M. Sc. Mathematics Examination, 2022

(1st Year, 2nd Semester)

Functional Analysis

Paper – Core-08

Time : Two Hours Full Marks : 40

Use a separate answer script for each Part.

Symbols / Notations have their usual meanings.

Part – I (20 Marks)

Answer ***any four*** questions. 4×5=20

1. a) Let X be a normed linear space. If the unit sphere $\{x \in X : \|x\| = 1\}$ is compact in X, then prove that X is finite dimensional. 3

   b) Let X be a Banach space under two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Prove that if there exists $\alpha > 0$ such that $\|x\|_1 \le \alpha \|x\|_2$ for all $x \in X$, then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. 2

2. a) Let X and Y be normed linear spaces. If Y is a Banach space, then prove that $B(X,Y)$ is a Banach space. 3

   b) Let X be a Banach space and $B(X)$ be the space of all bounded linear operators on X. For $A \in B(X)$, let

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Show that the series defining $\exp(A)$ is convergent in $B(X)$. 2

3. State and prove uniform boundedness principle. 5

4. Show that even a discontinuous linear map can have a closed graph. Does this contradict the closed graph theorem? 5

5. Let X_0 be a closed subspace of a normed linear space X. Prove that if $a \in X \setminus X_0$, then there exists $f \in X'$ such that $f(x)=0$ for all $x \in X_0$ and $f(a)=\operatorname{dist}(a, X_0)$, $\|f\|=1$.

   Hence deduce that if x is a nonzero element in a normed linear space X, then there exists a bounded linear functional f on X such that $f(x)=\|x\|$, $\|f\|=1$. 5

6. Suppose that the dual X' of a normed linear space X is separable. Show that X is separable. 5

Part – II (20 Marks)

Answer ***any four*** of the followings : 4×5=20

1. Considering the underlying linear space is real prove that a Banach space is a Hilbert space if and only if the parallelogram law hold. 5

2. i) Prove that if M and N are closed subspaces of a Hilbert space H such that $M \perp N$, then the subspace $M+N$ is closed.

   ii) Construct an orthonormal sequence in $L_2[0, 2\pi]$. 3+2

3. State and prove Bessel's inequality for a Hilbert space H. When the inequality will turn to equality? Justify. What is the name of the equality? 3+2

4. i) Prove that a continuous linear operator T on a Hilbert space H is self adjoint if and only if $\langle Tx, x\rangle$ is real, $\forall x \in H$. 3+2

   ii) Define normal operator on a Hilbert space. Give an example of a normal operator which is not selfadjoint.

5. i) If T is normal operator on a Hilbert space H, then prove that $Tx=\lambda x$ if and only if $T^* x=\bar{\lambda} x$ for $(\theta \neq) x \in H$ and λ is a scalar.

   ii) For a linear operator T on a Hilbert space H prove that $T^{**}=T$.

6. i) If T is a continuous linear operator on a Hilbert space H, then prove that T can be expressed uniquely in the form $T=A+iB$, where A and B are self-adjoint.

   ii) Why $l_p(p \neq 2)$ not a Hilbert space? Justify. 3+2