6. State Gauss's divergence theorem, and use it to evaluate the surface integral $\iint_S \vec{F} \cdot \vec{n} dS$, where $S \subset \mathbb{R}^3$ is the surface of the cube bounded by the planes $x = 0,\ x = 1$, $y = 0,\ y = 1,\ z = 0$ and $z = 1$ in $\mathbb{R}^3$, $\vec{n}$ is the unit outward drawn normal vector to the surface S, and $\vec{F}$ is the vector field on S defined by $\vec{F}(x, y, z) = \left(4xz, -y^2, yz\right)$. 1+4

Ex/SC/MATH/UG/CORE/TH/09/2022

B. Sc. Mathematics (Hons.) Examination, 2022

(2nd Year, 2nd Semester)

Paper – Core-9

Multivariate Calculus

Time : Two hours — Full Marks : 40

Use separate answer script for each Part.

Symbols and notaions have their usual meanings.

Part – I (Marks : 20)

Answer ***any four*** questions. 4×5=20

1. a) Prove that the function

$$f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & \text{when } x \neq y \\ 0, & \text{when } x = y \end{cases}$$

is not continuous at (0, 0). 2

b) Find the equations of the tangent plane and normal line to $x^2 y = 4ze^{x+y} - 35$ at (3, –3, 2). 3

2. a) If the directional derivative of the function $f(x, y) = y^2 e^{2x}$ at (2, –1) along the unit vector $\vec{b} = \alpha\hat{i} + \beta\hat{j}$ is zero, then find the value of $|\alpha + \beta|$. 2

b) Check whether the following function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & \text{when } x^2 + y^2 \neq 0 \\ 0, & \text{when } x^2 + y^2 = 0 \end{cases}$$

is differentiable at the point (0, 0) or not. 3

[Turn over

3. a) If $g(s,t) = f(s^2 - t^2,\, t^2 - s^2)$ and f is differentiable, then show that g satisfies the equation $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$. 2

b) If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, then prove that

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$ 3

4. a) Prove that in a neighbourhood of the point $\left(\sqrt{e},\, \frac{1}{\sqrt{e}}\right)$ the curve defined by the equation $xy = \log\frac{x}{y}$ is the graph of a function $y = f(x)$. 2

b) Let $F(x,y) = \sin x \cos y$. Prove that

$\frac{3}{4} = \frac{\pi}{3}\cos\frac{\pi\theta}{3}\cos\frac{\pi\theta}{6} - \frac{\pi}{6}\sin\frac{\pi\theta}{3}\sin\frac{\pi\theta}{6}$ for some $\theta \in (0,\, 1)$. 3

5. Find and classify the stationary points of the following function $f(x,y) = x^4 + y^4 - 4xy + 1$. 5

6. Find the maximum of $(x_1.x_2...x_n)^2$ under the restriction $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$.

Use the result to derive the following inequality, valid for positive real numbers $a_1, a_2, ..., a_n$:

$$(a_1.a_2...a_n)^{1/n} \le \frac{a_1 + a_2 + \cdots + a_n}{n}$$ 5

Part – II (Marks : 20)

Answer ***any four*** questions. 5×4=20

1. Let $S := \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 : y = 0 \text{ and } x \le 0\} \subset \mathbb{R}^2$. Show that the vector field f on S defined by

$$f(x,y) := \left(\frac{-y}{x^2 + y^2},\, \frac{x}{x^2 + y^2}\right),\ \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

is a gradient of a scalar field on S. 5

2. Find the volume of the part of the sphere of radius $a > 0$ that lies in the first octant in the Euclidean space $\mathbb{R}^3$. 5

3. Let C be the path $\mathbb{R}^2$ from (0, 0) to (2, 1) defined by the equation $x^4 - 6xy^3 - 4y^2 = 0$. Evaluate the integral $\int_C \left[(10x^4 - 2xy^3)dx - 3x^2y^2dy\right]$. 5

4. Use Green's theorem to evaluate the line integral $\int_C (5 - xy - y^2)dx - (2xy - x^2)dy$, where C is the square with vertices (0, 0), (1, 0), (1, 1) and (0, 1), traversed in the counterclockwise direction. 5

5. Let $\vec{F}$ be a vector field defined on an open subset $\mathcal{U}$ of $\mathbb{R}^3$. If F is a conservative vector field, show that $\nabla \times \vec{F} = 0$. Give an example to show that the converse may not hold if $\mathcal{U}$ is not simply connected. 2+3