7. i) Define convergence of a series $\sum_n x_n$. Let $\sum_{n=1}^{\infty} x_n$ be a convergent series then show that $\lim_{n\to\infty} x_n = 0$. Show that the converse may not be true. 3

ii) Discuss the convergency of following series:

a) $\frac{1}{3}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+\frac{1.2.3.4}{3.5.7.9}+.....$

b) $\frac{1}{\sqrt{2\cdot1}}+\frac{1}{\sqrt{3\cdot2}}+\frac{1}{\sqrt{4\cdot3}}+\frac{1}{\sqrt{5\cdot4}}+.....$ $3\frac{1}{2}+3\frac{1}{2}$

Bachelor of Engineering In Metallurgical and Materials Engineering Examination, 2022

(1st Year, 2nd Semester)

Mathematics II

Time : Three hours Full Marks : 100

(50 Marks for each Part)

(Use separate answer script for each Part)

(Symbols and notations have their usual meanings)

PART – I (50 Marks)

Answer any ***Five*** questions. 5×10

1. i) Give an example (with justification) of a continuous function $f:\mathbb{C}\to\mathbb{C}$ which is nowhere differentiable.

ii) Show that the function

$$f(z)=\begin{cases}\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & \text{for } x^2+y^2\neq 0\\ 0 & \text{for } x^2+y^2=0\end{cases}$$

satisfies CR-equations at the origin but $f'(0)$ does not exist. 3+7

2. i) If $f:\mathbb{C}\to\mathbb{R}$ is holomorphic then show that f is a constant function.

ii) Show that $u=\frac{1}{2}\ln\left(x^2+y^2\right)$ is harmonic. Find the harmonic conjugate of u. 3+7

[Turn over

3. i) Find the Laurent series expansion of $f(z)=\frac{z}{(z+1)(z+2)}$ about the point $z=-2$. Also discuss the nature of the singularity at $z=-2$.

ii) Using Cauchy's Residue Theorem, evaluate $\int_C \frac{dz}{z^2(z-1)}$ where $C:|z|=3$. 6+4

4. State Cauchy's Integral Formula for n^{th} derivative of an analytic function then evaluate

$\int_C \frac{e^z dz}{(z+1)^2(z-1)}$ where $C:|z|=\sqrt{2}$ 2+8

5. i) Find the Fourier series for the function $f:(-\pi,\pi)\to\mathbb{R}$ defined by $f(x)=e^{-ax}$.

Hence prove that

$$\frac{\pi}{\sinh\pi}=2\left(\frac{1}{2^2+1}+\frac{1}{3^2+1}+\frac{1}{4^2+1}+......\right).$$

ii) Determine the half range Fourier sine series for $f(x)=x(\pi-x)$ in $0<x<\pi$. 7+3

6. i) Find the Fourier Transformation of $e^{-|x|}$. Hence show that $\int_0^\infty \frac{x\sin mx}{1+x^2}dx=\frac{\pi}{2}e^{-m},\ m>0$.

b) Reverse the order of the integral $\int_{1/3}^{2/3} dx\int_{x^2}^{\sqrt{x}} f(x,y)\,dy$

c) Evaluate: $\int_0^{1/2} dy\int_y^1 e^{x^2}dx$. 4+2+4

5. i) Find the volume of the region bounded by $x=0$, $y=0$, $z=0$ and $x+y+z=1$ planes using b) triple integration. 4

ii) Evaluate $\iint_D (x^2-y^2)\,dxdy$, where D is the region bounded by $x-y=-1$, $y-x=0$ and $xy=1$, $xy=2$. 3

iii) Find the length of the arc of the parabola $y^2=16x$ from vertex to an extrimity of lotus return. 3

6. i) Define limit of a sequence.

ii) Define a bounded sequence. Show that a convergent sequence must be bounded. Show that the converse may not be true. 1+3

iii) Let $x_1>1$ and $x_{n+1}=2-\frac{1}{x_n}$ for $n\geq 1$. Show that $\{x_n\}$ is convergent sequence. Hence find the limit of the sequence. 6

b) Discuss the convergence of the following integrals:

i) $\int_0^{\pi} \frac{\sqrt{x}}{\sin x} dx$

ii) $\int_1^{\infty} \frac{dx}{x\sqrt{x^2+1}} dx$ 3+3

3. a) Define *Gamma function* $\Gamma(n)$.

Show that $\int_0^{\infty} \frac{x^c}{c^x} dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}},\ c>0$.

b) Show that $\Gamma(n+1) = n\Gamma(n)$ for $n>0$.

c) Define *Beta function* $B(m, n)$. Convert the integral $\int_0^1 \frac{dx}{\left(1-x^3\right)^{1/3}}$ as a scalar multiple of $B(m, n)$ for suitable m and n.

d) Show that $B(m,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$ 3+2+2+3

4. a) State the Leibnitz's rule for differentiation under the integral sign. Assuming the validity of differentiation under integral sign, show that $\int_0^1 \frac{x^m-1}{\log x} dx = \log(m+1),\ m>-1$.

ii) If $a>0$ then show that $\int_0^{\infty} \frac{x^2}{\left(a^2+x^2\right)^4} dx = \frac{\pi}{(2a)^5}$.

6+4

7. Evaluate :

i) $L^{-1}\left\{\frac{s}{(s+a)\left(s^2+1\right)}\right\}$ ii) $L^{-1}\left\{\ln\frac{s^2+1}{s(s+1)}\right\}$

iii) $Z^{-1}\left\{\ln\frac{2z^2+3z}{(z+2)(z-4)}\right\}$ 4+3+3

PART – II (50 Marks)

Answer any ***Five*** questions. 5×10

1. a) Let $I_{m,n} = \int \sin^m x \cos^n x dx$, where m and n being positive integers, greater than 1. Show that $I_{m,n} = \frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$ for $m>2$.

b) Deduce an appropriate reduction formula and hence show that $\int_0^{\pi/2} \sqrt{\sin x} \cos^5 x dx = \frac{64}{231}$. 5+5

2. a) Define *improper integral*. Is $\int_0^1 \frac{\sin x}{x} dx$ an improper integral? Justify. 2+2