M. Sc. Mathematics Examination, 2022

(2nd Year, 2nd Semester)

Operator Theory - II

Paper – DSE - 07 (B17)

Time : Two hours Full Marks : 40

Answer ***any four*** questions.

All questions carry equal marks.

1. i) Let T be bounded linear selfadjoint operator on a complex Hilbert space $\mathbb{H}$. Then prove that

$$\|T\| = \max\{|m|, |M|\},$$

where $m = \inf\{\langle Tx, x\rangle : \|x\| = 1\}$ and

$$M = \sup\{\langle Tx, x\rangle : \|x\| = 1\}.$$

ii) Let T be bounded linear selfadjoint operator on a complex Hilbert space $\mathbb{H}$. Then show that $m = \inf\{\langle Tx, x\rangle : \|x\| = 1\}$ lies in the spectrum of T.

2. Show that every positive selfadjoint bounded linear operator T defined on a Hilbert space has a unique positive square root.

3. Let $\{T_n\}$ be a sequence of bounded selfadjoint linear operator on a complex Hilbert space $\mathbb{H}$ such that

$$T_1 \le T_2 \le \ldots \le T_n \le \ldots \le K,$$

where K is a selfadjoint bounded linear operator on H. Suppose that any T_n commutes with K and with every T_m.

Then show that $\{T_n\}$ is strongly operator convergent and the limit operator T is linear, bounded, selfadjoint and satisfies $T \le K$.

4. Let A, B be two bounded selfadjoint linear operators on a complex Hilbert space $\mathbb{H}$ with $AB = BA$ and $A^2 = B^2$. Then show that there exists an orthogonal projection E such that

i) E commutes with any operator that commutes with $A - B$

ii) $Ax = 0$ implies $Ex = x$

iii) $A = (2E - I)B$.

5. Let T be bounded linear selfadjoint operator on a complex Hilbert space $\mathbb{H}$. Then show that T has the spectral representation $T = \int_{m-0}^{M} \lambda dE_\lambda$,

where $\varepsilon = (E_\lambda)$ is the spectral family associated with T, $m = \inf\{\langle Tx, x\rangle : \|x\| = 1\}$ and $M = \sup\{\langle Tx, x\rangle : \|x\| = 1\}$.

6. i) Let $T : D(T) \to H$ be a linear operator, where $D(T)$ is dense in the complex Hilbert space $\mathbb{H}$. Then show that T is symmetric if and only if $T \subset T^*$.

ii) Show that every non-unital C^*-algebra U is contained in a unital C^*-algebra $\tilde{U}$ as a maximal ideal of codimension one.