakian and β-Kenmotsu manifolds respectively.

In what follows, by a trans-Sasakian 3-manifold, we mean a 3-dimensional trans-Sasakian manifold (M, g, ϕ, ξ, η) of type (α, β) and we will use the notation (M, g) to denote it throughout this article. Now from the expression (5) it can be derived that

$$\begin{aligned} \nabla_X \xi &= -\alpha\phi(X) + \beta(X - \eta(X)\xi), \qquad (6) \\ (\nabla_X \eta)(Y) &= -\alpha g(\phi(X), Y) + \beta g(\phi(X), \phi(Y)), \qquad (7) \end{aligned}$$

for all vector fields X, Y in TM. Again from equation (20) of corollary 4.2 in the paper [19], the Riemannian curvature tensor in a trans-Sasakian 3-manifold (M, g) is given by

$$\begin{aligned} R(X,Y)Z = {} & \Big(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2)\Big)[g(Y, Z)X - g(X, Z)Y] \\ & - g(Y, Z)\Big[\Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\eta(X)\xi - \eta(X)(\phi(\mathrm{grad}\,\alpha) - \mathrm{grad}\,\beta) + (X\beta + (\phi X)\alpha)\xi\Big] \\ & + g(X, Z)\Big[\Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\eta(Y)\xi - \eta(Y)(\phi(\mathrm{grad}\,\alpha) - \mathrm{grad}\,\beta) + (Y\beta + (\phi Y)\alpha)\xi\Big] \\ & - \Big[(Z\beta + (\phi Z)\alpha)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(Z) + \Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\eta(Y)\eta(Z)\Big]X \\ & + \Big[(Z\beta + (\phi Z)\alpha)\eta(X) + (X\beta + (\phi X)\alpha)\eta(Z) + \Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\eta(X)\eta(Z)\Big]Y. \end{aligned}$$

Furthermore, in a trans-Sasakian 3-manifold (M, g), if the functions α, β are constants then, taking $Z = \xi$ (similarly for the second relation taking $X = \xi$ and taking $X = Z = \xi$ for the third relation) in the above equation, the following relations can easily be deduced

$$R(X,Y)\xi = (\alpha^2-\beta^2)[\eta(Y)X - \eta(X)Y], \tag{8}$$
$$R(\xi,X)Y = (\alpha^2-\beta^2)[g(X,Y)\xi - \eta(Y)X], \tag{9}$$
$$R(\xi,X)\xi = (\alpha^2-\beta^2)[\eta(X)\xi - X]. \tag{10}$$

Also, taking both α, β constant in corollary 4.2 in [19], we obtain the relations for the Ricci tensor

$$\begin{aligned} S(X,Y) &= \Big[\frac{r}{2} - (\alpha^2-\beta^2)\Big]g(X,Y) - \Big[\frac{r}{2} - 3(\alpha^2-\beta^2)\Big]\eta(X)\eta(Y), \\ S(X,\xi) &= 2(\alpha^2-\beta^2)\eta(X), \end{aligned} \tag{11}$$

for all vector fields X, Y in TM and where R is the curvature tensor and S is the Ricci tensor.

Definition 1. *A trans-Sasakian 3-manifold (M, g) is said to be an η-Einstein manifold if its Ricci tensor S is of the form*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

for all $X, Y \in TM$ and smooth functions a, b on the manifold (M, g).

2 η-Einstein solitons on trans-Sasakian 3-manifolds

Let us consider a trans-Sasakian 3-manifold (M, g) admitting an η-Einstein soliton given by the data (g, ξ, λ, μ). Then from equation (1) we can write

$$(\mathcal{L}_\xi g)(X,Y) + 2S(X,Y) + [2\lambda - r]g(X,Y) + 2\mu\eta(X)\eta(Y) = 0 \tag{12}$$

for all $X, Y \in TM$.

Again from the well-known formula $(\mathcal{L}_\xi g)(X,Y) = g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X)$ of Lie-derivative and using (6), we obtain for a trans-Sasakian 3-manifold

$$(\mathcal{L}_\xi g)(X,Y) = 2\beta[g(X,Y) - 2\beta\eta(X)\eta(Y)]. \tag{13}$$

Now in view of the equations (12) and (13) we get

$$S(X,Y) = \Big(\frac{r}{2} - \lambda - \beta\Big)g(X,Y) + (\beta - \mu)\eta(X)\eta(Y). \tag{14}$$

This shows that the manifold (M, g) is an η-Einstein manifold. Also from equation (14) replacing $Y = \xi$ we find that

$$S(X,\xi) = \Big(\frac{r}{2} - \lambda - \mu\Big)\eta(X). \tag{15}$$

Compairing the above equation (15) with (11) yields

$$r = 4(\alpha^2-\beta^2) + 2\lambda + 2\mu. \tag{16}$$

Again, considering an orthonormal basis $\{e_1, e_2, e_3\}$ of (M, g) and then setting $X = Y = e_i$ in equation (14) and summing over $i = 1, 2, 3$ we get

$$r = 6\lambda + 4\beta + 2\mu. \tag{17}$$

Finally combining equations (16) and (17) we arrive at

$$\lambda = (\alpha^2-\beta^2) - \beta. \tag{18}$$

Thus the above discussion leads to the following result.

