M. Sc. Mathematics Examination, 2022

(2nd Year, 2nd Semester)

Theory of Semigroups - II

Paper – DSE - 07 (B15)

Time : Two hours | Full Marks : 40

Notations / Symbols have their usual meanings.

Answer ***any four*** questions.

1. a) Define O-simple semigroups. Let S be a semigroup. Then prove that S is O-simple if and only if SaS=S, for evry $a \neq 0$ in S.

   b) If a is an element of a semigroup S, then prove that either

   i) Ja, the J-class containing a, is the Kernel of S,

   Or

   ii) the set $I(a)=\{b \in S : J_b < J_a\}$ is non-empty and is an ideal of $J(a)(=S'aS')$ such that $J(a)/I(a)$ is either O-simple or null semigroup.

   (1+4)+5

2. Define primitive idempotent of a semigroup. Define Rees Matrix semigroup over a O-group. Prove that this semigroup is completely O-simple, i.e., O-simple and has a primitive idempotent. 1+2+7

3. Let S be a O-simple semigroup containing at least one O-minimal left ideal and O-minimal right ideal. Prove that

for every O-minimal left ideal L of S there exists a O-minimal right ideal R such that

   i) LR = S

   ii) RL is a O-group

   iii) the identity element e of RL is a primitive idempotent. 10

4. Let S be a semigroup. Prove that the following are equivalent

   i) There exists a unary operation $S \to S$, $x \mapsto x^{-1}$ such that $x x^{-1} x = x$, $(x^{-1})^{-1} = x$, $xx^{-1}yy^{-1} = yy^{-1}xx^{-1}$ for all $x, y \in S$.

   ii) Every element of S has a unique inverse. 5+5

5. Prove that every completely regular semigroup is a semilattice of completely simple semigroups. Hence prove that every band is a semilattice of rectangular bands. 7+3

6. Let S be an inverse semigroup with semilattice E of idempotents. Suppose ρ is a congruence on S. Prove that $(Ker\rho,\ tr\rho)$ is a congruence pair. Conversely, if (N,τ) is a congruence pair, then the relation

$$\rho(N,\tau)=\{(a,b)\in S\times S : (a^{-1}a,\ b^{-1}b)\in\tau,\ ab^{-1}\in N\}$$

is a congruence on S. Moreover, $Ker\,\rho(N,\tau)=N$, $tr\rho(N,\tau)=\tau$ and $\rho(Ker\rho,\ tr\rho)=\rho$. 10