ii) $\Lambda^n(V)$ is a free K-module of rank 1 and

iii) for a linear operator T on V, there is a unique $c \in K$ such that $L(Tv_1.Tv_2. \ldots\ldots, Tv_n) = cL(v_1, v_2, \ldots\ldots, v_n)$ for all $L \in \Lambda^n(V)$. 5+2+3

M. Sc. Mathematics Examination, 2022

(2nd Year, 2nd Semester)

Advanced Algebra - II

Paper – 4.3 (A 2.1)

Time : Two hours | Full Marks : 50

Unexplained Symbols / Notations have their usual meanings.

Students are advised to quote the relevant result clearly whenever it is used in any argument.

The figures in the margin indicate full marks.

Answer ***any five*** questions. 10×5=50

1. i) Characterize the primitive rings in the class of commutative rings.

   ii) No $\mathbb{Z}$-module can be faithful and irreducible — Justify.

   iii) If M is a simple left R-module than prove that $R / Ann_R(M)$ is primitive ring. 5+2+3

2. i) Give an example of a primitive ring which is not simple. Answer with proper reasons.

   ii) Suppose V is a vector space over a division ring D and R is a dense ring of linear transformations of V. Prove that R is left Artinian if and only if dim$V$ is finite, in which case $R = \mathrm{End}_D(V)$. 5+5

3. i) Define radical property of a ring.

ii) List (no proof is required) the results regarding the Jacobson radical of a ring and other relevant results and use them to explain that left quasi regularity is a radical property.

iii) The homomorphic image and every ideal of a semisimple ring are semisimple ring are semisimple — Justify. 3+4+(2+1)

4. i) Let G be the cyclic group of order 3 with a as a generator. Let ρ_1, ρ_2, ρ_3, ρ_4 be the 2-dimensional representations of G over $\mathbb{C}$ given by

$$\rho_1 a = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, \quad \rho_2 a = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad \rho_3 a = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

$\rho_4 a = \begin{pmatrix} 1 & 0 \\ \omega^{-1} & \omega \end{pmatrix}$, where ω is a complex cube root of unity. Find the characters ψ_i of the representations ρ_i, $i = 1, 2, 3, 4$. Hence determine which of the ρ_i's are equivalent.

ii) Find the regular representation of S_3. 5+5

5. i) The following is a 4-dimensional real representation of the Quaterninon group

$$Q_8 =< a,b : a^4 = b^4 = 1,\ a^2 = b^2,\ ba = ab^{-1} >$$

$$a \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Determine the corresponding $\mathbb{R}Q_8$-module.

ii) The permutation matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ cannot be the matrix of the regular representation of a group. — Justiry.

iii) The number (up to isomorphism) of irreducible $\mathbb{C}\mathbb{Z}_{16}$-modules is 16. — Justify.

iv) Let $V = sp(v_1, v_2, v_3)$ be the permutation module for $S_3 = G$ over a field F. Then $U = sp(v_1 + v_2 + v_3)$ is an FG-submodule of V. — Justfy. 5+2+2+1

6. Let K be a commutative ring with identity and V be a unitary K-module. For any positive integer r, let $M^r(V)$ denote the set of all multilinear functions on V^r i.e., the set of all r-linear forms on V and $\Lambda^r(V)$ denote the set of all alternating r-linear forms on V. Prove that if V is a free K-module of rank n then.

i) $M^r(V)$ is a free K-module of rank n^r;