M. Sc. Mathematics Examination, 2022

(2nd Year, 2nd Semester)

Advanced Functional Analysis

Paper – DSE-05 (Unit 4.1)

Time : Two hours Full Marks : 40

Answer should be brief and clear.

Notations have their usual meaning.

Answer *Any Four* questions. 10×4=40

1. Let X be a topological vector space. Prove the following statements.

   i) If $A \subseteq X$ is compact and $B \subseteq X$ is closed, show that $A+B$ is closed in X. 4+3

   ii) Show that a subset $E \subseteq X$ is bounded if and only if every countable subset of E is bounded. 3

2. Let X and Y be two topological vector spaces over $\mathbb{R}$ and let $T : X \to Y$ be $\mathbb{R}$-linear map. Let $K \subseteq X$ be a closed $\mathbb{R}$-linear subspace of X. If $T(x) = 0,\ \forall\ x \in K$, show that there exists a unique $\mathbb{R}$-linear map $\tilde{T} : X/K \to Y$ such that $\tilde{T} \circ \pi = T$, where $\pi : X \to X/K$ is the quotient map defined by sending a point $x \in X$ to its left coset $x + K \in X/K$. Deduce that $\tilde{T}$ is continuous if and only if T is continuous, and also that $\tilde{T}$ is open if and only if T is open. 4+3+3

[Turn over

3. Let X and Y be two topological vector spaces over $\mathbb{R}$ with $\dim_{\mathbb{R}}(Y) < \infty$ Let $T : X \to Y$ be a surjective $\mathbb{R}$-linear map. Prove that T is an open map. If $\ker(T) : T^{-1}(0)$ is closed in X, show that T is continuous. 5+5

4. Let V be an open neighbourhood of 0 in a topological vector space X. Let X^* be the dual space of X. If $K := \{f \in X^* : |f(x)| \leq 1,\ \forall\ x \in V\}$, show that K is weak *-compact. 10

5. Let A be a convex absorbing subset of a $\mathbb{C}$-vector space X. For each $x \in X$, let

$$\mu_A(x) := \inf\{t \in \mathbb{R} : t > 0 \text{ and } x/t \in A\}.$$

Show that

i) $\mu_A(x+y) \leq \mu_A(x) + \mu_A(y),\ \forall\ x, y \in X$, 2

ii) $\mu_A(tx) = t\mu_A(x),\ \forall\ t > 0$ in $\mathbb{R}$, 3

iii) μ_A is a seminorm if A is balanced, 2

iv) if $B = \{x \in X : \mu_A(x) < 1\}$ and $C = \{x \in X : \mu_A(x) \leq 1\}$, then $B \subseteq A \subseteq C$ and $\mu_B = \mu_A = \mu_C$. 3

6. Let B be a closed unit ball in a normed linear space X. Let X^* be the dual space of X. For each $x^* \in X^*$, we define $\|x^*\| := \sup\{|<x, x^*>| : x \in B\}$. Prove the following:

i) The above definition of $\|x^*\|$ makes X^* a Banach space. 3

ii) If B^* is the closed unit ball in X^*, for each $x \in X$, $\|x\| = \sup\{|<x, x^*>| : x^* \in B^*\}$.

Consequently, the map $x^* \to <x, x^*>$ is a bounded linear functional on X^* of norm equal to $\|x\|$. 4

iii) B^* is weak *-compact. 3