Theorem 1. *If a trans-Sasakian 3-manifold (M,g) admits an η-Einstein soliton (g,ξ,λ,μ), then the manifold (M,g) becomes an η-Einstein manifold of constant scalar curvature $r = 6\lambda + 4\beta + 2\mu$. Furthermore, the soliton is shrinking, steady or expanding according as $\alpha^2 < \beta(\beta+1)$, $\alpha^2 = \beta(\beta+1)$, $\alpha^2 > \beta(\beta+1)$ respectively.*

Example 1. *Let us consider the 3-dimensional manifold $M = \{(u,v,w) \in \mathbb{R}^3 : w \neq 0\}$. Define a linearly independent set of vector fields $\{e_i : 1 \leq i \leq 3\}$ on the manifold M given by*

$$e_1 = e^{2w}\frac{\partial}{\partial u}, \quad e_2 = e^{2w}\frac{\partial}{\partial v}, \quad e_3 = \frac{\partial}{\partial w}.$$

Let us define the Riemannian metric g on M by

$$g(e_i,e_j) = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j, \end{cases}$$

for all $i,j = 1,2,3$. Now considering $e_3 = \xi$, let us take the 1-form η, on the manifold M, defined by $\eta(U) = g(U,e_3)$, for all $U \in TM$. Then it can be observed that $\eta(\xi) = 1$. Let us define the $(1,1)$ tensor field ϕ on M as

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$

Using the linearity of g and ϕ it can be easily checked that

$$\phi^2(U) = -U + \eta(U)\xi, \;\; g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V), \quad \forall U,V \in TM.$$

Hence the structure (g,ϕ,ξ,η) defines an almost contact metric structure on the manifold M. Now, using the definitions of Lie bracket, after some direct computations we get

$$[e_1,e_2] = 0, \quad [e_1,e_3] = -2e_1, \quad [e_2,e_3] = -2e_2.$$

Again the Riemannian connection ∇ of the metric g is defined by the well-known Koszul's formula, which is given by

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]).$$

Using the above formula one can easily calculate that

$$\begin{aligned} \nabla_{e_1}e_1 = 2e_3, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = -2e_1, \\ \nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = 2e_3, \quad \nabla_{e_2}e_3 = -2e_2, \\ \nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = 0. \end{aligned}$$

Thus from the above relations it follows that the manifold (M,g) is a trans-Sasakian 3-manifold. Now using the well-known formula $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ the non-vanishing components of the Riemannian curvature tensor R can be easily obtained as

$$\begin{aligned} R(e_1,e_2)e_2 = R(e_1,e_3)e_3 = -4e_1, \\ R(e_2,e_3)e_3 = R(e_3,e_1)e_1 = -4e_2, \\ R(e_3,e_2)e_2 = 4e_2, \quad R(e_2,e_1)e_1 = 4e_3. \end{aligned}$$

Hence we can calculate the components of the Ricci tensor as follows

$$S(e_1,e_1) = 0, \quad S(e_2,e_2) = 0, \quad S(e_3,e_3) = -8.$$

Therefore in view of the above values of the Ricci tensor, from the equation (1) *we can calculate* $\lambda = -2$ *and* $\mu = 6$. *Hence we can say that the data* $(g, \xi, -2, 6)$ *defines an* η*-Einstein soliton on the trans-Sasakian* 3*-manifold* (M, g). *Also we can see that the manifold* (M, g) *is a manifold of constant scalar curvature* $r = -8$ *and hence the theorem* 1 *is verified.*

Next we consider a trans-Sasakian 3-manifold (M, g) and assume that it admits an η-Einstein soliton (g, V, λ, μ) such that V is pointwise collinear with ξ, i.e. $V = b\xi$ for some function b. Then from the equation (1) it follows that

$$bg(\nabla_X\xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y\xi, X) + (Yb)\eta(X) \\ + 2S(X,Y) + (2\lambda - r)g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Then using the equation (6) in above we arrive at

$$(2b\beta + 2\lambda - r)g(X,Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X,Y) + 2(b\beta + \mu)\eta(X)\eta(Y) = 0. \quad (19)$$

Replacing $Y = \xi$ in the above equation yields

$$(Xb) + (\xi b)\eta(X) + 2S(X,\xi) + (2\lambda + 2\mu - r)\eta(X) = 0. \quad (20)$$

Again taking $X = \xi$ in (20) and by virtue of (11) we arrive at $2(\xi b) = (r - 2\lambda - 2\mu) - 4(\alpha^2 - \beta^2)$. Using this value in (20) and recalling (11), we can write

$$db = \Big[\frac{r}{2} - \lambda - \mu - 2(\alpha^2 - \beta^2)\Big]\eta. \quad (21)$$

Now taking exterior differentiation on both sides of (21) and using the famous Poincare's lemma, i.e. $d^2 = 0$, finally we arrive at $r = 2\lambda + 2\mu + 4(\alpha^2 - \beta^2)$. Therefore, the equation (21) gives us $db = 0$, i.e. the function b is constant. Then the equation (19) reduces to

$$S(X,Y) = \Big(\frac{r}{2} - \lambda - b\beta\Big)g(X,Y) + (b\beta - \mu)\eta(X)\eta(Y)$$

for all $X, Y \in TM$. Hence we can state the following result.

