M. Sc. Mathematics Examination, 2022

(2nd Year, 2nd Semester)

Advanced Rings and Modules - II

Paper – DSE - 06 (B2)

Time : Two hours Full Marks : 40

Answer ***any two*** questions.

(Notations / Symbols have their usual meanings)

Let R be a ring with identify 1.

1. i) Define Noetherian Module. Let M be a Noetherian left R-module and $f : M \to M$ be surjective R-linear mapping. Show that f is an R-isomorphism.

   ii) Does the above result hold if we replace surjective by injective? Justify.

   iii) Let M_1 and M_2 be two Artinian submodules of a left R-module M. Show that $M_1 + M_2$ is also Artinian module. (2+3)+2+3

2. Let M be left R-module and $End_R(M)$ be the endomorphism ring of M.

   i) If M is simple then show that $End_R(M)$ is a division ring.

   ii) If M is semisimple then show that $End_R(M)$ is a regular ring. 3+7

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3. i) Show that a ring R is semisimple if and only if R is Noetherian and regular.

   ii) Define group ring. Let G be a finite group. Show that the group ring RG forms a free R-module. 5+(2+3)

4. i) Define J - semisimple ring. Show that every regular ring is a J - semisimple ring. Is the converse true? Justify.

   ii) Show that a simple ring is both primitive ring and prime ring. Let D be a division ring. Is $M_n(D)$ both primitive and prime? Justify. (1+2+2)+(4+1)

5. i) Show that a ring R is isomorphic to a subdirect product of a family $\{R_i\}_{i\in I}$ of rings if and only if there exists a family of ideals $\{P_i\}_{i\in I}$ in R such that $\bigcap_{i\in I} P_i = \{0\}$ and $R/P_i \cong R_i$ for all $i \in I$. Hence conclude that the ring of integers is a subdirect product of fields.

   ii) What do you mean by lifting idempotent modulo an ideal of a ring? If I is a nil ideal of R then show that every idempotent of R/I can be lifted to R. (5+2)+(1+2)

6. i) Let V be a left vector space over a division ring D. What does it mean by dense subring of $End_D(V)$? Let

R be a left primitive ring and e be a non-zero idempotent in R. Show that eRe is a left primitive ring.

   ii) Show that a subdirectly irreducible integral domain is a field. Hence conclude that an integral domain cannot have exactly three ideals. (2+3)+(3+2)