NUMERICAL RADIUS INEQUALITIES OF HILBERT SPACE OPERATORS AND THEIR APPLICATIONS

Pintu Bhunia

(Index No.: 136/18/Math./26)

THIS THESIS IS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF DOCTOR OF PHILOSOPHY IN SCIENCE

DEPARTMENT OF MATHEMATICS

JADAVPUR UNIVERSITY

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যাদবপুর বিশ্ববিদ্যালয়

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**NUMERICAL RADIUS INEQUALITIES OF HILBERT SPACE OPERATORS AND THEIR APPLICATIONS**" submitted by **Pintu Bhunia** who got his name registered on 04/09/2018 (**Index No.: 136/18/Math./26**) for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon his own research work under the supervision of **Prof. Kallol Paul**, Department of Mathematics, Jadavpur University, Kolkata 700032, India and that neither this thesis nor any part of it has been submitted for any degree/diploma or any other academic award anywhere before.

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K Paul 25.02.2022

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(Prof. Kallol Paul)
(Signature of the Supervisor
and date with official seal)

Dedicated to my mother ***Mrs. Jyotsana Bhunia***
and my father ***Mr. Tapan Kumar Bhunia***

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Abstract

The numerical range $W(A)$ of a bounded linear operator A on a complex Hilbert space $\mathcal{H}$ is defined as the range of the continuous mapping $x \longmapsto \langle Ax, x\rangle$ on the unit sphere of the Hilbert space, i.e., $W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$. Clearly $W(A)$ is a bounded subset of the scalar field and its closure contains the spectrum of the operator. The bounds of the numerical range helps in estimating the spectrum of the operator. In this connection the numerical radius $w(A)$, which is defined as the radius of the smallest circle with center at the origin that contains the numerical range $W(A)$, plays a very important role. The main focus of this thesis is to develop stronger lower and upper bounds of the numerical radius using various technique. We obtain improvements and generalizations of the inequalities $w(A) \leq \frac{1}{2}\left(\|A\| + \|A^2\|^{1/2}\right)$ and $\frac{1}{4}\|A^*A + AA^*\| \leq w^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|$. Then we study the numerical radius inequality of the generalized commutator and anti-commutator operators which improves and generalizes the inequality $w(AB \pm BA) \leq 2\sqrt{2}\|B\|w(A)$. Next we present upper bounds for the numerical radius of bounded linear operators which generalize and improve on the well-known upper bound $w^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|$. We obtain an upper bound for the numerical radius of the sum of the product operators which generalizes and improves on the existing ones. We present equivalent conditions for the equality of $w(A) = \frac{\|A\|}{2}$ as well as $w^2(A) = \frac{1}{4}\|A^*A + AA^*\|$ in terms of the geometrical shape of the numerical range of A. Next we develop a number of inequalities using the properties of t-Aluthge transform. We show that the bounds obtained here are better than the existing ones. We also estimate the spectral radius of the sum of the product of n pairs of operators. Then we present upper and lower bounds for the numerical radius of 2×2 operator matrices. Applying the bounds obtained here, to Frobenius companion matrix of a complex monic polynomial $p(z)$ of degree greater than or equal to three, we obtain new bounds for the zeros of $p(z)$.

CONTENTS

CHAPTER 1

INTRODUCTION

The motivation of numerical range comes from the classical quadratic forms and in early days of Hilbert space studies quadratic forms were the object of chief interest. Later on the chief interest was shifted from quadratic questions to theory of operators and relevant notions like numerical ranges. The theories of numerical range and their applications appear in many branches of sciences including very recently grown quantum information system. The extension of quadratic forms to the setting of linear operator on both finite and infinite dimensional Hilbert spaces lead to the notion of numerical range or field of values of an operator. The numerical range of a bounded linear operator A on a complex Hilbert space $\mathcal{H}$, denoted by $W(A)$, is defined as the range of the continuous mapping $x \mapsto \langle Ax, x\rangle$ defined on the unit sphere of the Hilbert space $\mathcal{H}$. Readers can look at the two excellent books on numerical ranges in Hilbert space setting one by Halmos [44] and another by Gustafson and Rao [43]. The same in Banach space setting can be found in a book by Bonsall and Duncan [34]. The study of numerical range assists in understanding the behavior of a bounded linear operator. As for example, the spectrum of a bounded linear operator is always contained in the closure of the numerical range of that bounded linear operator. So, the spectral value of a bounded linear operator can be estimated if the numerical range of that bounded linear operator is known to us. The major role in this connection is played by the well-known constant numerical radius associated with the numerical range. The numerical radius of a bounded linear operator A, to be denoted by $w(A)$, is defined as the radius as the smallest circle with center at the origin that contains the numerical range $W(A)$. The classical bounds for the numerical radius is $\frac{1}{2}\|A\| \leq w(A) \leq \|A\|$ which was later on improved by many mathematicians. Out of those improvements a few are worth mentioning here, namely by Kittaneh [54, 55] and Yamazaki [75]. The key word of the title of this thesis is "Numerical Radius Inequalities". Having introduced "Numerical Radius" we now say a few words about "Inequalities". Over the years various inequalities have existed in different branches of Mathematics. In 1934, the first book "Inequalities" was written by G.H. Hardy, J.E. Littlewood and G. Pólya [45]. The second book on this topic was written by E.F. Beckenbach and R. Bellman [15] in 1961. These books have revolutionized the field of inequalities into a well organized field and provide motivations, ideas, techniques and applications for new research. The main purpose of this work is to develop numerical radius inequalities of Hilbert space operators and operator matrices with nice and simple form, which improve the existing lower and upper bounds. As applications of those bounds we give better estimations for the zeros of a complex polynomial of degree greater than or equal to three. For more on existing numerical radius inequalities we refer the readers to the monograph by Dragomir [37].

1.1 Introduction and preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ denote the field of real and complex numbers, respectively. First we define an inner product space which is one of the fundamental concept in Functional Analysis.

Definition 1.1. *Let $\mathcal{V}$ be a vector space over the field $\mathbb{F}\,(=\mathbb{R}$ or $\mathbb{C})$. An inner product on $\mathcal{V}$ is a function $\langle\cdot,\cdot\rangle \to \mathbb{F}$ such that for all $x, y, z \in \mathcal{V}$ and for all $\alpha, \beta \in \mathbb{F}$ the following are satisfied:*

$$\begin{aligned}
&(i)\quad \langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle,\\
&(ii)\quad \langle x, \alpha y + \beta z\rangle = \bar{\alpha}\langle x, y\rangle + \bar{\beta}\langle x, z\rangle,\\
&(iii)\quad \langle x, x\rangle \geq 0,\\
&(iv)\quad \langle x, x\rangle = 0 \text{ if and only if } x = 0,\\
&(v)\quad \langle x, y\rangle = \overline{\langle y, x\rangle},
\end{aligned}$$

where $\bar{\alpha}$ denotes the complex conjugate of the scalar α and $\alpha = \bar{\alpha}$ if α is real. If the vector space is considered over real field (complex field) then the pair $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ is called a real (complex) inner product space.

If $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ is an inner product space then it is easy to see that the function $\|\cdot\| : \mathcal{V} \to \mathbb{R}$ defined by $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$ for all $x \in \mathcal{V}$ satisfies the following:

- $\|x\| \geq 0$ for all $x \in \mathcal{V}$ (non-negativity), and $\|x\| = 0$ if and only if $x = 0$.

- $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathcal{V}$ (triangle inequality).
- $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{F}$ and for all $x \in \mathcal{V}$ (homegenity).

Therefore, the function $\|\cdot\|$ induced by the inner product $\langle \cdot, \cdot \rangle$ satisfies all the conditions of a norm and so $(\mathcal{V}, \|\cdot\|)$ is a normed linear space. In general, a vector space $\mathcal{V}$ is said to be a normed linear space if there is a function $\|\cdot\|$ on $\mathcal{V}$ satisfying the above three properties.

However, we concentrate our attention to an inner product space. First we note few properties on an inner product space.

1. (**Cauchy-Schwarz inequality**) Let $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ be an inner product space. Then

$$|\langle x, y \rangle| \leq \|x\|\|y\| \quad \text{for all} \quad x, y \in \mathcal{V}.$$

2. (**Parallelogram law**) Let $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ be an inner product space. Then

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \text{for all} \quad x, y \in \mathcal{V}.$$

3. (**Polarization identity**) Let $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ be an inner product space. Then

$$4\langle x, y \rangle = \|x+y\|^2 - \|x-y\|^2 + i\left(\|x+iy\|^2 - \|x-iy\|^2\right) \quad \text{for all} \quad x, y \in \mathcal{V}.$$

A Hilbert space is an inner product space $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ such that the space is complete with respect to the metric $d(x,y) = \|x-y\| = \langle x-y, x-y \rangle^{\frac{1}{2}}$ for all $x, y \in \mathcal{V}$, induced from the inner product $\langle \cdot, \cdot \rangle$. From now on, we reserve the symbol $\mathcal{H}$ for a complex Hilbert space with inner product $\langle ., . \rangle$. Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators acting on $\mathcal{H}$ with the identity I. The norm induced by the inner product $\langle ., . \rangle$ is denoted by $\|\cdot\|$. For $A \in \mathcal{B}(\mathcal{H})$, A^* stands for the adjoint of A and $|A|$ denotes the positive operator $(A^*A)^{1/2}$. We denote the real part and the imaginary part of an operator $A \in \mathcal{B}(\mathcal{H})$ by $\Re(A)$ and $\Im(A)$, respectively, that is, $\Re(A) = \frac{1}{2}(A+A^*)$ and $\Im(A) = \frac{1}{2i}(A-A^*)$. Therefore, the Cartesian decomposition of $A \in \mathcal{B}(\mathcal{H})$ is given by $A = \Re(A) + i\Im(A)$. The resolvent set of an operator $A \in \mathcal{B}(\mathcal{H})$ is defined as the collection of all scalars λ for which $(A-\lambda I)^{-1}$ exists as a bounded linear operator on the Hilbert space $\mathcal{H}$ and is denoted by $\rho(A)$. The spectrum of an operator $A \in \mathcal{B}(\mathcal{H})$, denoted by $\sigma(A)$, is defined as the complement of the resolvent set, i.e., $\sigma(A) = \mathbb{C} \setminus \rho(A)$. For $A \in \mathcal{B}(\mathcal{H})$, $\sigma(A)$ is a non-empty compact subset of $\mathbb{C}$. The spectral radius of an operator $A \in \mathcal{B}(\mathcal{H})$, denoted by $r(A)$, is defined by

$$r(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}.$$

The set $\sigma_p(A) = \{\lambda \in \sigma(A) : \ Ax = \lambda x \ \text{ for some non-zero } x \in \mathcal{H}\}$ is called the point spectrum of A. For a matrix A, $\sigma_p(A) = \sigma(A)$, since a linear operator on a finite-dimensional space is always bounded and is injective if and only if it is surjective. For $A \in \mathcal{B}(\mathcal{H})$, $\|A\|$ denotes the operator norm of A. Recall that $\|A\| = \sup\{\|Ax\| : x \in \mathcal{H}, \|x\| = 1\}$. It is known that $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$. If A is self-adjoint, then $\|A\|^2 = \|A^*A\| = \|A^2\|$ and so, by induction, $\|A\|^{2^n} = \|A^{2^n}\|$. Therefore, for the self-adjoint operator A, $r(A) = \lim_{n\to\infty} \|A^{2^n}\|^{1/2^n} = \|A\|$. In addition, at least one of $\|A\|$ or $-\|A\|$ is in $\sigma(A)$ and $\sigma(A) \subseteq [-\|A\|, \|A\|]$.

Let $A \in \mathcal{B}(\mathcal{H})$. The numerical range of A, denoted by $W(A)$, is defined as $W(A) = \big\{\langle Ax, x\rangle : x \in \mathcal{H}, \ \ \|x\| = 1\big\}$. The following properties of the numerical range $W(A)$ can be easily verified:

(i) $W(\alpha I + \beta A) = \alpha + \beta W(A)$ for all $\alpha, \beta \in \mathbb{C}$.

(ii) $W(A^*) = \big\{\bar{\lambda} : \lambda \in W(A)\big\}$.

(iii) $W(U^*AU) = W(A)$ for every *unitary* operator $U \in \mathcal{B}(\mathcal{H})$.

In the following theorems we state fundamental properties of the numerical range.

Theorem 1.1. (*Ellipse lemma, [43, Lemma 1.1-1]*) *If A is an operator on a two-dimensional space $\mathcal{H}$, then $W(A)$ is an ellipse whose foci are the eigenvalues of A.*

Theorem 1.2. (*Toeplitz-Hausdorff theorem, [43, Th. 1.1-2]*) *The numerical range of an operator is convex.*

Considering the continuous mapping $x \longmapsto \langle Ax, x\rangle$ from $\{x \in \mathcal{H} : \|x\| = 1\}$ to the scalar field $\mathbb{C}$, it is easy to see that $W(A)$ is a compact subset of $\mathbb{C}$ if $\mathcal{H}$ is finite dimensional. Further, we note the following characterization for the self-adjoint operators in $\mathcal{B}(\mathcal{H})$.

Proposition 1.1. (*[43, p. 7]*) *Let $A \in \mathcal{B}(\mathcal{H})$. The following statements hold:*

(i) A is self-adjoint if and only if $W(A)$ is real.

(ii) If A is self-adjoint and $W(A) = [m, M]$, then $\|A\| = \max\{|m|, |M|\}$.

(iii) If $\overline{W(A)} = [m, M]$, then $m, M \in \sigma(A)$.

Note that for a normal operator $A \in \mathcal{B}(\mathcal{H})$, the closure of $W(A)$, i.e., $\overline{W(A)}$ is the convex hull of the spectrum $\sigma(A)$ of A.

Now, we recall the following key notions of our study. The numerical radius of $A \in \mathcal{B}(\mathcal{H})$, denoted by $w(A)$, is defined as

$$w(A) \;\; = \;\; \sup\Big\{|\langle Ax, x\rangle| : x \in \mathcal{H}, \|x\| = 1\Big\} = \sup\Big\{|\lambda| : \lambda \in W(A)\Big\}.$$

Similarly, another numerical constant, the Crawford number of A, denoted by $c(A)$, is defined as

$$c(A) = \inf\Big\{|\langle Ax, x\rangle| : x \in \mathcal{H}, \|x\| = 1\Big\} = \inf\Big\{|\lambda| : \lambda \in W(A)\Big\}.$$

It is not difficult to verify that the numerical radius $w(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$ and for $A \in \mathcal{B}(\mathcal{H})$, we infer that

$$\frac{1}{2}\|A\| \le w(A) \le \|A\|. \tag{1.1}$$

Therefore, the numerical radius norm is equivalent to the operator norm on $\mathcal{B}(\mathcal{H})$. Let us note here that $w(\cdot)$ fails to be a norm if the Hilbert space is considered over the real field. The inequalities in (1.1) are sharp, $w(A) = \|A\|$ if A is normal (i.e., $A^*A = AA^*$) and $w(A) = \frac{1}{2}\|A\|$ if $A^2 = 0$.

The spectral inclusion theorem reads as follows.

Theorem 1.3. (*Spectral inclusion theorem, [43, p. 6]*) *Let $A \in \mathcal{B}(\mathcal{H})$. Then $\sigma(A)$ is contained in the closure of $W(A)$, that is, $\sigma(A) \subseteq \overline{W(A)}$.*

Therefore, the spectral radius $r(A)$ of A always satisfies $r(A) \le w(A)$. A basic property for the numerical radius is that it satisfies the power inequality, i.e., for $A \in \mathcal{B}(\mathcal{H})$, $w(A^n) \le w^n(A)$ *for all* $n \in \mathbb{N}$. Here, $\mathbb{N}$ denotes the set of all natural numbers.

For $A \in \mathcal{B}(\mathcal{H})$, let $A = U|A|$ be the polar decomposition of A. The Aluthge transform of A, denoted as $\widetilde{A}$, is defined as

$$\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}},$$

where U is the partial isometry associated with the polar decomposition of A and so $\ker A = \ker U$. It follows easily from the definition of $\widetilde{A}$ that $\|\widetilde{A}\| \le \|A\|$ and $r(\widetilde{A}) = r(A)$. Also, $w(\widetilde{A}) \le w(A)$ (see [51]). Okubo [62] generalized the Aluthge transform, known as the t-Aluthge transform as follows. The t-Aluthge transform of A, denoted by $\widetilde{A_t}$, is defined by

$$\widetilde{A_t} = |A|^tU|A|^{1-t}, \ t \in [0, 1].$$

Here, $|A|^0$ is defined as U^*U. In particular, $\widetilde{A_0} = U^*U^2|A|$, $\widetilde{A_1} = |A|UU^*U = |A|U$, $\widetilde{A_{\frac{1}{2}}} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}} = \widetilde{A}$.

Over the years, various eminent mathematicians have been tried to improve on the inequalities in (1.1), we refer to see [4, 7, 11, 12, 13, 19, 24, 25, 27, 32, 48, 50, 61, 63, 65, 66, 67, 68, 71, 72] and the references therein. Here we note some important improvement of the inequalities for the numerical radius of a bounded linear operator A on $\mathcal{H}$. Kittaneh [55] improved on the right hand inequality in (1.1) to prove that for $A \in \mathcal{B}(\mathcal{H})$,

$$w(A) \quad \leq \quad \frac{1}{2}\left(\|A\| + \sqrt{\|A^2\|}\right). \tag{1.2}$$

Further, Kittaneh [54] improved on both the inequalities in (1.1) to proved that for $A \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{4}\|A^*A + AA^*\| \leq w^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|. \tag{1.3}$$

Observe that for $A \in \mathcal{B}(\mathcal{H})$, the terms $\frac{1}{2}\left(\|A\| + \sqrt{\|A^2\|}\right)$ and $\sqrt{\frac{1}{2}\|A^*A + AA^*\|}$ are not comparable, in general. In [39], Dragomir obtained an another inequality, namely, for $A \in \mathcal{B}(\mathcal{H})$,

$$w^2(A) \quad \leq \quad \frac{1}{2}\left(\|A\|^2 + w(A^2)\right), \tag{1.4}$$

which improve on the right hand inequality in (1.1). Further, Abu-Omar and Kittaneh [1] obtained that for $A \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}c(A^2) + \frac{1}{4}\|A^*A + AA^*\| \leq w^2(A) \leq \frac{1}{4}\|A^*A + AA^*\| + \frac{1}{2}w(A^2). \tag{1.5}$$

Clearly, the first inequality in (1.5) is better than the first inequality in (1.3). Also, the second inequality in (1.5) is stronger than the corresponding inequalities in (1.2), (1.3) and (1.4). Using the Aluthge transform, Yamazaki [75] proved that if $A \in \mathcal{B}(\mathcal{H})$, then

$$w(A) \quad \leq \quad \frac{1}{2}\left(\|A\| + w(\widetilde{A})\right). \tag{1.6}$$

Since $w(\widetilde{A}) \leq \|\widetilde{A}\| \leq \sqrt{\|A^2\|}$, the inequality in (1.6) refines that in (1.2). After that, Abu-Omar and Kittaneh [6] improved on the inequality (1.6) by using t-Aluthge transform to prove that

$$w(A) \quad \leq \quad \frac{1}{2}\left(\|A\| + \min_{0\leq t\leq 1} w(\widetilde{A_t})\right). \tag{1.7}$$

In this thesis, we develop various new upper and lower bounds for the numerical radius of bounded linear operators defined on $\mathcal{H}$ which refine the bounds mentioned in (1.1) - (1.7). We next give a brief outline of the thesis.

1.2 Out line of the thesis

The thesis consists of seven chapters including the Introductory one. In the introductory chapter we provide a brief history of numerical range along with definitions and notations to be used throughout the thesis.

In Chapter 2, we present an improvement and generalization of the inequality in (1.2), that is, $w(A) \leq \frac{1}{2}\left(\|A\| + \|A^2\|^{1/2}\right)$. Further, we study the numerical radius inequality of the generalized commutator and anti-commutator operators which improves and generalizes the inequality $w(AB \pm BA) \leq 2\sqrt{2}\|B\|w(A)$, proved by Fong and Holbrook [40]. Recall that for linear operators A and B, the operator $AB-BA$ is called commutator operator and the operator $AB+BA$ is called anti-commutator operator.

In Chapter 3, we present upper bounds for the numerical radius of bounded linear operators which generalize and improve on the well-known upper bounds both in (1.2) and (1.3), that is, $w(A) \leq \frac{1}{2}\left(\|A\| + \|A^2\|^{1/2}\right)$ and $w^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|$. Further, we present an upper bound for the numerical radius of the sum of the product operators which generalizes and improves on the existing ones.

In Chapter 4, we develop norm inequalities for the sum of two bounded linear operators, from which we derive lower bounds for the numerical radius of bounded linear operators that strongly refine the lower bound in (1.3), that is, $\frac{1}{4}\|A^*A + AA^*\| \leq w^2(A)$. Further, we present upper bounds for the numerical radius of bounded linear operators by using operator convex functions which improve on the existing ones.

In Chapter 5, we establish new inequalities for the numerical radius of bounded linear operators. For $A \in \mathcal{B}(\mathcal{H})$, we obtain the following bounds:

$$\begin{aligned} w^2(A) &\leq \min_{0\leq\alpha\leq 1} \left\| \alpha|A|^2 + (1-\alpha)|A^*|^2 \right\|, \\ w^2(A) &\leq \min_{0\leq\alpha\leq 1} \left\{ \frac{\alpha}{2} w(A^2) + \left\| \frac{\alpha}{4}|A|^2 + \left(1 - \frac{3}{4}\alpha\right)|A^*|^2 \right\| \right\}. \end{aligned}$$

We show that the inequalities obtained here greneralize and improve on the existing well-known inequalities given in [1, 54, 55]. Further, we obtain lower bounds for the numerical radius of bounded linear operators which refine the well-known lower bounds $w(A) \geq \frac{\|A\|}{2}$ and $w^2(A) \geq \frac{1}{4}\|A^*A + AA^*\|$. We also present equivalent conditions for the equality of $w(A) = \frac{\|A\|}{2}$ as well as $w^2(A) = \frac{1}{4}\|A^*A + AA^*\|$ in terms of the geometrical shape of the numerical range of A. Further, applying the lower bounds obtained here, we obtain upper bounds for the numerical radius of commutators of bounded linear operators, which refine the existing ones in [40, 47].

In Chapter 6, we develop a number of inequalities using the properties of t-Aluthge transform. We show that the inequalities obtained here improve (1.2), (1.3), (1.4) and (1.6). We also obtain an upper bound for the numerical radius and show by an example that the bound is incomparable, in general, with that in (1.7).

In Chapter 7, we obtain an upper bound for the numerical radius of a bounded linear operator which improves on the upper bound in (1.5). Also we obtain a lower bound for the numerical radius of a bounded linear operator which improves on the lower bound in (1.3). We present an upper bound of the numerical radius in terms of $\|H_\theta\|$ and a lower bound of the numerical radius in terms of the spectral values of $\Re(A)$ and $\Im(A)$, which improves on the existing lower bounds. Here, $H_\theta = \Re(e^{i\theta}A)$ for $\theta \in \mathbb{R}$. We also estimate the spectral radius of the sum of the product of n pairs of operators. Further, we present upper and lower bounds for the numerical radius of 2×2 operator matrices. Applying the bounds obtained here, to Frobenius companion matrix of a complex monic polynomial $p(z)$ of degree greater than or equal to three, we provide bounds for the zeros of $p(z)$ which refine the existing ones.

Before we end this section we would like to mention that in the beginning of each of the following chapter we provide a brief motivation along with the relevant notations and terminologies necessary to keep each chapter independent for the convenience of the reader.

CHAPTER 2

FURTHERANCE OF NUMERICAL RADIUS INEQUALITIES

2.1 Introduction

The main focus of this chapter is to provide improvement and generalization of the inequality (1.2), i.e., $w(A) \leq \frac{1}{2}\left(\|A\| + \|A^2\|^{1/2}\right)$, obtained in [55]. Further, we study the numerical radius inequality of the generalized commutator and anti-commutator operators which improves and generalizes the inequality $w(AB \pm BA) \leq 2\sqrt{2}\|B\|w(A)$, obtained in [40]. Let us now introduce the following necessary notations and terminologies.

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle .,. \rangle$ and let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. As usual the norm induced by the inner product $\langle .,. \rangle$ is denoted by $\|\cdot\|$. For $A \in \mathcal{B}(\mathcal{H})$, A^* denotes the adjoint of A and $|A|, |A^*|$ respectively denote the positive square root of A^*A, AA^*, i.e., $|A| = (A^*A)^{\frac{1}{2}}, |A^*| = (AA^*)^{\frac{1}{2}}$. Let $S_{\mathcal{H}}$ denote the unit sphere of the Hilbert space $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, let $\|A\|$ be the operator norm of A, i.e., $\|A\| = \sup_{x \in S_{\mathcal{H}}} \|Ax\|$. The numerical range of A, denoted by $W(A)$, is defined as $W(A) = \left\{\langle Ax, x\rangle : x \in S_{\mathcal{H}}\right\}$. Considering the continuous mapping $x \longmapsto \langle Ax, x\rangle$ from $S_{\mathcal{H}}$ to

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the scalar field $\mathbb{C}$, it is easy to see that $W(A)$ is a compact subset of $\mathbb{C}$ if $\mathcal{H}$ is finite-dimensional. The famous Toeplitz-Hausdorff theorem states that the numerical range of a bounded linear operator is a convex subset of $\mathbb{C}$. The numerical radius and the Crawford number of A, denoted as $w(A)$ and $c(A)$, respectively, are defined as

$$w(A) = \sup_{x \in S_{\mathcal{H}}} |\langle Ax, x\rangle| \text{ and } c(A) = \inf_{x \in S_{\mathcal{H}}} |\langle Ax, x\rangle|.$$

The spectral radius of A, denoted as $r(A)$, is defined as $r(A) = \sup_{\lambda \in \sigma(A)} |\lambda|$, where $\sigma(A)$ is the spectrum of A. Since $\sigma(A) \subseteq \overline{W(A)}$, so $r(A) \leq w(A)$. It is well-known that $r(A) = w(A) = \|A\|$ if A is normal operator in $\mathcal{B}(\mathcal{H})$.

2.2 Bounds for the numerical radius of operators

An improvement of the inequality (1.2), is stated as the following theorem.

Theorem 2.1. *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$w(A) \leq \frac{1}{2}\left(\|A\| + \sqrt{r\left(|A||A^*|\right)}\right).$$

Remark 2.2. *If $A \in \mathcal{B}(\mathcal{H})$, then $r\left(|A||A^*|\right) \leq w\left(|A||A^*|\right) \leq \|\left(|A||A^*|\right)\| = \|A^2\|$. Hence, Theorem 2.1 improves (1.2). To show proper improvement we consider $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$. Then $|A| = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$ and $|A^*| = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$. It is easy to see that $r\left(|A||A^*|\right) = 9 < \|\left(|A||A^*|\right)\| = \|A^2\| = \sqrt{59 + 10\sqrt{34}} \approx 10.83$.*

In order to prove Theorem 2.1 we need the following sequence of lemmas. First lemma can be found in [58].

Lemma 2.1. *([58, Cor. 2]) Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. Then*

$$\|A + B\| \leq \max\{\|A\|, \|B\|\} + \left\|A^{1/2}B^{1/2}\right\|.$$

Second lemma which contains a mixed schwarz inequality, can be found in [44, pp. 75-76].

Lemma 2.2. *([44, pp. 75-76]) Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$|\langle Ax, x\rangle| \leq \langle |A|x, x\rangle^{1/2}\langle |A^*|x, x\rangle^{1/2}, \quad \forall \quad x \in \mathcal{H}.$$

Third lemma is as follows.

Lemma 2.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. Then*

$$\sqrt{r(AB)} = \left\| A^{1/2}B^{1/2} \right\|.$$

Proof. Using commutative property of the spectral radius, we infer that

$$\begin{aligned} r(AB) &= r\left(A^{1/2}A^{1/2}B^{1/2}B^{1/2}\right) = r\left(A^{1/2}B^{1/2}B^{1/2}A^{1/2}\right) \\ &= r\left(A^{1/2}B^{1/2}\left(A^{1/2}B^{1/2}\right)^*\right) = \left\| A^{1/2}B^{1/2}\left(A^{1/2}B^{1/2}\right)^* \right\| \\ &= \left\| A^{1/2}B^{1/2} \right\|^2, \end{aligned}$$

as required. ∎

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then by Lemma 2.2 we get,

$$\begin{aligned} |\langle Ax, x\rangle| &\leq \langle |A|x, x\rangle^{1/2} \ \langle |A^*|x, x\rangle^{1/2} \\ &\leq \frac{1}{2}(\langle |A|x, x\rangle + \langle |A^*|x, x\rangle) \\ &\leq \frac{1}{2}\| \ |A| + |A^*| \ \| \\ &\leq \frac{1}{2}\left(\|A\| + \left\| |A|^{1/2}|A^*|^{1/2} \right\|\right) \ \left(\textit{by Lemma 2.1}\right) \\ &= \frac{1}{2}\left(\|A\| + \sqrt{r(|A||A^*|)}\right) \ \left(\textit{by Lemma 2.3}\right). \end{aligned}$$

Therefore, taking supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$ we get,

$$w(A) \leq \frac{1}{2}\left(\|A\| + \sqrt{r(|A||A^*|)}\right),$$

as desired.

As an application of Theorem 2.1, we prove the following corollary.

Corollary 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$. If $r(|A||A^*|) = 0$, then $w(A) = \frac{\|A\|}{2}$.*

Proof. It follows from (1.1) and Theorem 2.1 that

$$\frac{\|A\|}{2} \leq w(A) \leq \frac{1}{2}\left(\|A\| + \sqrt{r\left(|A||A^*|\right)}\right).$$

This implies that if $r(|A||A^*|) = 0$, then $w(A) = \frac{\|A\|}{2}$. □

Remark 2.3. *It should be mentioned here that the converse of Corollary 2.1 does not hold if* $\dim(\mathcal{H}) \geq 3$. *As for example, we consider* $A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *Then we see that* $w(A) = \frac{3}{2} = \frac{\|A\|}{2}$, *but* $r(|A||A^*|) \neq 0$.

The following corollary is an immediate consequence of Theorem 2.1.

