Ex//BS/CE/MTH/T/122/2022

Bachelor of Engineering in Civil Engineering Examination, 2022

(1st Year, 2nd Semester)

Mathematics II

Time : Three hours | Full Marks : 100

(50 Marks for each Part)

(Use separate answer script for each Part)

Part-I
(50 Marks)

Answer any five questions.

1. (a) Derive the reduction formula for $I_n = \int_0^{\pi/4} tan^n x \, dx$, where n is a positive integer. Hence find the value of I_4. (5)

   (b) Test the convergence of the following integrals: (5)

   (i) $\int_1^{\infty} \frac{x^{3/2}}{3x^2+5}$ (ii) $\int_0^{\pi/2} \frac{dx}{(\cos x)^{1/n}}$, where $n > 1$

2. (a) Define $\beta(m,n)$ and $\Gamma(n)$ together with the condition for existence. Prove that $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ (6)

   (b) Evaluate $\int_0^{\pi} \sin^2\theta(1+\cos\theta)^4 d\theta$ (4)

3. (a) Use the rule for differentiation under integral sign to evaluate (4)

   $\int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x} dx$

   (b) Find the mass and the y-coordinate of the centre of mass of a triangular lamina with vertices $(0,0)$, $(2,1)$, $(0,3)$ and density function $\rho(x,y) = x+y$. (6)

4. (a) Evaluate by making an appropriate change of variables ~~in~~ $\iint_R (x+y)e^{x^2-y^2} dA$, where R is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$ and $x+y=3$. (5)

   (b) Evaluate $\iiint_E x^2 dV$, where E is the solid that lies within the cylinder $x^2+y^2=1$, above the plane $z=0$, and below the cone $z^2 = 4x^2+4y^2$. (5)

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5. (a) If $f(z) = \dfrac{((1-i)z + (1+i)\bar{z})^2}{z\bar{z}}$, Show that (6)

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(z)\right) = 4 = \lim_{x\to 0}\left(\lim_{y\to 0} f(z)\right).$$

Can we conclude that $\lim_{z\to 0} f(z) = 4$?. Justify your answer.

(b) If $f(z)$ is analytic in $\mathbb{C}$, then show that (4)

$$\nabla^2\left\{|f(z)|^p\right\} = p^2|f(z)|^{p-2}|f'(z)|^2$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ be the Laplace operator.

6. (a) Evaluate $\int_C |z|\bar{z}dz$, where C is (5)

i. the line segment from $-2i$ to $2i$, and

ii. the semi-circle $|z| = 2$ in the second and third quadrants.

(b) Show that the Cauchy-Riemann equation in polar form is (5)

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

7. (a) Expand $f(z) = \dfrac{3z-1}{z^2-2z-3}$ in Laurent series about the point $z = 0$ valid for (i) $|z| < 1$, (ii) $1 < |z| < 3$, (iii) $|z| > 3$. (6)

(b) Use Cauchy's residue theorem to evaluate $\int_C \dfrac{z^2+3z+2}{z^3-z^2}dz$, where C : $|z| = 2$. (4)

Part-II

(50 Marks)

Answer any five questions.

8. (a) Find Fourier series of $f(x) = \left(\dfrac{\pi - x}{2}\right)^2$. Hence deduce that, (7)

i. $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \dots$.

ii. $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \dots$.

(b) Find Laplace transform of $\dfrac{e^{-at}t^{n-1}}{(n-1)!}$. (3)

9. (a) Find z-transform of $f(n) = (b^n \cos an)\, u(n)$, where $u(n)$ is unit step function. (5)

(b) Apply convolution theorem to evaluate inverse Laplace transform of $\frac{p^2}{(p^2+4)(p^2+9)}$. (5)

10. (a) Find Fourier transform of $f(x) = e^{-x^2/2}$. (5)

(b) Find Fourier cosine transform of $\frac{e^{-px}}{x}$, where $p > 0$. (5)

11. (a) Using Laplace transform find the solution of $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 4e^{-t}$ if $y(0) = 0$ and $\frac{dy}{dt}|_{t=0} = 0$. (6)

(b) If $F(z)$ be z-transform of $f(n)$, defined for $r < |z|$ and for some integer p, $\lim_{|z|\to\infty} z^p F(z) = A$. Then prove that $f(p) = A$ when $f(n) = 0$ for $n < p$. (4)

12. (a) Find whether the following series is convergent or divergent. (6)

$$1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \frac{x^4}{17} + \ldots (x > 0)$$

(b) Examine the convergence of the series (4)

$$1 + \frac{1}{2}.\frac{1}{3} + \frac{1.3}{2.4}.\frac{1}{5} + \frac{1.3.5}{2.4.6}.\frac{1}{7} + \ldots$$

13. (a) Show that the sequence (5)

$$\sqrt{5}, \sqrt{5+\sqrt{5}}, \sqrt{5+\sqrt{5+\sqrt{5}}}, \ldots$$

converges to $\frac{(\sqrt{21}+1)}{2}$.

(b) Prove that (5)

$$\lim_{n\to\infty}\left[\left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 \ldots\ldots \left(\frac{n+1}{n}\right)^n\right]^{\frac{1}{n}} = e.$$

14. (a) Show that the series (4)

$$1 + \frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \ldots$$

is divergent.

(b) Prove that the series (6)

$$1 - \frac{1}{3.2^2} + \frac{1}{5.3^2} - \frac{1}{7.4^2} + \frac{1}{9.5^2} - \ldots$$

is convergent. Is it absolutely convergent?