Ex/SC/MATH/UG/CORE/TH/13/2022

B. Sc. Mathematics (Hons.) Examination, 2022

(3rd Year, 2nd Semester)

Group Theory - II

Paper – Core-13

Time : Two hours Full Marks : 40

All questions carry equal marks.

Answer any *four* questions : 4×10

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_n$ be the set of all integers modulo $n \in \mathbb{N}$.

1. a) Let G be a group. Define the group of *inner automorphisms I(G)* of G and the *center Z(G)* of G. Show that $I(G) \cong G/Z(G)$.

   b) Let G be a group and H be a subgroup of G such that $T(H) \subseteq H$ for all automorphisms T on G. Show that H is a normal subgroup of G.

2. a) Let G be a finite group of order n and p be a prime number such that p^m divides n for some $m \in \mathbb{N}$. Show that G has a subgroup of order p^m.

   b) Define *p-Sylow subgroups* of G. Let H be a normal subgroup of G such that p does not divide the index of H in G. Prove that H contains all p-Sylow subgroups of G.

3. a) Let $p < q$ be two distinct prime numbers such that p does not divide $q-1$. Show that any group of order pq is cyclic.

   b) Let $n \in \mathbb{N}$ be a product of distinct prime numbers. Show that any group of order n is either cyclic or non-abelian.

4. a) Define a *simple group*. Let G be a finite group and $a \in G$ such that $\left|\left\{xax^{-1} \middle| x \in G\right\}\right| = 2$. Then show that G is not simple.

   b) Prove that every group of order 108 has a normal subgroup of order 9 or 27.

5. a) Let G be a finite group such that every Sylow subgroup is normal in G. Prove that G is the internal direct product of its Sylow subgroups.

   b) Let G be group of order p^2, where p is a prime number. Prove that G is isomorphic to $(\mathbb{Z}_{p^2}, +)$ or $(\mathbb{Z}_p \times \mathbb{Z}_p, +)$.

6. a) Let G be a finite abelian group of order $n \in \mathbb{N}$. Let $m \in \mathbb{N}$ be a divisor of n. Show that G has a subgroup of order n.

   b) Define elementary divisors and invariant factors of a finite abelian group. Find the elementary divisors and the invariant factors of the group $(\mathbb{Z}_{50} \times \mathbb{Z}_{20} \times \mathbb{Z}_8, +)$.