Corollary 2.2. *Let* $A \in \mathcal{B}(\mathcal{H})$. *If* $w(A) = \frac{1}{2}\left(\|A\| + \sqrt{\|A^2\|}\right)$, *then* $r(|A||A^*|) = \|A^2\|$.

Proof. It follows from Theorem 2.1 and Remark 2.2 that

$$w(A) \leq \frac{1}{2}\left(\|A\| + \sqrt{r\left(|A||A^*|\right)}\right) \leq \frac{1}{2}\left(\|A\| + \sqrt{\|A^2\|}\right).$$

This implies that if $w(A) = \frac{1}{2}\left(\|A\| + \sqrt{\|A^2\|}\right)$, then $r(|A||A^*|) = \|A^2\|$. □

Remark 2.4. *It should be mentioned that the converse of Corollary 2.2 is not true. Considering the same example as in Remark 2.3, i.e.,* $A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *we see that* $r(|A||A^*|) = \|A^2\| = 1$, *but* $w(A) = \frac{3}{2} < 2 = \frac{1}{2}\left(\|A\| + \sqrt{\|A^2\|}\right)$.

Now we give a sufficient condition for $w(A) = \frac{1}{2}\left(\|A\| + \sqrt{r\left(|A||A^*|\right)}\right)$, when A is a complex $n \times n$ matrix.

Proposition 2.1. *Let A be a complex $n \times n$ matrix. Suppose A satisfies either one of the following conditions.*
(i) A *is unitarily similar to* $[\alpha] \oplus B$, *where* B *is an* $(n-1) \times (n-1)$ *matrix with* $\|B\| \leq |\alpha|$.
(ii) $r(|A||A^*|) = 0$.
Then, $w(A) = \frac{1}{2}\left(\|A\| + \sqrt{r\left(|A||A^*|\right)}\right)$.

Proof. Let (i) holds. Then $w(A) = |\alpha|$ and $\|A\| = |\alpha|$. Also it is not difficult to verify that $r(|A||A^*|) = |\alpha|^2$. Hence, $\frac{1}{2}\left(\|A\| + \sqrt{r\left(|A||A^*|\right)}\right) = |\alpha|$. Now let (ii) holds. Then from Corollary 2.1 we get, $w(A) = \frac{1}{2}\left(\|A\| + \sqrt{r\left(|A||A^*|\right)}\right) = \frac{\|A\|}{2}$. Thus, we complete the proof. □

Next we give a generalized result of Theorem 2.1. For this purpose we need the following lemma, which is the generalization of Lemma 2.2.

Lemma 2.4. *([59, Th. 5]). Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that $|A|B = B^*|A|$ and let f, g be non-negative continuous functions on $[0, \infty]$ satisfy $f(t)g(t) = t$, $\forall t \geq 0$. Then, $|\langle ABx, y\rangle| \leq r(B)\|f(|A|)x\|\|g(|A^*|)y\|$, $\forall x, y \in \mathcal{H}$.*

By using Lemma 2.4 and proceeding similarly as in Theorem 2.1, we can prove the following.

Theorem 2.5. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that $|A|B = B^*|A|$ and let f, g be as in Lemma 2.4. Then*

$$w(AB) \leq \frac{r(B)}{2}\Big(\max\left\{\|f(|A|)\|^2, \|g(|A^*|)\|^2\right\} + \|\ \ |f(|A|)||g(|A^*|)|\ \ \|\Big).$$

In particular, considering $f(t) = g(t) = \sqrt{t}$ in Theorem 2.5 we get the following corollary.

Corollary 2.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that $|A|B = B^*|A|$. Then*

$$\begin{aligned} w(AB) &\leq \frac{r(B)}{2}\left(\|A\| + \sqrt{r\left(|A||A^*|\right)}\right) \\ &\leq \frac{1}{4}\left(\|B\| + \sqrt{r\left(|B||B^*|\right)}\right)\left(\|A\| + \sqrt{r\left(|A||A^*|\right)}\right). \end{aligned}$$

Remark 2.6. *If $A, B \in \mathcal{B}(\mathcal{H})$ be such that $|A|B = B^*|A|$, then Alomari [7, Cor. 3.2] proved that*

$$w(AB) \leq \frac{1}{4}\left(\|B\| + \sqrt{\|B^2\|}\right)\left(\|A\| + \sqrt{\|A^2\|}\right). \tag{2.1}$$

Clearly, the inequalities in Corollary 2.3 improve on the inequality (2.1).

2.3 Bounds for the numerical radius of commutators of operators

To achieve our aim in this section we need the following inequality, which we obtained in [20, Cor. 2.3].

Lemma 2.5. *([20, Cor. 2.3]) Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\|AA^* + A^*A\| \leq 4\left[w^2(A) - \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}\right].$$

Theorem 2.7. *Let $A, B, X, Y \in \mathcal{B}(\mathcal{H})$. Then*

$$w(AXB \pm BYA) \leq 2\sqrt{2}\|B\| \max\{\|X\|, \|Y\|\} \sqrt{w^2(A) - \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}}.$$

Proof. First we assume that $\|X\| \leq 1$, $\|Y\| \leq 1$ and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then, we have

$$\begin{aligned}
|\langle (AX \pm YA)x, x\rangle| &\leq |\langle AXx, x\rangle| + |\langle YAx, x\rangle| \\
&= |\langle Xx, A^*x\rangle| + |\langle Ax, Y^*x\rangle| \\
&\leq \|A^*x\| + \|Ax\| \qquad \Big(\text{by Cauchy-Schwarz inequality}\Big) \\
&\leq \sqrt{2(\|A^*x\|^2 + \|Ax\|^2)} \quad \Big(\text{by convexity of } f(x) = x^2\Big) \\
&\leq \sqrt{2\|AA^* + A^*A\|} \\
&\leq 2\sqrt{2}\sqrt{w^2(A) - \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}} \quad \Big(\text{by Lemma 2.5}\Big).
\end{aligned}$$

Hence, by taking supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$ we get,

$$w(AX \pm YA) \leq 2\sqrt{2}\sqrt{w^2(A) - \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}}. \tag{2.2}$$

Now we consider the general case, i.e., $X, Y \in \mathcal{B}(\mathcal{H})$ be arbitrary. If $X = Y = 0$, then Theorem 2.7 holds trivially. Let $\max\{\|X\|, \|Y\|\} \neq 0$. Then clearly $\left\|\frac{X}{\max\{\|X\|,\|Y\|\}}\right\| \leq 1$ and $\left\|\frac{Y}{\max\{\|X\|,\|Y\|\}}\right\| \leq 1$. So, replacing X and Y by $\frac{X}{\max\{\|X\|,\|Y\|\}}$ and $\frac{Y}{\max\{\|X\|,\|Y\|\}}$, respectively, in (2.2) we get,

$$w(AX \pm YA) \leq 2\sqrt{2} \max\{\|X\|, \|Y\|\} \sqrt{w^2(A) - \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}}.$$

Now replacing X by XB and Y by BY in the above inequality we get,

$$w(AXB \pm BYA) \leq 2\sqrt{2} \max\{\|XB\|, \|BY\|\} \sqrt{w^2(A) - \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}},$$

which implies that

$$w(AXB \pm BYA) \leq 2\sqrt{2}\|B\| \max\{\|X\|, \|Y\|\} \sqrt{w^2(A) - \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}}.$$

This completes the proof. □

Based on Theorem 2.7, we obtain the following corollary.

Corollary 2.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$w(AB \pm BA) \leq 2\sqrt{2}\|B\|\sqrt{w^2(A) - \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}}. \tag{2.3}$$

and

$$w(AB \pm BA) \leq 2\sqrt{2}\|A\|\sqrt{w^2(B) - \frac{c^2(\Re(B)) + c^2(\Im(B))}{2}}. \tag{2.4}$$

Proof. By considering $X = Y = I$ in Theorem 2.7 we get, (2.3). Interchanging A and B in (2.3) we get, (2.4).

□

Remark 2.8. *Clearly, the inequality (2.3) is stronger than the inequality $w(AB \pm BA) \leq 2\sqrt{2}\|B\|w(A)$, obtained by Fong and Holbrook [40].*

As an application of the inequality (2.3) we prove the following result.

Corollary 2.5. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and let $B \neq 0$. If $w(AB \pm BA) = 2\sqrt{2}\|B\|w(A)$, then $0 \in \overline{W(\Re(A))} \cap \overline{W(\Im(A))}$.*

Proof. Let $w(AB \pm BA) = 2\sqrt{2}\|B\|w(A)$. Then it follows from (2.3) that

$$w(A) = \sqrt{w^2(A) - \frac{c^2(\Re(A)) + c^2(\Im(A))}{2}}.$$

Hence, $c^2(\Re(A)) + c^2(\Im(A)) = 0$, i.e., $c(\Re(A)) = c(\Im(A)) = 0$. Therefore, there exist norm one sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{H}$ such that $|\langle \Re(A)x_n, x_n\rangle| \to 0$ and $|\langle \Im(A)y_n, y_n\rangle| \to 0$ as $n \to \infty$. So, $0 \in \overline{W(\Re(A))} \cap \overline{W(\Im(A))}$. □

For our next result we need the following three lemmas, the first two of which can be found in [2] and [50], respectively.

Lemma 2.6. ([2, Remark 2.2]) *Let $A, B, X, Y \in \mathcal{B}(\mathcal{H})$. Then*

$$w^2(AX \pm BY) \leq \|AA^* + Y^*Y\| \;\; \|X^*X + BB^*\|.$$

Lemma 2.7. ([50, Th. 1.1]) *Let $A, B, X, Y \in \mathcal{B}(\mathcal{H})$. Then*

$$\left\| \begin{pmatrix} A & X \\ Y & B \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} \|A\| & \|X\| \\ \|Y\| & \|B\| \end{pmatrix} \right\|.$$

The next lemma is as follows.

Lemma 2.8. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then $\|AA^* + B^*B\| \leq \mu(A, B)$, where*

$$\mu(A, B) = \frac{1}{2}\left[\|A\|^2 + \|B\|^2 + \sqrt{(\|A\|^2 - \|B\|^2)^2 + 4\|BA\|^2}\right].$$

Proof. $AA^* + B^*B$ being a self-adjoint operator, we have

$$\begin{aligned}
\|AA^* + B^*B\| &= r(AA^* + B^*B) \\
&= r\begin{pmatrix} AA^* + B^*B & 0 \\ 0 & 0 \end{pmatrix} \\
&= r\left(\begin{pmatrix} |A^*| & |B| \\ 0 & 0 \end{pmatrix}\begin{pmatrix} |A^*| & 0 \\ |B| & 0 \end{pmatrix}\right) \\
&= r\left(\begin{pmatrix} |A^*| & 0 \\ |B| & 0 \end{pmatrix}\begin{pmatrix} |A^*| & |B| \\ 0 & 0 \end{pmatrix}\right) \quad \big(r(XY) = r(YX)\big) \\
&= r\begin{pmatrix} |A^*|^2 & |A^*||B| \\ |B||A^*| & |B|^2 \end{pmatrix} \\
&= \left\|\begin{pmatrix} |A^*|^2 & |A^*||B| \\ |B||A^*| & |B|^2 \end{pmatrix}\right\| \\
&\leq \left\|\begin{pmatrix} \|A\|^2 & \||A^*||B|\| \\ \||B||A^*|\| & \|B\|^2 \end{pmatrix}\right\| \quad \big(\textit{by Lemma 2.7}\big) \\
&= \left\|\begin{pmatrix} \|A\|^2 & \|BA\| \\ \|BA\| & \|B\|^2 \end{pmatrix}\right\| \\
&= \frac{1}{2}\left[\|A\|^2 + \|B\|^2 + \sqrt{(\|A\|^2 - \|B\|^2)^2 + 4\|BA\|^2}\right].
\end{aligned}$$

Hence, $\|AA^* + B^*B\| \leq \mu(A, B)$.

□

Remark 2.9. *Notice that $\mu(A, B) \leq \max\{\|A\|^2, \|B\|^2\} + \|BA\|$. In particular, if $A = B$ then $\mu(A, A) = \|A\|^2 + \|A^2\|$. Hence, we have $\|AA^* + A^*A\| \leq \|A\|^2 + \|A^2\|$.*

Now we are in a position to prove the following result.

Theorem 2.10. *Let $A, B, X, Y \in \mathcal{B}(\mathcal{H})$. Then*

$$w(AX \pm BY) \leq \sqrt{\mu(A, Y)\,\mu(B, X)}.$$

Proof. The proof follows from Lemma 2.6 and Lemma 2.8.

□

An application of Theorem 2.10, we get the following corollary.

Corollary 2.6. *Let* $A, B \in \mathcal{B}(\mathcal{H})$. *Then*

$$w(AB \pm BA) \leq \sqrt{\left(\|A\|^2 + \|A^2\|\right)\left(\|B\|^2 + \|B^2\|\right)}.$$

Remark 2.11. *Let* $A, B \in \mathcal{B}(\mathcal{H})$ *with* $A^2 = B^2 = 0$. *Then it follows from Corollary 2.6 that* $w(AB \pm BA) \leq \|A\|\|B\| < 2\sqrt{2}\|B\|w(A) = \sqrt{2}\|A\|\|B\|$.

CHAPTER 3

NEW UPPER BOUNDS FOR THE NUMERICAL RADIUS

3.1 Introduction

The main purpose of this chapter is to present upper bounds for the numerical radius of bounded linear operators which generalize and improve on the well-known upper bounds in (1.2) and (1.3), i.e., $w(A) \leq \frac{1}{2}\left(\|A\| + \|A^2\|^{1/2}\right)$ and $w^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|$, respectively. Further, we present an upper bound for the numerical radius of the sum of the product of operators which generalizes and improves on the existing ones. First we introduce the following necessary notations.

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle ., . \rangle$ and let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on $\mathcal{H}$. As usual the norm induced by the inner product $\langle ., . \rangle$ is denoted by $\| \cdot \|$. For $A \in \mathcal{B}(\mathcal{H})$, let $\|A\|$ be the operator norm of A, i.e., $\|A\| = \sup_{\|x\|=1} \|Ax\|$. For $A \in \mathcal{B}(\mathcal{H})$, A^* denotes the adjoint of A and $|A|$ denotes the positive square root of A^*A, i.e., $|A| = (A^*A)^{\frac{1}{2}}$. Let $S_{\mathcal{H}}$ denote the unit sphere of the Hilbert space $\mathcal{H}$. The numerical range of A, denoted by $W(A)$, is defined as $W(A) = \left\{\langle Ax, x \rangle : x \in \mathcal{H}, \ \ \|x\| = 1\right\}$. Considering the

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continuous mapping $x \longmapsto \langle Ax, x\rangle$ from $S_{\mathcal{H}}$ to the scalar field $\mathbb{C}$, it is easy to see that $W(A)$ is a compact subset of $\mathbb{C}$ if $\mathcal{H}$ is finite-dimensional. The numerical radius of A, denoted as $w(A)$, is defined as $w(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle|$. The numerical radius is a norm on $\mathcal{B}(\mathcal{H})$, satisfying that for $A \in \mathcal{B}(\mathcal{H})$, $\frac{1}{2}\|A\| \le w(A) \le \|A\|$. This implies that the numerical radius norm is equivalent to the operator norm on $\mathcal{B}(\mathcal{H})$. The above inequality is sharp, $w(A) = \frac{1}{2}\|A\|$ if $A^2 = 0$ and $w(A) = \|A\|$ if $AA^* = A^*A$.

3.2 Bounds for the numerical radius concerning one operator

We begin with the following sequence of lemmas which will be used to reach our goal in this present chapter. First lemma is known as a generalized mixed Cauchy-Schwarz inequality which involves two non-negative continuous functions.

Lemma 3.1. *([59, Th. 5]). Let $A \in \mathcal{B}(\mathcal{H})$. Let f and g be non-negative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$|\langle Ax, y\rangle| \le \|f(|A|)x\|\|g(|A^*|)y\|,$$

for all $x, y \in \mathcal{H}$.

Second lemma deals with positive operators.

Lemma 3.2. *([73, p. 20]). Let $A \in \mathcal{B}(\mathcal{H})$ be positive, i.e., $A \ge 0$. Then*

$$\langle Ax, x\rangle^r \le \langle A^r x, x\rangle,$$

for all $r \ge 1$ and for all $x \in \mathcal{H}$ with $\|x\| = 1$.

Third lemma is known as Buzano's inequality.

Lemma 3.3. *([35]) Let $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. Then*

$$|\langle x, e\rangle\langle e, y\rangle| \le \frac{1}{2}\left(\|x\|\|y\| + |\langle x, y\rangle|\right).$$

Fourth lemma is known as Bohr's inequality which deals with positive numbers.

Lemma 3.4. *([74]) For $i = 1, 2, \ldots, n$, let a_i be a positive real number. Then*

$$\left(\sum_{i=1}^{n} a_i\right)^r \le n^{r-1} \sum_{i=1}^{n} a_i^r,$$

for all $r \geq 1$.

Now, we are in a position to present our first inequality in this section.

Theorem 3.1. *Let* $A \in \mathcal{B}(\mathcal{H})$. *Then*

$$w^{2r}(A) \leq \frac{1}{4}\left\||A|^{2r} + |A^*|^{2r}\right\| + \frac{1}{2}w\left(|A|^r|A^*|^r\right),$$

for all $r \geq 1$.

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Considering $f(t) = g(t) = t^{\frac{1}{2}}$ in Lemma 3.1 we have that

$$|\langle Ax, x\rangle|^2 \leq \langle |A|x, x\rangle\langle |A^*|x, x\rangle.$$

It follows from Lemma 3.2 that

$$|\langle Ax, x\rangle|^{2r} \leq \langle |A|^r x, x\rangle\langle |A^*|^r x, x\rangle = \langle |A^*|^r x, x\rangle\langle x, |A|^r x\rangle.$$

From Lemma 3.3 we have,

$$\langle |A^*|^r x, x\rangle\langle x, |A|^r x\rangle \quad \leq \quad \frac{1}{2}\||A|^r x\|\,\||A^*|^r x\| + \frac{1}{2}\left|\langle |A^*|^r x, |A|^r x\rangle\right|.$$

$$\begin{aligned}
\text{So we get, } |\langle Ax, x\rangle|^{2r} &\leq \frac{1}{4}\left(\||A|^r x\|^2 + \||A^*|^r x\|^2\right) + \frac{1}{2}\left|\langle |A|^r|A^*|^r x, x\rangle\right| \\
&= \frac{1}{4}\left(\left\langle |A|^{2r}x, x\right\rangle + \left\langle |A^*|^{2r}x, x\right\rangle\right) + \frac{1}{2}\left|\langle |A|^r|A^*|^r x, x\rangle\right| \\
&= \frac{1}{4}\left\langle\left(|A|^{2r} + |A^*|^{2r}\right)x, x\right\rangle + \frac{1}{2}\left|\langle |A|^r|A^*|^r x, x\rangle\right| \\
&\leq \frac{1}{4}\left\||A|^{2r} + |A^*|^{2r}\right\| + \frac{1}{2}w\left(|A|^r|A^*|^r\right).
\end{aligned}$$

Therefore, taking supremum over $\|x\| = 1$ we get,

$$w^{2r}(A) \leq \frac{1}{4}\left\||A|^{2r} + |A^*|^{2r}\right\| + \frac{1}{2}w\left(|A|^r|A^*|^r\right),$$

as required. □

The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.1. *Let* $A \in \mathcal{B}(\mathcal{H})$. *Then*

$$w^2(A) \leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}w\left(|A||A^*|\right).$$

Remark 3.2. *(i) If* $|A||A^*| = 0$*, i.e.,* $A^2 = 0$*, then it follows from Corollary 3.1 and the left hand inequality of (1.3) that* $w^2(A) = \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| = \frac{1}{4}\|A^*A + AA^*\|$.
(ii) The inequality in Corollary 3.1 improves on the right hand inequality in (1.3). Clearly, $w\left(|A||A^*|\right) \leq \||A||A^*|\| = \left\|A^2\right\|$. *Also,* $2\|A^2\| \leq \|A^*A + AA^*\|$ *(see [57]). Therefore,*

$$\begin{aligned} w^2(A) &\leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}w\left(|A||A^*|\right) \\ &\leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}\left\|A^2\right\| \\ &\leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| \\ &= \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|. \end{aligned}$$

Thus, the inequality in Corollary 3.1 improves on the right hand inequality in (1.3).
(iii) The inequality in Corollary 3.1 improves on the inequality in (1.2).

$$\begin{aligned} \textit{Clearly,}\quad w^2(A) &\leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}w\left(|A||A^*|\right) \\ &\leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}\left\|A^2\right\| \\ &\leq \frac{1}{4}\left\|A^2\right\| + \frac{1}{4}\left\|A\right\|^2 + \frac{1}{2}\|A\|\left\|A^2\right\|^{\frac{1}{2}} \\ &= \left(\frac{1}{2}\|A\| + \frac{1}{2}\left\|A^2\right\|^{\frac{1}{2}}\right)^2. \end{aligned}$$

Thus, the inequality in Corollary 3.1 also improves on that in (1.2).

Next we obtain the following inequality for the numerical radius of the sum of n operators which generalizes Theorem 3.1.

Theorem 3.3. *Let* $A_i \in \mathcal{B}(\mathcal{H})$ *for* $i = 1, 2, \ldots, n$. *Then*

$$w^{2r}\left(\sum_{i=1}^{n} A_i\right) \leq \frac{n^{2r-1}}{4}\left\|\sum_{i=1}^{n}\left(|A_i|^{2r} + |A_i^*|^{2r}\right)\right\| + \frac{n^{2r-1}}{2}\left(\sum_{i=1}^{n} w\left(|A_i|^r|A_i^*|^r\right)\right),$$

for all $r \geq 1$.

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then from Lemma 3.4 we get,

$$\begin{aligned}
\left|\left\langle \left(\sum_{i=1}^{n} A_i\right) x, x \right\rangle\right|^{2r} &= \left|\sum_{i=1}^{n} \langle A_i x, x\rangle\right|^{2r} \\
&\leq \left(\sum_{i=1}^{n} |\langle A_i x, x\rangle|\right)^{2r} \\
&\leq n^{2r-1}\left(\sum_{i=1}^{n} |\langle A_i x, x\rangle|^{2r}\right).
\end{aligned}$$

Proceeding similarly as in the proof of Theorem 3.1 we get the required inequality. □

Our next result reads as follows:

Theorem 3.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Then*

$$\|A+B\| \leq \sqrt{w^2(A+\mathrm{i}B) + \|A\|\|B\| + w(BA)} \leq \|A\| + \|B\|.$$

Proof. Let $x \in \mathcal{H}$ be such that $\|x\| = 1$. Then we have,

$$\begin{aligned}
\|A+B\|^2 &= w^2(A+B) \\
&= \sup_{\|x\|=1} |\langle (A+B)x, x\rangle|^2 \\
&\leq \sup_{\|x\|=1} (|\langle Ax, x\rangle| + |\langle Bx, x\rangle|)^2 \\
&= \sup_{\|x\|=1} \left(|\langle Ax, x\rangle|^2 + |\langle Bx, x\rangle|^2 + 2|\langle Ax, x\rangle||\langle Bx, x\rangle|\right) \\
&= \sup_{\|x\|=1} \left(|\langle Ax, x\rangle + \mathrm{i}\langle Bx, x\rangle|^2 + 2|\langle Ax, x\rangle\langle x, Bx\rangle|\right) \\
&\leq \sup_{\|x\|=1} \left(|\langle (A+\mathrm{i}B)x, x\rangle|^2 + \|Ax\|\|Bx\| + |\langle Ax, Bx\rangle|\right) \left(\text{by Lemma 3.3}\right) \\
&= \sup_{\|x\|=1} \left(|\langle (A+\mathrm{i}B)x, x\rangle|^2 + \|Ax\|\|Bx\| + |\langle BAx, x\rangle|\right)^2 \\
&\leq w^2(A+\mathrm{i}B) + \|A\|\|B\| + w(BA).
\end{aligned}$$

Hence, $\|A+B\| \leq \sqrt{w^2(A+\mathrm{i}B) + \|A\|\|B\| + w(BA)}$.
It is easy to verify that $w^2(A+\mathrm{i}B) \leq \|A^2+B^2\|$. Therefore, we have

$$w^2(A+\mathrm{i}B) + \|A\|\|B\| + w(BA) \leq (\|A\| + \|B\|)^2.$$

This completes the proof. □

Remark 3.5. *We would like to remark that Theorem 3.4 gives better bound than the bound* *obtained by Moradi and Sababheh [60, Th. 2.4], namely, if* $A, B \in \mathcal{B}(\mathcal{H})$ *are self-adjoint, then* $\|A+B\| \le \sqrt{w^2(A+\mathrm{i}B)+2\|A\|\|B\|} \le \|A\|+\|B\|$.

3.3 Bounds for the numerical radius of the sum of the product operators

Our first result in this section reads as follows.

Theorem 3.6. *Let* $A_i, B_i, X_i \in \mathcal{B}(\mathcal{H})$ *for* $i = 1, 2, \ldots, n$. *Let* f *and* g *be two non-negative functions on* $[0, \infty)$ *which are continuous and satisfy the relation* $f(t)g(t) = t$ *for all* $t \in [0, \infty)$. *Then*

$$w^r\left(\sum_{i=1}^n A_i^* X_i B_i\right) \le \frac{n^{r-1}}{\sqrt{2}} w\left(\sum_{i=1}^n \left(\left[B_i^* f^2(|X_i|)B_i\right]^r + \mathrm{i}\left[A_i^* g^2(|X_i^*|)A_i\right]^r\right)\right),$$

for all $r \ge 1$.

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then we have,

$$\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^n A_i^* X_i B_i\right)x, x\right\rangle\right|^r \\
= \ & \left|\sum_{i=1}^n \langle A_i^* X_i B_i x, x\rangle\right|^r \\
\le \ & \left(\sum_{i=1}^n |\langle A_i^* X_i B_i x, x\rangle|\right)^r \\
\le \ & n^{r-1}\left(\sum_{i=1}^n |\langle A_i^* X_i B_i x, x\rangle|^r\right) \quad (\text{by Lemma 3.4}) \\
= \ & n^{r-1}\left(\sum_{i=1}^n |\langle X_i B_i x, A_i x\rangle|^r\right) \\
\le \ & n^{r-1}\left(\sum_{i=1}^n \|f(|X_i|)B_i x\|^r \, \|g(|X_i^*|)A_i x\|^r\right) \quad (\text{by Lemma 3.1}) \\
= \ & n^{r-1}\left(\sum_{i=1}^n \left\langle f^2(|X_i|)B_i x, B_i x\right\rangle^{\frac{r}{2}} \left\langle g^2(|X_i^*|)A_i x, A_i x\right\rangle^{\frac{r}{2}}\right) \\
= \ & n^{r-1}\left(\sum_{i=1}^n \left\langle B_i^* f^2(|X_i|)B_i x, x\right\rangle^{\frac{r}{2}} \left\langle A_i^* g^2(|X_i^*|)A_i x, x\right\rangle^{\frac{r}{2}}\right)
\end{aligned}$$

$$\leq \; n^{r-1}\left(\sum_{i=1}^{n}\left\langle\left[B_i^*f^2(|X_i|)B_i\right]^r x,x\right\rangle^{\frac{1}{2}}\left\langle\left[A_i^*g^2(|X_i^*|)A_i\right]^r x,x\right\rangle^{\frac{1}{2}}\right)$$

$$\Big(\text{by Lemma 3.2}\Big)$$

$$\leq \; \frac{n^{r-1}}{2}\left(\sum_{i=1}^{n}\left(\left\langle\left[B_i^*f^2(|X_i|)B_i\right]^r x,x\right\rangle+\left\langle\left[A_i^*g^2(|X_i^*|)A_i\right]^r x,x\right\rangle\right)\right)$$

$$\leq \; \frac{n^{r-1}}{\sqrt{2}}\left(\left|\sum_{i=1}^{n}\left\langle\left[B_i^*f^2(|X_i|)B_i\right]^r x,x\right\rangle+\mathrm{i}\sum_{i=1}^{n}\left\langle\left[A_i^*g^2(|X_i^*|)A_i\right]^r x,x\right\rangle\right|\right)$$

$$\Big(\text{as } \; |a+b|\leq\sqrt{2}|a+\mathrm{i}b|, \quad \forall a,b\in\mathbb{R}\Big)$$

$$= \; \frac{n^{r-1}}{\sqrt{2}}\left|\left\langle\left(\sum_{i=1}^{n}\left(\left[B_i^*f^2(|X_i|)B_i\right]^r+\mathrm{i}\left[A_i^*g^2(|X_i^*|)A_i\right]^r\right)\right)x,x\right\rangle\right|$$

$$\leq \; \frac{n^{r-1}}{\sqrt{2}}w\left(\sum_{i=1}^{n}\left(\left[B_i^*f^2(|X_i|)B_i\right]^r+\mathrm{i}\left[A_i^*g^2(|X_i^*|)A_i\right]^r\right)\right).$$

Therefore, taking supremum over $\|x\|=1$, we get

$$w^r\left(\sum_{i=1}^{n}A_i^*X_iB_i\right)\leq\frac{n^{r-1}}{\sqrt{2}}w\left(\sum_{i=1}^{n}\left(\left[B_i^*f^2(|X_i|)B_i\right]^r+\mathrm{i}\left[A_i^*g^2(|X_i^*|)A_i\right]^r\right)\right).$$

□

Remark 3.7. *Note that Theorem 3.6 indeed does not depend on the number n of summands in the case $r=1$.*

Considering $f(t)=t^{\alpha}$ and $g(t)=t^{1-\alpha}$, $0\leq\alpha\leq1$ in Theorem 3.6 we get the following corollary.

Corollary 3.2. *For $i=1,2,\ldots,n$, let $A_i,B_i,X_i\in\mathcal{B}(\mathcal{H})$. Then*

$$w^r\left(\sum_{i=1}^{n}A_i^*X_iB_i\right)\leq\frac{n^{r-1}}{\sqrt{2}}w\left(\sum_{i=1}^{n}\left(\left[B_i^*|X_i|^{2\alpha}B_i\right]^r+\mathrm{i}\left[A_i^*|X_i^*|^{2(1-\alpha)}A_i\right]^r\right)\right),$$

for all $r\geq1$.

The following corollary is an easy consequence of Theorem 3.6.

