M. Sc. Mathematics Examination, 2022

(2nd Year, 2nd Semester)

Stochastic Processes

Paper – DSE - 07 (B26)

Time : 2 hours Full Marks : 40

Symbols / Notations have their usual meaning.

Answer ***any Five*** questions.

Each question carries 8 marks.

1. Let x and y be distinct states of a Markov Chain having 10 states and suppose that x leads to y. Let n_0 be the smallest positive integer such that $P^{n_0}(x,y)>0$ and let $x_1,....,x_{n_0-1}$ be states such that

$$P(x,x_1)P(x_1,x_2)\ldots P(x_{n_0-2},x_{n_0-1})P(x_{n_0-1},y)>0$$

a) Show that $x,x_1,\ldots,x_{n_0-1},y$ are distinct states.

b) Use (a) to show that $n_0 \le 9$.

c) Conclude that $P_x(T_y \le 9)>0$.

2. Verify the following identities:

a) $P_x(T_y \le n+1) = P(x,y)+\sum_{x\ne y} P(x,z)P_z(T_y \le n),\ n\ge 0$;

b) $\rho_{xy} = P(x,y)+\sum_{z\ne y} P(x,z)\rho_{zy}$.

3. Consider a Markov chain having state space {0, 1, ..., 6} and transition matrix:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{2} & 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

a) Determine which states are transient and which states are recurrent.

b) Find ρ_{0y}, $y = 0, \ldots, 6$.

4. Consider a gambler's ruin chain on $\{0, 1, ..., d\}$. Find $P_x(T_0 < T_d)$, $0 < x < d$.

5. Consider a Markov chain on the non-negative integers having transition function P given by $P(x, x+1) = p$ and $P(x, 0) = 1 - p$, where $0 < p < 1$. Show that this chain has a unique stationary distribution π and find π.

6. Consider an irreducible Markov chain having finite state space $\mathcal{L}$, transition function P such that $P(x, x) = 0$, $x \in \mathcal{L}$ and stationary distribution π. Let p_x, $x \in \mathcal{L}$, be such that $0 < p_x < 1$ and let $Q(x, y)$, $x \in \mathcal{L}$ and $y \in \mathcal{L}$ be defined by

$$Q(x, x) = 1 - p_x$$

and

$$Q(x, y) = p_x P(x, y),\ y \neq x$$

Show that Q is the transition function of an irreducible Markov chain having state space $\mathcal{L}$ and stationary distribution π', defined by

$$\pi'(x) = \frac{p_x^{-1}\pi(x)}{\sum_{y \in \mathcal{L}} P_y^{-1}\pi(y)},\ x \in \mathcal{L}$$

Interpret the chain with transition function Q.