Ex/UG/SC/CORE/PHY/TH/08/2022

BACHELOR OF SCIENCE EXAMINATION, 2022

(2nd Year, 2nd Semester)

PHSYICS

[MATHEMATICAL PHYSICS - III]

PAPER – CORE 08

Time : Two hours — Full Marks : 40

Use separate answer-script for each group.

Group – A

Answer *any* ***two*** questions: 10×2=20

1. i) Evaluate $\int_c \left(z^2+2z+2\right)dz$ where c is the arc of the cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ between the points (0, 0) and $(\pi a, 2a)$. 4

   ii) State the relevant Cauchy Integral formula and evaluate $I=\oint \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3}dz$, where the closed curve is (a) $|z|=\frac{\pi}{9}$, (b) $|z|=\frac{\pi}{6}$ and (c) $|z|=\frac{\pi}{3}$. 6

2. i) State and derive Laurent's series for a function $f(z)$ of complex variable. 6

   ii) Find out the residue of the function $f(z)$ at its pole $z=z_0$ of order m. 4

3. i) State and prove Jordan's Lemma. 4

[Turn over

ii) Using contour integration and Jordan's Lemma evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2} dx$. 6

Group – B

Answer *any* ***two*** questions: 10×2=20

1. a) Given two functions f_1 and f_2, their *convolution* g is defined by :

$$g(x) := f_1 * f_2 = \int_{-\infty}^{\infty} f_1(y) f_2(x-y) dy$$

Show that the Fourier transform $\mathcal{F}$ for the convolution satisfies the simple product rule:

$$\mathcal{F}[g(x)] = \mathcal{F}[f_1(x)] \cdot \mathcal{F}[f_2(x)] \times \text{constant}$$

where the "constant" factor depends on the normalization conventions chosen.

b) Suppose $\{\vec{X}_1, \vec{X}_2, \vec{X}_3, ...\vec{X}_k\}$ is a set of vectors in $\mathbb{R}^N$. Show that the set of vectors is linearly dependent if $k > N$.

c) Check whether $-i\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ are Hermitian operators in the space of functions (on the real line) which vanish at the points $x = \pm\infty$. (Why is this vanishing condition necessary?) 4+2+4

2. a) Let $F(s)$ denote the Laplace transform of a real-valued function $f(x)$. Then show that the formula

for the Laplace transform ($\mathcal{L}$) of $f''(x)$ can be written as: $\mathcal{L}[f''(x)] = s^2 F(s) - sf(0) - f'(0)$.

b) Define an anti-Hermitian matrix. If $\mathcal{H}$ is anti-Hermitian, then show that its eigenvalues are all purely imaginary or zero. Show further that the eigenvectors corresponding to distinct eigenvalues are orthogonal (to each other).

c) Find the Laplace transform of x^n. Establish that $\mathcal{L}[x^n]$ is holomorphic (analytic) for Re(s) > 0 and has a pole at $s = 0$. 3+4+3

3. a) Using techniques from the theory of Laplace transforms, solve the following system of linear differential equations.

$$y'(t) - x'(t) + y(t) + 2x(t) = e^t$$
$$y'(t) + x'(t) + x(t) = e^{2t}$$
$$x(0) = y(0) = 0$$

b) Given the two vectors: $\vec{V}_1$ and $\vec{V}_2 \in \mathbb{R}^2$, check whether these may be considered as a *basis* for the vector space $\mathbb{R}^2$.

$$\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{V}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

If not then, using the Gram-Schmidt process, construct from these an orthonormal set of basis vectors for $\mathbb{R}^2$. 5+5