Corollary 3.3. *For $i=1,2,\ldots,n$, let $X_i\in\mathcal{B}(\mathcal{H})$. Let f and g be non-negative functions on $[0,\infty)$ which are continuous and satisfy the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. Then*

$$w^r\left(\sum_{i=1}^{n}X_i\right)\leq\frac{n^{r-1}}{\sqrt{2}}w\left(\sum_{i=1}^{n}\left(f^{2r}(|X_i|)+\mathrm{i}\;g^{2r}(|X_i^*|)\right)\right),$$

for all $r \geq 1$.

In particular, taking $n = 1$, $r = 1$ and $f(t) = g(t) = t^{\frac{1}{2}}$ in Corollary 3.3 we get the following inequality which refines the second inequality in (1.3).

Corollary 3.4. *Let* $A \in \mathcal{B}(\mathcal{H})$. *Then*

$$w(A) \leq \frac{1}{\sqrt{2}} w(|A| + \mathrm{i}|A^*|).$$

Remark 3.8. *It is easy to observe that* $w^2(|A| + \mathrm{i}|A^*|) \leq \left\| |A|^2 + |A^*|^2 \right\|$. *Therefore,*

$$w^2(A) \leq \frac{1}{2} w^2(|A| + \mathrm{i}|A^*|) \leq \frac{1}{2} \left\| |A|^2 + |A^*|^2 \right\|.$$

Hence, Corollary 3.4 is sharper than that in (1.3).

Next, we obtain an inequality which follows from Corollary 3.2.

Corollary 3.5. *Let* $A, B \in \mathcal{B}(\mathcal{H})$. *Then*

$$w^r(A^*B) \leq \frac{1}{2} w^2 \left(|B|^r + \mathrm{i}|A|^r \right),$$

for all $r \geq 2$.

Remark 3.9. *In [39], Dragomir proved that if* $A, B \in \mathcal{B}(\mathcal{H})$ *and* $r \geq 1$ *then*

$$w^r(A^*B) \leq \frac{1}{2} \left\| |B|^{2r} + |A|^{2r} \right\|.$$

For $r \geq 2$, *from Corollary 3.5 we get,*

$$w^r(A^*B) \leq \frac{1}{2} w^2 \left(|B|^r + \mathrm{i}|A|^r \right) \leq \frac{1}{2} \left\| |B|^{2r} + |A|^{2r} \right\|.$$

We would like to remark that for $r \geq 2$, *the inequality in Corollary 3.5 is stronger than the above Dragomir's inequality [39].*

Finally, we obtain the following estimation.

Theorem 3.10. *Let* $A_i, B_i, X_{ij} \in \mathcal{B}(\mathcal{H})$ *for* $i, j = 1, 2, \ldots n$. *Then*

$$w \left(\sum_{i,j=1}^{n} A_j^* X_{ij} B_i \right) \leq \frac{1}{2} \|\mathcal{X}\| \left\| \sum_{i=1}^{n} (A_i^* A_i + B_i^* B_i) \right\|,$$

where

$$\mathcal{X} = \begin{pmatrix} X_{11} & X_{21} & . & . & . & X_{n1} \\ X_{12} & X_{22} & . & . & . & X_{n2} \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ X_{1n} & X_{2n} & . & . & . & X_{nn} \end{pmatrix} \in \mathcal{B}\left(\sum_{i=1}^{n} \oplus \mathcal{H}\right).$$

Proof. Let $A = \begin{pmatrix} A_1 & 0 & . & . & . & 0 \\ A_2 & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ A_n & 0 & . & . & . & 0 \end{pmatrix}$, $B = \begin{pmatrix} B_1 & 0 & . & . & . & 0 \\ B_2 & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ B_n & 0 & . & . & . & 0 \end{pmatrix} \in \mathcal{B}\left(\sum_{i=1}^{n} \oplus \mathcal{H}\right)$. Then,

$A^*\mathcal{X}B = \begin{pmatrix} \sum_{i,j=1}^{n} A_j^* X_{ij} B_i & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & . & . & . & 0 \end{pmatrix}$ and so we have

$w\left(\sum_{i,j=1}^{n} A_j^* X_{ij} B_i\right) = w(A^*\mathcal{X}B)$. Now, by Cauchy-Schwarz inequality we get,

$$\begin{aligned} w(A^*\mathcal{X}B) &= \sup_{x \in S_{\mathcal{H}}} |\langle A^*\mathcal{X}Bx, x\rangle| = \sup_{x \in S_{\mathcal{H}}} |\langle \mathcal{X}Bx, Ax\rangle| \\ &\leq \sup_{x \in S_{\mathcal{H}}} \|\mathcal{X}Bx\|\|Ax\| \leq \sup_{x \in S_{\mathcal{H}}} \|\mathcal{X}\|\|Bx\|\|Ax\| \\ &\leq \sup_{x \in S_{\mathcal{H}}} \frac{1}{2}\|\mathcal{X}\|(\|Bx\|^2 + \|Ax\|^2) = \sup_{x \in S_{\mathcal{H}}} \frac{1}{2}\|\mathcal{X}\|\langle (B^*B + A^*A)x, x\rangle \\ &= \frac{1}{2}\|\mathcal{X}\|\|A^*A + B^*B\| = \frac{1}{2}\|\mathcal{X}\| \left\|\sum_{i=1}^{n} (A_i^*A_i + B_i^*B_i)\right\|. \end{aligned}$$

Thus, we have the desired inequality.

□

Remark 3.11. *We observe that the expression* $\sum_{i,j=1}^{n} A_j^* X_{ij} B_i$ *can also be written as* $\sum_{i=1}^{n^2} C_i^* X_i D_i$ *where* $C_i \in \{A_j : 1 \leq j \leq n\}, X_i \in \{X_{ij} : 1 \leq i,j \leq n\}, D_i \in \{B_j : 1 \leq j \leq n\}$ *for all* $i = 1, 2, \ldots n^2$. *So, one can estimate* $w\left(\sum_{i,j=1}^{n} A_j^* X_{ij} B_i\right)$ *as in Theorem 3.6.*

CHAPTER 4

REFINEMENTS OF NORM AND NUMERICAL RADIUS INEQUALITIES

4.1 Introduction

The main objective of this chapter is to develop norm inequalities for the sum of two bounded linear operators, from which we obtain lower bounds for the numerical radius of bounded linear operators that strongly refine the lower bound in (1.3), i.e., $\frac{1}{4}\|A^*A+AA^*\| \leq w^2(A)$, obtained by Kittaneh [54]. Further, we present upper bounds for the numerical radius of bounded linear operators by using operator convex functions which improve on the existing ones. First we introduce the following necessary notations and terminologies.

Let $\mathcal{H}$ denote a complex Hilbert space with inner product $\langle .,. \rangle$ and $\|\cdot\|$ denotes the norm induced by the inner product $\langle .,. \rangle$. Let $\mathcal{B}(\mathcal{H})$ denote the collection of all bounded linear operators on $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, A^* denotes the adjoint of A and $|A| = \sqrt{A^*A}$. For $A \in \mathcal{B}(\mathcal{H})$, let $\|A\|$ be the operator norm of A. Recall that $\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y \rangle|$. The numerical range of A, denoted by $W(A)$, is defined as $W(A) = \big\{\langle Ax, x\rangle : x \in \mathcal{H}, \ \ \|x\| = 1\big\}$. The two numerical constants, numerical radius $w(A)$ and Crawford number $c(A)$, associated

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with the numerical range $W(A)$, of A, are defined respectively as $w(A) = \sup_{\lambda \in W(A)} |\lambda|$ and $c(A) = \inf_{\lambda \in W(A)} |\lambda|$. The numerical radius is a norm on $\mathcal{B}(\mathcal{H})$ and is equivalent to the operator norm on $\mathcal{B}(\mathcal{H})$, satisfying $\frac{1}{2}\|A\| \leq w(A) \leq \|A\|$ for all $A \in \mathcal{B}(\mathcal{H})$.

4.2 Norm inequalities in estimating lower bound for the numerical radius

We begin with the introduction of two notations. Let $A = B + iC$ be the Cartesian decomposition of A, i.e., $B = \Re(A) = \frac{A+A^*}{2}$ and $C = \Im(A) = \frac{A-A^*}{2i}$. We observe that

$$\frac{1}{4}\|A^*A + AA^*\| = \frac{1}{2}\|B^2 + C^2\|. \tag{4.1}$$

By using the identity (4.1), we obtain our first refinement.

Theorem 4.1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} \frac{1}{4}\|A^*A + AA^*\| &\leq \frac{1}{8}\big(\|A + A^*\|^2 + \|A - A^*\|^2\big) \\ &\leq \frac{1}{8}\big(\|A + A^*\|^2 + \|A - A^*\|^2\big) + \frac{1}{8}c^2\big(A + A^*\big) + \frac{1}{8}c^2\big(A - A^*\big) \\ &\leq w^2(A). \end{aligned}$$

Proof. From the identity (4.1) we get,

$$\frac{1}{4}\|A^*A + AA^*\| = \frac{1}{2}\|B^2 + C^2\| \leq \frac{1}{2}(\|B\|^2 + \|C\|^2) = \frac{1}{8}(\|A + A^*\|^2 + \|A - A^*\|^2).$$

This is the first inequality of the theorem. The second inequality follows trivially. Now we prove the third inequality. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then from the Cartesian decomposition of A we get,

$$|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2 = |\langle Ax, x\rangle|^2. \tag{4.2}$$

From (4.2), we get the following two inequalities

$$c^2(B) + \|C\|^2 \leq w^2(A) \tag{4.3}$$

and

$$c^2(C) + \|B\|^2 \leq w^2(A). \tag{4.4}$$

It follows from the inequalities (4.3) and (4.4) that

$$c^2(B) + c^2(C) + \|B\|^2 + \|C\|^2 \leq 2w^2(A).$$

This implies that

$$\frac{1}{8}\Big(\|A + A^*\|^2 + \|A - A^*\|^2\Big) + \frac{1}{8}c^2\big(A + A^*\big) + \frac{1}{8}c^2\big(A - A^*\big) \leq w^2(A).$$

This completes the proof. □

Remark 4.2. *We note that the inequalities obtained in Theorem 4.1 refine*

$$\frac{1}{4}\|A + A^*\|\|A - A^*\| \quad \leq \quad w^2(A), \tag{4.5}$$

obtained by Omidvar and Moradi [64, Th. 2.1] and the first inequality in (1.3). Consider the matrix $A = \begin{pmatrix} 2+\mathrm{i} & 0 \\ 0 & 1+3\mathrm{i} \end{pmatrix}$*, then*

$$\begin{aligned}
& 5 = \frac{1}{4}\|A^*A + AA^*\| \\
< \quad & 6 = \frac{1}{4}\|A + A^*\|\|A - A^*\| \\
< \quad & 6.5 = \frac{1}{8}\big(\|A + A^*\|^2 + \|A - A^*\|^2\big) \\
< \quad & 7.5 = \frac{1}{8}\big(\|A + A^*\|^2 + \|A - A^*\|^2\big) + \frac{1}{8}c^2\big(A + A^*\big) + \frac{1}{8}c^2\big(A - A^*\big) \\
< \quad & 10 = w^2(A).
\end{aligned}$$

This shows that the inequalities obtained in Theorem 4.1 are proper.

The following corollary follows from Theorem 4.1.

Corollary 4.1. *Let* $A \in \mathcal{B}(\mathcal{H})$*. Then*

$$\frac{1}{4}\|A^*A + AA^*\| + \frac{1}{8}c^2\big(A + A^*\big) + \frac{1}{8}c^2\big(A - A^*\big) \leq w^2(A). \tag{4.6}$$

It should be mentioned here that the inequality (4.6) is weaker than the third inequality in Theorem 4.1.

In the next theorem we obtain a norm inequality which refines the triangle inequality.

Theorem 4.3. *Let $A, D \in \mathcal{B}(\mathcal{H})$. Then*

$$\|A+D\|^2 \leq \|A\|^2 + \|D\|^2 + \|A^*D + D^*A\| \leq (\|A\| + \|D\|)^2.$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then we have,

$$\begin{aligned}
\|(A+D)x\|^2 &= \langle (A+D)x, (A+D)x\rangle \\
&= \|Ax\|^2 + \|Dx\|^2 + \langle (A^*D + D^*A)x, x\rangle \\
&\leq \|A\|^2 + \|D\|^2 + \|A^*D + D^*A\|.
\end{aligned}$$

Taking supremum over $\|x\| = 1$ we get the first inequality of the theorem. The second inequality follows from the inequality $\|A^*D + D^*A\| \leq 2\|A\|\|D\|$. □

Remark 4.4. *We would like to note that if $\|A+D\| = \|A\| + \|D\|$ then it follows from Theorem 4.3 that $\|A^*D + D^*A\| = 2\|A\|\|D\|$. The converse is not true, in general. Consider $A = I$ and $D = -I$, then $\|A^*D + D^*A\| = 2\|A\|\|D\|$, but $\|A+D\| \neq \|A\| + \|D\|$.*

Next we need the following inequality, known as Buzano's inequality.

Lemma 4.1. *([35]) Let $x, e, y \in \mathcal{H}$ with $\|e\| = 1$. Then*

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}\left(\|x\|\|y\| + |\langle x, y\rangle|\right).$$

Now, we obtain another refinement of the triangle inequality.

Theorem 4.5. *Let $A, D \in \mathcal{B}(\mathcal{H})$. Then*

$$\|A+D\|^2 \leq \|A\|^2 + \|D\|^2 + \|A\|\|D\| + \min\left\{w(A^*D), w(AD^*)\right\} \leq (\|A\| + \|D\|)^2.$$

Proof. Let $x, y \in \mathcal{H}$ be two unit vectors. Then we get,

$$\begin{aligned}
|\langle (A+D)x, y\rangle|^2 &\leq (|\langle Ax, y\rangle| + |\langle Dx, y\rangle|)^2 \\
&= |\langle Ax, y\rangle|^2 + |\langle Dx, y\rangle|^2 + 2|\langle Ax, y\rangle\langle Dx, y\rangle| \\
&= |\langle Ax, y\rangle|^2 + |\langle Dx, y\rangle|^2 + 2|\langle Ax, y\rangle\langle y, Dx\rangle| \\
&\leq |\langle Ax, y\rangle|^2 + |\langle Dx, y\rangle|^2 + \|Ax\|\|Dx\| + |\langle Ax, Dx\rangle| \\
&\qquad \Big(\text{by Lemma 4.1}\Big) \\
&\leq \|A\|^2 + \|D\|^2 + \|A\|\|D\| + w(A^*D).
\end{aligned}$$

Taking supremum over $\|x\| = \|y\| = 1$ we get,

$$\|A + D\|^2 \leq \|A\|^2 + \|D\|^2 + \|A\|\|D\| + w(A^*D). \tag{4.7}$$

Replacing A by A^* and D by D^* in (4.7) we get,

$$\|A + D\|^2 \leq \|A\|^2 + \|D\|^2 + \|A\|\|D\| + w(AD^*). \tag{4.8}$$

Combining (4.7) and (4.8) we have the first inequality of the theorem. The second inequality follows from the observation that $\min\{w(A^*D), w(AD^*)\} \leq \|A\|\|D\|$. □

Remark 4.6. *It follows from Theorem 4.5 that if* $\|A+D\| = \|A\|+\|D\|$ *then* $w(A^*D) = \|A\|\|D\|$ *and* $w(AD^*) = \|A\|\|D\|$. *The converse is not true, in general. Consider* $A = I$ *and* $D = -I$, *then* $w(A^*D) = \|A\|\|D\|$ *and* $w(AD^*) = \|A\|\|D\|$, *but* $\|A + D\| \neq \|A\| + \|D\|$.

Now we need the following norm inequality.

Lemma 4.2. *([36]) Let* $A, D \in \mathcal{B}(\mathcal{H})$ *be positive. Then*

$$\|A + D\| \quad \leq \quad \max\{\|A\|, \|D\|\} + \|AD\|^{\frac{1}{2}}.$$

Next refinement of the first inequality in (1.3) is as follows.

Theorem 4.7. *Let* $A \in \mathcal{B}(\mathcal{H})$. *Then*

$$\begin{aligned} & \frac{1}{4}\|A^*A + AA^*\| \\ \leq \quad & \frac{1}{8}\left[\max\left\{\|A + A^*\|^2, \|A - A^*\|^2\right\} + \|A + A^*\|\|A - A^*\|\right] \leq w^2(A). \end{aligned}$$

Proof. From the identity (4.1) we get,

$$\begin{aligned} \frac{1}{4}\|A^*A + AA^*\| &= \frac{1}{2}\|B^2 + C^2\| \\ &\leq \frac{1}{2}\left[\max\{\|B\|^2, \|C\|^2\} + \|B^2C^2\|^{\frac{1}{2}}\right] \quad \Big(\text{by Lemma 4.2}\Big) \\ &\leq \frac{1}{2}\left[\max\{\|B\|^2, \|C\|^2\} + \|B\|\|C\|\right]. \end{aligned}$$

This implies the first inequality of the theorem. The second inequality follows from the observation that $\|B\| \leq w(A)$ and $\|C\| \leq w(A)$. □

Remark 4.8. *We note that the second inequality in Theorem 4.7 refines (4.5), obtained by Omidvar and Moradi [64, Th. 2.1].*

Next we need the following lemma, proved in [25, Th. 2.4], by Bhunia et al.

Lemma 4.3. *([25, Th. 2.4]) Let $A, D \in \mathcal{B}(\mathcal{H})$. Then*

$$\|A+D\|^2 \le 2\max\left\{\|A^*A+D^*D\|, \|AA^*+DD^*\|\right\}.$$

Based on the above lemma we obtain the following refinement of the first inequality in (1.3).

Theorem 4.9. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{4}\|A^*A+AA^*\| \le \frac{1}{4\sqrt{2}}\left[\left\|A+A^*\right\|^4 + \left\|A-A^*\right\|^4\right]^{\frac{1}{2}} \le w^2(A).$$

Proof. From the identity (4.1) we get,

$$\begin{aligned}\frac{1}{4}\|A^*A+AA^*\| &= \frac{1}{2}\|B^2+C^2\| \\ &\le \frac{1}{\sqrt{2}}\|B^4+C^4\|^{\frac{1}{2}} \quad \left(\text{by Lemma 4.3}\right) \\ &\le \frac{1}{\sqrt{2}}\left[\|B\|^4+\|C\|^4\right]^{\frac{1}{2}}.\end{aligned}$$

This implies the first inequality of the theorem. As before, the second inequality follows from the observation that $\|B\| \le w(A)$ and $\|C\| \le w(A)$. □

Remark 4.10. *The concavity of the function $f(t) = \sqrt{t}$ ensures that the first inequality in Theorem 4.9 is stronger than the first inequality in Theorem 4.1. We also note that the second inequality in Theorem 4.9 refines the inequality (4.5), obtained by Omidvar and Moradi [64, Th. 2.1].*

To obtain the next refinement of the first inequality in (1.3), we need the following lemma, proved in [25, Th. 2.10], by Bhunia et al.

Lemma 4.4. *([25, Th. 2.10]) Let $A, D \in \mathcal{B}(\mathcal{H})$. Then*

$$\|A+D\|^4 \le 2\max\left\{\|A^*A+D^*D\|^2 + 4w^2(D^*A), \|AA^*+DD^*\|^2 + 4w^2(AD^*)\right\}.$$

Theorem 4.11. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}&\frac{1}{4}\|A^*A+AA^*\| \\ \le\ &\frac{1}{8}\left[2\left(\|A+A^*\|^4+\|A-A^*\|^4\right)^2 + 8\|A+A^*\|^4\|A-A^*\|^4\right]^{\frac{1}{4}} \le w^2(A).\end{aligned}$$

Proof. From the identity (4.1) we get,

$$\begin{aligned}
\frac{1}{4}\|A^*A+AA^*\| &= \frac{1}{2}\|B^2+C^2\| \\
&\leq \frac{1}{2}\left[2\|B^4+C^4\|^2+8\max\{w^2(B^2C^2),w^2(C^2B^2)\}\right]^{\frac{1}{4}} \\
&\qquad \Big(\text{by Lemma 4.4}\Big) \\
&\leq \frac{1}{2}\left[2(\|B\|^4+\|C\|^4)^2+8\|B\|^4\|C\|^4\right]^{\frac{1}{4}}.
\end{aligned}$$

This implies the first inequality of the theorem. As before, the second inequality follows from the observation that $\|B\| \leq w(A)$ and $\|C\| \leq w(A)$. □

Now, we prove the following norm inequalities.

Theorem 4.12. *Let $A, D \in \mathcal{B}(\mathcal{H})$. Then the following inequalities hold:*

$$\|A+D\|^2 \leq \|A\|^2+\|D\|^2+\frac{1}{2}\|A^*A+D^*D\|+w(A^*D) \tag{4.9}$$

and

$$\|A+D\|^2 \leq \|A\|^2+\|D\|^2+\frac{1}{2}\|AA^*+DD^*\|+w(AD^*). \tag{4.10}$$

Proof. Let $x, y \in \mathcal{H}$ be two unit vectors. Then we get,

$$\begin{aligned}
|\langle (A+D)x,y\rangle|^2 &\leq (|\langle Ax,y\rangle|+|\langle Dx,y\rangle|)^2 \\
&= |\langle Ax,y\rangle|^2+|\langle Dx,y\rangle|^2+2|\langle Ax,y\rangle\langle Dx,y\rangle| \\
&= |\langle Ax,y\rangle|^2+|\langle Dx,y\rangle|^2+2|\langle Ax,y\rangle\langle y,Dx\rangle| \\
&\leq |\langle Ax,y\rangle|^2+|\langle Dx,y\rangle|^2+\|Ax\|\|Dx\|+|\langle Ax,Dx\rangle| \\
&\qquad \Big(\text{by Lemma 4.1}\Big) \\
&\leq |\langle Ax,y\rangle|^2+|\langle Dx,y\rangle|^2+\frac{1}{2}(\|Ax\|^2+\|Dx\|^2)+|\langle Ax,Dx\rangle| \\
&\leq |\langle Ax,y\rangle|^2+|\langle Dx,y\rangle|^2 \\
&\quad +\frac{1}{2}\langle (A^*A+D^*D)x,x\rangle+|\langle A^*Dx,x\rangle| \\
&\leq \|A\|^2+\|D\|^2+\frac{1}{2}\|A^*A+D^*D\|+w(A^*D).
\end{aligned}$$

Taking supremum over $\|x\|=\|y\|=1$ we get,

$$\|A+D\|^2 \leq \|A\|^2+\|D\|^2+\frac{1}{2}\|A^*A+D^*D\|+w(A^*D).$$

Replacing A by A^* and D by D^* in the above inequality we get,

$$\|A+D\|^2 \leq \|A\|^2 + \|D\|^2 + \frac{1}{2}\|AA^* + DD^*\| + w(AD^*).$$

This completes the proof. $\square$

Based on the norm inequalities obtained in Theorem 4.12 we obtain the following refinement of the first inequality in (1.3).

Theorem 4.13. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} & \frac{1}{4}\|A^*A + AA^*\| \\ \leq & \frac{1}{8}\Big[\big(\|A+A^*\|^2 + \|A-A^*\|^2\big)^2 + \frac{1}{2}\big(\|A+A^*\|^2 - \|A-A^*\|^2\big)^2\Big]^{\frac{1}{2}} \leq w^2(A). \end{aligned}$$

Proof. From the identity (4.1) we get,

$$\begin{aligned} \frac{1}{4}\|A^*A + AA^*\| &= \frac{1}{2}\|B^2 + C^2\| \\ &\leq \frac{1}{2}\Big[\|B\|^4 + \|C\|^4 + \frac{1}{2}\|B^4 + C^4\| + w(B^2C^2)\Big]^{\frac{1}{2}}, \\ &\qquad \Big(\text{by Theorem } 4.12\Big) \\ &\leq \frac{1}{2}\Big[\|B\|^4 + \|C\|^4 + \frac{1}{2}(\|B\|^4 + \|C\|^4) + \|B\|^2\|C\|^2)\Big]^{\frac{1}{2}}. \end{aligned}$$

This implies the first inequality of the theorem. The second inequality follows from the observation that $\|B\| \leq w(A)$ and $\|C\| \leq w(A)$. $\square$

Remark 4.14. *The first inequality in Theorem 4.13 is better than the first inequality in Theorem 4.1. We also note that the second inequality in Theorem 4.13 refines the inequality (4.5), obtained by Omidvar and Moradi [64, Th. 2.1].*

In [16], Bhatia and Kittaneh have obtained that if $A, D \in \mathcal{B}(\mathcal{H})$ be positive then

$$\|AD\| \leq \frac{1}{4}\|A+D\|^2. \tag{4.11}$$

Now, by using the inequality (4.11) we prove the following numerical radius inequality.

Theorem 4.15. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{4}\|A^*A + AA^*\| \leq \frac{1}{2}w^2(A) + \frac{1}{8}\Big\|(A+A^*)^2(A-A^*)^2\Big\|^{\frac{1}{2}} \leq w^2(A).$$

Proof. From the inequality (4.11) we have,

$$\|B^2C^2\| \leq \frac{1}{4}\|B^2 + C^2\|^2 \leq \frac{1}{4}(\|B\|^2 + \|C\|^2)^2.$$

It follows from the observation $\|B\| \leq w(A)$ and $\|C\| \leq w(A)$ that

$$w^2(A) \geq \|B^2C^2\|^{\frac{1}{2}} \geq \| \ |B||C| \ \| = \|BC\|.$$

Thus,

$$w^2(A) \geq \frac{1}{4}\left\|(A + A^*)^2(A - A^*)^2\right\|^{\frac{1}{2}} \geq \frac{1}{4}\left\|(A + A^*)(A - A^*)\right\|.$$

This implies that

$$w^2(A) \geq \frac{1}{2}w^2(A) + \frac{1}{8}\left\|(A + A^*)^2(A - A^*)^2\right\|^{\frac{1}{2}} \geq \frac{1}{2}w^2(A) + \frac{1}{8}\left\|(A + A^*)(A - A^*)\right\|.$$

From [64, Th. 2.1], we have

$$\frac{1}{2}w^2(A) + \frac{1}{8}\left\|(A + A^*)(A - A^*)\right\| \geq \frac{1}{4}\|A^*A + AA^*\|.$$

This completes the proof of the theorem. □

The following corollary is obvious.

Corollary 4.2. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{2}\|A^*A + AA^*\| - \frac{1}{4}\left\|(A + A^*)^2(A - A^*)^2\right\|^{\frac{1}{2}} \quad \leq \quad w^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|.$$

The following remark follows from the above corollary.

Remark 4.16. *If $(A + A^*)^2(A - A^*)^2 = 0$, then $w^2(A) = \frac{1}{2}\|A^*A + AA^*\|$.*

4.3 Numerical radius inequalities of bounded operators via operator convex function

The notion of operator convex function plays an important role in the development of norm and numerical radius inequalities. A real-valued continuous function f on an interval J is said to be operator convex if for all self-adjoint operators $A, D \in \mathcal{B}(\mathcal{H})$ whose spectra are contained in J satisfy $f\big((1-t)A+tD\big) \leq (1-t)f(A)+tf(D)$ for all $t \in [0,1]$. The function $f(t)=t^r$ is operator convex on $[0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$.

Bhatia and Kittaneh [18] have obtained a norm inequality, namely, for $A, D \in \mathcal{B}(\mathcal{H})$,

$$\|AD^*\| \quad \leq \quad \frac{1}{2}\|A^*A+D^*D\|\,. \tag{4.12}$$

Lemma 4.5. *Let $A \in \mathcal{B}(\mathcal{H})$ and let f be non-negative increasing operator convex function on $[0,\infty)$. Then*

$$f(w^2(A)) \quad \leq \quad \|f(\alpha|A|^2+(1-\alpha)|A^*|^2)\|, \quad \forall \alpha \in [0,1]. \tag{4.13}$$

Proof. In [30, Cor. 2.5], Bhunia and Paul obtained that

$$w^2(A) \leq \|\alpha|A|^2+(1-\alpha)|A^*|^2\|, \quad \forall \;\; \alpha \in [0,1]. \tag{4.14}$$

Therefore, for all $\alpha \in [0,1]$,

$$f(w^2(A)) \quad \leq \quad f(\|\alpha|A|^2+(1-\alpha)|A^*|^2\|) \leq \|f(\alpha|A|^2+(1-\alpha)|A^*|^2)\|.$$

This completes the proof. □

Lemma 4.6. *([38]) Let $f: J \to \mathbb{R}$ be an operator convex function on the interval J. Let A and D be two self-adjoint operators with spectra in J. Then*

$$f\left(\frac{A+D}{2}\right) \leq \int_0^1 f\Big((1-t)A+tD\Big)dt \leq \frac{1}{2}\big(f(A)+f(D)\big). \tag{4.15}$$

If f is non-negative then the operator inequality (4.15) can be reduced to the following norm inequality

$$\left\|f\left(\frac{A+D}{2}\right)\right\| \leq \left\|\int_0^1 f\Big((1-t)A+tD\Big)dt\right\| \leq \frac{1}{2}\big\|f(A)+f(D)\big\|. \tag{4.16}$$

Lemma 4.7. *([56]) Let $A, D \in \mathcal{B}(\mathcal{H})$ be positive. Then $\|A+D\| = \|A\|+\|D\|$ if and only if $\|AD\| = \|A\|\|D\|$.*

Now, we present the first theorem of this section.