Theorem 2. *If a trans-Sasakian* 3*-manifold* (M, g) *admits an* η*-Einstein soliton* (g, V, λ, μ) *such that* V *is pointwise collinear with* ξ*, then* V *is constant multiple of* ξ *and the manifold* (M, g) *becomes an* η*-Einstein manifold of constant scalar curvature* $r = 2\lambda + 2\mu + 4(\alpha^2 - \beta^2)$.

3 η-Einstein solitons on trans-Sasakian 3-manifolds with Codazzi type and cyclic parallel Ricci tensor

The purpose of this section is to study η-Einstein solitons in trans-Sasakian 3-manifolds having certain special types of Ricci tensor, namely Codazzi type Ricci tensor and cyclic parallel Ricci tensor.

Definition 2 ([9])**.** *A trans-Sasakian* 3*-manifold is said to have Codazzi type Ricci tensor if its Ricci tensor* S *is non-zero and satisfies the following relation*

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z), \quad \forall X, Y, Z \in TM. \quad (22)$$

Let us consider a trans-Sasakian 3-manifold having Codazzi type Ricci tensor and admits an η-Einstein soliton (g, ξ, λ, μ), then equation (14) holds. Now covariantly differentiating (14) with respect to an arbitrary vector field X and then using (7) we get

$$(\nabla_X S)(Y,Z) = 2(\beta-\mu)[\eta(Y)(-\alpha g(\phi X, Z) + \beta g(\phi X, \phi Z)) \\ + \eta(Z)(-\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y))]. \tag{23}$$

Similarly we can compute

$$(\nabla_Y S)(X,Z) = 2(\beta-\mu)[\eta(X)(-\alpha g(\phi Y, Z) + \beta g(\phi Y, \phi Z)) \\ + \eta(Z)(-\alpha g(\phi Y, X) + \beta g(\phi Y, \phi X))]. \tag{24}$$

Since the manifold has Codazzi type Ricci tensor, using (23) and (24) in the (22) and then recalling (4) we arrive at

$$2(\beta-\mu)[\eta(Y)(-\alpha g(\phi X, Z) + \beta g(X, Z)) \\ - \eta(X)(-\alpha g(\phi Y, Z) + \beta g(Y, Z)) - 2\alpha\eta(Z) g(\phi X, Y)] = 0. \tag{25}$$

Putting $Z = \xi$ in above and in view of (3) we finally obtain

$$4\alpha(\mu-\beta) g(\phi X, Y) = 0 \tag{26}$$

forall $X, Y \in TM$. Therefore from (25) we can conclude that either $\alpha = 0$ or $\mu = \beta$. Hence we have the following result.

Theorem 3. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ). If the Ricci tensor of the manifold is of Codazzi type then the manifold becomes a β-Kenmotsu manifold provided $\mu \neq \beta$.*

Now using $\alpha = 0$ in equation (18) we get $\lambda = -\beta(\beta+1)$. Thus we can state the following assertion.

Corollary 1. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ) with $\mu \neq \beta$. If the Ricci tensor of the manifold is of Codazzi type then the soliton is shrinking if $\beta < -1$ or $\beta > 0$; steady if $\beta = -1$ or $\beta = 0$; and expanding if $-1 < \beta < 0$ respectively.*

Again from the (25) we can write that $\mu = \beta$ if $\alpha \neq 0$. Then from equation (14) we obtain

$$S(X,Y) = \left(\frac{r}{2} - \lambda - \beta\right) g(X,Y) \tag{27}$$

for all $X, Y \in TM$. Then contracting the equation (26) we get $r = 6\lambda + 6\beta$. Hence in view of this and (26) we have the following result.

