Ex/SC/MATH/PG/4.1-2/2022

M. Sc. Mathematics Examination, 2022

(2nd Year, 2nd Semester)

Advanced Functional Analysis

Paper – 4.1

Time : Two hours Full Marks : 50

Symbols and notations have their usual meanings.

Answer **Q. No. 1** and ***any three*** questions from the rest.

1. a) Is the real number space $\mathbb{R}$ with cofinite topology a topological vector space? Answer with reasons.

   b) Give an example to show that the sum of two closed subsets of a topological vector space X need not be closed. 3+2

2. a) Let X be a topological vector space. Let K and C be compact and closed subsets of X respectively with $K \cap C = \phi$. Show that there is a nbd V of θ such that $(K+V) \cap (C+V) = \phi$. 5

   b) Prove that every convex nbd of θ in a topological vector space X contains a balanced convex nbd of θ. 6

   c) If V is nbd of θ in a topological vector space X and $0 < r_1 < r_2 < \ldots$, $r_n \to \infty$ as $n \to \infty$ then $X = \bigcup_{n=1}^{\infty} r_n V$. 4

3. a) Let Λ be a linear functional on a topological vector space X and assume that $\Lambda x \neq 0$ for some $x \in X$. Then prove that following are equivalent.

i) Λ is continuous.

ii) The null space $N(\Lambda)$ is closed.

iii) $N(\Lambda)$ is not dense in X.

iv) Λ is bounded in some nbd V of θ. 9

b) Prove that every locally compact subspace Y of a topological vector space X with the induced topology from X is a closed subspace of X. 6

4. a) Define a set of first category with a suitable example. 2

b) Let X, Y be two topological vector spaces, Γ is a collection of continuous linear mappings from X to Y and B is the set of all $x \in X$ whose orbits $\Gamma(x)$ are bounded in Y. If B is of second category in X then prove that $B = X$ and Γ is equicontinuous. 7

c) State and prove Baire's Theorem for a locally compact Hausdorff space. 6

5. a) Prove that every locally compact topological vector space is of finite dimension. 6

b) Let M be a subspace of a real topological vector space X,

i) $p : X \to R$ satisfies

$p(x+y) \leq p(x) + p(y)$ and $p(tx) = tp(x)$

for $x, y \in X$ and $t \geq 0$.

ii) $f : M \to R$ is linear and $f(x) \leq p(x)$ on M.

Then show that there exists a linear functional $\Lambda : X \to R$ such that $\Lambda(x) = f(x)$ for $x \in M$ and $-p(x) \leq \Lambda(x) \leq p(x)$ for all $x \in X$. 9

6. a) If f is a continuous linear functional on a subspace M of a locally convex topological vector space X, then prove that there exists $\Lambda \in X^*$ such that $\Lambda = f$ on M. 7

b) Suppose that X is a topological vector space and X_1 is a separating vector space of linear functionals on X. Then prove that the X_1-topology τ_1 makes X into a locally convex space whose dual space is X_1. 8