Theorem 4.17. *Let $A \in \mathcal{B}(\mathcal{H})$ and let f be non-negative increasing operator convex function* *on* $[0,\infty)$. *Then*

$$\begin{aligned} f(w^2(A)) &\le \left\| \int_0^1 f\Big((1-t)\big(\alpha|A|^2+(1-\alpha)|A^*|^2\big)+tw^2(A)I\Big)dt \right\| \\ &\le \big\| f(\alpha|A|^2+(1-\alpha)|A^*|^2)\big\|, \quad \forall \alpha\in[0,1]. \end{aligned}$$

Proof. For all $\alpha\in[0,1]$ we have,

$$\big\|\big(\alpha|A|^2+(1-\alpha)|A^*|^2\big)w^2(A)I\big\| = \big\|\alpha|A|^2+(1-\alpha)|A^*|^2\big\|\big\|w^2(A)I\big\|.$$

Thus, it follows from Lemma 4.7 that

$$\big\|\alpha|A|^2+(1-\alpha)|A^*|^2+w^2(A)I\big\| = \big\|\alpha|A|^2+(1-\alpha)|A^*|^2\big\|+w^2(A).$$

So, by using the inequality (4.14) we get,

$$w^2(A)\le \frac{1}{2}\big\|\alpha|A|^2+(1-\alpha)|A^*|^2+w^2(A)I\big\|.$$

Then,

$$\begin{aligned} f(w^2(A)) &\le f\left(\frac{1}{2}\big\|\alpha|A|^2+(1-\alpha)|A^*|^2+w^2(A)I\big\|\right) \\ &\le \left\| f\left(\frac{\alpha|A|^2+(1-\alpha)|A^*|^2+w^2(A)I}{2}\right)\right\| \\ &\le \left\| \int_0^1 f\Big((1-t)\big(\alpha|A|^2+(1-\alpha)|A^*|^2\big)+tw^2(A)I\Big)dt \right\| \\ &\qquad \Big(\text{by inequality (4.16)}\Big) \\ &\le \frac{1}{2}\big\| f(\alpha|A|^2+(1-\alpha)|A^*|^2)+f(w^2(A))I\big\| \Big(\text{by inequality (4.16)}\Big) \\ &= \frac{1}{2}\big\| f(\alpha|A|^2+(1-\alpha)|A^*|^2)\big\|+\frac{1}{2}f(w^2(A)) \ \Big(\text{by Lemma 4.7}\Big) \\ &\le \big\| f(\alpha|A|^2+(1-\alpha)|A^*|^2)\big\| \quad \Big(\text{by Lemma 4.5}\Big). \end{aligned}$$

This completes the proof. □

By considering $f(t)=t^2$ in Theorem 4.17, we get the following corollary.

Corollary 4.3. *Let* $A\in\mathcal{B}(\mathcal{H})$. *Then*

$$\begin{aligned} w^2(A) &\le \left\| \int_0^1 \Big((1-t)\big(\alpha|A|^2+(1-\alpha)|A^*|^2\big)+tw^2(A)I\Big)^2 dt \right\|^{\frac{1}{2}} \\ &\le \big\|\alpha|A|^2+(1-\alpha)|A^*|^2\big\|, \qquad \forall\alpha\in[0,1]. \end{aligned}$$

In particular, for $\alpha = \frac{1}{2}$

$$w^2(A) \leq \left\| \int_0^1 \left((1-t) \left(\frac{|A|^2 + |A^*|^2}{2} \right) + tw^2(A)I \right)^2 dt \right\|^{\frac{1}{2}} \leq \frac{1}{2} \left\| |A|^2 + |A^*|^2 \right\|.$$

This inequality can be written in the following form:

$$\begin{aligned} w^2(A) &\leq \frac{1}{\sqrt{3}} \left\| \left(\frac{|A|^2 + |A^*|^2}{2} \right)^2 + w^4(A)I + w^2(A) \left(\frac{|A|^2 + |A^*|^2}{2} \right) \right\|^{\frac{1}{2}} && (4.17) \\ &\leq \left\| \frac{|A|^2 + |A^*|^2}{2} \right\|. && (4.18) \end{aligned}$$

Remark 4.18. *We observe that the inequality in* (4.17) *is sharper than the second inequality in (1.3). We also remark that the first inequality in Corollary 4.3 improves on the inequality (4.14), obtained by Bhunia and Paul [30, Cor. 2.5].*

The following theorem again involves operator convex function.

Theorem 4.19. *Let $A \in \mathcal{B}(\mathcal{H})$ and let f be non-negative increasing operator convex function on $[0, \infty)$. Then*

$$\begin{aligned} f(w^2(A)) &\leq \left\| \int_0^1 f\left((1-t) \left(\alpha \left(\frac{|A| + |A^*|}{2} \right)^2 + (1-\alpha)|A|^2 \right) + tw^2(A)I \right) dt \right\| \\ &\leq \left\| f\left(\alpha \left(\frac{|A| + |A^*|}{2} \right)^2 + (1-\alpha)|A|^2 \right) \right\|, \quad \forall \alpha \in [0,1]. \end{aligned}$$

Proof. Following [30, Cor. 2.15] we have,

$$w^2(A) \leq \left\| \alpha \left(\frac{|A| + |A^*|}{2} \right)^2 + (1-\alpha)|A|^2 \right\|, \quad \forall \alpha \in [0,1]. \qquad (4.19)$$

Proceeding similarly as in Theorem 4.17 we get the required inequality. □

Considering $f(t) = t^2$ in Theorem 4.19, we get the following corollary.

Corollary 4.4. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} w^2(A) &\leq \left\| \int_0^1 \left((1-t) \left(\alpha \left(\frac{|A| + |A^*|}{2} \right)^2 + (1-\alpha)|A|^2 \right) + tw^2(A)I \right)^2 dt \right\|^{\frac{1}{2}} \\ &\leq \left\| \alpha \left(\frac{|A| + |A^*|}{2} \right)^2 + (1-\alpha)|A|^2 \right\|, \quad \forall \alpha \in [0,1]. \end{aligned}$$

Next, we prove the following theorem.

Theorem 4.20. *Let $A \in \mathcal{B}(\mathcal{H})$ and let f be non-negative increasing operator convex function on $[0,\infty)$. Then*

$$\begin{aligned} f(w^2(A)) &\leq \left\| \int_0^1 f\left((1-t)\left(\alpha\left(\frac{|A|+|A^*|}{2}\right)^2+(1-\alpha)|A^*|^2\right)+tw^2(A)I\right)dt\right\| \\ &\leq \left\| f\left(\alpha\left(\frac{|A|+|A^*|}{2}\right)^2+(1-\alpha)|A^*|^2\right)\right\|, \quad \forall \alpha \in [0,1]. \end{aligned}$$

Proof. Following [30, Cor. 2.15] we have,

$$w^2(A) \leq \left\| \alpha\left(\frac{|A|+|A^*|}{2}\right)^2+(1-\alpha)|A^*|^2\right\|, \quad \forall \alpha \in [0,1]. \tag{4.20}$$

The proof then follows by using the inequality (4.20) and proceeding similarly as in Theorem 4.17. □

By considering $f(t) = t^2$ in Theorem 4.20, we get the following corollary.

Corollary 4.5. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} w^2(A) &\leq \left\| \int_0^1 \left((1-t)\left(\alpha\left(\frac{|A|+|A^*|}{2}\right)^2+(1-\alpha)|A^*|^2\right)+tw^2(A)I\right)^2 dt\right\|^{\frac{1}{2}} \\ &\leq \left\| \alpha\left(\frac{|A|+|A^*|}{2}\right)^2+(1-\alpha)|A^*|^2\right\|, \quad \forall \alpha \in [0,1]. \end{aligned}$$

In particular, for $\alpha = 1$

$$w^2(A) \leq \left\| \int_0^1 \left((1-t)\left(\frac{|A|+|A^*|}{2}\right)^2+tw^2(A)I\right)^2 dt\right\|^{\frac{1}{2}} \leq \left\| \frac{|A|+|A^*|}{2}\right\|^2.$$

This inequality can be written in the following form:

$$w^2(A) \leq \frac{1}{\sqrt{3}}\left\| \left(\frac{|A|+|A^*|}{2}\right)^4+w^4(A)I+w^2(A)\left(\frac{|A|+|A^*|}{2}\right)^2\right\|^{\frac{1}{2}} \tag{4.21}$$

$$\leq \left\| \frac{|A|+|A^*|}{2}\right\|^2. \tag{4.22}$$

Remark 4.21. *The inequality (4.17) follows from the inequality (4.21) by using the operator convexity of the function $f(t) = t^2$. Clearly, the inequality (4.21) is also a refinement of the* *second inequality in (1.3).*

Next, we prove the following norm inequality.

Theorem 4.22. *Let $A, D \in \mathcal{B}(\mathcal{H})$ be positive and let f be non-negative increasing operator convex function on $[0,\infty)$. Then*

$$f(\|AD\|) \le \left\| \int_0^1 f\left((1-t)\left(\frac{A+D}{2}\right)^2 + t\|AD\|I \right) dt \right\| \le \left\| f\left(\left(\frac{A+D}{2}\right)^2 \right) \right\|.$$

Proof. Using the inequality (4.11) and proceeding similarly as in Theorem 4.17, we get the required inequality. □

In particular, if we consider $f(t) = t^2$ in Theorem 4.22, then we get the following corollary.

Corollary 4.6. *Let $A, D \in \mathcal{B}(\mathcal{H})$ be positive. Then*

$$\|AD\| \le \left\| \int_0^1 \left((1-t)\left(\frac{A+D}{2}\right)^2 + t\|AD\|I \right)^2 dt \right\|^{\frac{1}{2}} \le \frac{1}{4}\|A+D\|^2.$$

This inequality can be written in the following form:

$$\|AD\| \le \frac{1}{\sqrt{3}} \left\| \left(\frac{A+D}{2}\right)^4 + \|AD\|^2 I + \|AD\| \left(\frac{A+D}{2}\right)^2 \right\|^{\frac{1}{2}} \le \frac{1}{4}\|A+D\|^2.$$

Remark 4.23. *Clearly, the first inequality in Corollary 4.6 improves on the inequality (4.11), obtained by Bhatia and Kittaneh [16].*

The final result of this section is an improvement of the norm inequality (4.12), obtained by Bhatia and Kittaneh [18].

Theorem 4.24. *Let $A, D \in \mathcal{B}(\mathcal{H})$ and let f be non-negative increasing operator convex function on $[0,\infty)$. Then*

$$f(\|AD^*\|) \le \left\| \int_0^1 f\left((1-t)\left(\frac{|A|^2+|D|^2}{2}\right) + t\|AD^*\|I \right) dt \right\| \le \left\| f\left(\frac{|A|^2+|D|^2}{2} \right) \right\|.$$

Proof. Using the inequality (4.12) and proceeding similarly as in Theorem 4.17, we get the desired inequality. □

In particular, if we consider $f(t) = t^2$ in Theorem 4.24, then we get the following corollary.

Corollary 4.7. *Let $A, D \in \mathcal{B}(\mathcal{H})$. Then*

$$\|AD^*\| \leq \left\| \int_0^1 \left((1-t) \left(\frac{|A|^2 + |D|^2}{2} \right) + t\|AD^*\| I \right)^2 dt \right\|^{\frac{1}{2}} \leq \frac{1}{2} \left\| |A|^2 + |D|^2 \right\|.$$

This inequality can be written in the following form:

$$\begin{aligned} \|AD^*\| &\leq \frac{1}{\sqrt{3}} \left\| \left(\frac{|A|^2 + |D|^2}{2} \right)^2 + \|AD^*\|^2 I + \|AD^*\| \left(\frac{|A|^2 + |D|^2}{2} \right) \right\|^{\frac{1}{2}} \\ &\leq \frac{1}{2} \|A^*A + D^*D\|. \end{aligned}$$

Remark 4.25. *We would like to remark that the first inequality in Corollary 4.7 refines the inequality (4.12), obtained by Bhatia and Kittaneh [18]. Consider* $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ *and* $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *Then by elementary calculations we get,*

$$\frac{1}{\sqrt{3}} \left\| \left(\frac{|A|^2 + |D|^2}{2} \right)^2 + \|AD^*\|^2 I + \|AD^*\| \left(\frac{|A|^2 + |D|^2}{2} \right) \right\|^{\frac{1}{2}} = \sqrt{\frac{61}{12}} \approx 2.2546$$

and $\frac{1}{2} \|A^*A + D^*D\| = \frac{5}{2}$. *This shows that the inequality obtained in Corollary 4.7 is a proper refinement of the inequality (4.12).*

CHAPTER 5

DEVELOPMENT OF INEQUALITIES AND CHARACTERIZATION OF EQUALITY CONDITIONS FOR THE NUMERICAL RADIUS

5.1 Introduction

In this chapter, we establish new inequalities for the numerical radius of bounded linear operators. For a bounded linear operator A, we obtain the following inequalities

$$w^2(A) \leq \min_{0\leq\alpha\leq 1} \left\| \alpha|A|^2 + (1-\alpha)|A^*|^2 \right\|,$$

Content of this chapter is based on the following papers:
P. Bhunia, K. Paul; Proper improvement of well-known numerical radius inequalities and their applications, **Results Math.**, 76 (2021), no. 4, Paper No. 177, 12 pp. https://doi.org/10.1007/s00025-021-01478-3
P. Bhunia, K. Paul; Development of inequalities and characterization of equality conditions for the numerical radius, **Linear Algebra Appl.**, 630 (2021), 306–315. https://doi.org/10.1016/j.laa.2021.08.014

and

$$w^2(A) \leq \min_{0\leq\alpha\leq 1}\left\{\frac{\alpha}{2}w(A^2) + \left\|\frac{\alpha}{4}|A|^2 + \left(1-\frac{3}{4}\alpha\right)|A^*|^2\right\|\right\}.$$

We show that the inequalities obtained here generalize and improve on the existing well-known inequalities given in [1, 54, 55]. Further, we obtain lower bounds for the numerical radius of bounded linear operators which refine the well-known lower bound $w(A) \geq \frac{\|A\|}{2}$ and the bound $w^2(A) \geq \frac{1}{4}\|A^*A + AA^*\|$, obtained by Kittaneh [54, Th. 1]. We present equivalent conditions for the equality of $w(A) = \frac{\|A\|}{2}$ as well as $w^2(A) = \frac{1}{4}\|A^*A + AA^*\|$ in terms of the geometric shape of numerical range of A. Further, applying the lower bounds obtained here, we obtain upper bounds for the numerical radius of commutators of bounded linear operators, which refine the existing ones [40, 47]. For this purpose first we introduce the following notations and terminologies.

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle .,. \rangle$. As usual the norm induced by the inner product $\langle .,. \rangle$ is denoted by $\|\cdot\|$. For $A \in \mathcal{B}(\mathcal{H})$, let $W(A)$ denote the numerical range of A, which is defined as $W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$. For $A \in \mathcal{B}(\mathcal{H})$, let $w(A)$ and $\|A\|$ denote the numerical radius and the operator norm of A, respectively, defined as, $w(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}$ and $\|A\| = \sup\{\|Ax\| : x \in \mathcal{H}, \|x\| = 1\}$. It is easy to verify that $w(.)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the operator norm $\|\cdot\|$. In fact, for every $A \in \mathcal{B}(\mathcal{H})$, we have that $\frac{1}{2}\|A\| \leq w(A) \leq \|A\|$. The Crawford number of A, denoted by $c(A)$, is another important numerical constant associated with the numerical range and is defined as $c(A) = \inf\{|\langle Ax, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}$. The adjoint of an operator A is denoted by A^*. Clearly $w(A) = w(A^*)$ and $c(A) = c(A^*)$. For $A \in \mathcal{B}(\mathcal{H})$, the real part and imaginary part of A, denoted as $\Re(A)$ and $\Im(A)$, respectively, that is, $\Re(A) = \frac{A+A^*}{2}$ and $\Im(A) = \frac{A-A^*}{2i}$. Thus, $A = \Re(A) + i\Im(A)$ is the Cartesian decomposition of A. It is well known that, for $A \in \mathcal{B}(\mathcal{H})$, $w(A) = \sup_{\theta\in\mathbb{R}} \|\Re(e^{i\theta}A)\| = \sup_{\theta\in\mathbb{R}} \|\Im(e^{i\theta}A)\|$, see in [75].

5.2 Refined and generalized upper bounds for the numerical radius of bounded linear operators

We begin this section with the following proposition that gives an inequality involving the operator norm and the Crawford number of bounded linear operators.

Proposition 5.1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then the following inequality holds.*

$$\|A\|^2 + \max\left\{c(|A|^2), c(|A^*|^2)\right\} \leq \|A^*A + AA^*\|.$$

Proof. The proof follows from the observation that $\forall x \in \mathcal{H}$ with $\|x\| = 1$ we have, $\|Ax\|^2 + \|A^*x\|^2 = \langle (A^*A + AA^*)x, x\rangle \leq \|A^*A + AA^*\|$. □

To proceed further we need the following lemmas.

Lemma 5.1. *([44, pp. 75-76]) Let $A \in \mathcal{B}(\mathcal{H})$ and let $x \in \mathcal{H}$. Then*

$$|\langle Ax, x\rangle| \leq \langle |A|x, x\rangle^{1/2} \langle |A^*|x, x\rangle^{1/2}.$$

Lemma 5.2. *([73, p. 20]) Let $A \in \mathcal{B}(\mathcal{H})$ be positive and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle, \quad \forall \quad r \geq 1.$$

Also we need the well-known Heinz inequality.

Lemma 5.3 (Heinz inequality [52])**.** *Let $A \in \mathcal{B}(\mathcal{H})$. Then for all $x, y \in \mathcal{H}$,*

$$|\langle Ax, y\rangle|^2 \leq \langle |A|^{2\lambda}x, x\rangle \langle |A^*|^{2(1-\lambda)}y, y\rangle, \ \forall \ \ \lambda, \ 0 \leq \lambda \leq 1. \tag{5.1}$$

We note that Lemma 5.1 is a special case of the Heinz inequality. Now we prove our first theorem.

Theorem 5.1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$w^{2r}(A) \leq \left\| \frac{\alpha}{2}\left(|A|^{4\lambda r} + |A^*|^{4(1-\lambda)r}\right) + (1-\alpha)|A^*|^{2r} \right\| \tag{5.2}$$

and

$$w^{2r}(A) \leq \left\| \frac{\alpha}{2}\left(|A|^{4\lambda r} + |A^*|^{4(1-\lambda)r}\right) + (1-\alpha)|A|^{2r} \right\|, \tag{5.3}$$

$\forall \ \ r \geq 1$ and $\forall \ \alpha, \lambda$ with $0 \leq \alpha, \lambda \leq 1$.

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then by Cauchy-Schwarz inequality, we get

$$|\langle Ax, x\rangle| = \alpha|\langle Ax, x\rangle| + (1-\alpha)|\langle Ax, x\rangle| \leq \alpha|\langle Ax, x\rangle| + (1-\alpha)\|A^*x\|.$$

Therefore, by the convexity of the function $f(t) = t^{2r}$, we get

$$\begin{aligned}
|\langle Ax, x\rangle|^{2r} &\leq \alpha|\langle Ax, x\rangle|^{2r} + (1-\alpha)\|A^*x\|^{2r} \\
&\leq \alpha\langle |A|^{2\lambda}x, x\rangle^r \langle |A^*|^{2(1-\lambda)}x, x\rangle^r + (1-\alpha)\langle |A^*|^2 x, x\rangle^r \quad \Big(\text{by Lemma 5.3}\Big) \\
&\leq \alpha\langle |A|^{2\lambda r}x, x\rangle \langle |A^*|^{2(1-\lambda)r}x, x\rangle + (1-\alpha)\langle |A^*|^{2r} x, x\rangle \quad \Big(\text{by Lemma 5.2}\Big) \\
&\leq \frac{\alpha}{2}\Big(\langle |A|^{2\lambda r}x, x\rangle^2 + \langle |A^*|^{2(1-\lambda)r}x, x\rangle^2\Big) + (1-\alpha)\langle |A^*|^{2r} x, x\rangle \\
&\leq \frac{\alpha}{2}\Big(\langle |A|^{4\lambda r}x, x\rangle + \langle |A^*|^{4(1-\lambda)r}x, x\rangle\Big) + (1-\alpha)\langle |A^*|^{2r} x, x\rangle \Big(\text{by Lemma 5.2}\Big) \\
&= \Big\langle \Big\{\frac{\alpha}{2}\Big(|A|^{4\lambda r} + |A^*|^{4(1-\lambda)r}\Big) + (1-\alpha)|A^*|^{2r}\Big\} x, x\Big\rangle \\
&\leq \Big\|\frac{\alpha}{2}\Big(|A|^{4\lambda r} + |A^*|^{4(1-\lambda)r}\Big) + (1-\alpha)|A^*|^{2r}\Big\|.
\end{aligned}$$

Taking supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get (5.2). By similar arguments as above we can prove (5.3).

□

Based on Theorem 5.1 we prove the following inequality.

Corollary 5.1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$w^{2r}(A) \leq \left\|\alpha|A|^{2r} + (1-\alpha)|A^*|^{2r}\right\|, \quad \forall \ r \geq 1, \ \text{and} \ \forall \ \alpha, \ 0 \leq \alpha \leq 1.$$

Proof. Taking $\lambda = \frac{1}{2}$ in (5.2) and (5.3), respectively, we get

$$w^{2r}(A) \leq \left\|\frac{\alpha}{2}|A|^{2r} + (1-\frac{\alpha}{2})|A^*|^{2r}\right\|, \quad \forall \ r \geq 1, \quad \forall \ \alpha, \ 0 \leq \alpha \leq 1$$

and

$$w^{2r}(A) \leq \left\|\left(1-\frac{\alpha}{2}\right)|A|^{2r} + \frac{\alpha}{2}|A^*|^{2r}\right\|, \quad \forall \ r \geq 1, \quad \forall \ \alpha, \ 0 \leq \alpha \leq 1.$$

Combining the above two inequalities we get the desired inequality. □

As a consequence of Corollary 5.1 we easily get the following corollary.

Corollary 5.2. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$w^2(A) \leq \min_{0\leq\alpha\leq 1} \left\|\alpha|A|^2 + (1-\alpha)|A^*|^2\right\|. \tag{5.4}$$

Inequalities obtained in Corollary 5.1 and Corollary 5.2 generalize and improve on the second inequality in (1.3). In order to appreciate our inequality (5.4), we give the following examples which show that

$$\min_{0\leq\alpha\leq 1}\left\|\alpha|A|^2+(1-\alpha)\,|A^*|^2\right\| < \frac{1}{2}\left\||A|^2+|A^*|^2\right\|$$

and imply that our inequality (5.4) is a non-trivial improvement of the second inequality in (1.3).

Example 5.2. *(i) Let*

$$A=\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 0\end{pmatrix}.$$

Then

$$|A|^2=\begin{pmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 4\end{pmatrix} \quad \textit{and} \quad |A^*|^2=\begin{pmatrix} 1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Therefore,

$$\min_{0\leq\alpha\leq 1}\left\|\alpha|A|^2+(1-\alpha)\,|A^*|^2\right\| = \min_{0\leq\alpha\leq 1}\max\{1-\alpha, 4-3\alpha, 4\alpha\} = \frac{16}{7}$$

and

$$\frac{1}{2}\left\||A|^2+|A^*|^2\right\| = \frac{5}{2}.$$

Thus,

$$\min_{0\leq\alpha\leq 1}\left\|\alpha|A|^2+(1-\alpha)\,|A^*|^2\right\| < \frac{1}{2}\left\||A|^2+|A^*|^2\right\|.$$

(ii) Let

$$S=\begin{pmatrix} 0 & 2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Then

$$|S|^2=\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 4 & 0 & 0\\ 0 & 0 & 9 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \quad \textit{and} \quad |S^*|^2=\begin{pmatrix} 4 & 0 & 0 & 0\\ 0 & 9 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Therefore,

$$\min_{0\leq\alpha\leq 1}\left\|\alpha|S|^2+(1-\alpha)\,|S^*|^2\right\| = \frac{81}{14} < \frac{13}{2} = \frac{1}{2}\left\||S|^2+|S^*|^2\right\|.$$

In [1], Abu-Omar and Kittaneh proved that the following inequality

$$w^2(A) \leq \frac{1}{2}w(A^2) + \frac{1}{4}\left\||A|^2 + |A^*|^2\right\|. \tag{5.5}$$

In our next theorem we generalize and improve on the inequality (5.5). To do so we need the following inequality.

Lemma 5.4. *(Buzano's inequality, [35]) Let $a, e, b \in \mathcal{H}$ with $\|e\| = 1$. Then*

$$|\langle a, e\rangle\langle e, b\rangle| \leq \frac{1}{2}\left(\|a\|\|b\| + |\langle a, b\rangle|\right).$$

Using Buzano's inequality we first prove the following lemma.

Lemma 5.5. *Let $A \in \mathcal{B}(\mathcal{H})$ and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then*

$$|\langle Ax, x\rangle|^{2r} \leq \frac{1}{2}|\langle A^2x, x\rangle|^r + \frac{1}{4}\left\langle (|A|^{2r} + |A^*|^{2r})x, x\right\rangle, \quad \forall \quad r \geq 1. \tag{5.6}$$

Proof. Taking $a = Ax$, $b = A^*x$ and $e = x$ in Lemma 5.4, we get

$$|\langle Ax, x\rangle|^2 \leq \frac{1}{2}\left(|\langle A^2x, x\rangle| + \|Ax\| \;\; \|A^*x\|\right).$$

By convexity of the function $f(t) = t^r$ $(r \geq 1)$, we get

$$\begin{aligned}
|\langle Ax, x\rangle|^{2r} &\leq \frac{1}{2}\left(|\langle A^2x, x\rangle|^r + \|Ax\|^r \;\; \|A^*x\|^r\right) \\
&\leq \frac{1}{2}\left(|\langle A^2x, x\rangle|^r + \frac{1}{2}(\|Ax\|^{2r} + \|A^*x\|^{2r})\right)\Big(\text{by AM-GM inequality}\Big) \\
&= \frac{1}{2}\left(|\langle A^2x, x\rangle|^r + \frac{1}{2}(\langle |A|^2x, x\rangle^r + \langle |A^*|^2x, x\rangle^r)\right) \\
&\leq \frac{1}{2}\left(|\langle A^2x, x\rangle|^r + \frac{1}{2}(\langle |A|^{2r}x, x\rangle + \langle |A^*|^{2r}x, x\rangle)\right)\Big(\text{by Lemma 5.2}\Big) \\
&= \frac{1}{2}|\langle A^2x, x\rangle|^r + \frac{1}{4}\left\langle (|A|^{2r} + |A^*|^{2r})x, x\right\rangle.
\end{aligned}$$

This completes the proof. □

Now, we present the desired theorem.

Theorem 5.3. *Let $A \in \mathcal{B}(\mathcal{H})$. Then $\forall \;\; r \geq 1$ and $\forall \; \alpha, \;\; 0 \leq \alpha \leq 1$,*

$$(i)\; w^{2r}(A) \;\; \leq \;\; \frac{\alpha}{2}w^r(A^2) + \left\|\frac{\alpha}{4}|A|^{2r} + \left(1 - \frac{3}{4}\alpha\right)|A^*|^{2r}\right\|, \tag{5.7}$$

$$(ii)\ w^{2r}(A) \ \leq \ \frac{\alpha}{2}w^r(A^2) + \left\| \left(1-\frac{3}{4}\alpha\right)|A|^{2r} + \frac{\alpha}{4}|A^*|^{2r} \right\|. \tag{5.8}$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then by Cauchy-Schwarz inequality, we get

$$|\langle Ax, x\rangle| = \alpha|\langle Ax, x\rangle| + (1-\alpha)|\langle Ax, x\rangle| \leq \alpha|\langle Ax, x\rangle| + (1-\alpha)\|A^*x\|, \ \forall \ \ \alpha, 0 \leq \alpha \leq 1.$$

By convexity of the function $f(t) = t^{2r}$ $(r \geq 1)$, we get

$$\begin{aligned}
|\langle Ax, x\rangle|^{2r} &\leq \alpha|\langle Ax, x\rangle|^{2r} + (1-\alpha)\|A^*x\|^{2r} \\
&\leq \alpha|\langle Ax, x\rangle|^{2r} + (1-\alpha)\langle |A^*|^{2r}x, x\rangle \ \Big(\text{by Lemma 5.2}\Big) \\
&\leq \frac{\alpha}{2}|\langle A^2x, x\rangle|^r + \frac{\alpha}{4}\left\langle (|A|^{2r} + |A^*|^{2r})x, x\right\rangle + (1-\alpha)\langle |A^*|^{2r}x, x\rangle, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big(\text{by Lemma 5.4}\Big) \\
&= \frac{\alpha}{2}|\langle A^2x, x\rangle|^r + \left\langle \left\{ \frac{\alpha}{4}\left(|A|^{2r} + |A^*|^{2r}\right) + (1-\alpha)|A^*|^{2r} \right\} x, x \right\rangle \\
&= \frac{\alpha}{2}|\langle A^2x, x\rangle|^r + \left\langle \left\{ \frac{\alpha}{4}|A|^{2r} + \left(1-\frac{3}{4}\alpha\right)|A^*|^{2r} \right\} x, x \right\rangle \\
&\leq \frac{\alpha}{2}w^r(A^2) + \left\| \frac{\alpha}{4}|A|^{2r} + \left(1-\frac{3}{4}\alpha\right)|A^*|^{2r} \right\|.
\end{aligned}$$

Taking supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get the inequality (5.7). Replacing A by A^* in the inequality (5.7) we get the inequality (5.8). This completes the proof. □

As a consequence we get the following upper bound for the numerical radius.

Corollary 5.3. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$w^2(A) \leq \min\left\{\beta_1(A), \beta_2(A)\right\}, \tag{5.9}$$

where

$$\beta_1(A) = \min_{0\leq\alpha\leq 1}\left\{ \frac{\alpha}{2}w(A^2) + \left\| \frac{\alpha}{4}|A|^2 + \left(1-\frac{3}{4}\alpha\right)|A^*|^2 \right\| \right\}$$

and

$$\beta_2(A) = \min_{0\leq\alpha\leq 1}\left\{ \frac{\alpha}{2}w(A^2) + \left\| \left(1-\frac{3}{4}\alpha\right)|A|^2 + \frac{\alpha}{4}|A^*|^2 \right\| \right\}.$$

Proof. The proof follows easily by taking $r = 1$ in the inequalities (5.7) and (5.8). □

Inequalities obtained in Theorem 5.3 and Corollary 5.3 generalize and improve on the inequality (5.5). In order to appreciate our obtained inequality (5.9), we give the following examples, it shows that the inequality (5.9) is a non-trivial improvement of the inequality (5.5).

Example 5.4. *(i) Let*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then by elementary calculations, we get $\beta_1(A) = \frac{7}{4}$ *and* $\beta_2(A) = \frac{22}{13}$*. Therefore,*

$$\min\{\beta_1(A), \beta_2(A)\} = \frac{22}{13} < \frac{7}{4} = \frac{1}{2}w(A^2) + \frac{1}{4}\left\||A|^2 + |A^*|^2\right\|.$$

(ii) Let

$$S = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then by elementary calculations, we get $\beta_1(S) = \frac{19}{4}$ *and* $\beta_2(S) = \frac{37}{8}$*. Therefore,*

$$\min\{\beta_1(S), \beta_2(S)\} = \frac{37}{8} < \frac{19}{4} = \frac{1}{2}w(S^2) + \frac{1}{4}\left\||S|^2 + |S^*|^2\right\|.$$

Next, we prove the following theorem.