Theorem 4. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ). If the Ricci tensor of the manifold is of Codazzi type then the manifold becomes an Einstein manifold of constant scalar curvature $r = 6\lambda + 6\beta$ provided $\alpha \neq 0$.*

Definition 3 ([9]). *A trans-Sasakian 3-manifold is said to have cyclic parallel Ricci tensor if its Ricci tensor S is non-zero and satisfies the following relation*

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0 \quad \forall X, Y, Z \in TM. \tag{28}$$

Let us consider a trans-Sasakian 3-manifold, having cyclic parallel Ricci tensor, admits an η-Einstein soliton (g, ξ, λ, μ), then equation (14) holds. Now taking covariant differentiation of (14) and using (7) we obtain relations (23) and (24). In a similar manner we get the following

$$\begin{aligned}(\nabla_Z S)(X,Y) = 2(\beta-\mu)[\eta(X)(-\alpha g(\phi Z, Y) + \beta g(\phi Z, \phi Y)) \\ + \eta(Y)(-\alpha g(\phi Z, X) + \beta g(\phi Z, \phi X))].\end{aligned} \tag{29}$$

Now since the manifold has cyclic parallel Ricci tensor, using the values from (23), (24) and (29) in the equation (28) and then making use of (4) we arrive at

$$4\beta(\beta-\mu)[\eta(X)g(\phi Y, \phi Z) + \eta(Y)g(\phi Z, \phi X) + \eta(Z)g(\phi X, \phi Y)] = 0.$$

Replacing $Z = \xi$ in the above equation yields

$$4\beta(\beta-\mu)g(\phi X, \phi Y) = 0 \tag{30}$$

for all $X, Y \in TM$. Since $g(\phi X, \phi Y) \neq 0$ always, the above equation implies that either $\beta = 0$ or $\mu = \beta$. Thus we can state the following result.

Theorem 5. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ). If the manifold has cyclic parallel Ricci tensor, then the manifold becomes an α-Sasakian manifold provided $\mu \neq \beta$.*

Now using $\beta = 0$ in equation (18) we get $\lambda = \alpha^2 > 0$. Therefore we have the following assertion.

Corollary 2. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ) with $\mu \neq \beta$. If the manifold has cyclic parallel Ricci tensor then the soliton is expanding.*

Again if $\beta \neq 0$ then from (30) it follows that $\mu = \beta$. Therefore after a similar calculation like equation (27) we can state the following assertion.

Theorem 6. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ). If the manifold has cyclic parallel Ricci tensor, then the manifold becomes an Einstein manifold of constant scalar curvature $r = 6\lambda + 6\beta$ provided $\beta \neq 0$.*

4 η-Einstein solitons on trans-Sasakian 3-manifolds satisfying $R(\xi, X)\cdot S = 0$ and $W_2(\xi, X) \cdot S = 0$

Let us first consider a trans-Sasakian 3-manifold, which admits an η-Einstein soliton (g, ξ, λ, μ) and the manifold satisfies the curvature condition $R(\xi, X) \cdot S = 0$. Then for all $X, Y, Z \in TM$ we can write

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0. \tag{31}$$

Now using (14) in (31) we get

$$\begin{aligned}\left(\frac{r}{2} - \lambda - \beta\right) g(R(\xi, X)Y, Z) + (\beta-\mu)\eta(R(\xi, X)Y)\eta(Z) \\ + \left(\frac{r}{2} - \lambda - \beta\right) g(R(\xi, X)Z, Y) + (\beta-\mu)\eta(R(\xi, X)Z)\eta(Y) = 0.\end{aligned}$$

In view of (9) the previous equation becomes

$$(\alpha^2-\beta^2)(\beta-\mu)[g(X,Y)\eta(Z)+g(X,Z)\eta(Y)-2\eta(X)\eta(Y)\eta(Z)]=0.$$

Putting $Z=\xi$ in the above equation and recalling (4), we obtain

$$(\alpha^2-\beta^2)(\beta-\mu)g(\phi X,\phi Y)=0 \tag{32}$$

for all $X,Y\in TM$. Since $g(\phi X,\phi X)\neq 0$ always and for non-trivial case $\alpha^2\neq\beta^2$, we can conclude from (32) that $\mu=\beta$. Then from equation (14) we obtain

$$S(X,Y)=\Big(\frac{r}{2}-\lambda-\beta\Big)g(X,Y) \tag{33}$$

for all $X,Y\in TM$. Then contracting (33) we get $r=6\lambda+6\beta$. Hence in view of this and equation (33) we have the following result.

Theorem 7. *Let (M,g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g,ξ,λ,μ). If the manifold satisfies the curvature condition $R(\xi,X)\cdot S=0$, then the manifold becomes an Einstein manifold of constant scalar curvature $r=6\lambda+6\beta$.*

Our next result of this section is on W_2-curvature tensor. It is an important curvature tensor, which was introduced in 1970 by G.P. Pokhariyal and R.S. Mishra [16]. For this let us recall the definition of W_2-curvature tensor.

