Ex/BS/FTBE/MTH/T/122/2022

BACHELOR OF ENGINEERING IN FOOD TECHNOLOGY AND BIO-CHEMICAL ENGINEERING EXAMINATION, 2022

(1st Year, 2nd Semester)

MATHEMATICS II

Time : Three hours Full Marks : 100

(50 Marks for each Part)

(Use separate answer script for each Part)

(Symbols and notations have their usual meanings)

PART – I (50 Marks)

Answer ***Q. No. 1*** and any ***three*** from the rest.

1. Evaluate $\int_0^1 x^3\left(1-x^2\right)^{\frac{5}{2}} dx$. 5

2. a) Using a double integral, prove that

$$\beta(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)};\quad a,b>0.$$ 7

b) Examine the convergence of the following improper integrals:

i) $\int_0^1 \frac{dx}{(x+1)(x+2)\sqrt{x(1-x)}}$

ii) $\int_0^1 \frac{\sin x+\cos x}{\left(1-x^3\right)^{\frac{1}{5}}} dx$ 8

[Turn over

3. a) Evaluate $\int_0^\infty e^{-ax}\frac{\sin nx}{x}dx$, $a>0$ by using Leibnitz's rule and hence deduce that $\int_0^\infty \frac{\sin nx}{x}dx=\frac{\pi}{2}$. 5

b) Find the area of the cardioid $r=a(1+\cos\theta)$. 5

c) If $I_n=\int_0^{\pi/2} x^n \sin x dx$ and $n>1$, show that $I_n+n(n-1)I_{n-2}=n\left(\frac{\pi}{2}\right)^{n-1}$. 5

4. a) Evaluate $\iiint xyz dx dy dz$, the field of integration being the positive octant of the ellipsoid $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}\le 1$. 8

b) Give an example of a series $\sum x_n$ such that $\sum x_n$ is convergent but $\sum x_n^2$ is divergent. 2

c) Show that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is conditionally convergent. 5

5. a) Test the convergence of the series $\sum \frac{(-1)^{n+1}}{\sqrt{n}}$. 5

b) If $I_n=\int x^n\cos(ax)dx$ and $J_n=\int x^n\sin(ax)dx$, then show that $aI_n=x^n\sin(ax)-nJ_{n-1}$. 3

7. a) Prove the linearity property of Laplace Transform for $F_1(t)$ and $F_2(t)$.

b) If $L\{F(t)\}=f(s)$, then show that $L\{F(at)\}=\frac{1}{a}f\left(\frac{s}{a}\right)$.

c) If $L\{F(t)\}=f(s)$, then show that $L\left\{\int_0^t F(x)dx\right\}=\frac{1}{s}f(s)$. 3+3+4

8. a) Find the Laplace transform of the following:

i) $6\sin 2t-5\cos 2t$

ii) $te^{at}\sin at$

b) Show that $\int_0^\infty \frac{e^{-t}\sin t}{t}dt=\frac{\pi}{4}$. 3+3+4

4. a) If $f(z)=\dfrac{z^2+5z+6}{z-2}$, does Cauchy's theorem apply

i) when the path of integration C is a circle of radius 3 with origin as centre.

ii) when the path of integration C is circle of radius 1 with origin as centre.

b) Expand in the series for the function $f(z)=\dfrac{1}{z^2-3z+2}$ in the regions (i) $|z|<1$, (ii) $1<|z|<2$, (iii) $|z|>2$ 2+8

5. a) State Cauchy's residue theorem.

b) Using Cauchy's residue theorem, evaluate the integral $\int_C \dfrac{e^{2z}}{(z+1)^4}dz$, where $C:|z|=3$.

c) Show that $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)=4\dfrac{\partial^2}{\partial z.\partial \bar{z}}$, where $z=x+iy$ and $\bar{z}=x-iy$. 2+5+3

6. Find the Fourier series for the function $f(x)=x^2$ in $[-\pi,\pi]$ and hence deduce the following :

i) $\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{4^2}+\cdots=\dfrac{\pi^2}{6}$

ii) $\dfrac{1}{1^2}-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+\cdots=\dfrac{\pi^2}{12}$ 6+2+2

c) A sequence $\{x_n\}$ is given by $x_1=\sqrt{2}$ and $x_{n+1}=\sqrt{2x_n}$ for $n\geq 1$. Show that $\lim_{n\to\infty} x_n=2$. 4

d) Show that

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)=1.$$ 3

PART – II (50 Marks)

Answer any ***Five*** questions. 5×10

1. a) Let $f(z)=z^2$ and z_0 be any fixed point, then show that

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=2z_0.$$

b) Show that the function $f(z)=xy+iy$ is continuous everywhere but not differentiable.

c) Define Analytic function and Harmonic function.

3+3+4

2. Prove the necessary condition that the function $f(z)=u(x,y)+iv(x,y)$ is differentiable at a point $z_0=x_0+iy_0$ is that the partial derivatives u_x, u_y, v_x and v_y exists and $u_x=v_y$, $u_y=v_x$ at the point (x_0,y_0). 10

3. Prove that the function $u(x,y)=x^3-3xy^2+3x^2-3y^2+1$ satisfies Laplace's equation and find the corresponding analytic function $f(z)=u(x,y)+iv(x,y)$. 3+7