Theorem 5.5. *Let* $A \in \mathcal{B}(\mathcal{H})$*. Then* $\forall\ \ r \geq 1$ *and* $\forall\ \alpha,\ 0 \leq \alpha \leq 1$ *we have*

$$w^{2r}(A) \quad \leq \quad \left\|\alpha\left(\frac{|A| + |A^*|}{2}\right)^{2r} + (1-\alpha)\,|A^*|^{2r}\right\|. \tag{5.10}$$

$$w^{2r}(A) \quad \leq \quad \left\|\alpha\left(\frac{|A| + |A^*|}{2}\right)^{2r} + (1-\alpha)\,|A|^{2r}\right\|. \tag{5.11}$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then by Cauchy-Schwarz inequality, we get $\forall\ \ \alpha,\ \ 0 \leq \alpha \leq 1$,

$$|\langle Ax, x\rangle| = \alpha|\langle Ax, x\rangle| + (1-\alpha)|\langle Ax, x\rangle| \leq \alpha|\langle Ax, x\rangle| + (1-\alpha)\|A^*x\|.$$

By the convexity of the function $f(t) = t^{2r}$ $(r \geq 1)$, we get

$$\begin{aligned}
|\langle Ax, x\rangle|^{2r} &\leq \alpha|\langle Ax, x\rangle|^{2r} + (1-\alpha)\|A^*x\|^{2r} \\
&\leq \alpha|\langle Ax, x\rangle|^{2r} + (1-\alpha)\langle |A^*|^{2r}x, x\rangle \quad \Big(\text{by Lemma 5.2}\Big) \\
&\leq \alpha\left(\langle |A|x, x\rangle^{1/2}\langle |A^*|x, x\rangle^{1/2}\right)^{2r} + (1-\alpha)\langle |A^*|^{2r}x, x\rangle \quad \Big(\text{by Lemma 5.1}\Big) \\
&\leq \alpha\left(\frac{\langle |A|x, x\rangle + \langle |A^*|x, x\rangle}{2}\right)^{2r} + (1-\alpha)\langle |A^*|^{2r}x, x\rangle \\
&\qquad\qquad \Big(\text{by AM-GM inequality}\Big) \\
&= \alpha\left(\frac{\langle (|A| + |A^*|)x, x\rangle}{2}\right)^{2r} + (1-\alpha)\langle |A^*|^{2r}x, x\rangle \\
&\leq \alpha\left\langle \left(\frac{|A| + |A^*|}{2}\right)^{2r} x, x\right\rangle + (1-\alpha)\langle |A^*|^{2r}x, x\rangle \quad \Big(\text{by Lemma 5.2}\Big) \\
&= \left\langle \left\{\alpha\left(\frac{|A| + |A^*|}{2}\right)^{2r} + (1-\alpha)\,|A^*|^{2r}\right\} x, x\right\rangle \\
&\leq \left\|\alpha\left(\frac{|A| + |A^*|}{2}\right)^{2r} + (1-\alpha)\,|A^*|^{2r}\right\|.
\end{aligned}$$

Taking supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get the inequality (5.10). Replacing A by A^* in the inequality (5.10), we get that the inequality (5.11). □

The following corollary is an easy consequence of Theorem 5.5.

Corollary 5.4. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$w^2(A) \quad \leq \quad \min\{\gamma_1(A), \gamma_2(A)\}, \tag{5.12}$$

where

$$\gamma_1(A) = \min_{0\leq\alpha\leq 1}\left\|\alpha\left(\frac{|A| + |A^*|}{2}\right)^2 + (1-\alpha)\,|A^*|^2\right\|$$

and

$$\gamma_2(A) = \min_{0\leq\alpha\leq 1}\left\|\alpha\left(\frac{|A| + |A^*|}{2}\right)^2 + (1-\alpha)\,|A|^2\right\|.$$

Remark 5.6. *In [55], Kittaneh proved that the following inequality*

$$w^2(A) \leq \frac{1}{4}\left\||A| + |A^*|\right\|^2. \tag{5.13}$$

The inequalities obtained in Theorem 5.5 and Corollary 5.4 generalize and improve on the inequality (5.13). As before considering the operators A and S used in Example 5.2 we can *show that the inequality (5.12) is a non-trivial improvement of the inequality (5.13).*

5.3 Refined lower bounds for the numerical radius of bounded operators

We begin by noting an elementary equality of real numbers, $\max\{a,b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$ for all $a, b \in \mathbb{R}$. By using this maximum function we obtain the following lower bound for the numerical radius of bounded linear operators.

Theorem 5.7. *Let $A \in \mathcal{B}(\mathcal{H})$, then*

$$w(A) \geq \frac{\|A\|}{2} + \frac{\big|\ \|\Re(A)\| - \|\Im(A)\|\ \big|}{2}.$$

Proof. Let x be a unit vector in $\mathcal{H}$. Then it follows from the Cartesian decomposition of A that $|\langle Ax, x\rangle|^2 = |\langle \Re(A)x, x\rangle|^2 + |\langle \Im(A)x, x\rangle|^2$. This implies $w(A) \geq \|\Re(A)\|$ and $w(A) \geq \|\Im(A)\|$. Thus, we have

$$\begin{aligned} w(A) &\geq \max\{\|\Re(A)\|, \|\Im(A)\|\} \\ &= \frac{\|\Re(A)\| + \|\Im(A)\|}{2} + \frac{\big|\ \|\Re(A)\| - \|\Im(A)\|\ \big|}{2} \\ &\geq \frac{\|\Re(A) + \mathrm{i}\Im(A)\|}{2} + \frac{\big|\ \|\Re(A)\| - \|\Im(A)\|\ \big|}{2} \\ &= \frac{\|A\|}{2} + \frac{\big|\ \|\Re(A)\| - \|\Im(A)\|\ \big|}{2}, \end{aligned}$$

as desired.

□

Remark 5.8. *(i) We note that if A is Hermitian or skew Hermitian operator then the inequality in Theorem 5.7 becomes an equality.*
(ii) Clearly, the inequality in Theorem 5.7 is stronger than the first inequality in (1.1) when $\|\Re(A)\| \neq \|\Im(A)\|$.

As a consequence of Theorem 5.7 we prove the following corollary.

Corollary 5.5. *Let $A \in \mathcal{B}(\mathcal{H})$. If $w(A) = \frac{\|A\|}{2}$, then $\|\Re(A)\| = \|\Im(A)\| = \frac{\|A\|}{2}$.*

Proof. From Theorem 5.7, we have $w(A) \geq \frac{\|A\|}{2} + \frac{\big|\ \|\Re(A)\| - \|\Im(A)\|\ \big|}{2} \geq \frac{\|A\|}{2}$. This implies that if

$w(A) = \frac{\|A\|}{2}$, then $\|\Re(A)\| = \|\Im(A)\|$. Also

$$\|\Re(A)\| \leq w(A) = \frac{\|A\|}{2} = \frac{\|\Re(A) + \mathrm{i}\Im(A)\|}{2} \leq \frac{\|\Re(A)\| + \|\Im(A)\|}{2} = \|\Re(A)\|$$

and so $\|\Re(A)\| = \|\Im(A)\| = \frac{\|A\|}{2}$. □

Note that the converse of Corollary 5.5 is not true, in general. Now, we concentrate our study on an equivalent condition for $w(A) = \frac{\|A\|}{2}$.

Theorem 5.9. *Let $A \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent.*
(i) $w(A) = \frac{\|A\|}{2}$.
(ii) $\|\Re(e^{i\theta}A)\| = \|\Im(e^{i\theta}A)\| = \frac{\|A\|}{2}$, for all $\theta \in \mathbb{R}$.

Proof. (ii) implies (i) is trivial. We only prove (i) implies (ii). Let $w(A) = \frac{\|A\|}{2}$. Then from Corollary 5.5, we have $\|\Re(A)\| = \|\Im(A)\| = \frac{\|A\|}{2}$. Clearly, for all $\theta \in \mathbb{R}$, $e^{i\theta}A \in \mathcal{B}(\mathcal{H})$ and $w(e^{i\theta}A) = w(A)$, $\|e^{i\theta}A\| = \|A\|$. Thus, $w(A) = \frac{\|A\|}{2}$ implies $\|\Re(e^{i\theta}A)\| = \|\Im(e^{i\theta}A)\| = \frac{\|A\|}{2}$ for all $\theta \in \mathbb{R}$.

□

Next we prove that, $\|A\| = \sqrt{\|A^*A + AA^*\|} = \sqrt{\|A^*A - AA^*\|}$, if $w(A) = \frac{1}{2}\|A\|$. To do so we need the following lemma.

Lemma 5.6 ([14])**.** *Let $A, B \in \mathcal{B}(\mathcal{H})$ be non-zero operators. Then $\|A + B\| = \|A\| + \|B\|$ if and only if $\|A\|\|B\| \in \overline{W(A^*B)}$.*

Theorem 5.10. *Let $A \in \mathcal{B}(\mathcal{H})$. If $w(A) = \frac{\|A\|}{2}$, then*

$$\|A\|^2 = \|A^*A + AA^*\| = \|A^*A - AA^*\|.$$

Proof. We note that for all $\theta \in \mathbb{R}$, by Theorem 5.9,

$$\|A\| = \|\Re(e^{\mathrm{i}\theta}A) + \mathrm{i}\Im(e^{\mathrm{i}\theta}A)\| \leq \|\Re(e^{\mathrm{i}\theta}A)\| + \|\Im(e^{\mathrm{i}\theta}A)\| = \|A\|.$$

Then using Lemma 5.6 we get, $\|\Re(e^{\mathrm{i}\theta}A)\|\|\Im(e^{\mathrm{i}\theta}A)\| \in \overline{W(\mathrm{i}\ \Re(e^{\mathrm{i}\theta}A)\Im(e^{\mathrm{i}\theta}A))}$. Clearly, we have $\|\Re(e^{\mathrm{i}\theta}A)\|\|\Im(e^{\mathrm{i}\theta}A)\| \leq w(\mathrm{i}\ \Re(e^{\mathrm{i}\theta}A)\Im(e^{\mathrm{i}\theta}A)) \leq \|\mathrm{i}\ \Re(e^{\mathrm{i}\theta}A)\Im(e^{\mathrm{i}\theta}A)\| \leq \|\Re(e^{\mathrm{i}\theta}A)\|\|\Im(e^{\mathrm{i}\theta}A)\|$. Since $\|\Re(e^{\mathrm{i}\theta}A)\|\|\Im(e^{\mathrm{i}\theta}A)\| \in \mathbb{R}$,

$$\|\Re(e^{\mathrm{i}\theta}A)\|\|\Im(e^{\mathrm{i}\theta}A)\| = \left\|\Re\left(\mathrm{i}\ \Re(e^{\mathrm{i}\theta}A)\Im(e^{\mathrm{i}\theta}A)\right)\right\|.$$

Clearly, $\Re\left(\mathrm{i}\ \ \Re(e^{\mathrm{i}\theta}A)\Im(e^{\mathrm{i}\theta}A)\right) = \frac{1}{4}(A^*A - AA^*)$. Thus we get,

$$\|\Re(e^{\mathrm{i}\theta}A)\|\|\Im(e^{\mathrm{i}\theta}A)\| = \frac{1}{4}\|A^*A - AA^*\|, \quad \textit{for all} \;\; \theta \in \mathbb{R}. \tag{5.14}$$

From Theorem 5.9, we have $\|A\|^2 = \|A^*A - AA^*\|$. Now, by the first inequality in (1.3), $\|A^*A - AA^*\| = \|A\|^2 \leq \|A^*A + AA^*\| \leq 4w^2(A) = \|A\|^2$. Hence, $\|A\|^2 = \|A^*A - AA^*\| = \|A^*A + AA^*\|$. □

Remark 5.11. *Kittaneh [54, Prop. 1] proved that if $A^2 = 0$ then $\|A\|^2 = \|AA^* - A^*A\| = \|AA^* + A^*A\|$, whereas Theorem 5.10 says that if $w(A) = \frac{\|A\|}{2}$ then $\|A\|^2 = \|AA^* - A^*A\| = \|AA^* + A^*A\|$. Clearly, $\left\{A \in \mathcal{B}(\mathcal{H}) : A^2 = 0\right\} \subseteq \left\{A \in \mathcal{B}(\mathcal{H}) : w(A) = \frac{\|A\|}{2}\right\}$ is proper. Thus, Theorem 5.10 is applicable to a larger class of operators than [54, Prop. 1].*

In the next theorem we obtain another lower bound for the numerical radius which improves on that in (1.3).

Theorem 5.12. *Let $A \in \mathcal{B}(\mathcal{H})$, then*

$$w^2(A) \;\; \geq \;\; \frac{1}{4}\,\|A^*A + AA^*\| + \frac{1}{2}\left|\;\|\Re(A)\|^2 - \|\Im(A)\|^2\;\;\right|.$$

Proof. Let x be a unit vector in $\mathcal{H}$. Then it follows from the Cartesian decomposition of A that $|\langle Ax, x\rangle|^2 = |\langle \Re(A)x, x\rangle|^2 + |\langle \Im(A)x, x\rangle|^2$. This implies $w(A) \geq \|\Re(A)\|$ and $w(A) \geq \|\Im(A)\|$ and so,

$$\begin{aligned}
w^2(A) \;\; &\geq \;\; \max\left\{\|\Re(A)\|^2, \|\Im(A)\|^2\right\} \\
&= \;\; \frac{\|\Re(A)\|^2 + \|\Im(A)\|^2}{2} + \frac{\left|\|\Re(A)\|^2 - \|\Im(A)\|^2\right|}{2} \\
&= \;\; \frac{\|(\Re(A))^2\| + \|(\Im(A))^2\|}{2} + \frac{\left|\|\Re(A)\|^2 - \|\Im(A)\|^2\right|}{2} \\
&\geq \;\; \frac{\|(\Re(A))^2 + (\Im(A))^2\|}{2} + \frac{\left|\|\Re(A)\|^2 - \|\Im(A)\|^2\right|}{2} \\
&= \;\; \frac{1}{4}\,\|A^*A + AA^*\| + \frac{\left|\|\Re(A)\|^2 - \|\Im(A)\|^2\right|}{2},
\end{aligned}$$

as required.

□

Now, using Crawford number we obtain our next refinement.

Theorem 5.13. *Let $A \in \mathcal{B}(\mathcal{H})$, then*

$$\begin{aligned} w^2(A) &\geq \frac{1}{4}\|A^*A + AA^*\| + \frac{c^2(\Re(A)) + c^2(\Im(A))}{2} \\ &\quad + \left| \frac{\|\Re(A)\|^2 - \|\Im(A)\|^2}{2} + \frac{c^2(\Im(A)) - c^2(\Re(A))}{2} \right|. \end{aligned}$$

Proof. Let x be a unit vector in $\mathcal{H}$. Then it follows from the Cartesian decomposition of A that $|\langle Ax, x\rangle|^2 = |\langle \Re(A)x, x\rangle|^2 + |\langle \Im(A)x, x\rangle|^2$. This implies the following two inequalities: $w^2(A) \geq \|\Re(A)\|^2 + c^2(\Im(A))$ and $w^2(A) \geq \|\Im(A)\|^2 + c^2(\Re(A))$. Therefore, we have

$$\begin{aligned} w^2(A) &\geq \max\left\{\|\Re(A)\|^2 + c^2(\Im(A)), \|\Im(A)\|^2 + c^2(\Re(A))\right\} \\ &= \frac{\|\Re(A)\|^2 + c^2(\Im(A)) + \|\Im(A)\|^2 + c^2(\Re(A))}{2} \\ &\quad + \left| \frac{\|\Re(A)\|^2 + c^2(\Im(A)) - \|\Im(A)\|^2 - c^2(\Re(A))}{2} \right| \\ &= \frac{\|\Re(A)\|^2 + \|\Im(A)\|^2}{2} + \frac{c^2(\Re(A)) + c^2(\Im(A))}{2} \\ &\quad + \left| \frac{\|\Re(A)\|^2 - \|\Im(A)\|^2}{2} + \frac{c^2(\Im(A)) - c^2(\Re(A))}{2} \right| \\ &= \frac{\|(\Re(A))^2\| + \|(\Im(A))^2\|}{2} + \frac{c^2(\Re(A)) + c^2(\Im(A))}{2} \\ &\quad + \left| \frac{\|\Re(A)\|^2 - \|\Im(A)\|^2}{2} + \frac{c^2(\Im(A)) - c^2(\Re(A))}{2} \right| \\ &\geq \frac{\|(\Re(A))^2 + (\Im(A))^2\|}{2} + \frac{c^2(\Re(A)) + c^2(\Im(A))}{2} \\ &\quad + \left| \frac{\|\Re(A)\|^2 - \|\Im(A)\|^2}{2} + \frac{c^2(\Im(A)) - c^2(\Re(A))}{2} \right| \\ &= \frac{1}{4}\|A^*A + AA^*\| + \frac{c^2(\Re(A)) + c^2(\Im(A))}{2} \\ &\quad + \left| \frac{\|\Re(A)\|^2 - \|\Im(A)\|^2}{2} + \frac{c^2(\Im(A)) - c^2(\Re(A))}{2} \right|, \end{aligned}$$

as required.

□

Remark 5.14. *In [20, Cor. 2.3], Bhunia and Paul obtained that if $A \in \mathcal{B}(\mathcal{H})$, then*

$$w^2(A) \geq \frac{1}{4}\|A^*A + AA^*\| + \frac{1}{2}\left(c^2(\Re(A)) + c^2(\Im(A))\right). \tag{5.15}$$

Clearly, Theorem 5.13 is stronger than (5.15).

Next, we obtain an equivalent condition for the equality of $w(A) = \frac{1}{2}\sqrt{\|A^*A + AA^*\|}$.

Theorem 5.15. *Let $A \in \mathcal{B}(\mathcal{H})$. Then $w^2(A) = \frac{1}{4}\|A^*A + AA^*\|$ if and only if $\|\Re(e^{i\theta}A)\|^2 = \|\Im(e^{i\theta}A)\|^2 = \frac{1}{4}\|A^*A + AA^*\|$, for all $\theta \in \mathbb{R}$.*

Proof. The sufficient part is trivial, so we only prove the necessary part. Let $w^2(A) = \frac{1}{4}\|A^*A + AA^*\|$. Let $\theta \in \mathbb{R}$ be arbitrary. Then by simple computation we have, $\left(\Re(e^{i\theta}A)\right)^2 + \left(\Im(e^{i\theta}A)\right)^2 = \frac{A^*A+AA^*}{2}$. Now, we have

$$\begin{aligned}
\frac{1}{4}\|A^*A + AA^*\| &= \frac{1}{2}\left\|\left(\Re(e^{i\theta}A)\right)^2 + \left(\Im(e^{i\theta}A)\right)^2\right\| \\
&\leq \frac{1}{2}\left(\left\|\Re(e^{i\theta}A)\right\|^2 + \left\|\Im(e^{i\theta}A)\right\|^2\right) \\
&\leq \frac{1}{2}\left(w^2(A) + w^2(A)\right) \\
&= \frac{1}{4}\|A^*A + AA^*\|.
\end{aligned}$$

Thus, $\|\Re(e^{i\theta}A)\|^2 = \|\Im(e^{i\theta}A)\|^2 = w^2(A) = \frac{1}{4}\|A^*A + AA^*\|$, for all $\theta \in \mathbb{R}$. □

In the next theorem we characterize the numerical range of an operator when the numerical radius attains its lower bounds, namely, $w(A) = \frac{\|A\|}{2}$ and $w(A) = \frac{\sqrt{\|A^*A+AA^*\|}}{2}$.

Lemma 5.7. *Let $A \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:*
(i) $\|\Re(e^{i\theta}A)\| = k$, ($k$ is a constant) for all $\theta \in \mathbb{R}$.
(ii) $\overline{W(A)}$ is a circular disk with center at the origin and radius k.

Proof. $(i) \Rightarrow (ii)$. Since $w\left(\Re(e^{i\theta}A)\right) = k$ for all $\theta \in \mathbb{R}$, so, $\sup_{\|x\|=1} |\langle \Re(e^{i\theta}A)x, x\rangle| = k$, i.e., $\sup_{\|x\|=1} |Re(e^{i\theta}\langle Ax, x\rangle)| = k$ for all $\theta \in \mathbb{R}$. Thus, for each $\theta \in \mathbb{R}$, there exist a norm one sequence $\{x_n^\theta\}$ in $\mathcal{H}$ such that $|Re(e^{i\theta}\langle Ax_n^\theta, x_n^\theta\rangle)| \to k$. This implies that the boundary of $W(A)$ must be a circle with center at the origin and radius k. Since $W(A)$ is a convex subset of $\mathbb{C}$, so $\overline{W(A)}$ is a circular disk with center at the origin and radius k.
$(ii) \Rightarrow (i)$. Follows easily. □

Now, the desired characterizations follows easily from Theorem 5.9 and Theorem 5.15, respectively, by using Lemma 5.7.

Theorem 5.16. *Let $A \in \mathcal{B}(\mathcal{H})$. Then we have,*
(i) $w(A) = \frac{1}{2}\|A\|$ if and only if $\overline{W(A)}$ is a circular disk with center at the origin and radius $\frac{1}{2}\|A\|$.
(ii) $w(A) = \frac{1}{2}\sqrt{\|A^*A + AA^*\|}$ if and only if $\overline{W(A)}$ is a circular disk with center at the origin and radius $\frac{1}{2}\sqrt{\|A^*A + AA^*\|}$.

Remark 5.17. *For* $A \in \mathcal{B}(\mathcal{H})$*, we note* $w(A) = \frac{\|A\|}{2}$ *implies* $w(A) = \frac{1}{2}\sqrt{\|A^*A + AA^*\|}$*. However,* $w(A) = \frac{1}{2}\sqrt{\|A^*A + AA^*\|}$ *does not always imply* $w(A) = \frac{\|A\|}{2}$*. Consider* $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$*. Then,* $w(A) = \frac{1}{2}\sqrt{\|A^*A + AA^*\|} = \frac{1}{\sqrt{2}} > \frac{1}{2} = \frac{\|A\|}{2}$*.*

Our final inequality in this section is as follows.

Theorem 5.18. *Let* $A \in \mathcal{B}(\mathcal{H})$*, then*

$$\begin{aligned} w^4(A) &\geq \frac{1}{16}\left\|(A^*A + AA^*)^2 + 4\left(\Re(A^2)\right)^2\right\| + \frac{1}{2}\left|\|\Re(A)\|^4 - \|\Im(A)\|^4\right| \\ &\geq \frac{1}{16}\|A^*A + AA^*\|^2 + \frac{1}{4}c\left(\left(\Re(A^2)\right)^2\right) + \frac{1}{2}\left|\|\Re(A)\|^4 - \|\Im(A)\|^4\right|. \end{aligned}$$

Proof. It follows from $w^4(A) \geq \max\{\|\Re(A)\|^4, \|\Im(A)\|^4\}$ that

$$\begin{aligned} w^4(A) &\geq \frac{\|\Re(A)\|^4 + \|\Im(A)\|^4}{2} + \frac{\left|\ \|\Re(A)\|^4 - \|\Im(A)\|^4\ \right|}{2} \\ &\geq \frac{\|(\Re(A))^4 + (\Im(A))^4\|}{2} + \frac{\left|\ \|\Re(A)\|^4 - \|\Im(A)\|^4\ \right|}{2} \\ &= \frac{1}{16}\left\|(A^*A + AA^*)^2 + 4\left(\Re(A^2)\right)^2\right\| + \frac{\left|\ \|\Re(A)\|^4 - \|\Im(A)\|^4\ \right|}{2} \\ &\geq \frac{1}{16}\|A^*A + AA^*\|^2 + \frac{1}{4}c\left(\left(\Re(A^2)\right)^2\right) + \frac{\left|\ \|\Re(A)\|^4 - \|\Im(A)\|^4\ \right|}{2}. \end{aligned}$$

□

Remark 5.19. *Bag et al. [10, Th. 8] obtained that the following inequality:*

$$w^4(A) \geq \frac{1}{16}\|A^*A + AA^*\|^2 + \frac{1}{4}c\left(\left(\Re(A^2)\right)^2\right). \tag{5.16}$$

Bhunia et al. [25, Cor. 2.8] improved on the inequality (5.16) to

$$w^4(A) \geq \frac{1}{16}\left\|(A^*A + AA^*)^2 + 4\left(\Re(A^2)\right)^2\right\|. \tag{5.17}$$

Clearly, Theorem 5.18 improves on both the inequalities (5.16) and (5.17).

5.4 Application to estimate numerical radius bounds concerning commutators of operators

In this section we obtain upper bounds for the numerical radius of commutators of bounded linear operators, as applications of the lower bounds obtained in the previous section. First we prove the following theorem.

Theorem 5.20. *Let $A, B, X, Y \in \mathcal{B}(\mathcal{H})$, then*

$$w(AXB \pm BYA) \leq 2\sqrt{2}\|B\| \max\{\|X\|, \|Y\|\} \sqrt{w^2(A) - \nu},$$

where $\nu = \frac{c^2(\Re(A))+c^2(\Im(A))}{2} + \left|\frac{\|\Re(A)\|^2-\|\Im(A)\|^2}{2} + \frac{c^2(\Im(A))-c^2(\Re(A))}{2}\right|$.

Proof. First we assume that $\|X\| \leq 1$ and $\|Y\| \leq 1$. Let x be a unit vector in $\mathcal{H}$. Then, with $\|Xx\| \leq 1$ and $\|Y^*x\| \leq 1$, we have

$$\begin{aligned}
|\langle (AX \pm YA)x, x\rangle| &\leq |\langle Xx, A^*x\rangle| + |\langle Ax, Y^*x\rangle| \\
&\leq \|A^*x\| + \|Ax\| \\
&\leq \sqrt{2(\|A^*x\|^2 + \|Ax\|^2)} \\
&\leq \sqrt{2\|AA^* + A^*A\|} \\
&\leq 2\sqrt{2}\sqrt{w^2(A) - \nu},
\end{aligned}$$

where the last inequality follows from Theorem 5.13 with

$$\nu = \frac{c^2(\Re(A)) + c^2(\Im(A))}{2} + \left|\frac{\|\Re(A)\|^2 - \|\Im(A)\|^2}{2} + \frac{c^2(\Im(A)) - c^2(\Re(A))}{2}\right|.$$

Hence, by taking supremum over $\|x\| = 1$ we get,

$$w(AX \pm YA) \leq 2\sqrt{2}\sqrt{w^2(A) - \nu}. \tag{5.18}$$

Now we consider the general case, i.e., $X, Y \in \mathcal{B}(\mathcal{H})$ be arbitrary operators. If $X = Y = 0$, then Theorem 5.20 holds trivially. Let $\max\{\|X\|, \|Y\|\} \neq 0$. Then clearly $\left\|\frac{X}{\max\{\|X\|,\|Y\|\}}\right\| \leq 1$ and $\left\|\frac{Y}{\max\{\|X\|,\|Y\|\}}\right\| \leq 1$. So, replacing X and Y by $\frac{X}{\max\{\|X\|,\|Y\|\}}$ and $\frac{Y}{\max\{\|X\|,\|Y\|\}}$, respectively, in

(5.18) we get,

$$w(AX \pm YA) \leq 2\sqrt{2}\max\{\|X\|, \|Y\|\}\sqrt{w^2(A) - \nu}. \tag{5.19}$$

Now replacing X by XB and Y by BY in (5.19) we get,

$$w(AXB \pm BYA) \leq 2\sqrt{2}\max\{\|XB\|, \|BY\|\}\sqrt{w^2(A) - \nu}.$$

This implies that

$$w(AXB \pm BYA) \leq 2\sqrt{2}\|B\|\max\{\|X\|, \|Y\|\}\sqrt{w^2(A) - \nu},$$

as desired. □

Considering $X = Y = I$ in Theorem 5.20, we get the following inequality.

Corollary 5.6. *Let $A, B \in \mathcal{B}(\mathcal{H})$, then*

$$w(AB \pm BA) \quad \leq \quad 2\sqrt{2}\|B\|\sqrt{w^2(A) - \nu}, \tag{5.20}$$

where $\nu = \frac{c^2(\Re(A))+c^2(\Im(A))}{2} + \left|\frac{\|\Re(A)\|^2-\|\Im(A)\|^2}{2} + \frac{c^2(\Im(A))-c^2(\Re(A))}{2}\right|$.

Remark 5.21. *Fong and Holbrook [40] obtained that the following inequality*

$$w(AB + BA) \leq 2\sqrt{2}\|B\|w(A). \tag{5.21}$$

Hirzallah and Kittaneh [47] improved on the inequality (5.21) to prove that

$$w(AB \pm BA) \quad \leq \quad 2\sqrt{2}\|B\|\sqrt{w^2(A) - \frac{|\ \|\Re(A)\|^2 - \|\Im(A)\|^2\ |}{2}}. \tag{5.22}$$

Now,

$$\begin{aligned}
\nu &= \frac{c^2(\Re(A)) + c^2(\Im(A))}{2} + \left|\frac{\|\Re(A)\|^2 - \|\Im(A)\|^2}{2} + \frac{c^2(\Im(A)) - c^2(\Re(A))}{2}\right| \\
&\geq \frac{c^2(\Re(A)) + c^2(\Im(A))}{2} + \left|\frac{\|\Re(A)\|^2 - \|\Im(A)\|^2}{2}\right| - \left|\frac{c^2(\Im(A)) - c^2(\Re(A))}{2}\right| \\
&\geq \left|\frac{\|\Re(A)\|^2 - \|\Im(A)\|^2}{2}\right|.
\end{aligned}$$

Hence, Corollary 5.6 improves (5.22).

Proceeding similarly as in Corollary 5.6 and using Theorem 5.18, we get the following inequality.