Definition 4. *The W_2-curvature tensor in a trans-Sasakian 3-manifold (M,g) is defined as*

$$W_2(X,Y)Z=R(X,Y)Z+\frac{1}{2}\Big[g(X,Z)QY-g(Y,Z)QX\Big]. \tag{34}$$

Now assume that (M,g) is a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g,ξ,λ,μ) and also the manifold satisfies the curvature condition $W_2(\xi,X)\cdot S=0$. Then we can write

$$S(W_2(\xi,X)Y,Z)+S(Y,W_2(\xi,X)Z)=0,\quad \forall X,Y,Z\in TM.$$

In view of (14) the above equation becomes

$$\begin{aligned}\Big(\frac{r}{2}-\lambda-\beta\Big)&[g(W_2(\xi,X)Y,Z)+g(W_2(\xi,X)Z,Y)]\\&+(\beta-\mu)\big[\eta(W_2(\xi,X)Y)\eta(Z)+\eta(W_2(\xi,X)Z)\eta(Y)\big]=0.\end{aligned} \tag{35}$$

Again from (14) it follows that

$$QX=\Big(\frac{r}{2}-\lambda-\beta\Big)X+(\beta-\mu)\eta(X)\xi, \tag{36}$$

which implies

$$Q\xi=\Big(\frac{r}{2}-\lambda-\mu\Big)\xi. \tag{37}$$

Replacing $X=\xi$ in (34) and then using equations (9), (36) and (37) we obtain

$$W_2(\xi,Y)Z=Bg(Y,Z)\xi-A\eta(Z)Y+(A-B)\eta(Y)\eta(Z), \tag{38}$$

where $A = (\alpha^2 - \beta^2) - \frac{1}{2}(\frac{r}{2} - \lambda - \beta)$ and $B = (\alpha^2 - \beta^2) - \frac{1}{2}(\frac{r}{2} - \lambda - \mu)$. Taking inner product of (38) with respect to the vector field ξ yields

$$\eta(W_2(\xi, Y)Z) = B[g(Y, Z) - \eta(Y)\eta(Z)]. \tag{39}$$

Using (38) and (39) in (35) and then taking $Z = \xi$ we arrive at

$$(A - B)\Big[2B - \Big(\frac{r}{2} - \lambda - \beta\Big)\Big][g(X, Y) - \eta(X)\eta(Y)] = 0,$$

which in view of (4) implies

$$(A - B)\Big[2B - \Big(\frac{r}{2} - \lambda - \beta\Big)\Big]g(\phi X, \phi Y) = 0 \tag{40}$$

for all $X, Y \in TM$. Since $g(\phi X, \phi X) \neq 0$ always, we can conclude from (40) that either $A = B$ or $2B = \frac{r}{2} - \lambda - \beta$. Thus recalling the values of A and B it implies that either $\mu = \beta$ or

$$2(\alpha^2 - \beta^2) = r - 2\lambda - \mu - \beta. \tag{41}$$

Now for the case $\mu = \beta$, proceeding similarly as the equation (33) we can say that the manifold becomes an Einstein manifold. Again combining (41) with (16) we get

$$r = 2\lambda + 2\beta. \tag{42}$$

Therefore we can state the following assertion.

Theorem 8. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ). If the manifold satisfies the curvature condition $W_2(\xi, X) \cdot S = 0$, then either the manifold becomes an Einstein manifold or it is a manifold of constant scalar curvature $r = 2\lambda + 2\beta$.*

Again in view of (17), equation (42) implies $\lambda = -\frac{1}{2}(\mu + \beta)$. Hence we have the following result.

Corollary 3. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ) with $\mu \neq \beta$. If the manifold satisfies the curvature condition $W_2(\xi, X) \cdot S = 0$, then the soliton is expanding, steady or shrinking according as $\mu < -\beta$, $\mu = -\beta$ or $\mu > -\beta$, respectively.*

5 Einstein semi-symmetric trans-Sasakian 3-manifolds admitting η-Einstein solitons

Definition 5. *A trans-Sasakian 3-manifold (M, g) is called Einstein semi-symmetric [17] if $R.E = 0$, where E is the Einstein tensor given by*

$$E(X, Y) = S(X, Y) - \frac{r}{3}g(X, Y) \tag{43}$$

for all vector fields $X, Y \in TM$ and r is the scalar curvature of the manifold.

Now consider a trans-Sasakian 3-manifold is Einstein semi-symmetric, i.e. the manifold satisfies the curvature condition $R.E = 0$. Then for all vector fields $X, Y, Z, W \in TM$ we can write $E(R(X,Y)Z, W) + E(Z, R(X,Y)W) = 0$. In view of (43) the last equation becomes

$$S(R(X,Y)Z, W) + S(Z, R(X,Y)W) = \frac{r}{3}\Big[g(R(X,Y)Z, W) + g(Z, R(X,Y)W)\Big]. \tag{44}$$

Replacing $X = Z = \xi$ in (44) and using (9), (10) we arrive at

$$(\alpha^2 - \beta^2)S(Y, W) = (\alpha^2 - \beta^2)[\eta(Y)S(\xi, W) + \eta(W)S(\xi, Y) - g(Y, W)S(\xi, \xi)].$$

So, now in view of (11) the above equation finally yields

$$S(Y, W) = -2(\alpha^2 - \beta^2)g(Y, W) + 4(\alpha^2 - \beta^2)\eta(Y)\eta(W) \tag{45}$$

for all $Y, W \in TM$. This implies that the manifold is an η-Einstein manifold.

