Ex/SC/MATH/PG/CORE/TH/06/2022

M. Sc. Mathematics Examination, 2022

(1st Year, 2nd Semester)

Paper – Core-06

Linear Algebra and Module Theory

Time : Two Hours Full Marks : 40

Unexplained Symbols / Notations have their usual meanings.

The figures in the margin indicate full marks.

Answer ***any five*** questions. 8×5=40

1. Let V be a finite dimensional vector space over a field F and T a linear operator on V.

   i) Prove that the $F[x]$-module V via T is a finitely generated torsion module.

   ii) Write the result obtained by applying the structure theorem (invariant factor form) on finitely generated module over a PID to the $F[x]$-module V.

   iii) Suppose K is a field and W is a cyclic $K[x]$ module generated by $w \in W$ that satisifies the relation $f(x)w = 0$ where $f(x) \in K[x]$ is amonic polynomial. Choose a basis of W as a K-vector space and find the matrix of the linear operator multiplication by x with respect to this basis.

   iv) Using (ii) and (iii) prove that there is a basis of V with respect to which the matrix of T is a block diagonal matrix consisting of companion matrices of the invariant factors of T. 2+1+3+2

2. Suppose A is a complex matrix whose characteristic polynomial is $(x-2)^3(x-5)^3$ and such that the eigenspace corresponding to the eigenvalue 2 is two-dimensional while the eigenspace corresponding to the eigenvalue 5 is one-dimensional.

   i) Find the invariant factors, elementary divisors, minimal polynomial of A.

   ii) Is A diagonalizable? Answer with reasons.

   iii) Find the rational canonical forms (invariant factor form and elementary divisor form) and the Jordan form of A.

   iv) If the characteristic polynomial of A is $(x-2)^4(x-5)^3$ with the other conditions remaining the same is it possible to find unique Jordan form of A? Answer with precise reasons. 2+1+3+2

3. i) Suppose C_n denotes the vecto space of n times continuously differentiable complex valued functions of real variable and V denotes the subspace of C_n consisting of all the solutions of the differential equation

$$\frac{d^n f}{dt^n}+a_{n-1}\frac{d^{n-1} f}{dt^{n-1}}+a_{n-2}\frac{d^{n-2} f}{dt^{n-2}}+....+a_1\frac{df}{dt}+a_0 f=0$$

where A is a real or complex matrix of order $n\times n$ and X is a column matrix of n unknowns.

   a) State the relation between the solutions of the system of equations $\frac{dX}{dt}=AX$ and the matrix e^{tA}.

   b) Using (a) find the solutions where $X=\begin{pmatrix}x_1\\x_2\end{pmatrix}$ and A is 2×2 real matrix such that its eigenvalues are 6 and 1 with corresponding eigenvectors $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}3\\-2\end{pmatrix}$ respectively.

ii) Find the geometric significance of the real 2×2 matrices A satisfying $A^2=I$. 2+3+3

ii) Give examples, with proper reasons, of two matrices which are not similar but their minimal polynomials are identical and their characteristic polynomials are also identical.

iii) Suppose A is a real matrix of order n satisfying $A^2 + A + I = 0$. Prove that n is even. How many (up to similarity) matrices are there when $n = 6$? Are the results true when A is a complex matrix?

2+2+(1+1+2)

6. Give an example, with proper reasons, of

a) a square matrix over a field or of a linear operator on a finite dimensional vector space over a field which is not diagonalizable;

b) a square matrix over field or of a linear on a finite dimensional vector space over a field which has no Jordan form;

c) three 3×3 matrices over $\mathbb{Q}$ no two of which are similar such that –2 is the only eigenvalue;

d) a 4×4 real matrix whose eigenvalues as real matrix are eigenvalues as complex matrix are ± 1 and $\pm i$.

e) a matrix which is triangularizable but not diagonalizable.

1+1+2+2+2

7. i) Let $\dfrac{dX}{dt} = AX$ be a system of differential equations

where $a_i \in \mathbb{C}\left(i = 0,1,2,.....,n-1\right)$.

a) Prove that V is invariant under the differentiation operator $D : C_n \to C_n$.

b) By applying the generalized version of Primary Decomposition Theorem prove that the dimension of V is n.

ii) Suppose T is a linear operator on a finite dimensional vector space V over a field F such such that rank T=1. Prove that either T is digonalizable or nilpotent but not both.

1+4+3

4. i) Let V be a finite dimensional vector space over a field F. Suppose T is a linear operator on V such that T commutes with every projection of V. Prove that

a) every subspace of V is invariant under T;

b) every nonzero vector of V is an eigenvector of T;

c) T has only one eigenvalue;

d) T is a scalar multiple of identity operator on V.

ii) Prove that the conclusions of (i) are also true if T commutes with every diagonalizable operator on V.

(2+1+2+2)+1

5. i) Suppose F is a field. Prove that two matrices $A, B \in M_n\left(F\right)$ are similar if and only if they have the same invariant factors.

ii) Give examples, with proper reasons, of two matrices which are not similar but their minimal polynomials are identical and their characteristic polynomials are also identical.

iii) Suppose A is a real matrix of order n satisfying $A^2 + A + I = 0$. Prove that n is even. How many (up to similarity) matrices are there when $n = 6$? Are the results true when A is a complex matrix?

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