commutative, has identity as well as inverse for functions f with $f(1) \neq 0$. Also prove that the Möbius function μ has Dirichlet inverse.

ii) Using the results of (i), establish the Möbius inversion formula. 3+2

6. i) Determine all the primitive Pythagorean triples i.e., the positive solutions of $x^2 + y^2 = z^2$.

ii) If $2^n + 1$ is an odd prime for some integer $n \geq 1$ then prove that n is a power of 2. 4+1

Ex/SC/MATH/UG/DSE/TH/03/A/2022

B. Sc. Mathematics (Hons.) Examination, 2022

(3rd Year, 2nd Semester)

Number Theory

Paper – DSE-3A

Time : Two hours Full Marks : 40

Use separate answer script for each Part.

Unexplained Symbols / Notations have their usual meaning.

The figures in the margin indicate full marks.

Part – I (marks : 20)

Answer any *Five* questions. **4×5=20**

Each question carries four marks.

1. Determine all solutions in the positive integer of the following Diophantine equation: $18x + 5y = 48$

2. Let a and b be integers, not both zero. Then show that a and b are relatively prime if and only if there exist integers x and y such that $1 = ax + by$.

3. Prove that there are infinite number of primes.

4. Solve the following problem:

   $x \equiv 5 \pmod{6}$, $x \equiv 4 \pmod{11}$, $x \equiv 3 \pmod{17}$.

5. From Fermat's theorem deduce that, for any integer $n \geq 10$, $11^{12n+6} + 1$ is divisible by 13.

6. i) For $n > 2$, prove that $\phi(n)$ is an even integer.

ii) Show that $2^{20} - 1$ is divisible by 41.

7. If the integer $n > 1$ has prime factorization $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$, then show that

$$\phi(n) = \left(p_1^{k_1} - p_1^{k_1-1}\right)\left(p_2^{k_2} - p_2^{k_2-1}\right)\ldots\left(p_r^{k_r} - p_r^{k_r-1}\right)$$

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_r}\right)$$

Hence calculate the value of $\phi(360)$.

8. If f is a multiplicative function and F is defined by $F(n) = \sum_{d|n} f(d)$, then show that F is also multiplicative.

Part – II

Answer any *Four* questions. **5×4=20**

1. Suppose m is a positive integer and b is any integer such that $(b, m) = 1$.

   i) Prove that b is a primitive root modulo m if and only if $b, b^2, \ldots\ldots, b^{\phi(m)}$ form a reduced residue system modulo m.

   ii) 2 is a primitive root modulo 13. Reducing to an equivalent linear congruence check whether the congruence $4x^9 \equiv 7(\text{mod}\,13)$ has solutions. 3+2

2. i) Suppose p is an odd prime. Write, in terms of a

primitive root mod p, the quadratic residues mod p and the quadratic nonresidues mod p. Illustrate the result taking $p = 19$ and knowing that 2 is a primitive root of 19.

   ii) State Euler's criterion on quadratic residues of an odd prime and illustrate with an example. 3+2

3. i) Solve the quadratic congruence $3x^2 + 9x + y \equiv 0(\text{mod}\,13)$ by reducing to the form $y^2 \equiv d(\text{mod}\,13)$.

   ii) Evaluate the Legendre $\left(\frac{-46}{17}\right)$ symbol and use this to check whether the congruence $x^2 \equiv -46(\text{mod}\,17)$ is solvable. 3+2

4. i) Suppose p is an odd prime. Prove that $\sum_{a=1}^{p-1}\left(\frac{a}{p}\right) = 0$ and hence prove that there are $\frac{p-1}{2}$ quadratic residues and $\frac{p-1}{2}$ quadratic non-residues mod$p$.

   ii) Using Quadratic reciprocity law check whether $x^2 \equiv 5(\text{mod}\,103)$ has solutions. 4+1

5. i) Prove that the Dirichlet product or Dirichlet convolution of arithmetic functions is associative,