Lemma 1. *An Einstein semi-symmetric trans-Sasakian 3-manifold is an η-Einstein manifold.*

Now let us assume that the Einstein semi-symmetric trans-Sasakian 3-manifold (M, g) admits an η-Einstein soliton (g, ξ, λ, μ). Then equation (14) holds and combining (14) with the above equation (45) we get $r = 2\lambda + \mu + \beta$. Again recalling (17) in the last equation we have $\lambda = -\frac{1}{4}(\mu + 3\beta)$.

Theorem 9. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ). If the manifold is Einstein semi-symmetric, then the manifold becomes an η-Einstein manifold of constant scalar curvature $r = 2\lambda + \mu + \beta$ and the soliton is expanding, steady or shrinking according as $\mu < 3\beta$, $\mu = 3\beta$ or $\mu > 3\beta$ respectively.*

6 η-Einstein solitons on trans-Sasakian 3-manifolds satisfying $B(\xi,X)\cdot S=0$

In 1949, S. Bochner [4] introduced the concept of the well-known Bochner curvature tensor merely as a Kähler analogue of the Weyl conformal curvature tensor but the geometric significance of it in the light of Boothby-Wangs fibration was presented later by D.E. Blair [2]. The notion of C-Bochner curvature tensor in a Sasakian manifold was introduced by M. Matsumoto, G. Chūman [13] in 1969. The C-Bochner curvature tensor in trans-Sasakian 3-manifold (M, g) is given by

$$\begin{aligned} B(X,Y)Z = {} & R(X,Y)Z + \frac{1}{6}\Big[g(X,Z)QY - S(Y,Z) - g(Y,Z)QX + S(X,Z)Y + g(\phi X, Z)Q\phi Y \\ & - S(\phi Y, Z)\phi X - g(\phi Y, Z)Q\phi X + S(\phi X, Z)\phi Y + 2S(\phi X, Y)\phi Z + 2g(\phi X, Y))Q\phi Z \\ & + \eta(Y)\eta(Z)QX - \eta(Y)S(X,Z)\xi + \eta(X)S(Y,Z)\xi - \eta(X)\eta(Z)QY\Big] \\ & - \frac{D+2}{6}\Big[g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X + 2g(\phi X, Y)\phi Z\Big] \\ & + \frac{D}{6}\Big[\eta(Y)g(X,Z)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y - \eta(X)g(Y,Z)\xi\Big] \\ & - \frac{D-4}{6}\Big[g(X,Z)Y - g(Y,Z)X\Big], \end{aligned} \tag{46}$$

where $D = \frac{r+2}{4}$.

Let us consider a trans-Sasakian 3-manifold (M,g), which admits an η-Einstein soliton (g,ξ,λ,μ) and also the manifold satisfies the curvature condition $B(\xi,X)\cdot S=0$. Then for all $X,Y,Z\in TM$ we can write

$$S(B(\xi,X)Y,Z)+S(Y,B(\xi,X)Z)=0. \tag{47}$$

Now using (14) in (47) we get

$$\Big(\frac{r}{2}-\lambda-\beta\Big)[g(B(\xi,X)Y,Z)+g(B(\xi,X)Z,Y)] \\ +(\beta-\mu)[\eta(B(\xi,X)Y)\eta(Z)+\eta(B(\xi,X)Z)\eta(Y)]=0. \tag{48}$$

Again from (14) it follows that $QX=(\frac{r}{2}-\lambda-\beta)X+(\beta-\mu)\eta(X)\xi$, which implies

$$Q\xi=\Big(\frac{r}{2}-\lambda-\mu\Big)\xi. \tag{49}$$

Also taking $X=\xi$ in (46) we obtain

$$B(\xi,Y)Z=R(\xi,Y)Z+\frac{1}{6}\Big[S(\xi,Z)Y-g(Y,Z)Q\xi+\eta(Y)\eta(Z)Q\xi-\eta(Y)S(\xi,Z)\xi\Big] \\ +\frac{4}{6}\Big[\eta(Z)Y-g(Y,Z)\xi\Big].$$

Using equations (9), (15) and (49) in the above equation yields

$$B(\xi,Y)Z=\Big[(\alpha^2-\beta^2)-\frac{1}{6}\Big(\frac{r}{2}-\lambda-\mu\Big)-\frac{4}{6}\Big][g(Y,Z)\xi-\eta(Z)Y]. \tag{50}$$

In view of (50) the equation (48) becomes

$$\Big[(\alpha^2-\beta^2)-\frac{1}{6}\Big(\frac{r}{2}-\lambda-\mu\Big)-\frac{4}{6}\Big](\beta-\mu)[g(X,Y)\eta(Z)+g(X,Z)\eta(Y)-2\eta(X)\eta(Y)\eta(Z)]=0.$$

Replacing $Z=\xi$ in the above equation and recalling (4), finally we arrive at

$$\Big[(\alpha^2-\beta^2)-\frac{1}{6}\Big(\frac{r}{2}-\lambda-\mu\Big)-\frac{4}{6}\Big](\beta-\mu)g(\phi X,\phi Y)=0 \tag{51}$$

for all vector fields $X,Y\in TM$. Hence from (51) we can conclude that either

$$\Big[(\alpha^2-\beta^2)-\frac{1}{6}\Big(\frac{r}{2}-\lambda-\mu\Big)-\frac{4}{6}\Big]=0 \tag{52}$$

or $\mu=\beta$. Also for $\mu=\beta$ proceeding similarly as equation (27) it can be easily shown that the manifold becomes an Einstein manifold. Again if $\mu\neq\beta$ using (18) in (52) we have

$$r=10\lambda+2\mu+12\beta-8, \tag{53}$$

which implies that the manifold becomes a manifold of constant scalar curvature. Therefore we can state the following assertion.

