Ex/SC/MATH/UG/CORE/TH/03/2022

B. Sc. Mathematics (Hons.) Examination, 2022

(1st Year, 2nd Semester)

Real Analysis

Paper – Core-3

Time : Two hours Full Marks : 40

Use separate answer script for each Part.

Part – I (Marks : 24)

Answer **Q. No. 1** and ***any four*** from the rest.

1. Show that [0, 1] and [0, 1] × [0, 1] have the same cardinality. 4

2. Let F be an Archimedean ordered field. Show that if F satisfies Cantor's nested interval property then F satisfies Dedekind Completeness property. 5

3. Show that the set $\left\{m+n\sqrt{2} : m, n \in Z\right\}$ is dense in $\mathbb{R}$. 5

4. Prove that in $\mathbb{R}$, finite intersection of open sets is open. Give examples to show that arbitrary intersection of open sets may not be open. 3+2

5. Let S be a non-empty subset of $\mathbb{R}$. If S is a clopen set then show that $S = \mathbb{R}$. Give an example of a clopen set S in $\mathbb{Q}$ such that $S \neq \varnothing$, $S \neq \mathbb{Q}$. 4+1

6. Show that every closed and bounded set in $\mathbb{R}$ is compact. Does the same hold in $\mathbb{Q}$? If not, then give an example. 4+1

[Turn over

7. Justify from the definition of a compact set whether the following sets are compact or not: (i) The set of natural numbers $\mathbb{N}$ (ii) (0, 1) 2+3

Part – II (Marks : 16)

Answer ***any four*** questions. 4×4=16

All questions carry equal marks.

1. i) Give an example of a sequence of rational numbers that converges to an irrational number.

   ii) Give an example of a sequence of irrational numbers that converges to a rational number.

   iii) Give an example of divergent sequences $\{u_n\}$ and $\{v_n\}$ such that the sequence $\{u_n + v_n\}$ is convergent.

   iv) Give an example of divergent $\{u_n\}$ and $\{v_n\}$ such that the sequence $\{u_n v_n\}$ is convergent.

   1+1+1+1=4

2. i) Show that the sequence $\{u_n\}$ defined by $0 < u_1 < u_2$ and $u_{n+2} = \frac{1}{2}(u_n + u_{n+1})$, is convergent. 3

   ii) Find the limit. 1

3. Using Cauchy's general principle of convergence

   i) Prove that the sequence $\left\{\frac{n}{n+1}\right\}$ is convergent.

   ii) Prove that the sequence $\{u_n\}$, where $u_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$ is not convergent.

4. i) Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p > 0$. 4

5. i) Show that the series

$$1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots,$$

   where $\alpha, \beta, \gamma, x > 0$ is convergent if $0 < x < 1$ and divergent if $x > 1$. 2

   ii) Also, when $x = 1$, show that the series is convergent if $\gamma > \alpha + \beta$ and divergent if $\gamma \le \alpha + \beta$. 2

6. Show that the series

$$\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \cdots, \; a > 0$$

   is

   i) absolutely convergent if $p > 1$. 2

   ii) conditionally convergent if $0 < p \le 1$. 2