Bachelor of Engineering in Chemical Engineering Examination, 2022

(1st Year, 2nd Semester)

Mathematics II

Time : Three hours Full Marks : 100

(50 Marks for each Part)

(Use separate answer script for each Part)

(Unexplained Symbols/Notations have their usual meanings)

Special credit will be given for precise answer

PART – I (50 Marks)

Answer any ***Five*** questions. 10×5=50

1. a) Give an example (with proper explanation) of a bounded sequence which is not convergent. 2

   b) Check if the sequence $x_n = 1 - \frac{1}{n}$ is monotone increasing or decreasing or oscillatory and and hence discuss its convergence. 2

   c) Apply sandwich theorem to show that $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{(n+k)^2} = 0$. 3

   d) Using Cauchy's General Principle of Convergence show that the sequences $(x_n)_{n \geq 1}$ can not converge, where $x_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$. 3

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2. a) Determine the limits of the double integral $\iint_R f(x,y)\,dxdy$ where R is the region bounded by the straight lines $y=x$ and $3x+4y=12$ and the X-axis considering

i) x as the first variable

ii) y as the first variable.

b) Using double integral find the area of the region R as mentioned above. 1+1+2

c) Change the order of integration of the integral

$$\int_0^{\frac{12}{7}}\int_0^x f(x,y)\,dxdy+\int_{\frac{12}{7}}^4\int_0^{\frac{12-3x}{4}} f(x,y)\,dxdy.$$ 2

d) Evaluate $\iint_R (x+y)^2\,dxdy$ where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation $u=x+y$ and $v=x-2y$. 4

3. a) Why are the following integrals improper? When do they converge (no proof is required)? What are their names when they converge?

i) $\int_0^\infty e^{-x}x^{n-1}dx$, n is a real number.

ii) $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, m, n are real numbers. 2+2

Deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+.....=\frac{\pi^2}{8}$.

b) Obtain a half-range cosine series for

$$f(x)=\begin{cases} x, & 0\le x\le \frac{l}{2} \\ l-x, & \frac{l}{2}\le x\le l \end{cases}$$ 6+4

7. a) If Laplace transform of $f(t)$ in $L[f(t)]=\bar{f}(s)$, then prove that $L\left[e^{at}f(t)\right]=\bar{f}(s-a)$.

Hence or otherwise find the Laplace transform of i) $e^{2t}\cos^2 t$, ii) $\sqrt{t}\,e^{3t}$.

b) Find the inverse Laplace transform of $\frac{s}{s^4+4a^4}$.

6+4

3. a) Evaluate $\int_{1-i}^{2+i}(2x+iy+1)\,dz$ along the two paths

i) $x = t+1,\ y = 2t^2 - 1$

ii) the straight line joining $1-i$ and $2+i$.

b) If $f(z)$ is a regular function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left|f(z)\right|^2 = 4\left|f'(z)\right|^2.$$ 6+4

4. a) State and prove Lawrent's theorem.

b) Expand $\dfrac{1}{z\left(z^2-3z+2\right)}$ for the regions

i) $0<|z|<1$ ii) $1<|z|<2$ iii) $|z|>2$ 5+5

5. a) Evaluate $\int_c \dfrac{z-3}{z^2+2z+5}\,dz$ where c is the circle

i) $|z|=1$ ii) $|z+1-i|=2$ iii) $|z+1+i|=2$

b) Apply the calculus of residues to evaluate

$$\int_0^{2\pi}\frac{\cos 2\theta\, d\theta}{1-2a\cos\theta+a^2},\ |a|<1$$ 5+5

6. a) Find the Fourier series expansion for $f(x)$, if

$$f(x)=\begin{cases}-\pi, & -\pi\le x<0\\ x, & 0\le x\le\pi\end{cases}$$

b) If the series $\sum_{n=1}^{\infty} x_n$ converges then prove that $\lim_{n\to\infty} x_n = 0$. By using this result show that the series $\sum_{n=1}^{\infty} x_n$ is divergent, where $x_n = \dfrac{n}{n+1}$. 3

c) Define geometric series and discuss its convergence. 3

4. a) Write the triple integral that represents the volume of the sphere $x^2+y^2+z^2=16$ and evaluate it changing to spherical coordinates. 5

b) Changing to cylindrical coordinates evaluate $\iiint_W \left(z^2x^2+z^2y^2\right)dxdydz$, where W is the cylindrical region determined by $x^2+y^2\le 1,\ -1\le z\le 1$. 5

5. a) State Leibniz's rule and Generalized Leibniz's rule (no proof is required) related with differentiation under the sign of integration. 2

b) Discuss the convergence of the improper integrals $\int_a^b (b-x)^{-p}\,dx$ and $\int_a^b (x-a)^{-p}\,dx$. 5

c) Write the relation between Beta function and Gamma function and use it to evaluate $\Gamma\left(\frac{1}{2}\right)$. 3

6. a) Find the radius of convergence (R) of the following power series and check the behaviour at R, $\sum_{n=0}^{\infty}(n+1)x^n$. 5

b) What is p-series? When does it converge (no proof is required)? 2

c) What is limit comparison theorem related with infinite series of nonnegative terms? Using this test the convergence of the series $\sum_{n=1}^{\infty}\frac{2n^2+n+2}{5n^3+3n}$. 3

7. a) Write the trigonometric form of Beta function. 2

b) Reduce $\iint_R [xy(1-x-y)]^{\frac{1}{2}}\,dxdy$ in terms of Beta function, where R is the region enclosed by the lines $x=0$, $y=0$ and $x+y=1$ in the positive quadrant. 6

c) State 1st Form and 2nd Form of the Fundamental theorem of integral calculus. 2

PART – II (50 Marks)

Answer any ***Five*** questions. 10×5=50

All questions carry equal marks.

1. a) State the necessary and sufficient conditions for the derivative of the function $f(z)$ to exist for all values of $z(=x+iy)$ in a region R. Prove only the necessary part.

b) Prove that the function $f(z)$ defined by

$$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}\ (z\neq 0),\ f(0)=0$$

is continuous and Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist. 4+6

2. a) State and prove Cauchy's theorem and extend it to a multiply connected region.

b) If $u-v=(x-y)(x^2+4xy+y^2)$ and $f(z)=u+iv$ is an analytic function of $z(=x+iy)$, find $f(z)$ in terms of z. 5+5

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