Theorem 10. *Let (M,g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g,ξ,λ,μ). If the manifold satisfies the curvature condition $B(\xi,X)\cdot S=0$, then either the manifold is an Einstein manifold or it is a manifold of constant scalar curvature $r=10\lambda+2\mu+12\beta-8$.*

Now for the case $\mu\neq\beta$, using the equation (17) in (53) we obtain $\lambda=2(1-\beta)$.

Corollary 4. *Let (M,g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g,ξ,λ,μ) with $\mu\neq\beta$. If the manifold satisfies the curvature condition $B(\xi,X)\cdot S=0$, then the soliton is expanding, steady or shrinking according as $\beta<1$, $\beta=1$ or $\beta>1$ respectively.*

7 η-Einstein solitons on trans-Sasakian 3-manifolds satisfying $S(\xi,X)\cdot R=0$

In this section, we study the curvature condition $S(\xi, X) \cdot R = 0$, where by $\cdot$ we denote the derivation of the tensor algebra at each point of the tangent space as follows

$$\begin{aligned} S((\xi, X) \cdot R)(Y, Z)W &:= ((\xi \wedge_S X) \cdot R)(Y, Z)W \\ &:= (\xi \wedge_S X)R(Y, Z)W + R((\xi \wedge_S X)Y, Z)W \\ &\quad + R(Y, (\xi \wedge_S X)Z)W + R(Y, Z)(\xi \wedge_S X)W, \end{aligned} \tag{54}$$

where the endomorphism $X \wedge_S Y$ is defined by $(X \wedge_S Y)Z := S(Y, Z)X - S(X, Z)Y$.

Now let us consider a trans-Sasakian 3-manifold (M, g), which admits an η-Einstein soliton (g, ξ, λ, μ) and also the manifold satisfies the curvature condition $S(\xi, X) \cdot R = 0$. Then using this condition and the equation (54) we can write

$$\begin{aligned} S(X, R(Y, Z)W)\xi - S(\xi, R(Y, Z)W)X + S(X, Y)R(\xi, Z)W - S(\xi, Y)R(X, Z)W \\ + S(X, Z)R(Y, \xi)W - S(\xi, Z)R(Y, X)W \\ + S(X, W)R(Y, Z)\xi - S(\xi, W)R(Y, Z)X = 0 \end{aligned} \tag{55}$$

for all vector fields $X, Y, Z, W \in TM$. Taking inner product of (55) with the vector field ξ and then replacing $W = \xi$ we obtain

$$\begin{aligned} S(X, R(Y, Z)\xi) - S(\xi, R(Y, Z)\xi)\eta(X) + S(X, Y)\eta(R(\xi, Z)\xi) - S(\xi, Y)\eta(R(X, Z)\xi) \\ + S(X, Z)\eta(R(Y, \xi)\xi) - S(\xi, Z)\eta(R(Y, X)\xi) \\ + S(X, \xi)\eta(R(Y, Z)\xi) - S(\xi, \xi)\eta(R(Y, Z)X) = 0. \end{aligned}$$

In view of (8) and (10) the foregoing equation becomes

$$\begin{aligned} (\alpha^2 - \beta^2)[S(X, Y)\eta(Z) - S(X, Z)\eta(Y) - S(\xi, Y)\eta(X)\eta(Z) + S(\xi, Z)\eta(X)\eta(Y)] \\ - S(\xi, \xi)\eta(R(Y, Z)X) = 0. \end{aligned} \tag{56}$$

Putting $Y = \xi$ in (56) and then recalling (9) we get

$$\begin{aligned} (\alpha^2 - \beta^2)\big[S(X, \xi)\eta(Z) - S(X, Z) - S(\xi, \xi)\eta(X)\eta(Z) \\ + S(\xi, Z)\eta(X) - S(\xi, \xi)[g(X, Z) - \eta(X)\eta(Z)]\big] = 0. \end{aligned}$$

Using equations (14) and (15) in the previous equation yields

$$(\alpha^2 - \beta^2)[(r - 2\lambda - 2\mu + \beta)\eta(X)\eta(Z) - (r - 2\lambda - \mu - \beta)g(X, Z)] = 0.$$

Replacing $X = \xi$ in above, we arrive at

$$(\alpha^2 - \beta^2)(2\beta - \mu)\eta(X) = 0, \quad \forall X \in TM. \tag{57}$$

Since for non-trivial case $\alpha^2 \neq \beta^2$, from the above equation (57) it follows that $\mu = 2\beta$. Therefore in view of this and recalling (17) we finally obtain $r = 6\lambda + 8\beta$. Therefore we can state the following assertion.

