M. Sc. Mathematics Examination, 2022

(1st Year, 2nd Semester)

Ordinary Differential Equations and Special Functions

Paper – Core-09

Time : Two Hours | Full Marks : 40

Use a separate answer script for each Part.

Symbols / Notations have their usual meanings.

Part – I (20 Marks)

Answer ***any four*** questions. 5×4=20

1. Find the Green's function of the boundary value problem $\frac{d^2y}{dx^2}+\alpha^2 y = f(x)$, $0 \le x \le a$ subject to the boundary conditions: $y(0)=0$ and $y'(a)=0$.

2. Let $G(x,\xi)$ be the Green's function of the second order ordinary differential equation $Ly = f(x)$ in $[0,a]$ subject to $h_1 y(0)+h_2 y'(0)=0$ with $h_1^2+h_2^2 \neq 0$ and $H_1 y(a)+H_2 y'(a)=0$ with $H_1^2+H_2^2 \neq 0$.

   If L is self-adjoint, then prove that $G(x,\xi)=G(\xi,x)$.

3. a) Show that the Legendre equation of order n is self adjoint.

   b) Let $L = a_0(x)\frac{d^2}{dx^2}+a_1(x)\frac{d}{dx}+a_2(x)$. Find a function $\mu = \mu(x)$ such that the operator μL is self-adjoint. 2+3

4. Let *f* be a solution of the ODE

$$\frac{d}{dt}\left[P(t)\frac{dx}{dt}\right] + Q(t)x = 0$$

on $a \le t \le b$. If f has an infinite number of zeros on $a \le t \le b$, then prove that $f(t) = 0$ for all $t \in [a,b]$.

5. Find the series solution of the ordinary differential equation

$$z^4 w'' + 2z^3 w' + w = 0 \text{ at } z = \infty .$$

6. Prove the following Rodrigue's formula for $P_n(z)$:

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}\left(z^2 - 1\right)^n .$$

Part – II (20 Marks)

Answer ***any two*** questions.

1. Show that the necessary conditions for the point $z = z_0$ to be a regular singular point of the differential equation $\frac{d^2 w}{dz^2} = p_1(z)\frac{dw}{dz} + p_2(z)w$ are that $p_1(z)$ and $p_2(z)$ should have poles of order not higher than 1 and 2 respectively at point $z = z_0$. 10

2. Show that Bessel's equation

$$z^2 \frac{d^2 w}{dz^2} + z\frac{dw}{dz} + \left(z^2 - n^2\right)w = 0$$

has contour integral solutions of the form

$$H_\lambda^{(1)}(z) = \frac{1}{\pi}\int_{L_1} e^{-iz\sin\zeta + i\lambda\zeta} d\zeta$$

$$H_\lambda^{(1)}(z) = -\frac{1}{\pi}\int_{L_2} e^{-iz\sin\zeta + i\lambda\zeta} d\zeta$$

for $\mathrm{Re}(z) > 0$ and L_1, L_2 are two contours to be defined by you. 10

3. Answer either (a) **Or** (b)

a) Find the general form of an ordinary differential equation of second order with only three singularities (all regular) at $z = a,\ b,\ c$ with $\alpha,\ \alpha',\ \beta,\ \beta',\ \gamma,\ \gamma'$ exponents respectively. 10

b) Find a contour integral solution of

$$z(1-z)\frac{d^2 w}{dz^2} + \{\gamma - (\alpha + \beta + 1)z\}\frac{dw}{dz} - \alpha\beta w = 0$$

in the form

$$w = \int_\Gamma (t - z)^{-\alpha} t^{\alpha - \gamma} (t - 1)^{\gamma - \beta - 1} dt$$

where Γ is a closed contour to be defined by you. 10