Corollary 5.7. *Let $A, B \in \mathcal{B}(\mathcal{H})$, then*

$$\begin{aligned} w(AB \pm BA) &\leq 2\sqrt{2}\|B\| \left(w^4(A) - \frac{1}{4}c\left(\left(\Re(A^2)\right)^2 \right) - \frac{\left|\ \|\Re(A)\|^4 - \|\Im(A)\|^4\ \right|}{2} \right)^{\frac{1}{4}} \\ &\leq 2\sqrt{2}\|B\| \left(w^4(A) - \frac{\left|\ \|\Re(A)\|^4 - \|\Im(A)\|^4\ \right|}{2} \right)^{\frac{1}{4}}. \end{aligned}$$

Remark 5.22. *Clearly, Corollary 5.7 is an improvement of the inequality (5.21).*

CHAPTER 6

BOUNDS FOR THE NUMERICAL RADIUS OF BOUNDED OPERATORS VIA T-ALUTHGE TRANSFORM

6.1 Introduction

The aim of this chapter is to develop a number of inequalities using the properties of t-Aluthge transform. We show that the inequalities obtained here improve (1.2), (1.3), (1.4) and (1.6). We also obtain an upper bound for the numerical radius and show by an example that the bound is better than that in (1.7) for certain operators. Let us first introduce the following necessary notations, definitions and terminologies.

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators defined on a complex Hilbert space $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, the numerical range of T is defined as $W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$. The numerical radius, $w(T)$, is defined as the radius of the smallest circle with center at the origin and containing the numerical range, i.e., $w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$. The Crawford number of T is defined as $c(T) = \inf\{|\lambda| : \lambda \in W(T)\}$. The Cartesian decomposition of T is

Content of this chapter is based on the following paper:
S. Bag, **P. Bhunia**, K. Paul; Bounds of numerical radius of bounded linear operator using t-Aluthge transform, **Math. Inequal. Appl.**, 23 (2020), no. 3, 991–1004. dx.doi.org/10.7153/mia-2020-23-76

given by $T = \Re(T) + \mathrm{i}\,\Im(T)$, where $\Re(T) = \frac{T+T^*}{2}$ and $\Im(T) = \frac{T-T^*}{2\mathrm{i}}$. The spectral radius of T is defined as $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ where $\sigma(T)$ is the collection of all spectral values of T. It is well-known that $w(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the operator norm $\|\cdot\|$, satisfying that $\frac{1}{2}\|T\| \le w(T) \le \|T\|$. The first inequality becomes an equality if $T^2 = 0$ and the second inequality becomes an equality if T is normal. For $T \in \mathcal{B}(\mathcal{H})$, the Aluthge transform of T, denoted as $\widetilde{T}$, is defined as

$$\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}},$$

where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the partial isometry associated with the polar decomposition of T and so $T = U|T|$, $\ker T = \ker U$. It follows easily from the definition of $\widetilde{T}$ that $\|\widetilde{T}\| \le \|T\|$ and $r(\widetilde{T}) = r(T)$. Also $w(\widetilde{T}) \le w(T)$ (see [51]). Okubo [62] generalized the Aluthge transform, known as the t-Aluthge transform as follows: For $t \in [0, 1]$, the t-Aluthge transform is defined by,

$$\widetilde{T_t} = |T|^t U |T|^{1-t}.$$

Here, $|T|^0$ is defined as U^*U. In particular, $\widetilde{T_0} = U^*U^2|T|$, $\widetilde{T_1} = |T|UU^*U = |T|U$, $\widetilde{T_{\frac{1}{2}}} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}} = \widetilde{T}$.

6.2 Numerical radius inequalities using t-Aluthge transform

We begin this section with two notations H_θ and K_θ, defined as follows: For $T \in \mathcal{B}(\mathcal{H})$ and $\theta \in \mathbb{R}$, $H_\theta = \Re(e^{\mathrm{i}\theta} T)$ and $K_\theta = \Im(e^{\mathrm{i}\theta} T)$. The following lemma (see [75]) will be used repeatedly to reach our goal in this chapter.

Lemma 6.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w(T) = \sup_{\theta \in \mathbb{R}} \|H_\theta\| = \sup_{\theta \in \mathbb{R}} \|\Re(e^{\mathrm{i}\theta} T)\|.$$

Replacing T by iT *in the above equation, we have*

$$w(T) = \sup_{\theta \in \mathbb{R}} \|K_\theta\| = \sup_{\theta \in \mathbb{R}} \|\Im(e^{\mathrm{i}\theta} T)\|.$$

We next prove the following proposition which states that $T^2 = 0$ and $\widetilde{T_t} = 0$ for any $t \in [0, 1]$ are equivalent. To achieve it, we need the Heinz inequality (see [46]) given below.

Lemma 6.2. *[46] Let $A, B, X \in \mathcal{B}(\mathcal{H})$ where A and B be positive operators. Then*

$$\|A^r X B^r\| \leq \|AXB\|^r \|X\|^{1-r},$$

for $r \in [0,1]$.

Proposition 6.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then (i) $T^2 = 0$ and (ii) $\widetilde{T}_t = 0$ for $t \in [0,1]$ are equivalent.*

Proof. We first prove the easier part $(ii) \Rightarrow (i)$. It follows from the fact that $T^2 = U|T|U|T| = U|T|^{1-t}|T|^t U|T|^{1-t}|T|^t = U|T|^{1-t}\widetilde{T}_t|T|^t$ for any $t \in [0,1]$.
We next prove $(i) \Rightarrow (ii)$. We claim that

$$\|\widetilde{T}_t\| \leq \begin{cases} \|T^2\|^t \|T\|^{1-2t}, & 0 \leq t \leq \frac{1}{2} \\ \|T^2\|^{1-t}\|T\|^{2t-1}, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Consider $0 \leq t \leq \frac{1}{2}$. Then $\|\widetilde{T}_t\| = \||T|^t U|T|^{1-t}\|$. Using Lemma 6.2, we get

$$\|\widetilde{T}_t\| \leq \||T|^t U|T|^t\| \||T|^{1-2t}\| \leq \||T|U|T|\|^t \|U\|^{1-t}\|T\|^{1-2t} = \|T^2\|^t\|T\|^{1-2t}.$$

Next consider $\frac{1}{2} \leq t \leq 1$. Then using Lemma 6.2, we get

$$\|\widetilde{T}_t\| = \||T|^t U|T|^{1-t}\| \leq \||T|^{2t-1}\|\||T|^{1-t}U|T|^{1-t}\| \leq \|T\|^{2t-1}\||T|U|T|\|^{1-t}\|U\|^t = \|T\|^{2t-1}\|T^2\|^{1-t}.$$

The proof now easily follows from the claim established.

□

Next we present the following numerical radius inequality in terms of the Aluthge transform, which improves on the upper bound obtained by Yamazaki in [75, Th. 2.1].

Theorem 6.1. *(i) Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w(T) \leq \min_{t\in[0,1]} \left\{ \frac{1}{2} w(\widetilde{T}_t) + \frac{1}{4}\left(\|T\|^{2t} + \|T\|^{2-2t}\right) \right\}.$$

In particular,

$$w(T) \leq \frac{1}{2} w(\widetilde{T}) + \frac{1}{2}\|T\|.$$

(ii) If dim $\mathcal{H} < \infty$, then the equalities hold in the above inequalities if and only if T is either unitarily similar to $[a] \oplus B$, $\|B\| \leq |a|$ or to $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \oplus C$, $\|C\| \leq (|a|^2 + |b|^2)^{\frac{1}{2}}$ and $w(\widetilde{C_t}) \leq |a|$.

(iii) When $\mathcal{H}$ is an arbitrary Hilbert space, then the equalities hold if $T^2 = 0$ or T is normaloid, i.e., $w(T) = \|T\|$.

Proof. (i) It follows from arithmetic-geometric mean inequality that $2\|T\| \leq \|T\|^{2t} + \|T\|^{2-2t}$ for all $t \in [0,1]$. Using this and inequality (1.7), we get

$$w(T) \leq \min_{t\in[0,1]} \left\{ \frac{1}{2} w(\widetilde{T}_t) + \frac{1}{4}\left(\|T\|^{2t} + \|T\|^{2-2t}\right) \right\}.$$

Considering $t = \frac{1}{2}$, we get

$$w(T) \leq \frac{1}{2} w(\widetilde{T}) + \frac{1}{2}\|T\|.$$

(ii) Let us assume that T is an $n \times n$ matrix. Then following [42, Th. 4.2] we can conclude that the equalities hold if and only if T is either unitarily similar to $[a] \oplus B$, $\|B\| \leq |a|$ or to $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \oplus C$, $\|C\| \leq (|a|^2 + |b|^2)^{\frac{1}{2}}$ and $w(\widetilde{C_t}) \leq |a|$.

(iii) The proof is obvious.

□

Next we prove the following inequality for the numerical radius which improves on the upper bound obtained by Kittaneh in [55, Th. 1].

Theorem 6.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w^2(T) \leq \frac{1}{2}\|T\| \left(\min_{t\in[0,1]} \|\widetilde{T}_t\| \right) + \frac{1}{4}\|T^*T + TT^*\|.$$

In particular,

$$w^2(T) \leq \frac{1}{2}\|T\|\|\widetilde{T}\| + \frac{1}{4}\|T^*T + TT^*\|.$$

Proof. Since $H_\theta = \frac{1}{2}(e^{\mathrm{i}\theta}T + e^{-\mathrm{i}\theta}T^*)$ for all $\theta \in \mathbb{R}$, we have

$$\begin{aligned}
4{H_\theta}^2 &= e^{2\mathrm{i}\theta}T^2 + e^{-2\mathrm{i}\theta}T^{*2} + T^*T + TT^* \\
&= e^{2\mathrm{i}\theta}U|T|U|T| + e^{-2\mathrm{i}\theta}|T|U^*|T|U^* + T^*T + TT^* \\
&= e^{2\mathrm{i}\theta}U|T|^{1-t}|T|^tU|T|^{1-t}|T|^t + e^{-2\mathrm{i}\theta}|T|^t|T|^{1-t}U^*|T|^t|T|^{1-t}U^* \\
&\quad +(T^*T + TT^*) \\
&= e^{2\mathrm{i}\theta}U|T|^{1-t}\widetilde{T}_t|T|^t + e^{-2\mathrm{i}\theta}|T|^t\widetilde{T_t^*}|T|^{1-t}U^* + T^*T + TT^*.
\end{aligned}$$

Hence,

$$\begin{aligned} 4\|H_\theta\|^2 &\leq \|e^{2\mathrm{i}\theta}U|T|^{1-t}\widetilde{T}_t|T|^t\| + \|e^{-2\mathrm{i}\theta}|T|^t\widetilde{T_t^*}|T|^{1-t}U^*\| + \|T^*T+TT^*\| \\ &\leq 2\|T\|\|\widetilde{T}_t\| + \|T^*T+TT^*\|. \end{aligned}$$

Therefore,

$$\|H_\theta\|^2 \leq \frac{1}{2}\|T\|\|\widetilde{T}_t\| + \frac{1}{4}\|T^*T+TT^*\|.$$

Taking supremum over $\theta \in \mathbb{R}$ in the above inequality and then using Lemma 6.1, we get

$$w^2(T) \leq \frac{1}{2}\|T\|\|\widetilde{T}_t\| + \frac{1}{4}\|T^*T+TT^*\|.$$

This inequality holds for all $t \in [0,1]$, and so taking minimum we get,

$$w^2(T) \leq \frac{1}{2}\|T\|\big(\min_{t\in[0,1]}\|\widetilde{T}_t\|\big) + \frac{1}{4}\|T^*T+TT^*\|.$$

Considering the case $t = \frac{1}{2}$, we get

$$w^2(T) \leq \frac{1}{2}\|T\|\|\widetilde{T}\| + \frac{1}{4}\|T^*T+TT^*\|.$$

□

Remark 6.3. *If $T^2 = 0$ or T is a normaloid operator then inequalities in Theorem 6.2 become equalities. If $T^2 = 0$ then $w(T) = \frac{1}{2}\sqrt{\|T^*T+TT^*\|}$ (see [33, Th. 2.3]) and $\frac{1}{2}\|T\|\big(\min_{t\in[0,1]}\|\widetilde{T}_t\|\big)$ $+\frac{1}{4}\|T^*T+TT^*\| = \frac{1}{4}\|T^*T+TT^*\|$. Thus we get the equalities if $T^2 = 0$. Note that $w^2(T) \leq \frac{1}{2}\|T\|\big(\min_{t\in[0,1]}\|\widetilde{T}_t\|\big) + \frac{1}{4}\|T^*T+TT^*\| \leq \|T\|^2$ and so normaloid condition forces the inequalities to be equalities.*

Remark 6.4. *Kittaneh in [55, Th. 1] proved that for $T \in \mathcal{B}(\mathcal{H})$,*

$$w(T) \leq \frac{1}{2}\left(\|T\| + \|T^2\|^{\frac{1}{2}}\right).$$

Since, $\|\widetilde{T}\| \leq \|T^2\|^{\frac{1}{2}}$ (see the proof of Proposition 6.1) and $\|T^*T+TT^*\| \leq \|T\|^2 + \|T^2\|$ (see [22, Remark 3.9]), so from Theorem 6.2, we get

$$w^2(T) \leq \frac{1}{2}\|T\|\|T^2\|^{\frac{1}{2}} + \frac{1}{4}\left(\|T\|^2 + \|T^2\|\right).$$

Hence,

$$w(T) \leq \frac{1}{2}\left(\|T\| + \|T^2\|^{\frac{1}{2}}\right).$$

Thus the bound obtained in Theorem 6.2 is better than bound (1.2) obtained by Kittaneh in [55, Th. 1]. Also there are operators for which bound obtained by us in Theorem 6.2 is better than that in (1.7) obtained by Abu-Omar and Kittaneh [6, Th. 3.2]. As for example we consider $T = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *It is easy to see that* $\|T\| = 2$ *and* T *has the polar decomposition* $T = U|T|$, *where* $|T| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *and* $U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *Hence* $\widetilde{T}_t = |T|^t U |T|^{1-t} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *for all* $t \in [0,1]$, *and so* $w(\widetilde{T}_t) = \|\widetilde{T}_t\| = 1$ *for all* $t \in [0,1]$. *It follows that*

$$\frac{1}{2}\big(\|T\| + \min_{t\in[0,1]} w(\widetilde{T}_t)\big) = \frac{1}{2}(2+1) = \frac{3}{2},$$

$$\frac{1}{2}\|T\|\big(\min_{t\in[0,1]} \|\widetilde{T}_t\|\big) + \frac{1}{4}\|T^*T + TT^*\| = \frac{1}{2} \times 2 \times 1 + \frac{1}{4} \times 4 = 2.$$

Therefore, Theorem 6.2 gives $w(T) \leq \sqrt{2}$, *whereas (1.7) gives* $w(T) \leq \frac{3}{2}$.

Next we obtain an upper bound for the numerical radius which improves on the bound (1.2). To achieve it, we need the following inequality obtained by Abu-Omar and Kittaneh [6].

Theorem 6.5. *[6, Th. 2.2] Let* $A_1, A_2, B_1, B_2 \in \mathcal{B}(\mathcal{H})$. *Then*

$$\begin{aligned} r(A_1B_1 + A_2B_2) \;\; \leq \;\; & \frac{1}{2}\left(w(B_1A_1) + w(B_2A_2)\right) \\ & + \frac{1}{2}\sqrt{\left(w(B_1A_1) - w(B_2A_2)\right)^2 + 4\|B_1A_2\|\|B_2A_1\|}. \end{aligned}$$

Now, we prove the following theorem.

Theorem 6.6. *Let* $T \in \mathcal{B}(\mathcal{H})$. *Then*

$$w^2(T) \leq \min_{t\in[0,1]} \left(\frac{1}{4} w(\widetilde{T}_t^{\,2}) + \frac{1}{4}\|T\|\|\widetilde{T}_t\|\right) + \frac{1}{4}\|T^*T + TT^*\|.$$

In particular,

$$w^2(T) \leq \frac{1}{4} w(\widetilde{T}^2) + \frac{1}{4}\|T\|\|\widetilde{T}\| + \frac{1}{4}\|T^*T + TT^*\|.$$

Proof. Since $H_\theta = \frac{1}{2}(e^{i\theta}T + e^{-i\theta}T^*)$ for all $\theta \in \mathbb{R}$, we have

$$\begin{aligned} 4{H_\theta}^2 &= e^{2i\theta}T^2 + e^{-2i\theta}T^{*2} + T^*T + TT^* \\ &= e^{2i\theta}U|T|U|T| + e^{-2i\theta}|T|U^*|T|U^* + T^*T + TT^* \\ &= e^{2i\theta}U|T|^{1-t}|T|^tU|T|^{1-t}|T|^t + e^{-2i\theta}|T|^t|T|^{1-t}U^*|T|^t|T|^{1-t}U^* \\ &\quad + T^*T + TT^*. \end{aligned}$$

Hence,

$$\begin{aligned} 4\|H_\theta\|^2 &\leq \|e^{2i\theta}U|T|^{1-t}|T|^tU|T|^{1-t}|T|^t + e^{-2i\theta}|T|^t|T|^{1-t}U^*|T|^t|T|^{1-t}U^*\| \\ &\quad + \|T^*T + TT^*\| \\ &= r\big(e^{2i\theta}U|T|^{1-t}|T|^tU|T|^{1-t}|T|^t + e^{-2i\theta}|T|^t|T|^{1-t}U^*|T|^t|T|^{1-t}U^*\big) \\ &\quad + \|T^*T + TT^*\| \quad \Big(r(S) = \|S\| \ \textit{for hermitian operator} \ S\Big) \\ &= r\big(A_1B_1 + A_2B_2\big) + \|T^*T + TT^*\|, \end{aligned}$$

where $A_1 = e^{2i\theta}U|T|^{1-t}|T|^tU|T|^{1-t}, B_1 = |T|^t, A_2 = e^{-2i\theta}|T|^t$ and $B_2 = |T|^{1-t}U^*|T|^t|T|^{1-t}U^*$. Then using Theorem 6.5, we get

$$\begin{aligned} 4\|H_\theta\|^2 &\leq w(\widetilde{T}_t^{\,2}) + \sqrt{\||T|^{2t}\|\|\widetilde{T}_t^{\,*}|T|^{1-t}U^*U|T|^{1-t}\widetilde{T}_t\|} + \|T^*T + TT^*\| \\ &= w(\widetilde{T}_t^{\,2}) + \sqrt{\|T\|^{2t}\|\widetilde{T}_t^{\,*}|T|^{2-2t}\widetilde{T}_t\|} + \|T^*T + TT^*\| \\ &\leq w(\widetilde{T}_t^{\,2}) + \sqrt{\|T\|^{2t}\|\widetilde{T}_t\|^2\|T\|^{2-2t}} + \|T^*T + TT^*\| \\ &= w(\widetilde{T}_t^{\,2}) + \|T\|\|\widetilde{T}_t\| + \|T^*T + TT^*\|. \end{aligned}$$

Taking supremum over $\theta \in \mathbb{R}$ in the above inequality and then using Lemma 6.1, we get

$$w^2(T) \leq \frac{1}{4}w(\widetilde{T}_t^{\,2}) + \frac{1}{4}\|T\|\|\widetilde{T}_t\| + \frac{1}{4}\|T^*T + TT^*\|.$$

This holds for all $t \in [0,1]$, and so taking minimum we get,

$$w^2(T) \leq \min_{t\in[0,1]}\left(\frac{1}{4}w(\widetilde{T}_t^{\,2}) + \frac{1}{4}\|T\|\|\widetilde{T}_t\|\right) + \frac{1}{4}\|T^*T + TT^*\|.$$

Considering the case $t = \frac{1}{2}$, we get

$$w^2(T) \leq \frac{1}{4}w(\widetilde{T}^2) + \frac{1}{4}\|T\|\|\widetilde{T}\| + \frac{1}{4}\|T^*T + TT^*\|.$$

□

Remark 6.7. *We observe that* $\min_{t\in[0,1]}\left(\frac{1}{4}w(\widetilde{T}_t^{\,2})+\frac{1}{4}\|T\|\|\widetilde{T}_t\|\right)=0$ *if* $T^2=0$. *Also, as discussed in Remark 6.3, if* $T^2=0$ *or* T *is a normaloid operator then inequalities in Theorem 6.6 become equalities.*

Remark 6.8. *It is easy to observe that the inequality obtained by us in Theorem 6.6 is sharper than the inequality obtained in Theorem 6.2 and so it is sharper than inequality (1.2) obtained in [55, Th. 1]. Also if we take the same matrix* $T=\begin{pmatrix}0&2&0\\0&0&0\\0&0&1\end{pmatrix}$ *as in Remark 6.4 then Theorem 6.6 gives* $w(T)\leq\sqrt{\frac{7}{4}}$, *whereas (1.7) gives* $w(T)\leq\frac{3}{2}$. *Thus for this matrix, our inequality obtained in Theorem 6.6 is better than inequality (1.7) obtained by Abu-Omar and Kittaneh [6, Th. 3.2]. In fact, if we consider* $T=\begin{pmatrix}0&a&0\\0&0&0\\0&0&b\end{pmatrix}$ *where* $a,b\in\mathbb{C}$, *then we see that the bound in Theorem 6.6 is always less than or equal to the bound* (1.7) *given in [6, Th. 3.2].*

Now, by using Theorem 6.6 we obtain the following inequality for the numerical radius in terms of iterated t-Aluthge transform. For a non-negative integer n, we denote the nth iterated t-Aluthge transform $\widetilde{T}_{t_n}$, i.e., $\widetilde{T}_{t_n}=\widetilde{\widetilde{T}}_{t_{n-1}}$ and $\widetilde{T}_{t_0}=T$.

Theorem 6.9. *Let* $T\in\mathcal{B}(\mathcal{H})$. *Then*

$$w^2(T)\leq\sum_{n=1}^{\infty}\frac{1}{4^n}\left(\|\widetilde{T}_{t_{n-1}}\|\|\widetilde{T}_{t_n}\|+\|\widetilde{T}^*_{t_{n-1}}\widetilde{T}_{t_{n-1}}+\widetilde{T}_{t_{n-1}}\widetilde{T}^*_{t_{n-1}}\|\right),$$

for all $t\in[0,1]$.

Proof. By using Theorem 6.6 repeatedly, we get

$$\begin{aligned}
w^2(T) &\leq \frac{1}{4}\left(\|T\|\|\widetilde{T}_t\| + \|T^*T + TT^*\|\right) + \frac{1}{4}w(\widetilde{T}_t^2) \\
&\leq \frac{1}{4}\left(\|T\|\|\widetilde{T}_t\| + \|T^*T + TT^*\|\right) + \frac{1}{4}w^2(\widetilde{T}_t) \\
&\leq \frac{1}{4}\left(\|T\|\|\widetilde{T}_t\| + \|T^*T + TT^*\|\right) \\
&\quad + \frac{1}{4^2}\left(\|\widetilde{T}_t\|\|\widetilde{T}_{t_2}\| + \|\widetilde{T}_t^*\widetilde{T}_t + \widetilde{T}_t\widetilde{T}_t^*\|\right) + \frac{1}{4^2}w(\widetilde{T}_2^2) \\
&\leq \frac{1}{4}\left(\|T\|\|\widetilde{T}_t\| + \|T^*T + TT^*\|\right) \\
&\quad + \frac{1}{4^2}\left(\|\widetilde{T}_t\|\|\widetilde{T}_{t_2}\| + \|\widetilde{T}_t^*\widetilde{T}_t + \widetilde{T}_t\widetilde{T}_t^*\|\right) + \frac{1}{4^2}w^2(\widetilde{T}_2) \\
&\leq \frac{1}{4}\left(\|T\|\|\widetilde{T}_t\| + \|T^*T + TT^*\|\right) \\
&\quad + \frac{1}{4^2}\left(\|\widetilde{T}_t\|\|\widetilde{T}_{t_2}\| + \|\widetilde{T}_t^*\widetilde{T}_t + \widetilde{T}_t\widetilde{T}_t^*\|\right) \\
&\quad + \frac{1}{4^3}\left(\|\widetilde{T}_{t_2}\|\|\widetilde{T}_{t_3}\| + \|\widetilde{T}_{t_2}^*\widetilde{T}_{t_2} + \widetilde{T}_{t_2}\widetilde{T}_{t_2}^*\|\right) + \frac{1}{4^3}w(\widetilde{T}_3^2) \\
&\leq \ldots \\
&\leq \sum_{n=1}^{\infty}\frac{1}{4^n}\left(\|\widetilde{T}_{t_{n-1}}\|\|\widetilde{T}_{t_n}\| + \|\widetilde{T}_{t_{n-1}}^*\widetilde{T}_{t_{n-1}} + \widetilde{T}_{t_{n-1}}\widetilde{T}_{t_{n-1}}^*\|\right).
\end{aligned}$$

□

Now, based on Theorem 6.9, we obtain the following inequality.

Corollary 6.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w^2(T) \leq \frac{1}{2}\left[\|T^2\|^{\frac{1}{2}}\left(\frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}}\right) + \frac{1}{2}\|T^*T + TT^*\|\right].$$

Proof. Let $\widetilde{T}_n$ be the n-th iterated Aluthge transform. Then from Theorem 6.9 (*for* $t = \frac{1}{2}$), we get

$$
\begin{aligned}
w^2(T) &\leq \sum_{n=1}^{\infty} \frac{1}{4^n}\left(\|\widetilde{T}_{n-1}\|\|\widetilde{T}_n\| + \|\widetilde{T}_{n-1}^*\widetilde{T}_{n-1} + \widetilde{T}_{n-1}\widetilde{T}_{n-1}^*\|\right) \\
&= \frac{1}{4}\left(\|T\|\|\widetilde{T}\| + \|T^*T + TT^*\|\right) \\
&\qquad + \sum_{n=2}^{\infty} \frac{1}{4^n}\left(\|\widetilde{T}_{n-1}\|\|\widetilde{T}_n\| + \|\widetilde{T}_{n-1}^*\widetilde{T}_{n-1} + \widetilde{T}_{n-1}\widetilde{T}_{n-1}^*\|\right) \\
&\leq \frac{1}{4}\left(\|T\|\|\widetilde{T}\| + \|T^*T + TT^*\|\right) + \sum_{n=2}^{\infty} \frac{1}{4^n}\left(\|\widetilde{T}_{n-1}\|\|\widetilde{T}_n\| + 2\|\widetilde{T}_{n-1}\|^2\right) \\
&\leq \frac{1}{4}\left(\|T\|\|\widetilde{T}\| + \|T^*T + TT^*\|\right) + \sum_{n=2}^{\infty} \frac{1}{4^n}\left(3\|\widetilde{T}\|^2\right) \\
&\qquad \left(\textit{using } \ \|\widetilde{T}_n\| \leq \|\widetilde{T}_{n-1}\|, \quad n \geq 2\right) \\
&\leq \frac{1}{4}\left(\|T\|\|T^2\|^{\frac{1}{2}} + \|T^*T + TT^*\|\right) + \sum_{n=2}^{\infty} \frac{1}{4^n}\left(3\|T^2\|\right) \\
&\qquad \left(\textit{using } \|\widetilde{T}\| \leq \|T^2\|^{\frac{1}{2}}\right) \\
&= \frac{1}{4}\left(\|T\|\|T^2\|^{\frac{1}{2}} + \|T^*T + TT^*\|\right) + \frac{3}{4^2}\|T^2\|\sum_{n=0}^{\infty} \frac{1}{4^n} \\
&= \frac{1}{4}\left(\|T\|\|T^2\|^{\frac{1}{2}} + \|T^*T + TT^*\|\right) + \frac{1}{4}\|T^2\| \\
&= \frac{1}{2}\left[\|T^2\|^{\frac{1}{2}}\left(\frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}}\right) + \frac{1}{2}\|T^*T + TT^*\|\right].
\end{aligned}
$$

□

Remark 6.10. *The inequality in Corollary 6.1 is better than inequality (1.2), it follows from the fact that*

$$
\begin{aligned}
&\frac{1}{2}\left[\|T^2\|^{\frac{1}{2}}\left(\frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}}\right) + \frac{1}{2}\|T^*T + TT^*\|\right] \\
= &\ \frac{1}{4}\|T^2\|^{\frac{1}{2}}\|T\| + \frac{1}{4}\|T^2\| + \frac{1}{4}\|T^*T + TT^*\| \\
\leq &\ \frac{1}{4}\|T^2\|^{\frac{1}{2}}\|T\| + \frac{1}{4}\|T^2\| + \frac{1}{4}\|T^2\| + \frac{1}{4}\|T\|^2 \\
\leq &\ \frac{1}{2}\|T^2\|^{\frac{1}{2}}\|T\| + \frac{1}{4}\|T^2\| + \frac{1}{4}\|T\|^2 \\
= &\ \left(\frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}}\right)^2.
\end{aligned}
$$

We also observe that bound obtained in Corollary 6.1 is sharper than that in the right hand inequality of (1.3), if $\|T\|\|T^2\|^{\frac{1}{2}} + \|T^2\| \leq \|TT^* + T^*T\|$.

Next we obtain an upper bound for the numerical radius and give an example to show that this bound improves on bound (1.6).