Theorem 11. *Let (M, g) be a trans-Sasakian 3-manifold admitting an η-Einstein soliton (g, ξ, λ, μ). If the manifold satisfies the curvature condition $S(\xi, X) \cdot R = 0$, then it becomes a manifold of constant scalar curvature $r = 6\lambda + 8\beta$.*

8 η-Einstein solitons on trans-Sasakian 3-manifolds with torse-forming vector field

This section is devoted to study the nature of η-Einstein solitons on trans-Sasakian 3-manifolds with torse-forming vector field.

Definition 6. *A vector field V on a trans-Sasakian 3-manifold is said to be torse-forming vector field [20] if*

$$\nabla_X V = fX + \gamma(X)V, \tag{58}$$

where f is a smooth function and γ is a 1-form.

Now let (g, ξ, λ, μ) be an η-Einstein soliton on a trans-Sasakian 3-manifold (M, g) and assume that the Reeb vector field ξ of the manifold is a torse-forming vector field. Then ξ being a torse-forming vector field, by definiton from equation (58) we have

$$\nabla_X \xi = fX + \gamma(X)\xi, \qquad \forall X \in TM, \tag{59}$$

f being a smooth function and γ is a 1-form.

Recalling (6) and taking inner product on both sides with ξ we have

$$g(\nabla_X \xi, \xi) = (\beta - 1)\eta(X). \tag{60}$$

Again from the equation (59), applying inner product with ξ we obtain

$$g(\nabla_X \xi, \xi) = f\eta(X) + \gamma(X). \tag{61}$$

Combining (60) and (61) we get $\gamma = (\beta - 1 - f)\eta$. Thus from (59) it implies that for torse-forming vector field ξ in a trans-Sasakian 3-manifold we have

$$\nabla_X \xi = f(X - \eta(X)\xi) + (\beta - 1)\eta(X)\xi. \tag{62}$$

Therefore using (62) from the formula of Lie differentiation it follows

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 2f[g(X, Y) - \eta(X)\eta(Y)] + 2(\beta - 1)\eta(X)\eta(Y). \tag{63}$$

Since (g, ξ, λ, μ) is an η-Einstein soliton, the equation (1) holds. So in view of (63), equation (1) reduces to

$$S(X, Y) = \Big(\frac{r}{2} - \lambda + f\Big)g(X, Y) + (f - \mu - \beta + 1)\eta(X)\eta(Y). \tag{64}$$

This implies that the manifold is an η-Einstein manifold. Again putting $Y = \xi$ in (64) we get

$$S(X, \xi) = \Big(\frac{r}{2} - \lambda - \mu - \beta + 1\Big)\eta(X). \tag{65}$$

Combining (65) with (11) gives us

$$\Big(\frac{r}{2} - \lambda - \mu - \beta + 1\Big) = 2(\alpha^2 - \beta^2). \tag{66}$$

Again tracing out the equation (64) we obtain

$$r = 6\lambda + 2\mu + 4f + 2\beta - 2. \tag{67}$$

Using the above equation (67) in (66), finally we get $\lambda = f - (\alpha^2 - \beta^2)$. Therefore we have the following result.

Theorem 12. *Let (g, ξ, λ, μ) be an η-Einstein soliton on a trans-Sasakian 3-manifold (M, g) with torse-forming vector field ξ. Then the manifold becomes an η-Einstein manifold and the soliton is expanding, steady or shrinking according as $f > (\alpha^2 - \beta^2)$, $f = (\alpha^2 - \beta^2)$ or $f < (\alpha^2 - \beta^2)$ respectively.*

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Ґанґулі Д., Дей С., Бхаттачарія А. *Про транс-Сасакянові 3-многовиди як η-солітони Айнштайна* // Карпатські матем. публ. — 2021. — Т.13, №2. — С. 460–474.

Ця стаття має на меті обговорення класу 3-вимірних транс-Сасакянових многовидів, що допускають η-солітони Айнштайна. Ми вивчили η-солітони Айнштайна на 3-вимірних транс-Сасакянових многовидах, де тензори Річчі мають тип Кодацці та є циклічно паралельними. Ми також обговорили деякі умови кривизни, що допускають η-солітони Айнштайна на 3-вимірних транс-Сасакянових многовидах, а векторне поле торцеутворююче. Ми також показали приклад 3-вимірного транс-Сасакянового многовиду відносно η-солітону Айнштайна для перевірки наших результатів.

Ключові слова і фрази: солітон Айнштайна, η-солітон Айнштайна, транс-Сасакяновий многовид, тензор Річчі типу Кодацці, C-тензор кривизни Бохнера.