Ex/SC/MATH/UG/CORE/TH/10/2022

Bachelor of Science Examination, 2022

(2nd Year, 2nd Semester)

Mathematics (Hons.)

Paper – Core - 10

Ring Theory and Linear Algebra - II

Time : Two hours Full Marks : 40

Use separate answer script for each Part.

Symbols and notations have their usual meanings.

The figures in the margin indicate full marks.

Part – I (20 Marks)

Answer ***any four*** questions. 4×5=20

1. i) Let F be a field and $f(x) \in F[x]$ such that $f(x)$ divides $g(x)$ for every nonconstant polynomial $g(x) \in F[x]$. Show that $f(x)$ is a constant polynomial.

   ii) If a monic polynomial with integer coefficients has a rational root then show that this root must be an integer.

   iii) Show that $\sqrt[5]{2}$ is irrational by using Rational Root Theorem. 1+2+2

2. i) Show that every Euclidean Domain (ED) is a Principal Ideal Domain (PID).

   ii) Show that 3 is a prime element in $\mathbb{Z}[i]$ but 5 is not a prime element in $\mathbb{Z}[i]$. 3+2

[Turn over

3. i) Let R be a Unique Factorization Domain (UFD) and consider a linear polynomial $f(x)$ in $R[x]$. Is $f(x)$ irreducible in $R[x]$? Justify.

 ii) Let R be a PID and P be a prime ideal of R. Show that R/P is PID. Does this result hold if P is an ideal of R? Justify. 2+(2+1)

4. Let R be a commutative ring with identity. Show that $\frac{R[x]}{<x>} \cong R$. Hence conclude that $<x>$ is a prime ideal of $\mathbb{Z}[x]$ where as $<x>$ is a maximal ideal of $\mathbb{Q}[x]$. 3+2

5. i) Let $R[x]$ be the ring of polynomials and $I[x]=\{f(x)\in R[x]: f(0)=0=f(1)\}$. Is $I[x]$ a maximal ideal of $R[x]$? Justify.

 ii) Show that the polynomial $x^3+8ix-6x-1+3i$ is irreducible in $(\mathbb{Z}[i])[x]$. 2+3

6. Let $F_1=\mathbb{Q}[x]/<x^2-2>$ and $F_2=\mathbb{Q}[x]/<x^2-3>$. Show that both F_1 and F_2 are fields but F_1 is not isomorphic to F_2. 2+3

Part – II (20 Marks)

Answer ***any five*** questions. 4×5=20

1. Let V be a finite dimensional vector space over a field F. Let $\mathfrak{B}=\{v_1, v_2, \ldots\ldots, v_n\}$ be a basis for V and

7. a) Let $A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\in M_2(\mathbb{C})$ (θ is real). Without solving the characteristic polynomial how can you say that the eigenvalues of A are of unit modulus?

 b) Suppose A is a real matrix of order $m\times n$ with $m>n$, $b\in\mathbb{R}^m$ such that the over determined system of linear equations $AX=b$ is inconsistent and rank$A=n$. Let W be the column space of A and b_0 be the orthogonal projection of b onto W ($\mathbb{R}^n, \mathbb{R}^m$ are assumed to be with the standard inner product). Find the least square approximate solution of $AX=b$. 1+3

4. a) Find all real 2×2 matrices such that $A^2 = I$ and describe geometrically the way they operate on $\mathbb{R}^2$ by left multiplication.

b) Let A be a real 2×2 matrix such that 3 is an eigenvalue with $\begin{pmatrix}1\\0\end{pmatrix}$ as a corresponding eigenvector and 2 is another eigenvalue with $\begin{pmatrix}1\\-1\end{pmatrix}$ as a corresponding eigenvector. Find A. State the relevant results that have been used to determine A. 2+2

5. a) Find the possible Jordan Forms of a linear operator T on a finite dimensional vector space whose characteristic polynomial is $(x-2)^3(x-5)^5$ and minimal polynomial is $(x-2)^2(x-5)^3$.

b) Give examples, with proper reasons, of two matrices which are not similar but their minimal polynomials are identical and their characteristic polynomials are also identical. 3+1

6. a) Find an inner product $<,>$ on $\mathbb{R}^2$ such that $<(1,0),(0,1)>=3$.

b) Let A be a real or complex square matrix of order n. Assuming that $F^n\left(F=\mathbb{R} \text{ or } \mathbb{C}\right)$ is with the standard inner product prove that rows of A are orthonormal iff its columns are orthonormal. 1+3

$\mathfrak{B}^* = \{f_1, f_2, \ldots\ldots, f_n\}$ be the corresponding dual basis for V^*.

a) Prove that for any $v \in V$, $v = \sum_{i=1}^{n} f_i(v) v_i$.

b) Prove that if V is an inner product space and $\mathfrak{B}$ is an orthonormal basis then for all $i = 1, 2, \ldots\ldots, n, f_i(v) = < v, v_i >$ for all $v \in V$. 2+2

2. a) Suppose V and W are two vector spaces over the same field F. Let $T: V \to W$ be a linear transformation. Define T^t, the transpose of T. If V and W are finite dimensional then prove that $\text{rank} T = \text{rank}\, T^t$.

b) Define the canonical mapping from a vector space V over a field F to its double dual V^{**}. 3+1

3. Let T be a linear operator on a vector space V over a field F.

a) Prove that every eigenspace of T is T-invariant.

b) Prove that every one dimensional T-invariant subspace is an eigenspace of T.

c) What is the geometric multiplicity of λ in an elementary Jordan matrix of order n corresponding to λ? 1+2+1