b) If $L\{f(t)\} = \bar{f}(s)$ then prove that

$$L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$ 6+4

7. Obtain the Fourier series expansion of the function $f(x) = x + x^2$ on $(-\pi, \pi)$, and show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$ 7+3

8. a) Find the Fourier transform of

$$f(x) = \begin{cases} 1 - x^2 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

and hence evaluate the following integral:

$$\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx$$

b) Find the Fourier transform of the function

$$f(x) = e^{-a|x|}, \; -\infty < x < \infty$$ 7+3

Bachelor of Engineering in Power Engineering Examination, 2022

(1st Year, 2nd Semester)

Mathematics II

Time : Three hours | Full Marks : 100

(50 Marks for each Part)

(Use separate answer script for each Part)

(Notations / Symbols have their usual meanings.)

PART – I (50 Marks)

Answer ***any five*** questions. (5×10=50)

1. a) State the Bonnet's form of second Mean value theorem.

b) Use Bonnet's form of second Mean value theorem to prove that

$$\left|\int_a^b \sin x^2 dx\right| \le \frac{1}{a}$$

if $0 < a < b < \infty$.

c) Test the convergence of $\int_0^\infty \frac{1}{e^{x+2}} dx$. 2+4+4

2. a) Find the value of $\int_0^1 x^5 \left(1 - x^3\right)^3 dx$.

b) Show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

c) Evaluate $\frac{d}{dx}\int_0^x \left(x^2 - t^2\right)\cos t \, dt$. 3+3+4

[Turn over

3. a) Find the area of the region which is bounded by $x^2 = y$ and $y = 2 - x$.

   b) Evaluate

$$I = \iiint_E (x + y + z + 1)^2 \, dxdydz,$$

   where E is the region defined by $x \geq 0,\ y \geq 0,\ z \geq 0,\ x + y + z \leq 1$. 5+5

4. a) Prove that

$$\iint_E e^{-x^2 - y^2} dxdy = \frac{\pi}{4}\left(1 - e^{-a^2}\right)$$

   where E is the region defined by $x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq a^2$.

   b) Find the reduction formula for

$$\int \frac{dx}{\left(x^2 + a^2\right)^n},$$

   n being a positive integer greater than 1. 5+5

5. a) If $\sum_{n=1}^{\infty} u_n$ be a series of positive real numbers and $v_n = \dfrac{u_1 + u_2 + \cdots + u_n}{n}$, prove that $\sum_{n=1}^{\infty} v_n$ is divergent.

   b) Test the convergence of the series $1 + \dfrac{3}{2!} + \dfrac{5}{3!} + \dfrac{7}{4!} + \cdots$. 5+5

   b) If $f(z)$ is a analytic function of z, then show that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$. 5+5

4. a) Find the Laurent series expansion in power of z of the function

$$f(z) = \frac{z^2 - 4}{(z+1)(z+4)} \text{ for } 1 < |z| < 4.$$

   b) Show that $u = \dfrac{1}{2}\log\left(x^2 + y^2\right)$ is harmonic. Find its harmonic conjugate. 4+6

5. a) Use unilateral Z transformation to evaluate $Z\{a^n\}$. Use this result to find $Z\{r^n \cosh n\theta\}$ and $Z\{r^n \sinh n\theta\}$.

   b) Evaluate : $L\left\{\dfrac{1 - e^{2t}}{t}\right\}$ 7+3

6. a) Use Laplace transformation to determine the solution of the following ODE :

$$y'' + 6y' + 8y = 2$$

   subject to initial conditions $y(0) = 0$, $y'(0) = 0$

PART – II (50 marks)

Answer ***any five*** questions.

1. a) Show that the function

$$f(z)=\begin{cases}\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & , x^2+y^2\neq 0\\ 0 & , x^2+y^2=0\end{cases}$$

satisfies Cauchy-Riemann Equations at the origin but $f'(0)$ does not exist.

b) If $f(z)$ is analytic and $f'(z)=0$ in D then show that $f(z)=$ a constant in *D*. 6+4

2. a) Evaluate $\int_0^{1+i}\left(x-y+ix^2\right)dz$ along the straight line from $z=0$ to $z=1+i$.

b) Use Cauchy's Integral formula to evaluate the following integral :

$$\int_C \frac{e^{2z}}{(z-1)(z-2)}dz \text{ where } C:|z|=3.$$ 5+5

3. a) Use Cauchy's Residue theorem to evaluate

$$\int_C \frac{1}{z^2(z-1)(z+1)}dz \text{ where } C:|z|=3.$$

6. a) Show that the series

$$\frac{1}{(1+a)^p}-\frac{1}{(2+a)^p}+\frac{1}{(3+a)^p}-\cdots,\ a>0, \text{ is}$$

i. absolutely convergent if $p>1$,

ii. conditionally convergent if $0<p\leq 1$.

b) Test the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$.

6+4

7. a) Test the convergence of the series $\sum_{n=1}^{\infty}\sin\frac{1}{n}$.

b) Find the radius of convergence of the power series $\sum_{n=0}^{\infty}a_n x^n$ where $a_n=\frac{(-1)^n n^n}{n!2^n}$, $n=1, 2, \ldots$ and $a_0=0$.

c) A sequence $\{u_n\}$ is defined by $u_{n+2}=\frac{1}{2}(u_{n+1}+u_n)$ for $n\geq 1$ and $0<u_1<u_2$. Prove that the sequence $\{u_n\}$ converges to $\frac{u_1+2u_2}{3}$. 3+3+4