B. Sc. Mathematics (Hons.) Examination, 2022

(3rd Year, 2nd Semester)

Dynamical Systems

Paper – DSE-4E

Time : Two hours Full Marks : 40

The figures in the margin indicate full marks.

Symbols / Notations have their usual meanings.

Part – I (Marks: 20)

Answer any *TWO* questions.

1. The equation of motion of a nonuniform oscillator is given by $\dot{\theta} = w - a \sin\theta$.

   Explain the dynamics for different values of a and w. Draw the vector field on a line. Draw the same on a circle. Find and classify all fixed points. Determine the point at which saddle-node bifurcation occurs. Calculate the period of oscillation (T), when it overturns. Draw the graph of T as function of a. Hence explain the phenomenon of ghost and bottleneck. 10

2. Prove that

   a) the Cantor set K is uncountable. 2

   b) all elements in K can be represented using only digits 0 and 2 in base 3. 3

   c) K has measure zero. 3

   d) the box counting dimension of K is log(2)/log(3). 2

[Turn over

3. a) Consider the logistic map $x_{n+1} = rx_n(1-x_n)$ for $0 \le x_n \le 1$ and $0 \le r \le 4$. Find all the fixed points and determine their stability. 3

b) Show that the logistic map has stable 2-cycle orbit for $r > 3$. Find the value of r at which this 2-cycle orbit becomes unstable. 4

c) Draw a rough sketch of the bifurcation diagram for the above map for $2{\cdot}4 < r < 3{\cdot}6$ indicating the bifurcation points. 2

d) What types of bifurcation occur at critical points? 1

Part – II (Marks: 20)

Answer any *FOUR* questions.

1. Show that there exists a pitchfork bifurcation in the system $\dot{x} = \xi x + x^3,\ x \in \mathbf{R}$, when $\xi \in \mathbf{R}$ is smoothly varied. What is the critical value of bifurcation? 4+1

2. Show that the following system of differential equations has at least one periodic orbit:

$$\dot{x} = y,$$
$$\dot{y} = -x + y\left(1 - x^2 - 2y^2\right).$$ 5

3. a) What do you mean by α and ω limit sets of a system of differential equations?

b) Define a limit cycle. When a limit cycle is said to be stable?

c) Consider the system

$$\dot{r} = r(1-r)(2-r)(3-r),\ \dot{\theta} = 1.$$

Assume two points $P = \left(\frac{1}{2}, 0\right)$ and $Q = (4,0)$ on the plane. Find the α and ω limit sets of these points. 2+1+2

4. Convert the following second order differential equation

$$\ddot{x} + 2a\dot{x} + x + x^3 = 0,\ a \in \mathbf{R}$$

into a system of first order differential equations. Determine the nature of the equilibrium points of this system for different values of a. 1+4

5. The matrix A of the linear system $\dot{x} = Ax$, where $x \in \mathbf{R}^2$, $\det A \neq 0$, has a pair of eigenvalues. Discuss the stability of the origin (0, 0) and draw all possible phase diagrams of the trajectories. 5