Ex/SC/MATH/PG/CORE/TH/07/2022

M. Sc. Mathematics Examination, 2022

(1st Year, 2nd Semester)

Complex Analysis and Partial Differential Equations

Paper – Core-07

Time : Two Hours | Full Marks : 40

Use a separate answer script for each Part.

Symbols / Notations have their usual meanings.

Part – I

(Complex Analysis)

(20 Marks)

Answer ***any four*** questions. 4×5=20

1. Let $w = f(z) = \left(z^2+1\right)^{1/2}$. (i) Show that $z = \pm i$ are branch points of $f(z)$. (ii) Show that a complete circuit around both the branch points produces no change in the branches of $f(z)$. 2+3

2. Let $f(z)$ be analytic inside and on a simple closed curve except for a finite number of poles inside C. Suppose that $f(z) \neq 0$ on C. If N and P are, respectively, the number of zeros and poles of $f(z)$ inside C, counting multiplicities. Prove that $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)} \cdot dz = N - P$ 5

3. State Rouché's Theorem on the number of zeros of two analytic functions *f* and *g* inside a simple closed curve *C*. If $a > e$, use Rouché's Theorem to prove that the equation $e^z = az^n$ has *n* roots inside the circle $|z| = 1$. 1+4=5

4. Suppose that *f* is analytic inside and on a simple close curve *C*. Prove that the maximum value of $|f(z)|$ occurs on *C*, unless *f* is a constant. 5

5. State Cauchy's Residue Theorem. Using this theorem, show that the trigonometric integral $\int_0^{2\pi} \frac{d\theta}{1+3\cos^2\theta} = \pi$. 1+4=5

6. i) Define Conformal mapping. Show that the function $f(z) = z^2$ is not conformal at $z = 0$. 1+1

   ii) Define Bilinear transformation. Find the Bilinear transformation which maps the points $z_1 = 2$, $z_2 = i$ and $z_3 = -2$ into the points $w_1 = 1$, $w_2 = i$ and $w_3 = -1$. 1+2

Part – II

(Partial Differential Equations)

(20 Marks)

Answer ***any two*** questions. 2×10=20

1. a) Without solving the initial boundary value problem

$$u_t = 4u_{xx}, \text{ for } 0 \le x \le 1,\ t \ge 0,$$

subject to

$$u(0,t) = u(2,t) = 0 \text{ for } t \ge 0$$

and $u(x,0) = 4x^3 - 5x^2 + x$ for $0 \le x \le 1$,

find the numerical bounds on $u(x,t)$.

   b) Derive the Cauchy integral solution of the initial value problem

$$\theta_t = \theta_{xx},\ -\infty < x < \infty,\ t \ge 0$$

subject to $\theta(x,0) = f(x)$. 6

2. Solve the boundary value problem

$$\nabla^2 u = 0,\ 0 \le r \le 10,\ 0 \le \theta \le \pi,$$

subject to

$$u(10,\theta) = \frac{400}{\pi}\left(\pi\theta - \theta^2\right),\ 0 \le \theta \le \pi$$

$$u(r,0) = 0 = u(r,\pi),\ 0 \le r \le 10.$$

3. a) Derive d'Alembert solution of Cauchy problem given by $u_{tt} - c^2 u_{xx} = 0$, $-\infty < x < \infty$, $t \ge 0$, together with $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $-\infty < x < \infty$. Explain why the solution is called a travelling wave solution. 6

   b) Give physical significance of the solution for $g(x) = 0$ and

$$f(x) = \begin{cases} b + x, & -b \le x \le 0 \\ b - x, & 0 \le x \le b \\ 0, & x \ge b \text{ or } x < -b \end{cases}$$ 4