B. Sc. Mathematics Examination, 2022

(3rd Year, 2nd Semester, Supplementary)

Dynamical Systems

Paper – DSE-4E

Time : Two hours Full Marks : 40

Use a separate Answer scripts for each part.

The figures in the margin indicate full marks.

(Symbols/Notations have their usual meanigns)

PART I (20 Marks)

Answer **any two** questions.

1. (a) The equation of a damped pendulum is given by (6)

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma.$$

Approximate the equation for overdamped limit. Give an example of physical situation in which the above appromimation is valid. Find the position of equlibrium points and discuss their stability considering the forces acting on the system.

(b) For which real values of a does the equation $\dot{\theta} = \sin a\theta$ give a well-defined vector field on a circle? Find and classify the fixed points when it is defined. Sketch also the phase portrait on the circle. (4)

2. (a) Prove that the set of real numbers between 0 and 1 is uncountable. (2)

(b) Show that the set of natural numbers $\{1, 2, 3, ...\}$ has measure zero. (2)

(c) Consider a unit square. Delete a symmetric cross from the middle, leaving 4 corner squares with side length $1/3$. Repeat this step infinitely to form the limiting set 'Sierpinski Carpet'.

i. Show that the 'Sierpinski Carpet' has zero area. (2)

ii. Define box-dimension of a fractal. (2)

iii. Find the box-dimension of the 'Sierpinski Carpet' . (2)

3. (a) What do you mean by sensitive dependence on initial condition? (1)

(b) Use the above concept to define Lyapunov exponent (λ). (2)

(c) Give significance of the negative value of λ. (1)

(d) Prove that the map $f(x) = 2x(mod\ 1)$ defined on the real line $\mathbb{R}$

i. has an orbit which is not asymptotically periodic. (4)

ii. has an orbit with positive Lyapunov exponent. (2)

Part - II (20 Marks)

*Answer any **FOUR** questions*

1. *The matrix A of the linear system $\dot{x} = Ax$, where $x \in \mathbb{R}^2$, $\det A \neq 0$, has a pair of real and unequal eigenvalues. Discuss the stability of the origin $(0,0)$ and draw all possible phase diagrams of the trajectories.* [5]

2. *Investigate the behaviour of the system*

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= (\alpha - 1)x - \alpha y\end{aligned}$$

for different values of the parameter $\alpha \in \mathbb{R}$. [5]

3. *State and prove Bendixson criterion.* [5]

4. *Convert the second order differential equation*

$$\frac{d^2 Z}{dt^2} = l(1 - Z^2)\frac{dZ}{dt} - Z$$

to an equivalent planar system of first order equations. Determine the nature and stability of the trivial equilibrium with respect to the parameter $l \in \mathbb{R}$. [5]

5. *Find the nature of bifurcation of the one-dimensional system*

$$\frac{dW}{dt} = gW - W^2$$

with respect to the parameter $g \in \mathbb{R}$. Draw a rough sketch of the bifurcation and indicate the bifurcation point. [5]