Bachelor of Science Examination, 2022

(2nd Year, 2nd Semester)

Mathematics (Hons.)

Paper – Core - 08

Riemann Integration and Series of Functions

Time : Two hours Full Marks : 40

Use separate answer script for each Part.

Symbols and notaions have their usual meanings.

Part – I (20 Marks)

Answer ***any two*** questions. 2×10=20

All questions carry equal marks.

1. i) If f is continuous on $[a,b]$ then show that f is Riemann integrable on $[a,b]$. Is the converse true? Justify your answer.

   ii) Examine the convergence of the integral $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$ 4+2+4

2. i) If f is Riemann integrable on $[a,b]$ then show that $|f|$ is Riemann integrable on $[a,b]$. Is the converse true? Justify your answer.

   ii) Show that $\Gamma\left(n+\frac{1}{2}\right)=\frac{\Gamma(2n+1)\sqrt{\pi}}{2^{2n}\Gamma(n+1)}$ 4+2+4

3. i) State and prove Fundamental Theorem of Integral Calculus.

[Turn over

ii) Evaluate $\int_0^1 x^{\alpha+k-1}(t-x)^{\beta+k-1}\,dx$

and find the value when $\alpha=\beta=\frac{1}{2}$ 6+4

Part – II (20 Marks)

Answer ***any four*** questions. 4×5=20

All questions carry equal marks.

1. i) Define pointwise and uniform convergence of sequence of functions. 1+1=2

ii) Show that the sequence of functions $\{f_n\}$ which is defined by $f_n(x)=\dfrac{nx}{1+n^2x^2},\ 0\le x\le 1$, is not uniformly convergent. 3

2. Prove that the series of functions

$$\frac{x}{x+1}+\frac{x}{(x+1)(2x+1)}+\frac{x}{(2x+1)(3x+1)}+\cdots,\ x\ge 0$$

is convergent on $[0,\infty)$ but the convergence is not uniform on $[0,\infty)$. 5

3. Prove that

i) the series $\sum_{n=1}^{\infty}\dfrac{1}{n^3+n^4x^2}$ is uniformly convergent for all real x. 2.5

ii) The series $1-\dfrac{e^{-2x}}{2^2-1}+\dfrac{e^{-4x}}{4^2-1}-\dfrac{e^{-6x}}{6^2-1}+\cdots$ converges uniformly for all $x\ge 0$. 2.5

4. Let $R(>0)$ be the radius of convergence of the power series $\sum_{n=0}^{\infty}a_nz^n$. Then the radii of convergence of the power series $a_0x+\dfrac{a_1}{2}x^2+\dfrac{a_2}{3}x^3+\cdots$ obtained by term-by-term integrating and the power series $a_1+2a_2x+3a_3x^2+\cdots$ obtained by term-by-term differentiation are R. 5

5. Expand in Fourier series of $f(x)=x+x^2$ on $-\pi<x<\pi$ and deduce that $\dfrac{\pi^2}{6}=1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots$ 5

6. Find the Fourier series of the periodic function f with period 2π, where $f(x)=\begin{cases}0, & -\pi<x<a,\\ 1, & a\le x\le b,\\ 0, & b<x<\pi\end{cases}$

Find the sum of the series for $x=4\pi+a$ and deduce that $\sum_{n=1}^{\infty}\dfrac{\sin n(b-a)}{n}=\dfrac{\pi-b+a}{2}$.