Theorem 6.11. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w^4(T) \le \frac{1}{16} \min_{t\in[0,1]} \left(w(\widetilde{T_t}^2) + \|T\| \|\widetilde{T_t}\| \right)^2 + \frac{1}{8} w(T^2P + PT^2) + \frac{1}{16}\|P\|^2,$$

where $P = T^*T + TT^*$. In particular,

$$w^4(T) \le \frac{1}{16} \left(w(\widetilde{T}^2) + \|T\| \|\widetilde{T}\| \right)^2 + \frac{1}{8} w(T^2P + PT^2) + \frac{1}{16}\|P\|^2.$$

Proof. Since $H_\theta = \frac{1}{2}(e^{i\theta}T + e^{-i\theta}T^*)$ for all $\theta \in \mathbb{R}$, we have

$$\begin{aligned} 4{H_\theta}^2 &= e^{2i\theta}T^2 + e^{-2i\theta}T^{*2} + P \\ \Rightarrow 16{H_\theta}^4 &= \left(e^{2i\theta}T^2 + e^{-2i\theta}T^{*2}\right)^2 + 2\Re(e^{2i\theta}(T^2P + PT^2)) + P^2. \end{aligned}$$

Hence,

$$\begin{aligned} 16\|H_\theta\|^4 &\le \|e^{2i\theta}T^2 + e^{-2i\theta}T^{*2}\|^2 + 2\|\Re(e^{2i\theta}(T^2P + PT^2))\| + \|P\|^2 \\ &\le r^2\left(e^{2i\theta}T^2 + e^{-2i\theta}T^{*2}\right) + 2w(T^2P + PT^2) + \|P\|^2, \\ &\qquad \Big(r(S) = \|S\| \ \ \textit{for hermitian operator} \ \ S\Big) \\ &= r^2\left(e^{2i\theta}U|T|U|T| + e^{-2i\theta}|T|U^*|T|U^*\right) + 2w(T^2P + PT^2) + \|P\|^2. \end{aligned}$$

Then using the same technique as in Theorem 6.6, we get

$$\|H_\theta\|^4 \le \frac{1}{16}\left(w(\widetilde{T_t}^2) + \|T\| \|\widetilde{T_t}\|\right)^2 + \frac{1}{8} w(T^2P + PT^2) + \frac{1}{16}\|P\|^2.$$

Taking supremum over $\theta \in \mathbb{R}$ in the above inequality and then using Lemma 6.1, we get

$$w^4(T) \le \frac{1}{16}\left(w(\widetilde{T_t}^2) + \|T\| \|\widetilde{T_t}\|\right)^2 + \frac{1}{8} w(T^2P + PT^2) + \frac{1}{16}\|P\|^2.$$

This holds for all $t \in [0, 1]$, and so taking minimum we get,

$$w^4(T) \le \frac{1}{16} \min_{t\in[0,1]} \left(w(\widetilde{T_t}^2) + \|T\| \|\widetilde{T_t}\|\right)^2 + \frac{1}{8} w(T^2P + PT^2) + \frac{1}{16}\|P\|^2.$$

Considering the case $t = \frac{1}{2}$, we get

$$w^4(T) \le \frac{1}{16}\left(w(\widetilde{T}^2) + \|T\| \|\widetilde{T}\|\right)^2 + \frac{1}{8} w(T^2P + PT^2) + \frac{1}{16}\|P\|^2.$$

□

Remark 6.12. *We observe that as discussed in Remark 6.3, if $T^2 = 0$ or T is a normaloid operator then inequalities in Theorem 6.11 become equalities.*

Now, we give an example to show that the bound obtained in Theorem 6.11 improves on bound (1.6) obtained by Yamazaki in [75, Th. 2.1].

Example 6.13. *We consider $T = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$. Then it is easy to see that $P = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 9 \end{pmatrix}$, $|T| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and $U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, where U is the partial isometry in the polar decomposition of T, i.e., $T = U|T|$. So,*

$$\widetilde{T}_t = |T|^t U |T|^{1-t} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2^t 3^{1-t} \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $w(\widetilde{T}_t) = \frac{2^t 3^{1-t}}{2}$, $\|\widetilde{T}_t\| = 2^t 3^{1-t}$, $\|P\| = 13$ and $w(T^2P + PT^2) = 39$. So, the inequality obtained by us in Theorem 6.11 gives $w(T) \leq 2.05076838$. But inequality (1.6) obtained by Yamazaki in [75, Th. 2.1] gives $w(T) \leq 2.11237244$.

6.3 Bounds for the numerical radius of operators

Our aim in this section is to improve on both upper and lower bounds for the numerical radius of bounded operators, obtained by Kittaneh in [54, Th. 1], i.e.,

$$\frac{1}{4}\|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|.$$

Before doing so, we first give an alternative proof of the above inequalities.

Theorem 6.14. *[54, Th. 1] Let $T \in \mathcal{B}(\mathcal{H})$, then*

$$\frac{1}{4}\|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|.$$

Proof. Since $H_\theta = \frac{1}{2}(e^{i\theta}T + e^{-i\theta}T^*)$ and $K_\theta = \frac{1}{2i}(e^{i\theta}T - e^{-i\theta}T^*)$ for all $\theta \in \mathbb{R}$, we have $H_\theta^2 + K_\theta^2 = \frac{1}{2}(T^*T + TT^*)$ and so $\frac{1}{2}\|T^*T + TT^*\| = \|H_\theta^2 + K_\theta^2\| \leq \|H_\theta\|^2 + \|K_\theta\|^2 \leq 2w^2(T)$, using Lemma 6.1. Thus $\frac{1}{4}\|T^*T + TT^*\| \leq w^2(T)$. This completes the proof of the first inequality.
Again, from $H_\theta^2 + K_\theta^2 = \frac{1}{2}(T^*T + TT^*)$ we get, $H_\theta^2 - \frac{1}{2}(T^*T + TT^*) = -K_\theta^2 \leq 0$. Thus $H_\theta^2 \leq \frac{1}{2}(T^*T + TT^*)$ and so $\|H_\theta^2\| \leq \frac{1}{2}\|T^*T + TT^*\|$. Taking supremum over $\theta \in \mathbb{R}$ and then using Lemma 6.1, we get $w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|$.

□

Now, we prove the desired inequality which improves on inequality (1.3).

Theorem 6.15. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{4}c\big(\left(\Re(T^2)\right)^2\big) + \frac{1}{16}\|T^*T + TT^*\|^2 \leq w^4(T) \leq \frac{1}{2}w^2(T^2) + \frac{1}{8}\|T^*T + TT^*\|^2.$$

Proof. We first prove the left hand inequality. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Since $H_\theta = \frac{1}{2}(e^{i\theta}T + e^{-i\theta}T^*)$ and $K_\theta = \frac{1}{2i}(e^{i\theta}T - e^{-i\theta}T^*)$ for all $\theta \in \mathbb{R}$, we have

$$\begin{aligned}
\frac{1}{8}\left[4\left(\Re(e^{2i\theta}T^2)\right)^2 + (T^*T + TT^*)^2\right] &= H_\theta^4 + K_\theta^4 \\
\Rightarrow \frac{1}{2}\langle\left(\Re(e^{2i\theta}T^2)\right)^2 x, x\rangle + \frac{1}{8}\langle(T^*T + TT^*)^2 x, x\rangle &= \langle H_\theta^4 x, x\rangle + \langle K_\theta^4 x, x\rangle \\
\Rightarrow \frac{1}{2}\langle\left(\Re(e^{2i\theta}T^2)\right)^2 x, x\rangle + \frac{1}{8}\langle(T^*T + TT^*)^2 x, x\rangle &\leq 2w^4(T).
\end{aligned}$$

This inequality holds for all $\theta \in \mathbb{R}$. So taking $\theta = 0$, we infer that

$$\begin{aligned}
\frac{1}{2}\langle\left(\Re(T^2)\right)^2 x, x\rangle + \frac{1}{8}\langle(T^*T + TT^*)^2 x, x\rangle &\leq 2w^4(T) \\
\Rightarrow \frac{1}{2}c\big((\Re(T^2))^2\big) + \frac{1}{8}\langle(T^*T + TT^*)^2 x, x\rangle &\leq 2w^4(T).
\end{aligned}$$

Taking supremum over $x \in \mathcal{H}, \|x\| = 1$, we get

$$\frac{1}{2}c\big(\left(\Re(T^2)\right)^2\big) + \frac{1}{8}\|T^*T + TT^*\|^2 \leq 2w^4(T).$$

Thus,

$$\frac{1}{4}c\big(\left(\Re(T^2)\right)^2\big) + \frac{1}{16}\|T^*T + TT^*\|^2 \leq w^4(T).$$

This completes the proof of the left hand inequality. We next prove the right hand inequality.

As before, we have

$$H_\theta^4 + K_\theta^4 = \frac{1}{8}\left[4\left(\Re(e^{2i\theta}T^2)\right)^2 + (T^*T+TT^*)^2\right]$$

and so

$$\frac{1}{8}\left[4\left(\Re(e^{2i\theta}T^2)\right)^2 + (T^*T+TT^*)^2\right] - H_\theta^4 = K_\theta^4 \geq 0.$$

Hence,

$$H_\theta^4 \leq \frac{1}{8}\left[4\left(\Re(e^{2i\theta}T^2)\right)^2 + (T^*T+TT^*)^2\right].$$

Therefore,

$$\begin{aligned}\|H_\theta\|^4 &\leq \frac{1}{8}\left\|\left[4\left(\Re(e^{2i\theta}T^2)\right)^2 + (T^*T+TT^*)^2\right]\right\| \\ &\leq \frac{1}{8}\left[4\|\Re(e^{2i\theta}T^2)\|^2 + \|T^*T+TT^*\|^2\right] \\ &\leq \frac{1}{8}\left[4w^2(T^2) + \|T^*T+TT^*\|^2\right] \quad \left(\text{using Lemma 6.1}\right).\end{aligned}$$

Taking supremum over $\theta \in \mathbb{R}$ in the above inequality and then using Lemma 6.1, we get

$$w^4(T) \leq \frac{1}{2}w^2(T^2) + \frac{1}{8}\|T^*T+TT^*\|^2.$$

□

Remark 6.16. *Clearly the left hand inequality obtained in Theorem 6.15 is sharper than that of (1.3) obtained by Kittaneh in [54, Th. 1]. To claim the same for the right hand inequality we first note that $2\|T^2\| \leq \|T^*T+TT^*\|$ (see [57]). From the right hand inequality obtained in Theorem 6.15 we get,*

$$\begin{aligned}w^4(T) &\leq \frac{1}{2}w^2(T^2) + \frac{1}{8}\|T^*T+TT^*\|^2 \\ &\leq \frac{1}{2}\|T^2\|^2 + \frac{1}{8}\|T^*T+TT^*\|^2 \\ &= \frac{1}{8}(2\|T^2\|)^2 + \frac{1}{8}\|T^*T+TT^*\|^2 \\ &\leq \frac{1}{8}\|T^*T+TT^*\|^2 + \frac{1}{8}\|T^*T+TT^*\|^2 \\ &= \frac{1}{4}\|T^*T+TT^*\|^2.\end{aligned}$$

Thus, the right hand inequality in Theorem 6.15 is sharper than that of (1.3) obtained by Kit-

taneh [54, Th. 1].

Next, we concentrate our attention to the bounds that are, not comparable, in general. The following numerical examples will illustrate the incomparability of some of the upper bounds of the numerical radius.

Example 6.17.

(i) ***Incomparability of*** $\frac{1}{2}\left(\|T\|+\|T^2\|^{\frac{1}{2}}\right)$ ***and*** $\sqrt{\frac{1}{2}\|T^*T+TT^*\|}$. *Consider* $T=\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ *then* $\frac{1}{2}\left(\|T\|+\|T^2\|^{\frac{1}{2}}\right)=\frac{3+\sqrt{5}}{4}$, *whereas* $\sqrt{\frac{1}{2}\|T^*T+TT^*\|}=\sqrt{\frac{3}{2}}$. *Again if we consider* $T=\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ *then* $\frac{1}{2}\left(\|T\|+\|T^2\|^{\frac{1}{2}}\right)=\frac{2+\sqrt{2}}{2}$, *whereas* $\sqrt{\frac{1}{2}\|T^*T+TT^*\|}=\sqrt{3}$. *This shows that upper bounds in (1.2) and (1.3) are, not comparable, in general.*

(ii) ***Incomparability of*** $\left(\frac{1}{2}w^2(T^2)+\frac{1}{8}\|T^*T+TT^*\|^2\right)^{\frac{1}{4}}$ ***and*** $\frac{1}{2}\Big(\min_{t\in[0,1]} w(\widetilde{T}_t)+\|T\|\Big)$. *Consider* $T=\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *then* $\left(\frac{1}{2}w^2(T^2)+\frac{1}{8}\|T^*T+TT^*\|^2\right)^{\frac{1}{4}}=\sqrt{\sqrt{\frac{5}{2}}}$, *whereas* $\frac{1}{2}\left(\|T\|+\min_{t\in[0,1]} w(\widetilde{T}_t)\right)=\frac{3}{2}$.

Again, if we consider $T=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *then* $\left(\frac{1}{2}w^2(T^2)+\frac{1}{8}\|T^*T+TT^*\|^2\right)^{\frac{1}{4}}=\sqrt{\sqrt{\frac{1}{8}}}$, *whereas* $\frac{1}{2}\left(\|T\|+\min_{t\in[0,1]} w(\widetilde{T}_t)\right)=\frac{1}{2}$. *This shows that the upper bounds in (1.7) and Theorem 6.15 are, not comparable, in general.*

We observe that inequality (1.7) is sharper than (1.2) and the inequality obtained in Theorem 6.15 is sharper than (1.3). Similarly, using the same matrices one can conclude that upper bound in (1.3) is not comparable, in general, with the inequalities in (1.6) and (1.7).

CHAPTER 7

NUMERICAL RADIUS INEQUALITIES AND ITS APPLICATIONS IN ESTIMATION OF ZEROS OF POLYNOMIALS

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7.1 Introduction

In this chapter, we aim to develop an upper bound for the numerical radius of a bounded linear operator which improves on the existing upper bound in (1.5) , i.e., $w^2(A) \leq \frac{1}{4}\|A^*A + AA^*\| + \frac{1}{2}w(A^2)$. We obtain a lower bound for the numerical radius of a bounded linear operator which

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P. Bhunia, S. Bag, K. Paul; Numerical radius inequalities and its applications in estimation of zeros of polynomials, **Linear Algebra Appl.**, 573 (2019), 166–177. https://doi.org/10.1016/j.laa.2019.03.017
P. Bhunia, S. Bag, K. Paul; Numerical radius inequalities of operator matrices with applications, **Linear Multilinear Algebra**, 69 (2021), no. 9, 1635–1644. https://doi.org/10.1080/03081087.2019.1634673
P. Bhunia, S. Bag, K. Paul; Bounds for zeros of a polynomial using numerical radius of Hilbert space operators, **Ann. Funct. Anal.**, 12 (2021), no. 2, Paper No. 21, 14 pp. https://doi.org/10.1007/s43034-020-00107-4

improves on the existing lower bound (1.3), obtained in [54, Th. 1]. We also estimate the spectral radius of the sum of the product of n pairs of operators. Further, we present upper and lower bounds for the numerical radius of 2×2 operator matrices. As an application of the numerical radius inequalities of 2×2 operator matrices, we estimate bounds for the zeros of a monic polynomial with complex coefficients. First we introduce the following necessary notations and terminologies.

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle .,. \rangle$. Let $T \in \mathcal{B}(\mathcal{H})$ and $W(T)$, $w(T)$, $c(T)$, $\|T\|$ be the numerical range, numerical radius, Crawford number, operator norm of T, respectively, defined as follows:

$$\begin{aligned} W(T) &= \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}, \\ w(T) &= \sup\{|\lambda| : \lambda \in W(T)\}, \\ c(T) &= \inf\{|\lambda| : \lambda \in W(T)\}, \\ \|T\| &= \sup\{\|Tx\| : x \in \mathcal{H}, \|x\| = 1\}. \end{aligned}$$

It is well known that $w(\cdot)$ is a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\|\cdot\|$ and satisfies the inequality $\frac{1}{2}\|T\| \leq w(T) \leq \|T\|$. The first inequality becomes an equality if $T^2 = 0$ and the second inequality becomes an equality if T is normal. The bounded linear operator T can be represented as $T = \Re(T) + i\Im(T)$, the Cartesian decomposition, where $\Re(T)$ and $\Im(T)$ are the real part of T and the imaginary part of T, respectively, i.e., $\Re(T) = \frac{T+T^*}{2}$ and $\Im(T) = \frac{T-T^*}{2i}$, T^* denotes the adjoint of T. It is well known that $w(T) = \sup_{\theta \in \mathbb{R}} \|H_\theta\|$, where $H_\theta = \Re(e^{i\theta}T)$ (see in [75]). Let $r(T)$ be the spectral radius of T, i.e., $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$, where $\sigma(T)$ denotes the spectrum of T. Also it is well known that $\sigma(T) \subseteq \overline{W(T)}$, so $r(T) \leq w(T)$.

The direct sum of two copies of $\mathcal{H}$ is denoted by $\mathcal{H} \oplus \mathcal{H}$. If $A, B, C, D \in \mathcal{B}(\mathcal{H})$, then the operator matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ can be considered as an operator on $\mathcal{H} \oplus \mathcal{H}$, and is defined by $\begin{pmatrix} A & B \\ C & D \end{pmatrix} x = \begin{pmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{pmatrix}, \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H}$.

7.2 Upper bounds for the numerical radius of bounded operators

We begin this section with the following inequality.

Theorem 7.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w^4(T) \leq \frac{1}{4}w^2(T^2) + \frac{1}{8}w(T^2P + PT^2) + \frac{1}{16}\|P\|^2,$$

where $P = T^*T + TT^*$.

Proof. We know that $w(T) = \sup_{\theta\in\mathbb{R}} \|H_\theta\|$ where $H_\theta = \Re(e^{i\theta}T)$. Then,

$$\begin{aligned}
H_\theta &= \frac{1}{2}(e^{i\theta}T + e^{-i\theta}T^*) \\
\Rightarrow 4{H_\theta}^2 &= e^{2i\theta}T^2 + e^{-2i\theta}T^{*2} + P \\
\Rightarrow 16{H_\theta}^4 &= \big(e^{2i\theta}T^2 + e^{-2i\theta}T^{*2} + P\big)\big(e^{2i\theta}T^2 + e^{-2i\theta}T^{*2} + P\big) \\
&= \big(e^{2i\theta}T^2 + e^{-2i\theta}T^{*2}\big)^2 + \big(e^{2i\theta}T^2 + e^{-2i\theta}T^{*2}\big)P \\
&+ P\big(e^{2i\theta}T^2 + e^{-2i\theta}T^{*2}\big) + P^2 \\
&= 4\big(\Re(e^{2i\theta}T^2)\big)^2 + 2\Re(e^{2i\theta}(T^2P + PT^2)) + P^2 \\
\Rightarrow \|{H_\theta}^4\| &\leq \frac{1}{4}\big\|\Re(e^{2i\theta}T^2)\big\|^2 + \frac{1}{8}\big\|\Re(e^{2i\theta}(T^2P + PT^2))\big\| + \frac{1}{16}\|P\|^2.
\end{aligned}$$

Now taking the supremum over $\theta \in \mathbb{R}$ in the above inequality we get,

$$\Rightarrow w^4(T) \leq \frac{1}{4}w^2(T^2) + \frac{1}{8}w(T^2P + PT^2) + \frac{1}{16}\|P\|^2.$$

□

Remark 7.2. *It is easy to check that $w(T^2P + PT^2) \leq 2w(T^2)\|P\|$, (see [40]) and so the bound obtained in Theorem 7.1 improves on the bound (1.5) obtained by Abu-Omar and Kittaneh [1], namely,*

$$w^4(T) \leq \frac{1}{4}w^2(T^2) + \frac{1}{4}w(T^2)\|P\| + \frac{1}{16}\|P\|^2.$$

Abu-Omar and Kittaneh [1] also proved that this bound is better than the bounds (1.2) and (1.3) obtained in [55, 54]. The inequality (1.4) obtained by Dragomir [39], namely, $w^2(T) \leq \frac{1}{2}\left(w(T^2) + \|T\|^2\right)$, i.e., $w^4(T) \leq \frac{1}{4}w^2(T^2) + \frac{1}{2}w(T^2)\|T\|^2 + \frac{1}{4}\|T\|^4$ which is weaker than the bound (1.5). Thus the bound obtained in Theorem 7.1 improves on all the existing upper bounds in (1.2), (1.3), (1.4) and (1.5).

Next, we prove the following inequality.

Theorem 7.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w^3(T) \leq \frac{1}{4}w(T^3) + \frac{1}{4}w(T^2T^* + T^*T^2 + TT^*T).$$

Moreover,

if $T^2 = 0$, *then* $w(T) = \frac{1}{2}\sqrt{\|TT^* + T^*T\|}$, *and*
if $T^3 = 0$, *then* $w(T) = \left[\frac{1}{4}w(T^2T^* + T^*T^2 + TT^*T)\right]^{\frac{1}{3}}$.

Proof. We note that $w(T) = \sup_{\theta\in\mathbb{R}} \|H_\theta\|$ where $H_\theta = \Re(e^{i\theta}T)$. Then,

$$\begin{aligned}
H_\theta &= \frac{1}{2}(e^{i\theta}T + e^{-i\theta}T^*) \\
\Rightarrow 4{H_\theta}^2 &= e^{2i\theta}T^2 + e^{-2i\theta}T^{*2} + T^*T + TT^* \\
\Rightarrow 8H_\theta^3 &= \big(e^{2i\theta}T^2 + e^{-2i\theta}T^{*2} + T^*T + TT^*\big)(e^{i\theta}T + e^{-i\theta}T^*) \\
\Rightarrow H_\theta^3 &= \frac{1}{4}\Re(e^{3i\theta}T^3) + \frac{1}{4}\Re(e^{i\theta}(T^2T^* + T^*T^2 + TT^*T) \\
\Rightarrow \|H_\theta^3\| &\leq \frac{1}{4}\|\Re(e^{3i\theta}T^3)\| + \frac{1}{4}\|\Re(e^{i\theta}(T^2T^* + T^*T^2 + TT^*T)\|.
\end{aligned}$$

Taking the supremum over $\theta \in \mathbb{R}$ in the above inequality we have the desired inequality. If $T^2 = 0$ then $4{H_\theta}^2 = T^*T + TT^*$ and so $w(T) = \frac{1}{2}\sqrt{\|TT^* + T^*T\|}$. If $T^3 = 0$ then $H_\theta^3 = \frac{1}{4}\Re(e^{i\theta}(T^2T^* + T^*T^2 + TT^*T)$ and so $w^3(T) = \frac{1}{4}w(T^2T^* + T^*T^2 + TT^*T)$. □

Remark 7.4. *The inequality obtained in Theorem 7.3 gives a better bound for the numerical radius of the matrix* T *than the upper bound in* (1.5) *obtained in [1], where* $T = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. *In particular,* $w(T) \leq 1.863$ *if we follow the inequality obtained in Theorem 7.3, whereas* $w(T) \leq 1.989$ *if we follow the bound* (1.5).

Next, we prove the following inequality.

Theorem 7.5. *Let* $T \in \mathcal{B}(\mathcal{H})$. *Then for each* $r \geq 1$,

$$w^{2r}(T) \leq \frac{1}{2}w^r(T^2) + \frac{1}{4}\big\|(T^*T)^r + (TT^*)^r\big\|.$$

Proof. We note that $w(T) = \sup_{\theta\in\mathbb{R}} \|H_\theta\|$ where $H_\theta = \Re(e^{i\theta}T)$. Now,

$$\begin{aligned}
H_\theta &= \frac{1}{2}(e^{i\theta}T + e^{-i\theta}T^*) \\
\Rightarrow 4{H_\theta}^2 &= e^{2i\theta}T^2 + e^{-2i\theta}T^{*2} + T^*T + TT^* \\
\Rightarrow {H_\theta}^2 &= \frac{1}{2}\Re(e^{2i\theta}T^2) + \frac{1}{4}(T^*T + TT^*) \\
\Rightarrow \|{H_\theta}^2\| &\leq \frac{1}{2}\big\|\Re(e^{2i\theta}T^2)\big\| + \frac{1}{4}\big\|T^*T + TT^*\big\|
\end{aligned}$$

For $r \geq 1$, t^r and $t^{\frac{1}{r}}$ are convex and concave operator functions respectively and using that we get,

$$\begin{aligned}
\|{H_\theta}^2\|^r &\leq \left\{\frac{1}{2}\big\|\Re(e^{2i\theta}T^2)\big\| + \frac{1}{2}\left\|\frac{T^*T+TT^*}{2}\right\|\right\}^r \\
&\leq \frac{1}{2}\big\|\Re(e^{2i\theta}T^2)\big\|^r + \frac{1}{2}\left\|\frac{T^*T+TT^*}{2}\right\|^r \\
&\leq \frac{1}{2}\big\|\Re(e^{2i\theta}T^2)\big\|^r + \frac{1}{2}\left\|\left(\frac{(T^*T)^r+(TT^*)^r}{2}\right)^{\frac{1}{r}}\right\|^r \\
&= \frac{1}{2}\big\|\Re(e^{2i\theta}T^2)\big\|^r + \frac{1}{2}\left\|\frac{(T^*T)^r+(TT^*)^r}{2}\right\|.
\end{aligned}$$

Now taking the supremum over $\theta \in \mathbb{R}$ in the above inequality we get,

$$w^{2r}(T) \leq \frac{1}{2}w^r(T^2) + \frac{1}{4}\big\|(T^*T)^r + (TT^*)^r\big\|.$$

□

Remark 7.6. *For $A, B \in \mathcal{B}(\mathcal{H})$, Sattari et. al. [71] proved that $w^r(B^*A) \leq \frac{1}{4}\|(AA^*)^r + (BB^*)^r\| + \frac{1}{2}w^r(AB^*)$. When $A = B^*$ then $w^r(A^2) \leq \frac{1}{4}\|(AA^*)^r + (A^*A)^r\| + \frac{1}{2}w^r(A^2)$. Thus for the case $A = B^*$ our bound obtained in theorem 7.5 is better than the bound obtained by Sattari et. al. [71].*

Next we give another upper bound for the numerical radius $w(T)$ in terms of $\|H_\phi\|$.

Theorem 7.7. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w(T) \leq \inf_{\phi \in \mathbb{R}} \sqrt{\|H_\phi\|^2 + \|H_{\phi+\frac{\pi}{2}}\|^2}$$

where $H_\phi = \Re(e^{i\phi}T)$.

Proof. We have, $H_\theta = \Re(e^{i\theta}T) = \cos\theta\Re(T) - \sin\theta\Im(T)$. Then for $\phi \in [0, 2\pi]$, we get

$$\begin{aligned}
H_{\theta+\phi} &= \cos(\theta+\phi)\Re(T) - \sin(\theta+\phi)\Im(T) \\
&= \cos\theta[\cos\phi\Re(T) - \sin\phi\Im(T)] - \sin\theta[\sin\phi\Re(T) + \cos\phi\Im(T)] \\
&= \cos\theta[\cos\phi\Re(T) - \sin\phi\Im(T)] - \sin\theta[-\cos(\phi+\frac{\pi}{2})\Re(T) \\
&\quad + \sin(\phi+\frac{\pi}{2})\Im(T)] \\
&= \cos\theta\Re(e^{i\phi}T) + \sin\theta\Re(e^{i(\phi+\frac{\pi}{2})}T) \\
&= H_\phi\cos\theta + H_{\phi+\frac{\pi}{2}}\sin\theta \\
\Rightarrow \|H_{\theta+\phi}\| &\leq \|H_\phi\cos\theta\| + \|H_{\phi+\frac{\pi}{2}}\sin\theta\| \\
\Rightarrow \|H_{\theta+\phi}\| &\leq \sqrt{\|H_\phi\|^2 + \|H_{\phi+\frac{\pi}{2}}\|^2}.
\end{aligned}$$

Taking supremum over $\theta \in \mathbb{R}$ in the above inequality, we get

$$w(T) \leq \sqrt{\|H_\phi\|^2 + \|H_{\phi+\frac{\pi}{2}}\|^2}.$$

This is true for any $\phi \in \mathbb{R}$ and so we get,

$$w(T) \leq \inf_{\phi\in\mathbb{R}} \sqrt{\|H_\phi\|^2 + \|H_{\phi+\frac{\pi}{2}}\|^2}.$$

□

Remark 7.8. *Noting that for $\phi = 0, \|H_\phi\| = \|\Re(T)\|$ and $\|H_{\phi+\pi/2}\| = \|\Im(T)\|$, it follows from Theorem 7.7 that $w(T) \leq \sqrt{\|\Re(T)\|^2 + \|\Im(T)\|^2}$. Also, this inequality follows directly from the definition of the numerical radius by considering the Cartesian decomposition of T.*

Next we give an upper bound for the numerical radius of $n \times n$ operator matrices which follows from [3, Theorem 2 and Remark 1].

Theorem 7.9. *Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ be Hilbert spaces and $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$. If $A = (A_{ij})$ be an $n \times n$ operator matrix acting on $\mathcal{H}$ with $A_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$, then*

$$w(A) \leq \max_{1\leq i\leq n} \left\{ w(A_{ii}) + \frac{1}{2}\sum_{j=1, j\neq i}^{n} (\|A_{ij}\| + \|A_{ji}\|) \right\}.$$

By using Theorem 7.9 we can estimate the spectral radius of the sum of the product of n pairs of operators as follows.

Theorem 7.10. *Let $A_i, B_i \in \mathcal{B}(\mathcal{H})$. The spectral radius of $\sum_{i=1}^n A_iB_i$ satisfies the following inequality*

$$r\left(\sum_{i=1}^n A_iB_i\right) \le \max_{1\le i\le n}\left\{w(B_iA_i) + \frac{1}{2}\sum_{j=1, j\neq i}^n (\|B_iA_j\| + \|B_jA_i\|)\right\}.$$

Proof. We have

$$\begin{aligned}
r\left(\sum_{i=1}^n A_iB_i\right) &= r\left(\begin{pmatrix} \sum_{i=1}^n A_iB_i & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & . & . & . & 0 \end{pmatrix}\right) \\
&= r\left(\begin{pmatrix} A_1 & A_2 & . & . & . & A_n \\ 0 & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & . & . & . & 0 \end{pmatrix}\begin{pmatrix} B_1 & 0 & . & . & . & 0 \\ B_2 & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ B_n & 0 & . & . & . & 0 \end{pmatrix}\right) \\
&= r\left(\begin{pmatrix} B_1 & 0 & . & . & . & 0 \\ B_2 & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ B_n & 0 & . & . & . & 0 \end{pmatrix}\begin{pmatrix} A_1 & A_2 & . & . & . & A_n \\ 0 & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & . & . & . & 0 \end{pmatrix}\right) \\
&= r\left(\begin{pmatrix} B_1A_1 & B_1A_2 & . & . & . & B_1A_n \\ B_2A_1 & B_2A_2 & . & . & . & B_2A_n \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ B_nA_1 & B_nA_2 & . & . & . & B_nA_n \end{pmatrix}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad w\left(\begin{pmatrix} B_1A_1 & B_1A_2 & . & . & . & B_1A_n \\ B_2A_1 & B_2A_2 & . & . & . & B_2A_n \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ B_nA_1 & B_nA_2 & . & . & . & B_nA_n \end{pmatrix}\right) \\
&\leq \quad \max_{1\leq i\leq n}\left\{ w(B_iA_i) + \frac{1}{2}\sum_{j=1, j\neq i}^{n} (\|B_iA_j\| + \|B_jA_i\|)\right\}.
\end{aligned}
$$

□

7.3 Lower bounds for the numerical radius of bounded operators

We begin this section with the following inequality on lower bound of numerical radius.

Theorem 7.11. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$w^4(T) \geq \frac{1}{4}C^2(T^2) + \frac{1}{8}c(T^2P + PT^2) + \frac{1}{16}\|P\|^2,$$

where $P = T^*T + TT^*, C(T) = \inf_{x\in\mathcal{H}, \|x\|=1} \inf_{\phi\in\mathbb{R}} \|\Re(e^{i\phi}T)x\|$.

Proof. We know that $w(T) = \sup_{\phi\in\mathbb{R}} \|H_\phi\|$ where $H_\phi = \Re(e^{i\phi}T)$. Let x be a unit vector in H and let θ be a real number such that $e^{2i\theta}\langle(T^2P + PT^2)x, x\rangle = |\langle(T^2P + PT^2)x, x\rangle|$. Then

$$
\begin{aligned}
H_\theta &= \frac{1}{2}(e^{i\theta}T + e^{-i\theta}T^*) \\
\Rightarrow 4{H_\theta}^2 &= e^{2i\theta}T^2 + e^{-2i\theta}T^{*2} + P \\
\Rightarrow 16{H_\theta}^4 &= \big(e^{2i\theta}T^2 + e^{-2i\theta}T^{*2} + P\big)\big(e^{2i\theta}T^2 + e^{-2i\theta}T^{*2} + P\big) \\
&= \big(e^{2i\theta}T^2 + e^{-2i\theta}T^{*2}\big)^2 + \big(e^{2i\theta}T^2 + e^{-2i\theta}T^{*2}\big)P \\
&\quad +P\big(e^{2i\theta}T^2 + e^{-2i\theta}T^{*2}\big) + P^2 \\
&= 4\big(\Re(e^{2i\theta}T^2)\big)^2 + 2\Re(e^{2i\theta}(T^2P + PT^2)) + P^2
\end{aligned}
$$

$$\begin{aligned}
\Rightarrow 16w^4(T) &\geq \|4\big(\Re(e^{2i\theta}T^2)\big)^2 + 2\Re(e^{2i\theta}(T^2P + PT^2)) + P^2\| \\
&\geq |\langle \big(4\big(\Re(e^{2i\theta}T^2)\big)^2 + 2\Re(e^{2i\theta}(T^2P + PT^2)) + P^2\big)x, x\rangle| \\
&= |\, 4\langle \big(\Re(e^{2i\theta}T^2)\big)^2 x, x\rangle + 2\Re(e^{2i\theta}\langle (T^2P + PT^2)x, x\rangle) + \langle P^2x, x\rangle| \\
&= 4\|\big(\Re(e^{2i\theta}T^2)\big)x\|^2 + 2|\langle (T^2P + PT^2)x, x\rangle| + \|Px\|^2 \\
&\geq 4C^2(T^2) + 2c(T^2P + PT^2) + \|Px\|^2 \\
\Rightarrow 16w^4(T) &\geq 4C^2(T^2) + 2c(T^2P + PT^2) + \sup_{\|x\|=1} \|Px\|^2 \\
&= 4C^2(T^2) + 2c(T^2P + PT^2) + \|P\|^2 \\
\Rightarrow w^4(T) &\geq \frac{1}{4}C^2(T^2) + \frac{1}{8}c(T^2P + PT^2) + \frac{1}{16}\|P\|^2.
\end{aligned}$$

This completes the proof. □

Remark 7.12. *Kittaneh[54, Th. 1] proved that* $w^2(T) \geq \frac{1}{4}\|T^*T + TT^*\| = \frac{1}{4}\|P\|$, *which easily follows from Theorem 7.11.*

Next, we prove the following inequalities involving $\Re(T)$ and $\Im(T)$.

Theorem 7.13. *Let* $T \in \mathcal{B}(\mathcal{H})$. *Then*

$$w(T) \geq \sqrt{\|\Re(T)\|^2 + c^2(\Im(T))} \text{ and } w(T) \geq \sqrt{\|\Im(T)\|^2 + c^2(\Re(T))}.$$

Proof. First we assume $\|\Re(T)\| = |\lambda|$. Therefore, there exists a sequence $\{x_n\}$ in $\mathcal{H}$ with $\|x_n\| = 1$ such that $\langle \Re(T)x_n, x_n\rangle \to \lambda$. Now

$$\begin{aligned}
\langle Tx_n, x_n\rangle &= \langle (\Re(T) + i\Im(T))x_n, x_n\rangle \\
\Rightarrow \langle Tx_n, x_n\rangle &= \langle \Re(T)x_n, x_n\rangle + i\langle \Im(T)x_n, x_n\rangle \\
\Rightarrow |\langle Tx_n, x_n\rangle|^2 &= (\langle \Re(T)x_n, x_n\rangle)^2 + (\langle \Im(T)x_n, x_n\rangle)^2 \\
\Rightarrow |\langle Tx_n, x_n\rangle|^2 &\geq (\langle \Re(T)x_n, x_n\rangle)^2 + m^2(\Im(T)) \\
\Rightarrow w^2(T) &\geq \lambda^2 + c^2(\Im(T)) \\
\Rightarrow w(T) &\geq \sqrt{\|\Re(T)\|^2 + c^2(\Im(T))}.
\end{aligned}$$

The proof of other inequality follows in the same way. □

Note that if $\Re(T)$ and $\Im(T)$ are unitarily equivalent to scalar operators then $\|\Re(T)\| = c(\Re(T))$ and $\|\Im(T)\| = c(\Im(T))$ respectively. Therefore from Remark 7.8 and Theorem 7.13 we get the following equality.

Corollary 7.1. *Let $T \in \mathcal{B}(\mathcal{H})$. If either $\Re(T)$ or $\Im(T)$ is unitarily equivalent to a scalar operator, then* $w(T) = \sqrt{\|\Re(T)\|^2 + \|\Im(T)\|^2}$.

Remark 7.14. *For $T \in \mathcal{B}(\mathcal{H})$, Kittaneh et. al. [53] proved that $w(T) \geq \|\Re(T)\|$ and $w(T) \geq \|\Im(T)\|$. For any bounded linear operators these bounds are weaker than the bounds obtained in Theorem 7.13.*

7.4 Bounds for the numerical radius of 2×2 operator matrices

We begin this section with the following lemmas which are used to reach our goal in this present section.

Lemma 7.1 ([48])**.** *Let $X \in \mathcal{B}(\mathcal{H}_1), Y \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1), Z \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $W \in \mathcal{B}(\mathcal{H}_2)$. Then the following results hold:*

(i) $w\begin{pmatrix} X & 0 \\ 0 & W \end{pmatrix} = \max\{w(X), w(W)\}$.

(ii) $w\begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} = w\begin{pmatrix} 0 & Z \\ Y & 0 \end{pmatrix}$.

(iii) $w\begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} = \sup_{\theta \in \mathbb{R}} \frac{1}{2}\|e^{i\theta}Y + e^{-i\theta}Z^*\|$.

(iv) If $\mathcal{H}_1 = \mathcal{H}_2$, *then* $w\begin{pmatrix} 0 & Y \\ Y & 0 \end{pmatrix} = w(Y)$.

Lemma 7.2 ([40])**.** *Let $C, T \in \mathcal{B}(\mathcal{H})$. Then $w(TC + C^*T) \leq 2w(T)$, where C is any contraction ($i.e., \|C\| \leq 1$).*

Now, we are ready to prove the following inequality for the numerical radius of 2×2 operator matrices which improves on the existing inequalities.

Theorem 7.15. *Let $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1), Y \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then*

$$w^4\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \leq \frac{1}{16}\|S\|^2 + \frac{1}{4}w^2(YX) + \frac{1}{8}w(YXS + SYX),$$

where $S = |X|^2 + |Y^*|^2$.

Proof. Let $f(\theta) = \frac{1}{2}\|e^{i\theta}X + e^{-i\theta}Y^*\|$. Therefore,

$$\begin{aligned}
f(\theta) &= \frac{1}{2}\|(e^{i\theta}X + e^{-i\theta}Y^*)^*(e^{i\theta}X + e^{-i\theta}Y^*)\|^{\frac{1}{2}} \\
&= \frac{1}{2}\|(e^{-i\theta}X^* + e^{i\theta}Y)(e^{i\theta}X + e^{-i\theta}Y^*)\|^{\frac{1}{2}} \\
&= \frac{1}{2}\|S + 2\Re(e^{2i\theta}YX)\|^{\frac{1}{2}} \\
&= \frac{1}{2}\|(S + 2\Re(e^{2i\theta}YX))^2\|^{\frac{1}{4}} \\
&= \frac{1}{2}\|S^2 + 4(\Re(e^{2i\theta}YX))^2 + 2\Re(e^{2i\theta}(YXS + SYX))\|^{\frac{1}{4}} \\
\Rightarrow f^4(\theta) &\leq \frac{1}{16}\|S\|^2 + \frac{1}{4}\|\Re(e^{2i\theta}YX)\|^2 + \frac{1}{8}\|\Re(e^{2i\theta}(YXS + SYX))\|.
\end{aligned}$$

Now taking supremum over $\theta \in \mathbb{R}$ in the above inequality and then from Lemma 7.1 (iii) we get,

$$w^4\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \leq \frac{1}{16}\|S\|^2 + \frac{1}{4}w^2(YX) + \frac{1}{8}w(YXS + SYX).$$

This completes the proof. □

Now using Lemma 7.1 (ii) and Theorem 7.15 we get the following inequality.

Corollary 7.2. *Let $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1), Y \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then*

$$w^4\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \leq \frac{1}{16}\|P\|^2 + \frac{1}{4}w^2(XY) + \frac{1}{8}w(XYP + PXY),$$

where $P = |X^*|^2 + |Y|^2$.

Again using Lemma 7.1 (i) and Theorem 7.15 we get the following inequality.

Corollary 7.3. *Let $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1), Y \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then*

$$w(XY) \leq \frac{1}{4}\sqrt{\|S\|^2 + 4w^2(YX) + 2w(YXS + SYX)}$$

where $S = |X|^2 + |Y^*|^2$.

Proof. We have,

$$\begin{aligned}
w(XY) &\leq \max\{w(XY), w(YX)\} \\
&= w\begin{pmatrix} XY & 0 \\ 0 & YX \end{pmatrix} \\
&= w\left(\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}^2\right) \\
&\leq w^2\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \\
&\leq \frac{1}{4}\sqrt{\|S\|^2 + 4w^2(YX) + 2w(YXS + SYX)}.
\end{aligned}$$

□

Remark 7.16. *Using Lemma 7.2, it is easy to observe that the bound obtained in Theorem 7.15 is better than the second inequality in [3, Th. 3].*

Remark 7.17. *Here we note that when $\mathcal{H}_1 = \mathcal{H}_2$ and $Y = X$ then it follows from Theorem 7.15 and Lemma 7.1 (iv) that $w^4(X) \leq \frac{1}{16}\|R\|^2 + \frac{1}{4}w^2(X^2) + \frac{1}{8}w(X^2R + RX^2)$, where $R = |X|^2 + |X^*|^2$. This inequality also obtained in Theorem 7.1, i.e., [33, Th. 2.1].*

Next we prove a lower bound for the numerical radius of 2×2 operator matrices.

Theorem 7.18. *Let $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1), Y \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then*

$$w^4\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \geq \frac{1}{16}\|S\|^2 + \frac{1}{4}C^2(YX) + \frac{1}{8}c(YXS + SYX),$$

where $S = |X|^2 + |Y^*|^2$, $C(YX) = \inf_{\theta \in \mathbb{R}} \inf_{x \in \mathcal{H}_2, \|x\|=1} \|\Re(e^{i\theta}YX)x\|$.

Proof. Let $x \in \mathcal{H}_2$ with $\|x\| = 1$ and θ be a real number such that $e^{2i\theta}\langle(YXS + SYX)x, x\rangle =$

$|\langle (YXS+SYX)x,x\rangle|$. Then from Lemma 7.1 (iii) we get,

$$\begin{aligned}
w\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} &\geq \frac{1}{2}\|e^{i\theta}X+e^{-i\theta}Y^*\| \\
&\geq \frac{1}{2}\|(e^{i\theta}X+e^{-i\theta}Y^*)^*(e^{i\theta}X+e^{-i\theta}Y^*)\|^{\frac{1}{2}} \\
&\geq \frac{1}{2}\|(e^{-i\theta}X^*+e^{i\theta}Y)(e^{i\theta}X+e^{-i\theta}Y^*)\|^{\frac{1}{2}} \\
&\geq \frac{1}{2}\|S+2\Re(e^{2i\theta}YX)\|^{\frac{1}{2}} \\
&\geq \frac{1}{2}\|(S+2\Re(e^{2i\theta}YX))^2\|^{\frac{1}{4}} \\
&\geq \frac{1}{2}\|S^2+4(\Re(e^{2i\theta}YX))^2+2\Re(e^{2i\theta}(YXS+SYX))\|^{\frac{1}{4}} \\
&\geq \frac{1}{2}|\langle \big(S^2+4(\Re(e^{2i\theta}YX))^2+2\Re(e^{2i\theta}(YXS+SYX))\big)x,x\rangle|^{\frac{1}{4}} \\
&\geq \frac{1}{2}|\langle S^2x,x\rangle+4\langle (\Re(e^{2i\theta}YX))^2x,x\rangle+2\Re\langle (e^{2i\theta}(YXS+SYX))x,x\rangle|^{\frac{1}{4}} \\
&= \frac{1}{2}\big[\|Sx\|^2+4\|\Re(e^{2i\theta}YX)x\|^2+2|\langle (YXS+SYX)x,x\rangle\big]^{\frac{1}{4}} \\
&\geq \frac{1}{2}\big[\|Sx\|^2+4C^2(YX)+2c(YXS+SYX)\big]^{\frac{1}{4}}.
\end{aligned}$$

Now taking supremum over $x\in\mathcal{H}_2$ with $\|x\|=1$ in the above inequality we get,

$$w^4\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \geq \frac{1}{16}\|S\|^2+\frac{1}{4}C^2(YX)+\frac{1}{8}c(YXS+SYX).$$

This completes the proof. □

Now using Lemma 7.1(ii) and Theorem 7.18 we get the following inequality.

Corollary 7.4. *Let $X\in\mathcal{B}(\mathcal{H}_2,\mathcal{H}_1), Y\in\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$. Then*

$$w^4\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \geq \frac{1}{16}\|P\|^2+\frac{1}{4}C^2(XY)+\frac{1}{8}c(XYP+PXY),$$

where $P=|X^*|^2+|Y|^2$.

Remark 7.19. *Here we note that when $\mathcal{H}_1=\mathcal{H}_2$ and $Y=X$ then it follows from Theorem 7.18 and Lemma 7.1 (iv) that $w^4(X)\geq\frac{1}{16}\|R\|^2+\frac{1}{4}c^2(X^2)+\frac{1}{8}m(X^2R+RX^2)$, where $R=|X|^2+|X^*|^2$. Also, this inequality obtained in 7.11, i.e., [33, Th. 3.1].*

Next we state the following lemma which can be found in [17, p. 107].

Lemma 7.3. *Let $X, Y, Z, W \in \mathcal{B}(\mathcal{H})$. Then*

$$w\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \geq w\begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix}$$

and

$$w\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \geq w\begin{pmatrix} X & 0 \\ 0 & W \end{pmatrix}.$$

Now we are ready to prove an upper bound and a lower bound for the numerical radius of an operator matrix $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, where $X, Y, Z, W \in \mathcal{B}(\mathcal{H})$.

Corollary 7.5. *Let $X, Y, Z, W \in \mathcal{B}(\mathcal{H})$. Then*

$$w\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \leq \max\{w(X), w(W)\} + \left[\frac{1}{16}\|S\|^2 + \frac{1}{4}w^2(ZY) + \frac{1}{8}w(ZYS + SZY)\right]^{\frac{1}{4}}$$

and

$$w\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \geq \max\left\{w(X), w(W), \left[\frac{1}{16}\|S\|^2 + \frac{1}{4}C^2(ZY) + \frac{1}{8}c(ZYS + SZY)\right]^{\frac{1}{4}}\right\},$$

where $S = |Y|^2 + |Z^*|^2$, $C(ZY) = \inf_{\theta \in \mathbb{R}} \inf_{x \in \mathcal{H}, \|x\|=1} \|\Re(e^{i\theta} ZY)x\|$.

Proof. The proof follows easily from Theorem 7.15, Theorem 7.18 and Lemma 7.3. □

Now, we prove the following theorem.

Theorem 7.20. *Let $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, $Y \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then*

$$w^2\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \geq \frac{1}{4}\max\left\{\|XX^* + Y^*Y\|, \|X^*X + YY^*\|\right\},$$

$$w^2\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \leq \frac{1}{2}\max\left\{\|XX^* + Y^*Y\|, \|X^*X + YY^*\|\right\}.$$

Proof. Let $T = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$ and $H_\theta = \Re(e^{i\theta}T)$, $K_\theta = \Im(e^{i\theta}T)$. An easy calculation gives

$$H_\theta^2 + K_\theta^2 = \frac{1}{2}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where $A = XX^* + Y^*Y$, $B = X^*X + YY^*$. Therefore,

$$\frac{1}{2}\left\| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\| = \|H_\theta^2 + K_\theta^2\| \leq \|H_\theta\|^2 + \|K_\theta\|^2 \leq 2w^2(T).$$

This shows that

$$\frac{1}{2}\max\left\{\|A\|, \|B\|\right\} \leq 2w^2(T).$$

This completes the proof of the first inequality of the theorem.

Again, from $H_\theta^2 + K_\theta^2 = \frac{1}{2}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, we have

$$H_\theta^2 - \frac{1}{2}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = -K_\theta^2 \leq 0.$$

Therefore,

$$H_\theta^2 \leq \frac{1}{2}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

and so,

$$\|H_\theta\|^2 \leq \frac{1}{2}\left\| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\| = \frac{1}{2}\max\left\{\|A\|, \|B\|\right\}.$$

Taking supremum over $\theta \in \mathbb{R}$, we get

$$w^2(T) \leq \frac{1}{2}\max\left\{\|A\|, \|B\|\right\}.$$

This completes the proof of the second inequality of the theorem. □

Corollary 7.6. *Let $X, Y, Z, W \in \mathcal{B}(\mathcal{H})$. Then*

$$w\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \leq \max\Big\{w(X), w(W)\Big\} + \left(\frac{1}{2}\max\Big\{\|YY^* + Z^*Z\|, \|Y^*Y + ZZ^*\|\Big\}\right)^{\frac{1}{2}},$$

$$w\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \geq \max\left\{w(X), w(W), \left(\frac{1}{4}\max\Big\{\|YY^* + Z^*Z\|, \|Y^*Y + ZZ^*\|\Big\}\right)^{\frac{1}{2}}\right\}.$$

Proof. The proof follows easily from Theorem 7.20 and Lemma 7.3. □

Remark 7.21. *We would like to remark that the first inequality of Corollary 7.6 is valid even if we consider $X \in \mathcal{B}(\mathcal{H}_1), Y \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1), Z \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2), W \in \mathcal{B}(\mathcal{H}_2)$.*

7.5 Application to estimate bounds for the zeros of polynomials

We consider a monic polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ of degree n, with complex coefficients $a_0, a_1, \ldots, a_{n-1}$. When n varies from 1 to 4, we can exactly compute the zeros of the polynomial $p(z)$. But for $n \geq 5$, there is no general method to compute the zeros of the polynomial $p(z)$ and for this reason the estimation of bounds for the zeros of polynomials becomes more interesting. One of the important technique to obtain bounds for the zeros of the polynomial $p(z)$ is to obtain bounds for the numerical radius of the Frobenius companion matrix $C(p)$ of $p(z)$, where

$$C(p) = \begin{pmatrix} -a_{n-1} & -a_{n-2} & . & . & . & -a_1 & -a_0 \\ 1 & 0 & . & . & . & 0 & 0 \\ 0 & 1 & . & . & . & 0 & 0 \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ 0 & 0 & . & . & . & 1 & 0 \end{pmatrix}_{n,n}.$$

It is well-know that the zeros of the polynomial $p(z)$ are exactly the eigenvalues of $C(p)$. Therefore, if λ is a zero of the polynomial $p(z)$, then $|\lambda| \leq w(C(p))$.

Many eminent mathematicians, over the years, have estimated the zeros of the polynomial, some of them are mentioned below. Let λ be a zero of the polynomial $p(z)$.

(1) Cauchy [49] proved that

$$|\lambda| \leq 1 + \max\Big\{|a_0|, |a_1|, \ldots, |a_{n-1}|\Big\}.$$

(2) Carmichael and Mason [49] proved that

$$|\lambda| \leq \Big(1 + |a_0|^2 + |a_1|^2 + \ldots + |a_{n-1}|^2\Big)^{\frac{1}{2}}.$$

(3) Montel [49] proved that

$$|\lambda| \leq \max\Big\{1, |a_0| + |a_1| + \ldots + |a_{n-1}|\Big\}.$$

(4) Fujii and Kubo [41] proved that

$$|\lambda| \leq \cos\frac{\pi}{n+1} + \frac{1}{2}\left[\left(\sum_{j=0}^{n-1}|a_j|^2\right)^{\frac{1}{2}} + |a_{n-1}|\right].$$

(5) Alpin et. al. [9] proved that

$$|\lambda| \leq \max_{1\leq k\leq n}\Big[(1+|a_{n-1}|)(1+|a_{n-2}|)\ldots(1+|a_{n-k}|)\Big]^{\frac{1}{k}}.$$

(6) Paul and Bag [65] proved that

$$|\lambda| \leq \frac{1}{2}\left[w(A) + \cos\frac{\pi}{n-1} + \sqrt{\left(w(A) - \cos\frac{\pi}{n-1}\right)^2 + \left(1 + \sqrt{\sum_{k=3}^{n}|a_{n-k}|^2}\right)^2}\right],$$

where $A = \begin{pmatrix} -a_{n-1} & -a_{n-2} \\ 1 & 0 \end{pmatrix}$.

(7) Abu-Omar and Kittaneh [5] proved that

$$|\lambda| \leq \frac{1}{2}\left[\frac{1}{2}(|a_{n-1}|+\alpha) + \cos\frac{\pi}{n+1} + \sqrt{\left(\frac{1}{2}(|a_{n-1}|+\alpha) - \cos\frac{\pi}{n+1}\right)^2 + 4\alpha'}\right],$$

where $\alpha = \sqrt{\sum_{j=0}^{n-1}|a_j|^2}$ and $\alpha' = \sqrt{\sum_{j=0}^{n-2}|a_j|^2}$.

(8) M. Al-Dolat et. al. [8] proved that

$$\lambda \leq \max\left\{w(A), \cos\frac{\pi}{n+1}\right\} + \frac{1}{2}\left(1 + \sqrt{\sum_{j=0}^{n-3}|a_j|^2}\right),$$

where $A = \begin{pmatrix} -a_{n-1} & -a_{n-2} \\ 1 & 0 \end{pmatrix}$.

To obtain our desired bounds we first we need the following lemma.

Lemma 7.4. *[43, pp. 8–9] If* $D_n = \begin{pmatrix} 0 & 0 & . & . & . & 0 \\ 1 & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & . & . & 1 & 0 \end{pmatrix}_{n,n}$ *, then* $w(D_n) = \cos\frac{\pi}{n+1}$.

Now, we prove the following theorem.

Theorem 7.22. *Let* λ *be any zero of* $p(z)$*. Then*

$$|\lambda| \quad \leq \quad \left|\frac{a_{n-1}}{n}\right| + \cos\frac{\pi}{n} + \frac{1}{2}\left[(1+\alpha)^2 + 4\alpha + 4\sqrt{\alpha}(1+\alpha)\right]^{\frac{1}{4}},$$

where

$$\alpha_r \quad = \quad \sum_{k=r}^{n} {}^kC_r\big(-\frac{a_{n-1}}{n}\big)^{k-r} a_k, \quad r = 0, 1, \ldots, n-2, a_n = 1, {}^0C_0 = 1,$$
$$\alpha \quad = \quad \sum_{i=0}^{n-2} |\alpha_i|^2.$$

Proof. Putting $z = \eta - \frac{a_{n-1}}{n}$ in the polynomial $p(z)$ we get, a polynomial
$q(\eta) = \eta^n + \alpha_{n-2}\eta^{n-2} + \alpha_{n-3}\eta^{n-3} + \ldots + \alpha_1\eta + \alpha_0$,
where $\alpha_r = \sum_{k=r}^{n} {}^kC_r\big(-\frac{a_{n-1}}{n}\big)^{k-r} a_k$, $r = 0, 1, \ldots, n-2$, $a_n = 1$ and ${}^0C_0 = 1$.
Now the Frobenius companion matrix of the polynomial $q(\eta)$ is $C(q) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where
$A = (0)_{1,1}$, $B = (-\alpha_{n-2} \quad -\alpha_{n-3} \quad \ldots \quad -\alpha_1 \quad -\alpha_0)_{1,n-1}$, $C^t = (1 \quad 0 \quad \ldots \quad 0 \quad 0)_{1,n-1}$,

$$D = \begin{pmatrix} 0 & 0 & . & . & . & 0 \\ 1 & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & . & . & 1 & 0 \end{pmatrix}_{n-1,n-1}.$$

Now using Lemma 7.1(i) and Lemma 7.4 we get,

$$
\begin{aligned}
w\begin{pmatrix} A & B \\ C & D \end{pmatrix} &\leq w\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + w\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \\
&= w(D) + w\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \\
&= \cos\frac{\pi}{n} + w\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.
\end{aligned}
$$

Therefore, if η is any zero of the polynomial $q(\eta)$ then $|\eta| \leq \cos\frac{\pi}{n} + w\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. Therefore if λ is any zero of the polynomial $p(z)$ then $|\lambda| \leq |\frac{a_{n-1}}{n}| + \cos\frac{\pi}{n} + w\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. Now using Theorem 7.15 in the above inequality we get,

$$
\begin{aligned}
|\lambda| &\leq \left|\frac{a_{n-1}}{n}\right| + \cos\frac{\pi}{n} + \left[\frac{1}{16}\|S\|^2 + \frac{1}{4}w^2(CB) + \frac{1}{8}w(CBS + SCB)\right]^{\frac{1}{4}}, \\
&\quad \text{where } \ S = B^*B + CC^* \\
&\leq \left|\frac{a_{n-1}}{n}\right| + \cos\frac{\pi}{n} + \frac{1}{2}\left[\|S\|^2 + 4\|B\|^2 + 4\|B\|\|S\|\right]^{\frac{1}{4}} \ \left(\text{using Lemma 7.2}\right) \\
&\leq \left|\frac{a_{n-1}}{n}\right| + \cos\frac{\pi}{n} + \frac{1}{2}\left[(1+\alpha)^2 + 4\alpha + 4\sqrt{\alpha}(1+\alpha)\right]^{\frac{1}{4}}.
\end{aligned}
$$

This completes the proof of the theorem. □

We illustrate with numerical examples to show that the above bound obtained by us in Theorem 7.22 is better than the existing bounds.

Example 7.23. *Consider the polynomial $p(z) = z^5 + 2z^4 + z + 1$. Then the upper bounds of the zeros of this polynomial $p(z)$ estimated by different mathematicians are as shown in the following table.*

Cauchy [49]	3.000
Montel [49]	4.000
Carmichael and Mason [49]	2.645
Fujii and Kubo [41]	3.090
Alpin et. al. [9]	3.000
Paul and Bag [65]	2.810
Abu-Omar and Kittaneh [7]	2.914
M. Al-Dolat et. al. [8]	3.325

But our bound obtained in Theorem 7.22 gives $|\lambda| \leq 2.625$ which is better than all the estimations mentioned above.

Next, we obtain another bound for the zeros of the polynomial $p(z)$.

Theorem 7.24. *Let λ be any zero of $p(z)$. Then*

$$|\lambda| \leq \max\left\{|a_{n-1}|, \cos\frac{\pi}{n}\right\} + \sqrt{\frac{1}{2}\left(1+\sum_{j=2}^{n}|a_{n-j}|^2\right)}.$$

Proof. Let $C(p) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$,
where $A = (-a_{n-1})_{1,1}$, $B = (-a_{n-2} \ \ -a_{n-3} \ \ \ldots \ \ -a_1 \ \ -a_0)_{1,n-1}$,

$$C^* = (1 \ 0 \ \ldots \ 0 \ 0)_{1,n-1} \text{ and } D = D_{n-1} = \begin{pmatrix} 0 & 0 & . & . & . & 0 & 0 \\ 1 & 0 & . & . & . & 0 & 0 \\ 0 & 1 & . & . & . & 0 & 0 \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ 0 & 0 & . & . & . & 1 & 0 \end{pmatrix}_{n-1,n-1}.$$

Therefore, using Lemma 7.4 and Corollary 7.6, we get

$$\begin{aligned} w(C(p)) &\leq \max\left\{|a_{n-1}|, \cos\frac{\pi}{n}\right\} + \sqrt{\frac{1}{2}\max\left\{\|B^*B+CC^*\|, \|BB^*+C^*C\|\right\}} \\ &\leq \max\left\{|a_{n-1}|, \cos\frac{\pi}{n}\right\} + \sqrt{\frac{1}{2}\left(\|B\|^2+\|C\|^2\right)}. \end{aligned}$$

Thus,

$$|\lambda| \leq \max\left\{|a_{n-1}|, \cos\frac{\pi}{n}\right\} + \sqrt{\frac{1}{2}\left(1+\sum_{j=2}^{n}|a_{n-j}|^2\right)},$$

as required. □

We would like to note that the existing bounds for the zeros of the polynomial $p(z)$ are not always better than the one in Theorem 7.24, and vice versa.

Clearly, the zeros of the polynomial $\frac{z^n}{a_0}p(\frac{1}{z})$ are the reciprocal of the zeros of $p(z)$, if $a_0 \neq 0$ (see in [21]). Therefore, lower bound for the zeros of $p(z)$ can be obtained by considering the polynomial $\frac{z^n}{a_0}p(\frac{1}{z})$ and using Theorem 7.24. This enables us to describe annuli in the complex plane containing all the zeros of $p(z)$.

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